MAT

Serie A

Conferencias, seminarios y trabajos de Matemática

ISSN:1515-4904

17

An Introduction to Black-Scholes Modeling and Numerical Methods in Derivatives Pricing

Carlos Vázquez

Departamento de Matemática, Rosario, Argentina

Julio 2010

UNIVERSIDAD AUSTRAL

FACULTAD DE CIENCIAS EMPRESARIALES

MAT

Serie A: CONFERENCIAS, SEMINARIOS Y TRABAJOS DE MATEMATICA

ISSN: 1515-4904
Propiedad de ACES

MAT es una publicación del Departamento de Matemática de la Facultad de Ciencias Empresariales de la Universidad Austral (FCE-UA) cuyo objetivo es contribuir a la difusión de conocimientos y resultados matemáticos. Se compone de dos series:

- Serie A: CONFERENCIAS, SEMINARIOS Y TRABAJOS DE MATEMATICA.
- Serie B: CURSOS Y SEMINARIOS PARA EDUCACION MATEMATICA.

La Serie A contiene trabajos originales de investigación y/o recapitulación que presenten una exposición interesante y actualizada de algunos aspectos de la Matemática, además de cursos, conferencias, seminarios y congresos realizados en el Departamento de Matemática. El Director, los miembros del Comité Editorial y Científico y/o los árbitros que ellos designen serán los encargados de dictaminar sobre los merecimientos de los artículos que se publiquen. La Serie B se compone de cursos especialmente diseñados para profesores de Matemática de cada uno de los niveles de educación: Primaria, Secundaria, Terciaria y Universitaria.
Los trabajos publicados en MAT-Serie A están indexados en las bases de datos del Mathematical Reviews, MathSciNet y Zentralblatt für Mathematik. Todos los trabajos, en formato pdf, pueden ser bajados de:
http://web.austral.edu.ar/cienciasEmpresariales-investigacion-mat-A.asp

ISSN 1515-4904

MAT

SERIE A: CONFERENCIAS, SEMINARIOS Y TRABAJOS DE MATEMATICA

No. 17

AN INTRODUCTION TO BLACK-SCHOLES MODELING AND NUMERICAL METHODS IN DERIVATIVES PRICING

CARLOS VÁZQUEZ

Departament of Mathematics
University of A Coruña
Faculty of Informatics, Campus Elviña s/n, 15071-A Coruña, Spain
E-mail: carlosv@udc.es

Rosario - July 2010

Abstract. The technique of dynamic hedging, combined with the application of Ito calculus and the absence of arbitrage hypothesis, provides a methodology for the valuation of financial derivatives by models of partial differential equations of Black-Scholes type. This document is intended to summarize in a simple way the concepts and techniques used in this methodology up to get the prices of the more traditional products. The document is divided into two main parts: the models and numerical methods. Prior to both, the lognormal stochastic model for the underlying asset is briefly recalled. In the models section, first the dynamic hedging technique is described to deduce the European vanilla options pricing models and the popular Black-Scholes formula. The methodology is extended to the case of American options, Asian options and options on various assets. The modeling part concludes with the statement of bonds pricing models as an example of interest rate derivatives. Finite differences and finite elements numerical methods are first described for European and American options. Then, some indications are given about its application to Asian options and bond models. Finally, some basic ideas on the technique of Monte Carlo simulation for European options are presented.

Resumen. La técnica de la cobertura dinámica, combinada con el cálculo de Ito y la hipótesis de ausencia de arbitraje, proporciona una metodología para la valoración de derivados financieros mediante modelos de ecuaciones en derivadas parciales de tipo Black-Scholes. En este documento se pretende resumir de modo sencillo los conceptos y técnicas empleadas en esta metodología hasta llegar a obtener los precios de los productos derivados más clásicos. El documento se divide en dos partes principales: los modelos y los métodos numéricos. Previamente a ambas, se recuerda brevemente el modelo estocástico lognormal para el subyacente. En el apartado de modelos, primero se describe la técnica de cobertura dinámica para deducir los modelos de opciones vainilla europeas y la popular fórmula de Black-Scholes. La metodología se extiende al caso de las opciones americanas, asiáticas y sobre varios activos. Los parte de modelos termina con la valoración de bonos como ejemplo de derivados de tipos de interés. Los métodos numéricos de diferencias y elementos finitos se describen primero para opciones europeas y americanas. A continuación, se dan indicaciones sobre la aplicación a los modelos de opciones asiáticas y bonos. Finalmente, se dan algunas ideas básicas sobre la técnica de simulación de Monte Carlo para opciones europeas.

Keywords: options, interest rate derivatives, pricing, stochastic models, dynamic hedging, Ito calculus, Black-Scholes models, numerical methods

Palabras claves: opciones financieras, derivados de tipos de interés, valoración, modelos estocásticos, cobertura dinámica, cálculo de Ito, modelos de Black-Scholes, métodos numéricos

AMS Mathematics Subject Classification (MSC) 2010: 91G20; 91G30; 91G60; 91G80; 65M06; 65M60; 65K15; 35K51; 35K20; 35K85.

These Notes are the lecture notes containing a large summary of the course given by Prof. Carlos Vázquez at the Department of Mathematics of FCE-UA, Rosario, on September 2006. They are intended to contain the basics of Black Scholes type mathematical models for derivatives pricing.

The manuscript has been received and accepted on March 2010.

AN INTRODUCTION TO BLACK-SCHOLES MODELING AND NUMERICAL METHODS IN DERIVATIVES PRICING

Carlos Vázquez [1]

1. Introduction

Financial derivatives are financial instruments whose value depends on another financial instrument (assets, two currencies exchange rates, interest rates or bonds, for example). This second financial instrument is termed as underlying. Obviously, also there exist derivatives with nonfinancial underlying. Amongst all possible financial derivatives, in this document we will restrict to options and bonds, where the bonds are considered an example of interest rate derivatives. In the following paragraphs we indicate the main financial concepts related to both of them.

A vanilla option is a contract that gives to the owner the right to buy or sell the underlying at a prescribed price within a specific period of time. The owner has the right but not the obligation to carry out the transaction associated to the contract. Call options give the right to buy while put options give the right to sell. The prescribed price is known as strike or exercise price. The expiry or maturity date is when the option contract expires. European style options are those ones where the right (and therefore the option) can be only exercised at expiry date, while American style options can also be exercised at any time before maturity date. Between them, Bermudan options can be exercised at a finite number of prescribed dates before maturity. The payoff of the option is the value of the option at expiry date. Besides vanilla options, according to the characteristics of the contract, more complex options are traded on the markets. Thus, if the payoff depends on certain average asset value we obtain the class of Asian option. If the option is activated or canceled depending on that the asset reaches certain price (barrier) we enter to the class of barrier options, where the barrier hitting can be monitored discretely or continuously. The main utilities of options are related to speculation and hedging.

An ordinary bond is a (paid up front) contract that gives to the bondholder some quantities of money in the future. These payments by the bond issuer can take place only at the maturity date or may take place at specific dates during the life of the bond. In the first case (zero coupon bond), the bondholder just receives the face value at maturity while in the second case he/she receives coupon payments at certain intermediate dates and the last coupon plus the face value at maturity. As in the case of options, there exist different classes of bonds according to some specifications in the contracts (with fixed or floating coupons, callable, puttable or convertible bonds, for example). The main objective of the bond issuers is to raise capital for their investments.

Although the price of derivatives clearly depends on the balance between supply and demand as in any market, derivatives pricing theory aims to obtain a fair price under certain hypotheses on the market. Among different existing pricing methodologies, we present in this document the one based on dynamic hedging techniques, combined with

[1]Department of Mathematics, University of A Coruña, Faculty of Informatics, Campus Elviña s/n, 15071-A Coruña, Spain

Ito calculus and arbitrage free assumption, to pose suitable Black-Scholes pricing models for the different derivatives here treated. In most cases, the determination of the prices in the practice requires the use of appropriate numerical methods to solve the models.

These Notes are organized as follows: in Section 2 the most classical stochastic model for the asset prices dynamics is introduced; Section 3 contains the basic ideas of dynamic hedging and Black-Scholes methodology which is applied to obtain pricing models for European, American and Asian options, as well as in the case of several underlying assets. Also, dynamic hedging methodology is applied to obtain Black-Scholes bond pricing models, as an example of interest rate derivatives. In Section 4 different numerical methods are applied to the pricing models: finite differences, finite elements and Monte Carlo simulation.

The author of this document knows the existence of a lot of excellent books concerning the here addressed topics, some of them are included in the references. The number of suitable existing textbooks and the expected size of this document have prevented the author from being exhaustive in the list of references. Having this in view, the main objective in these lecture notes is to summarize some of the basic ideas and possibly serve as an introduction in the subject to the interested readers.

It is a pleasure to acknowledge Prof. Domingo Tarzia for encouraging the author to write this document as well as for his kind hospitality during the unforgettable stay at Rosario.

2. A stochastic model for the underlying asset price

Any derivatives pricing theory needs first modeling the evolution of the underlying (asset, index, interest rates, commodities, for example). It is usually assumed that these prices move randomly according to efficient market hypotheses, which can be formulated in different ways and lead to two main statements:

- The present asset price only depends on the most recent past price
- The market answers instantaneously to any new arriving information

The two previous assumptions justify that the changes in asset prices follow a Markov process. By taking this into account, in 1973 Black and Scholes proposed the following stochastic differential equation for the underlying prices evolution in [6]:

$$dS_t = \mu S_t \, dt \; + \; \sigma S_t \, dX_t \,, \tag{2.1}$$

where μ and σ denote the drift and volatility of the asset prices and X_t represents a Wiener process (Brownian motion). The stochastic process S_t is a geometric Brownian motion and can be also understood as the continuous limit of the corresponding discrete version. The statement of the previous equation requires the consideration of an appropriate framework given by a filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$, where $\mathcal{F}_t$ represents the filtration and P denotes the probability measure (real world probability measure). As S_t verifies (2.1) then it is a particular example of Ito process and Ito's lemma can be applied to the function $F(S) = \log S$, so that the stochastic equation (2.1) can be solved to obtain [15]:

$$S_t = S_0 \, \exp\left((\mu - \frac{\sigma^2}{2}) \, t \; + \; \sigma \, X_t \right) . \tag{2.2}$$

The previous expression allows to simulate the trajectories of the lognormal process S_t in terms of the data S_0, μ and σ, just by simulating the normal distribution of $dX_t \in \mathcal{N}(0, \sqrt{dt})$. More precisely, after introducing the time discretization of interval $[0, T]$ with step $dt = T/N$ and $t_i = idt, \; i = 0, \ldots, N$, we proceed in two steps

- Obtain $X_{t_i} = X_{t_{i-1}} + dX_{t_i}$ with $dX_{t_i} \in \mathcal{N}(0, \sqrt{dt})$ for $i = 1, \ldots, N$

- Obtain $S_{t_i} = S_{t_{i-1}} \ \exp\left((\mu - \frac{\sigma^2}{2})\, dt \, + \, \sigma\, X_{t_i} \right)$ for $i = 1, \ldots, N$

In Fig. 1 a simulated trajectory corresponding to the lognormal random walk satisfying equation (2.1) is obtained with the above algorithm. For practical purposes, parameter

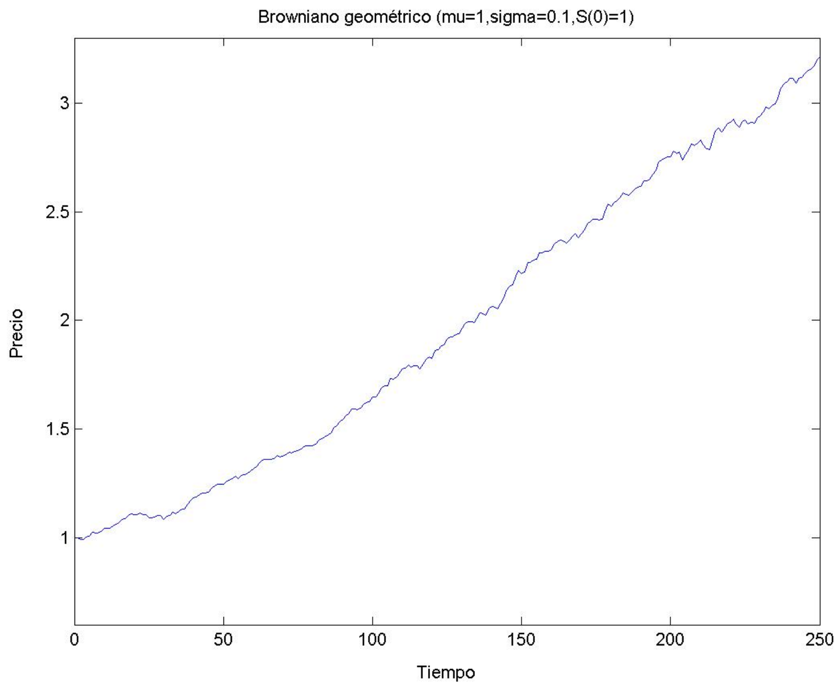

Figure 1. A lognormal random walk trajectory simulation corresponding to $S_0 = 1$, $\mu = 1$ and $\sigma = 0.3$.

estimation tools are required to get the drift and volatility from market data. Notice that for the riskless case $\sigma = 0$ exponential returns are obtained.

The trajectories of the lognormal process S_t are continuous. In order to consider the possibility of jumps in the evolution of asset prices, the jump-diffusion models incorporate additional Lévi processes in equation (2.1) (for example, see [8] and the references therein).

3. Dynamic hedging and Black-Scholes models

3.1. **General assumptions and Black-Scholes equation.** In Section 3, by means of the classical dynamic hedging methodology, the statement of Black-Scholes models for European and American vanilla options is posed. For this purpose, we first recall the usual hypotheses in Black-Scholes framework [23]:

(H1) The asset (or underlying) price, S, follows a lognormal random walk.
(H2) The risk free interest rate, r, and the volatility, σ, are assumed to be deterministic functions of time (although for simplicity we will consider them as constants).
(H3) No arbitrage assumption: all risk free portfolios have the same return which is equal to the risk free interest rate.
(H4) Continuous market: a non necessarily integer number of assets can be bought or sold in continuous time.
(H5) Markets without transaction costs are assumed (frictionless markets).
(H6) Initially we assume that assets do not pay any dividends, but later on both cases of a continuous dividend yield and discrete in time dividend payments are considered.

Despite the agreement to consider the previous hypotheses in Black-Scholes modeling, several limitations are recognized world wide. Concerning hypothesis (H1), the lognormal random walk dynamics implies that the price trajectories are continuous, the presence of jumps should require the consideration of more general jump-diffusion models in which Lévi processes are included. In practice, the volatilities and risk free interest rates are not known in advance and more sophisticated models consider themselves as following also stochastic dynamics. Concerning arbitrage free assumption, traders are always trying to take advantage of arbitrage opportunities and there exist studies that measure the presence of arbitrage in real markets. Nevertheless, these opportunities occur during very small periods of time. Concerning hypothesis (H4), it is clear that markets do not operate in a continuous timetable and you cannot by a half or a third of an asset. With respect to frictionless hypothesis, in practice the transaction costs would make the continuous dynamic hedging strategy very expensive. Nevertheless, some Black-Scholes models include transaction costs. Although in a particular firm it is difficult to know in advance the future policy of dividend payments, we will make some simplifying assumptions to deal with known continuous dividend yields or dividend payments at prescribed dates. The consideration of stochastic payments at stochastic dates would give rise to far more complex models.
Having in view the limitations of Black-Scholes hypotheses pointed out in the previous paragraph, we proceed to the statement of the model.

As assumed in hypothesis (H1), the asset price, S, is governed by the stochastic differential equation (2.1). Next, assuming that the option price, V, is a function depending on asset price S and time t, we can apply Ito's lemma to the function $V(t,S)$ (see Mikosh [15], for example), so that

$$dV = \left(\mu S \frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + \frac{\partial V}{\partial t} \right) \, dt \; + \; \sigma S \frac{\partial V}{\partial S} \, dX \, . \tag{3.1}$$

Now, by combining processes S and V we define the portfolio π as follows:

$$\pi = V - \Delta S \, .$$

This portfolio is intended to be risk free in the time interval $[t, t + dt]$, and this can be obtained in practice by selling Δ assets and buying 1 call option at time t, with Δ appropriately chosen. The change of the portfolio value, $\pi = V - \Delta S$, in the interval $[t, t + dt]$, is due to the change in the option price and in the underlying asset. By using (3.1) the portfolio price variation can be obtained as

$$d\pi = \left(\mu S \frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + \frac{\partial V}{\partial t} - \mu \Delta S \right) \, dt + \; \sigma S \left(\frac{\partial V}{\partial S} - \Delta \right) \, dX \, . \tag{3.2}$$

Therefore, the portfolio results to be risk free for the particular choice $\Delta = \dfrac{\partial V}{\partial S}$. Moreover, in this case we have

$$d\pi = \left(\frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + \frac{\partial V}{\partial t} \right) \, dt \, . \tag{3.3}$$

Next, by using the no arbitrage assumption (H3), we have that

$$d\pi = r\pi dt \tag{3.4}$$

and, using the previous choice of Δ, we get

$$d\pi = r\left(V - S\frac{\partial V}{\partial S}\right) dt\,. \tag{3.5}$$

Then, by identifying equations (3.3) and (3.5), we obtain

$$\left(\frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2} + \frac{\partial V}{\partial t}\right) = r\left(V - S\frac{\partial V}{\partial S}\right)$$

or equivalently, the following classical Black-Scholes equation for the option price:

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0\,. \tag{3.6}$$

Notice that Equation (3.6) is a parabolic second order partial differential equation and it is valid for any derivative security which is paid upfront, particularly for a call or a put vanilla option. In order to obtain an associated well posed problem, an additional final condition has to be imposed at expiry date. This final condition follows from the expression of the particular derivative payoff. Thus, for a general payoff function, G, this final condition is

$$V(T,S) \;=\; G(S)\,. \tag{3.7}$$

The case of a vanilla call option corresponds to the choice

$$V(T,S) \;=\; \max(S-E,0)\,, \tag{3.8}$$

while for the put option we consider

$$V(T,S) \;=\; \max(E-S,0)\,, \tag{3.9}$$

where E denotes the strike price which is a given constant price.
Notice that the asset drift, μ, does not appear in the equation so that the option price results to be independent of this drift. From the financial point of view, as the derivative is created to hedge the risk of the underlying asset, its value depends only on the uncertainty of the asset price, which is governed by the volatility σ.

3.2. **No arbitrage and Black-Scholes equation.** The statement of Black-Scholes equation (3.6) mainly relies on the identification of the risk free portfolio return in equations (3.3) and (3.4) due to the no arbitrage assumption. It is important to notice that in case that the strict inequalities

$$r\pi dt < \left(\frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2} + \frac{\partial V}{\partial t}\right)\, dt \tag{3.10}$$

or

$$r\pi dt > \left(\frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2} + \frac{\partial V}{\partial t}\right)\, dt \tag{3.11}$$

hold, then arbitrage opportunities arise. More precisely, if condition (3.10) holds we can borrow money from the bank and buy the portfolio π at time t, then wait until time $t+dt$ to sell the portfolio, give back the money to the bank and pay the associated interests, thus obtaining for sure a benefit without risk. Conversely, if condition (3.11) holds, we would sell the riskless portfolio and put the money at the bank since time t until $t+dt$. At time $t+dt$ we recover the principal plus the associated revenue and buy back the portfolio. Again, this strategy allows us to obtain a riskless benefit for sure (arbitrage opportunity). These previous arguments prove that the identity (instead of the strict inequatilies (3.10) or (3.11)) follows from the absence of arbitrage assumption.

3.3. **Black-Scholes for European vanilla options.** In this section, the PDE models for European vanilla call and put options are posed. Moreover, the well known Black-Scholes formulas for both options are obtained.
For this purpose, let $D = [0,T] \times [0,\infty)$ be the time-asset domain and C denote the call option price. Then, the pricing problem consists of finding the call price function C such that:

$$\frac{\partial C}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} + rS\frac{\partial C}{\partial S} - rC \; = \; 0 \quad \text{in} \;\; D\,, \tag{3.12}$$

$$C(T,S) = max(S-E,0), \qquad S>0\,, \tag{3.13}$$

where r, σ, E and T denote the risk free interest rate, the asset volatility, the exercise price and the expiry date, respectively. The solution of problem (3.12)-(3.13) is given by the following well known Black-Scholes formula for call options:

$$C(t,S) = S\;N(d_1) \;-\; E\;exp(-r(T-t))N(d_2)\,, \tag{3.14}$$

with

$$\begin{aligned} d_1 &= \frac{log(S/E) + (r+\sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}\,, \\ d_2 &= \frac{log(S/E) + (r-\sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}\,, \\ N(x) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} exp(-y^2/2)\,dy\,. \end{aligned}$$

In what follows we give the main ideas and some computations to deduce formula (3.14). First, we consider the new variables:

$$x = \log(S/E),\; -\infty < x < +\infty; \;\; \tau = \frac{\sigma^2}{2}(T-t),\; 0 \le \tau \le \frac{\sigma^2 T}{2}, \tag{3.15}$$

and the new unknown

$$v(\tau,x) \;=\; C(t(\tau,x), S(,\tau,x))/E\,, \tag{3.16}$$

so that the function v verifies the following equation with constant coefficients:

$$\frac{\partial v}{\partial \tau} = \frac{\partial^2 v}{\partial x^2} + (k-1)\frac{\partial v}{\partial x} - k\,v, \tag{3.17}$$

jointly with the initial condition

$$v(0,x) = \max\left(e^x - 1, 0\right). \tag{3.18}$$

Notice that in the previous change of variables we reversed time direction by replacing the *physical* time, t, by a kind of scaled time to maturity, τ. Moreover, the new variable x represents logarithmic scaled prices. In the new unknown, v, the option price has been normalized by the strike and it is written in the new variables.
Next, we consider a new change of unknown to remove the first order terms in equation (3.17). More precisely, we define

$$u(\tau,x) = e^{(\alpha x + \beta \tau)}\, v(\tau,x), \tag{3.19}$$

to pose an initial problem associated to a heat equation. Therefore, we choose

$$\alpha = \frac{k-1}{2}, \quad \beta = \frac{(k+1)^2}{4}$$

or, equivalently, the new unknown

$$u(\tau,x)=e^{\frac{x}{2}(k-1)+\frac{\tau}{4}(k+1)^2}\,v(\tau,x)\,, \tag{3.20}$$

that verifies the following heat equation

$$\frac{\partial u}{\partial \tau}-\frac{\partial^2 u}{\partial x^2}=0, \tag{3.21}$$

jointly with the initial condition

$$u(0,x)=\max\left(e^{\frac{x}{2}(k+1)}-e^{\frac{x}{2}(k-1)},0\right)\,. \tag{3.22}$$

Next, using the classical expression for the solution of the heat equation we get

$$u(\tau,x)=\frac{1}{2\sqrt{\pi\tau}}\int_{-\infty}^{+\infty}u(0,s)\,e^{-\frac{1}{4\tau}(x-s)^2}\,ds. \tag{3.23}$$

Then, by introducing the change of variable, $x'=(s-x)/\sqrt{2\tau}$, after some easy computations we have

$$\begin{aligned} u(\tau,x) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}u(0,x+x'\sqrt{2\tau})\,e^{-\frac{x'^2}{2}}\,dx' = \\ &= \frac{1}{\sqrt{2\pi}}\int_{-x/\sqrt{2\tau}}^{+\infty} e^{\frac{(k+1)}{2}(x+x'\sqrt{2\tau})-\frac{x'^2}{2}}\,dx' - \frac{1}{\sqrt{2\pi}}\int_{-x/\sqrt{2\tau}}^{+\infty} e^{\frac{(k-1)}{2}(x+x'\sqrt{2\tau})-\frac{x'^2}{2}}\,dx' = \\ &= I_1-I_2\,. \end{aligned}$$

In order to compute I_1, we use the relation $\delta^2-(\gamma-\delta)^2=-\gamma^2+2\gamma\delta$ for the exponents $\gamma=x'$ and $\delta=(k+1)\sqrt{2\tau}/2$. Therefore, we obtain

$$\begin{aligned} I_1 &= \frac{e^{\frac{x}{2}(k+1)}}{\sqrt{2\pi}}\int_{-x/\sqrt{2\tau}}^{+\infty} e^{\frac{(k+1)^2}{4}\tau-\frac{1}{2}(x'-(k+1)\sqrt{2\tau}/2)^2}\,dx' = \\ &= \frac{e^{\frac{x}{2}(k+1)+\frac{(k+1)^2}{4}\tau}}{\sqrt{2\pi}}\int_{-x/\sqrt{2\tau}}^{+\infty} e^{-\frac{1}{2}(x'-(k+1)\sqrt{2\tau}/2)^2}\,dx' = \\ &= \frac{e^{\frac{x}{2}(k+1)+\frac{(k+1)^2}{4}\tau}}{\sqrt{2\pi}}\int_{-d_1}^{+\infty} e^{-\frac{\rho^2}{2}}\,d\rho\,, \end{aligned}$$

with $\rho=x'-(k+1)\sqrt{2\tau}/2$ and $d_1=x/\sqrt{2\tau}+(k+1)\sqrt{2\tau}/2$. Finally, we get

$$I_1=\frac{e^{\frac{x}{2}(k+1)+\frac{(k+1)^2}{4}\tau}}{\sqrt{2\pi}}\int_{-\infty}^{d_1} e^{-\frac{\rho^2}{2}}\,d\rho=e^{\frac{x}{2}(k+1)+\frac{(k+1)^2}{4}\tau}\,N(d_1). \tag{3.24}$$

Analogously, for the integral I_2 we get

$$I_2=e^{\frac{x}{2}(k-1)+\frac{(k-1)^2}{4}\tau}\,N(d_2), \tag{3.25}$$

with $d_2=x/\sqrt{2\tau}+(k-1)\sqrt{2\tau}/2$.

So, by using the previous expressions for I_1 and I_2, we deduce

$$\begin{aligned} v(\tau,x) &= e^{-\frac{x}{2}(k-1)-\frac{\tau}{4}(k+1)^2}\,(I_1-I_2) = \\ &= e^{-\frac{x}{2}(k-1))-\frac{\tau}{4}(k+1)^2}\left(e^{\frac{x}{2}(k+1)+\frac{(k+1)^2}{4}\tau}\,N(d_1)-e^{\frac{x}{2}(k-1)+\frac{(k+1)^2}{4}\tau}\,N(d_2)\right) = \\ &= e^x\,N(d_1)\;-\;N(d_2). \end{aligned}$$

Finally, coming back to the initial financial variables, we obtain

$$\begin{aligned} C(t,S) &= E\,v\left(\frac{\sigma^2}{2}(T-t),\log(S/E)\right) = \\ &= E\frac{S}{E}\,N(d_1)\;-\;E\,N(d_2) = \\ &= S\,N(d_1)\;-\;E\,e^{-r(T-t)}\,N(d_2)\,, \end{aligned}$$

with

$$d_1 = x/\sqrt{2\tau}+(k+1)\sqrt{2\tau}/2 = \frac{log(S/E)+(r+\sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}\,,$$
$$d_2 = x/\sqrt{2\tau}+(k-1)\sqrt{2\tau}/2 = \frac{log(S/E)+(r-\sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}\,.$$

Analogously to the European call option, the put option price, P, verifies the final value problem

$$\frac{\partial P}{\partial t}+\frac{1}{2}\sigma^2S^2\frac{\partial^2 P}{\partial S^2}+rS\frac{\partial P}{\partial S}-rP \quad = \quad 0 \quad \text{in} \quad D\,, \tag{3.26}$$

$$P(T,S)=max(E-S,0), \qquad S>0\,, \tag{3.27}$$

the solution of which is given by the Black-Scholes formula

$$P(t,S)=E\;exp(-r(T-t))N(-d_2)\;-\;S\;N(-d_1)\,. \tag{3.28}$$

At this point, we notice that by means of no arbitrage arguments, a useful formula relating the prices of European vanilla call and put options can be obtained (call-put parity formulas). More precisely, we consider the following portfolio:

$$\pi(t,S)=S+P(t,S)-C(t,S), \tag{3.29}$$

the value of which at expiry date is given by $\pi(T,S)=E$, so that it does not depend on S. Therefore, the discounted portfolio value at time $t<T$ is

$$\pi(t,S)=E\;exp(-r(T-t)), \tag{3.30}$$

so that by identifying expression in (3.29) and (3.30) the call-parity formula

$$P(t,S)=C(t,S)-S+E\;exp(-r(T-t)) \tag{3.31}$$

allows to obtain the European put option price in terms of the call one.

3.4. **Sensitivities of option prices: the Greeks.** Also very important for their interest in hedging strategies are the *Greeks* or sensitivities of an option, which measure the variation of the option price with respect to different variables or parameters. Several factors affect the value of the option and the wrong estimation of anyone of them represents a source of risk for the derivatives trader. On the other hand, the value of these parameters changes with time and affects the option price. For both reasons it is important to obtain not only the option price but also their sensitivities with respect to the involved factors. The following Greeks associated to the variables in Black-Scholes formula are identified:

- *Delta* (Δ): measures the sensitivity of the option price with respect to the underlying price (S). In the case of European call an put options they can be exactly computed by taking derivatives in the respective Black-Scholes formula and their values are:
$$\Delta_C = \frac{\partial C}{\partial S} = N(d_1),\ \Delta_P = \frac{\partial P}{\partial S} = N(d_1) - 1,$$
where the subindex C and P refer to the call and put cases, respectively.
- *Gamma* (Γ): measures the sensitivity of the option price with respect to the delta (Δ), i.e.:
$$\Gamma = \frac{\partial \Delta}{\partial S} = \frac{\partial^2 V}{\partial S^2}$$
and can be exactly computed in the case of European vanilla options.
- *Theta* (Θ): measures the sensitivity of the option price with respect to time:
$$\Theta = \frac{\partial V}{\partial t}$$
and can be exactly computed for European vanilla options.

Concerning the Greeks with respect to parameters, we identify *Vega*, that measures the sensitivity with respect to volatility, and *Rho* which represents the sensitivity with respect to the risk free interest rate. In both cases exact formulas can be obtained for European vanilla options.
In forthcoming cases where there is no analytical formula for the option price and numerical methods are required to approximate its value, then also numerical methods have to be designed to approximate the Greeks. As it is detailed later in the section of numerical methods, when using finite differences to approximate the option price, we obtain a finite set of prices at the finite differences mesh points. The same occurs in the case of the finite element approximation methods. Therefore, appropriate numerical derivation formulas can be used to obtain the derivatives with respect to variables, thus computing *Delta*, *Gamma* and *Theta*. In order to obtain *Vega* in the finite difference setting, we can use the following approximation at a generic finite differences mesh point (t, S):
$$\frac{\partial V}{\partial \sigma}(t, S) \approx \frac{V_{\sigma+d\sigma}(t, S) - V_{\sigma}(t, S)}{d\sigma},$$
where $d\sigma$ denotes a small enough increment of σ. This approach requires the numerical solution of the option price problem for two volatility values, σ and $\sigma + d\sigma$. The same technique can be developed to approximate the value of *Rho*.

3.5. **Some examples of European options prices.** In Table 1 an example of European call options is presented, where the option data and the computed values for the option price and the different greeks are shown. The analogous results for the put option are presented in Table 2. Moreover in Figure 2 a typical graph of the call option price in terms of time and asset price is displayed. For particular dates, the different curves representing the call option prices with respect to asset price are shown in Figure 3. The analogous representations for the put option are displayed in Figures 4 and 5.

Option data		**Prices**	
Expiry date:	01-02-10	**Option:**	2,3277
Exercise price:	01-07-10	**Delta:**	0,8515
Strike price:	15,00	**Gamma:**	0,0988
Interest rate:	3 %	**Theta:**	-1,2568
Volatility:	25 %	**Vega:**	2,1710
Pricing date:	12-03-10	**Rho:**	3,6944
Spot price:	17,00		

TABLE 1. Example of European call option data, prices and Greeks.

Option data		**Prices**	
Expiry date:	01-02-10	**Option:**	0,1915
Exercise price:	01-07-10	**Delta:**	-0,1485
Strike price:	15,00	**Gamma:**	0,0988
Interest rate:	3 %	**Theta:**	-0,8109
Volatility:	25 %	**Vega:**	2,1710
Pricing date:	12-03-10	**Rho:**	-0,8529
Spot price:	17,00		

TABLE 2. Example of European put option data prices and Greeks.

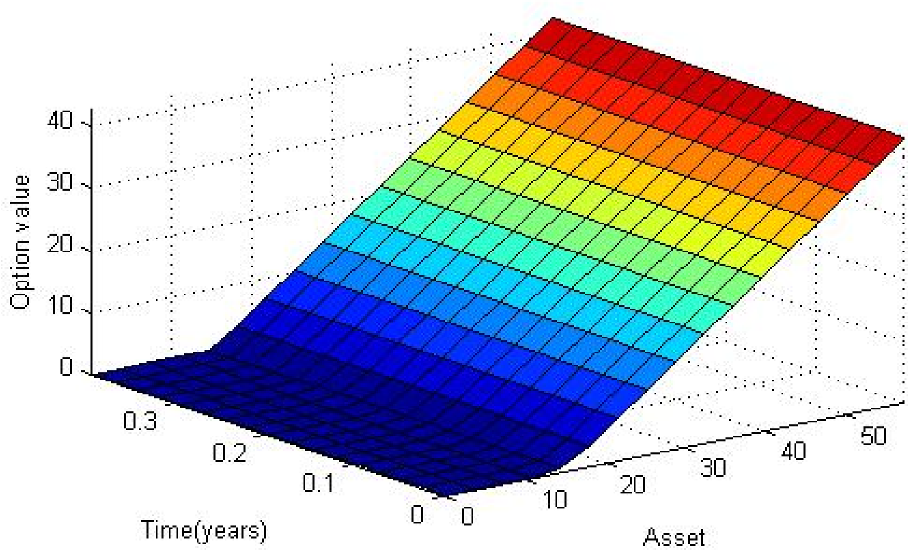

FIGURE 2. European call option price as a function of time and asset.

3.6. **Black-Scholes equations for the case of assets with dividend payments.** Some firms payback to their shareholders certain amounts of money at certain dates. These quantities are known as dividends. Actually, in most cases the amounts and dates of dividends are not known for sure in advance. An accurate approach to dividend payment modeling would lead us to very complex problems by considering stochastic payment dates

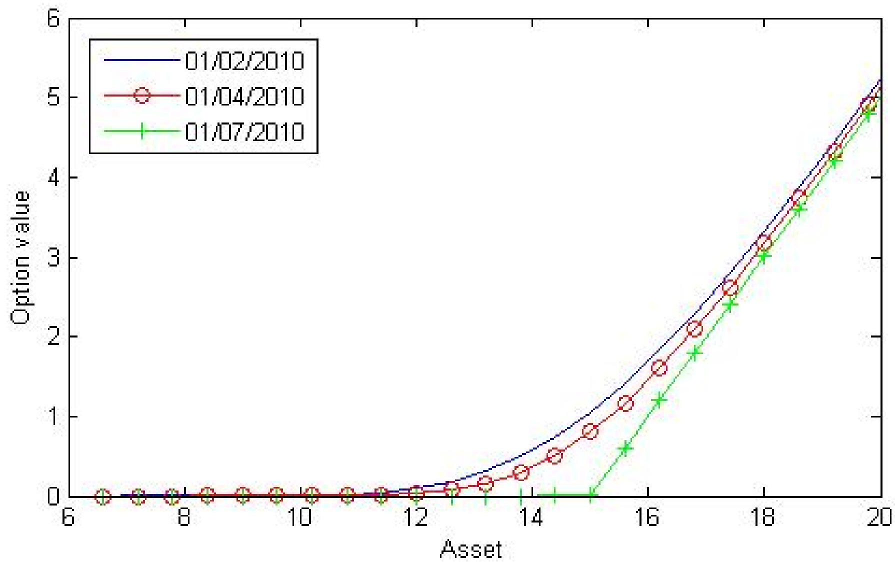

FIGURE 3. European call option price as a function of asset for different times.

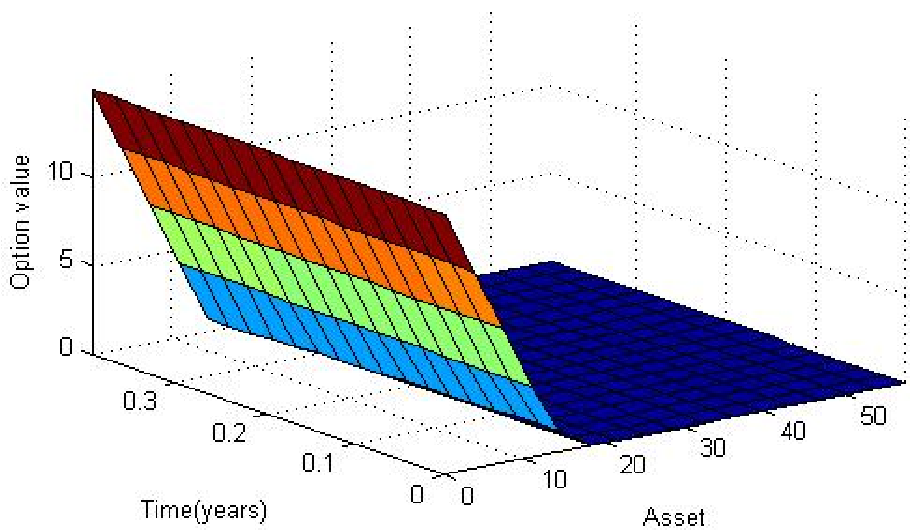

FIGURE 4. European put option price as a function of time and asset.

and quantities. Nevertheless in this notes we will take into account the dividend policy in two simple cases.

Some firms have a regular policy of dividend payments in terms of quantity and time. This fact motivates one of the cases we consider: known dividend payments at prescribed dates. On the other hand, in the case of options on stock indexes it can be appropriate to associate a continuous dividend yield to the index in order to account for the set of particular dividend payments associated to each asset included the index. This argument motivates the consideration of the second case: the continuous dividend yield. Taking into account the mathematical complexity of both cases, we first consider the dividend yield

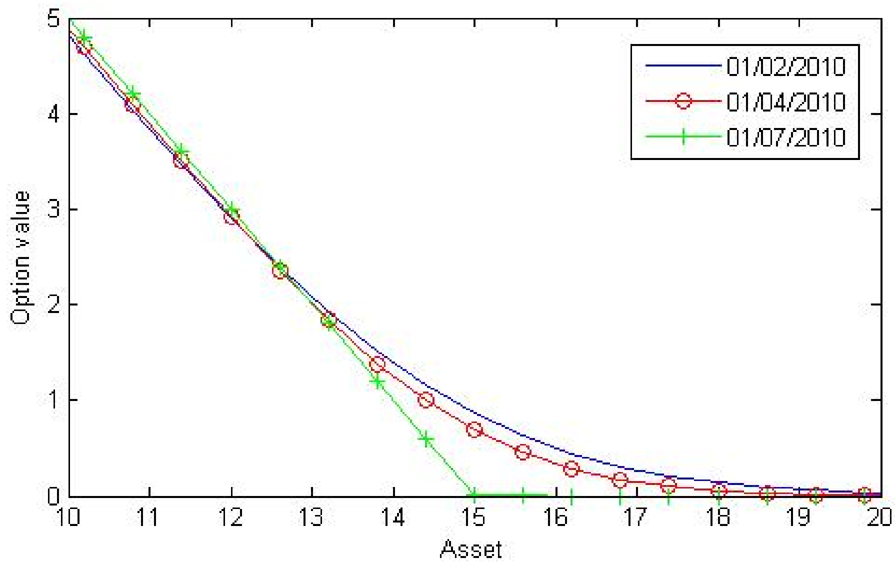

FIGURE 5. European put option price as a function of asset for different times.

scenario, for which there exists a Black-Scholes formula, and then the discrete dividend payment case, for which numerical methods are required.

3.6.1. *Black-Scholes for the dividend yield case.* If the underlying of the option has an associated continuous dividend yield payment to the its holder, then we assume that in the interval $[t, t+dt]$ the shareholder receives the quantity $D_0 S dt$, where D_0 denotes the constant given fraction of the asset value associated to the dividend yield. Therefore, the variation of S in $[t, t+dt]$ is governed by the following stochastic differential equation for S:

$$dS_t \;=\; (\mu - D_0) S_t \, dt \;+\; \sigma S_t \, dX_t.$$

Notice that for the case $D_0 = 0$ we recover the already analyzed case without dividends. A more general expression for the dividend yield could be $D_0(t, S)$ instead of $D_0 S$. The consideration of the latter one will allow us to obtain analytical Black-Scholes formulas, although the more general one can be handled by numerical methods in order to approximate the solution of the resulting partial differential equation problem.

Thus, when applying again the dynamic hedging methodology, we first build the portfolio

$$\pi = V - \Delta S,$$

the variation of which in $[t, t+dt]$ is given by

$$d\pi = dV - \Delta dS - D_0 \Delta S dt.$$

Therefore, after the appropriate choice of Δ, we get the following Black-Scholes equation:

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r - D_0) S \frac{\partial V}{\partial S} - rV = 0.$$

Therefore, the price of a put option, P, verifies on $D = [0, T] \times [0, \infty)$ the final value problem:

$$\begin{aligned}
\frac{\partial P}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 P}{\partial S^2} + (r - D_0) S \frac{\partial P}{\partial S} - rP \;&=\; 0 \quad \text{in } D, \\
P(T, S) = max(E - S, 0), \qquad & S > 0,
\end{aligned}$$

the exact solution of which can be obtained by analogous tools to the ones in the case without dividends. This methodology leads to the following Black-Scholes formula:

$$P(t,S) = E\,exp(-r(T-t))\,N(-d_2)\;-\;S\,exp(-D_0(T-t))\,N(-d_1)\,, \tag{3.32}$$

where

$$\begin{aligned} d_1 &= \frac{log(S/E)+(r-D_0+\sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}\,,\\ d_2 &= \frac{log(S/E)+(r-D_0-\sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}\,. \end{aligned}$$

In Table 3 the data and results for an example of put option on an asset with associated dividend yield are shown and can be compared with those ones for the same example without dividend yield in Table 2.

Option data		**Prices**	
Expiry date:	01-02-10	**Option:**	0,2033
Exercise price:	01-07-10	**Delta:**	-0,1556
Strike price:	15,00	**Gamma:**	0,1018
Interest rate:	3 %	**Theta:**	-0,8732
Volatility:	25 %	**Vega:**	2,2358
Dividend yield:	1,5 %	**Rho:**	-0,8661
Pricing date:	12-03-10		
Spot price:	17,00		

TABLE 3. Example of European put option data, prices and Greeks in the presence of dividend yield.

Analogously to the put option case, in the presence of dividend yield the call price is the solution of the following problem in the domain $D = [0,T]\times[0,\infty)$:

$$\begin{aligned} \frac{\partial C}{\partial t}+\frac{1}{2}\sigma^2 S^2\frac{\partial^2 C}{\partial S^2}+(r-D_0)S\frac{\partial C}{\partial S}-rC &= 0 \quad \text{in } D,\\ C(T,S)=max(S-E,0), &\qquad S>0. \end{aligned}$$

Therefore, the call price is given by

$$C(t,S) = S\,exp(-D_0(T-t))\,N(d_1)\;-\;E\,exp(-r(T-t))\,N(d_2)\,,$$

where

$$\begin{aligned} d_1 &= \frac{log(S/E)+(r-D_0+\sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}\,,\\ d_2 &= \frac{log(S/E)+(r-D_0-\sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}\,. \end{aligned}$$

Table 4 shows an example of call option with dividend yield.

Option data		**Prices**	
Expiry date:	01-02-10	**Option:**	2,2662
Exercise price:	01-07-10	**Delta:**	0,8399
Strike price:	15,00	**Gamma:**	0,1018
Interest rate:	3 %	**Theta:**	-1,0653
Volatility:	25 %	**Vega:**	2,2358
Dividend yield:	1,5 %	**Rho:**	3,6541
Pricing date:	12-03-10		
Spot price:	17,00		

TABLE 4. Example of European call option prices and Greeks in the presence of dividend yield.

3.6.2. *Black-Scholes equation for the discrete dividend payments case.* In this section we assume that at a given time, t_d, the shareholder receives a dividend payment $y_d S$. Therefore, the asset price variation when passing through the dividend payment date t_d is equal to

$$S(t_d^+) = S(t_d^-) - y_d S(t_d^-) = (1 - y_d) S(t_d^-),$$

where t_d^- and t_d^+ denote the moments just before and after t_d, respectively. Also y_d denotes the fraction of the asset price which is paid as dividend. It is important to notice that although the asset price is discontinuous in time at t_d the option price remains to be continuous in the sense that

$$V(t_d^-, S(t_d^-)) = V(t_d^+, S(t_d^+)).$$

Nevertheless, if we fix the asset price, the option price is discontinuous as a function of time at $t = t_d$, that is

$$V(t_d^-, S) = V(t_d^+, (1 - y_d)S) \neq V(t_d^+, S). \tag{3.33}$$

Notice that analogous results hold if the payment is y_d instead of $y_d S(t_d^-)$, the use of the second one is just accordingly to what we have already done in the case of a dividend yield. In any case, the treatment of discrete dividend payments requires the use of numerical methods. The consideration of time discretization meshes that included the dividend payment dates facilitates the implementation of the numerical methods applied to the corresponding Black-Scholes equations. This topic will be addressed in the section of numerical methods.

Table 5 shows an example of call option with discrete dividend payments. The option prices and Greeks have been obtained by numerical methods. The dividend payment dates are 01-03-2010, 01-04-2010 and 01-05-2010. In each date the asset owner receives a quantity of 0.8 currency units. Figure 6 displays the option prices in terms of asset and time, the small jump discontinuities at the dividend dates can be devised. These discontinuities are better illustrated by Figure 7, in which the time evolution for the fixed asset price $S = 17$ is shown. The jump condition (3.33) has to be implemented in the backward in time numerical method as indicated in the section of numerical methods in this document.

3.7. **Black-Scholes models for American vanilla options.** The main difference between European and American vanilla options comes from the fact that American ones can also be exercised at any time before expiry date while European ones can only be exercised at expiry date. Therefore, it is clear from the financial point of view that for the same

Option data		**Prices**	
Expiry date:	01-02-10	**Option:**	2,2662
Exercise price:	01-07-10	**Delta:**	0,8399
Strike price:	15,00	**Gamma:**	0,1018
Interest rate:	3 %	**Theta:**	-1,0653
Volatility:	25 %	**Vega:**	2,2358
Dividend payment:	0,8	**Rho:**	3,6541
Dividend dates:	01-(03,04,05)-10		
Pricing date:	12-03-10		
Spot price:	17,00		

TABLE 5. Example of European call option prices and greeks in the presence of discrete dividend payments

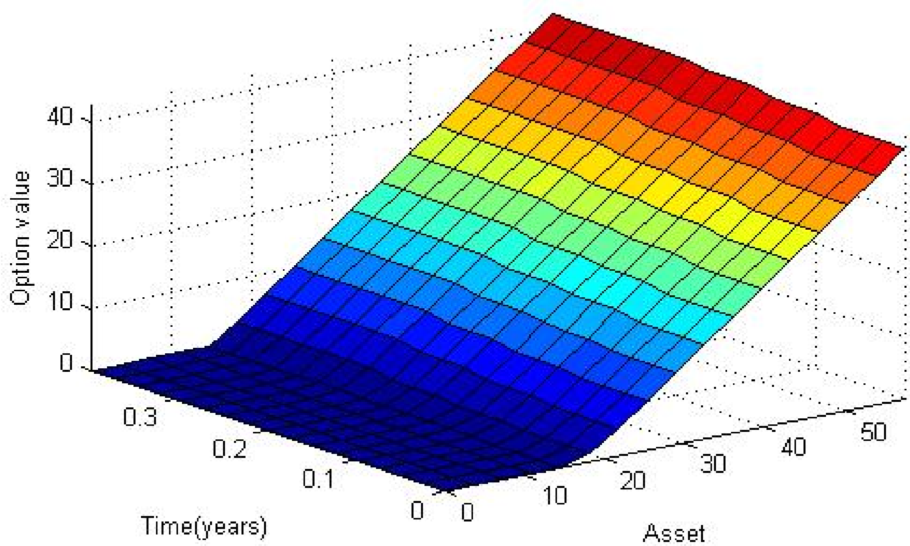

FIGURE 6. European call option price as a function of time and asset in the case of discrete dividend payments

parameter data an American option is worthier than the corresponding European one (as it gives more rights to the owner). Despite the small difference concerning the rights of its owner, the mathematical models are very different. Moreover, numerical methods are required for any American option, even in the simplest case of vanilla ones, as there are no exact formulas to price them. In the following paragraph the complementarity formulation of the vanilla American option pricing problem is posed.
First, notice the arbitrage free hypothesis implies that

$$P(t, S) \geq \max(E - S, 0) \tag{3.34}$$

in the case of American options. Otherwise, at time t we can buy the option at the price $P(t, S) < \max(E - S, 0)$ and exercise it immediately to receive $\max(E - S, 0)$, thus obtaining for sure a profit without risk and violating the absence of arbitrage assumption.

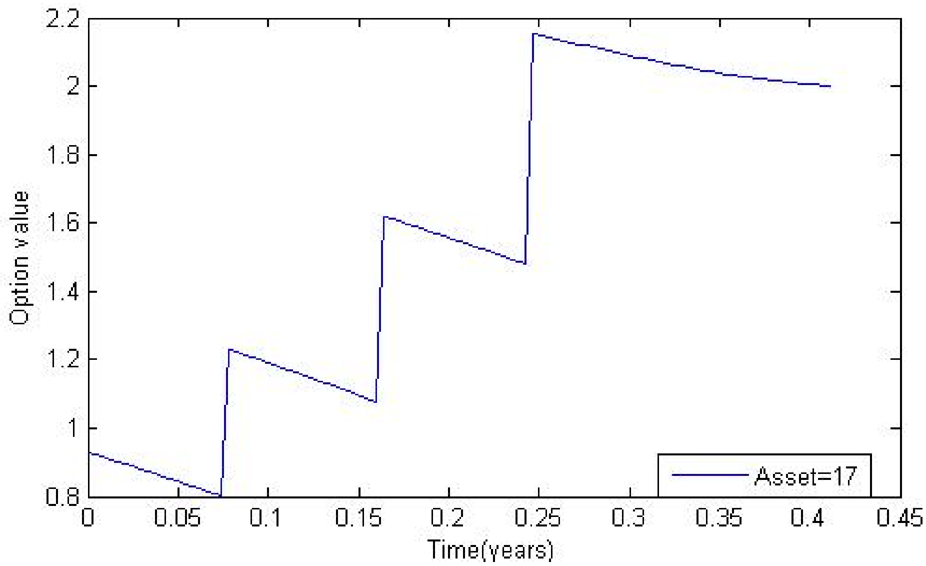

FIGURE 7. European call option price as a function of time in the case of discrete dividend payments for the asset spot value equal to 17

Also, using the same arguments as in the case of European options, the absence of arbitrage implies that

$$r\pi dt \geq \left(\frac{\partial P}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 P}{\partial S^2} \right) . \tag{3.35}$$

Nevertheless, unlike the European case, the inequality

$$r\pi dt > \left(\frac{\partial P}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 P}{\partial S^2} \right)$$

does not lead to an arbitrage opportunity. Actually, as it has been previously argued, in the European case the arbitrage is achieved by selling the portfolio containing European options and assets at time t and putting the obtained money on the bank at time t. Then, at time $t + dt$ the money and the revenue it produces is recovered from the bank to buy back the sold portfolio. In the American case, this procedure is risky as the portfolio contains American options. Notice that the portfolio owner can exercise the included American options between t and $t + dt$. Therefore the portfolio seller has to guarantee this possibility and cannot afford to have the money in a bank deposit between t and $t + dt$.

In view of the previous arguments, the Black-Scholes model for American vanilla put options can be written as follows:

For $D = [0, T] \times [0, \infty)$, find the function $P = P(t, S)$ such that:

$$L(P) = \frac{\partial P}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 P}{\partial S^2} + rS\frac{\partial P}{\partial S} - rP \leq 0 \quad \text{in } D \tag{3.36}$$

$$P(t, S) \geq max(E - S, 0)\,, \qquad \text{in } D \tag{3.37}$$

$$L(P) \cdot (P - max(E - S, 0)) = 0, \qquad \text{in } D \tag{3.38}$$

$$P(T, S) = max(E - S, 0), \qquad S > 0 \tag{3.39}$$

$$P(t, S) \to 0, \qquad t \in [0, T],\ S \to \infty \tag{3.40}$$

$$P(t, 0) = E\, exp(-r(T - t)), \qquad t \in [0, T] \tag{3.41}$$

The above problem is referred in the mathematical literature as a linear complementarity problem and also as moving boundary problem of obstacle type [10].
It is important to notice that in the American options pricing problem, besides the computation of the option price for each time asset value we need to decide whether it is better to maintain the option or to exercise it. Thus, we distinguish a set of points (t, S) where it is optimal to maintain the option from a set where it is optimal to exercise it. Both sets are *a priori* unknown as well as the curve in the $t - S$ plane which separates them. More precisely, for each time t, the set

$$\Omega^0(t) = \{S \in [0, \infty) / P(t, S) = max(E - S, 0)\}$$

is know as exercise region in the financial literature and as coincidence set in the mathematical one, while the set

$$\Omega^+(t) = \{S \in [0, \infty) / P(t, S) > max(E - S, 0)\}$$

is termed as no exercise region (or hold region) and non coincidence set, respectively. The free boundary between both sets is named as optimal exercise boundary in financial applications and free boundary in the mathematical literature, and it is denoted by

$$\Sigma(t) = \partial\Omega^+(t) \cap \partial\Omega^0(t).$$

In Figure 8 a sketch of the different sets is shown.
The complementarity condition (3.38) implies that when the option price is strictly greater than the exercise price (non coincidence region) then it verifies the same PDE than in the European case. Moreover, boundary condition (3.40) follows from the fact that when the asset price tends to infinity then the right to sell it at price E is worthless. On the other hand, in view of stochastic equation (2.1) if the asset price is zero at any time t the it remains to be zero after time t, in particular at expiry date, so that $P(T, 0) = E$ and the condition (3.41) is just obtained by discounting $P(T, 0) = E$ backwards to time t.
The solution of the American pricing problem requires the use of numerical methods, these methods mainly combine the usual ones for time and space discretization of the associated parabolic partial differential equation (finite differences, finite elements, finite volumes,...) and the numerical techniques for solving the resulting discrete linear complementarity problem (projected methods, penalization, duality methods,...). The application of one of these methods leads to the computed values presented in Table 6.

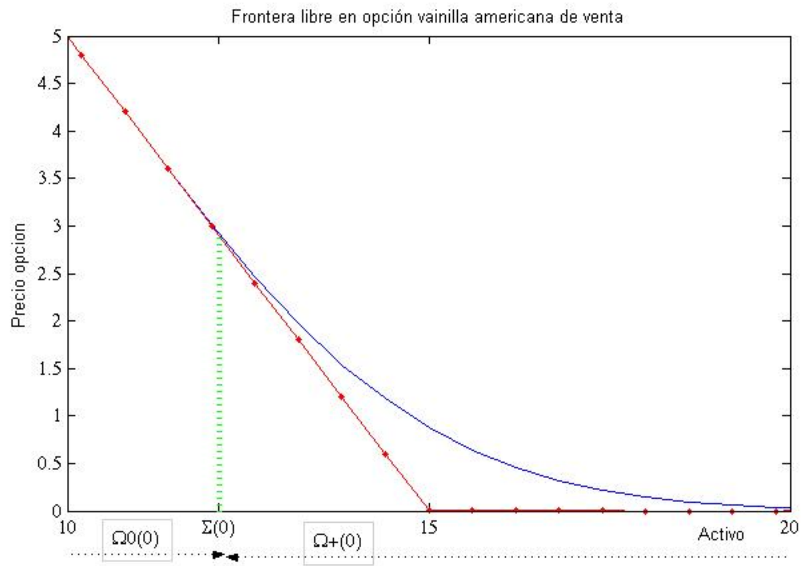

FIGURE 8. Scheme of the different sets appearing in the American put pricing problem.

Option data		**Prices**	
Expiry date:	01-02-10	**Option:**	0,1978
Exercise price:	01-07-10	**Delta:**	-0,1534
Strike price:	15,00	**Gamma:**	0,1000
Interest rate:	3 %	**Theta:**	-0,8205
Volatility:	25 %	**Vega:**	2,1761
Dividend yield:	0 %	**Rho:**	-0,6108
Pricing date:	12-03-10		
Spot price:	17,00		

TABLE 6. Example of American put option data, prices and Greeks.

In Figure 9 the computed prices for different times and assets are shown and not too much difference is appreciated with respect to European case with the same parameters shown in Figure 4. Nevertheless, when comparing Figures 5 and 10 the fact that the European put option price fails bellow its exercise value for low prices of the asset shows the main difference between both cases. In the case of call options, it can be proved that the prices of European and American options are the same in the absence of dividends [9].

3.8. **Black Scholes models for Asian options.** Asian options are a particular example of exotic options. In Asian options the payoffs of depend on certain average values of the underlying asset over some prescribed period of time. These options are strongly path-dependent and their prices can be modeled in the Black Scholes framework by introducing an additional space-like variable in the PDE.
In comparison with the behavior of vanilla options, Asian options result useful to protect from prices manipulations near the maturity of the contract and their payoff functions are generally less volatile. For example, Asian options are interesting for a company that works with products not very much traded in the markets, as the commodities; these products have to be bought every year at a certain moment and must be sold regularly

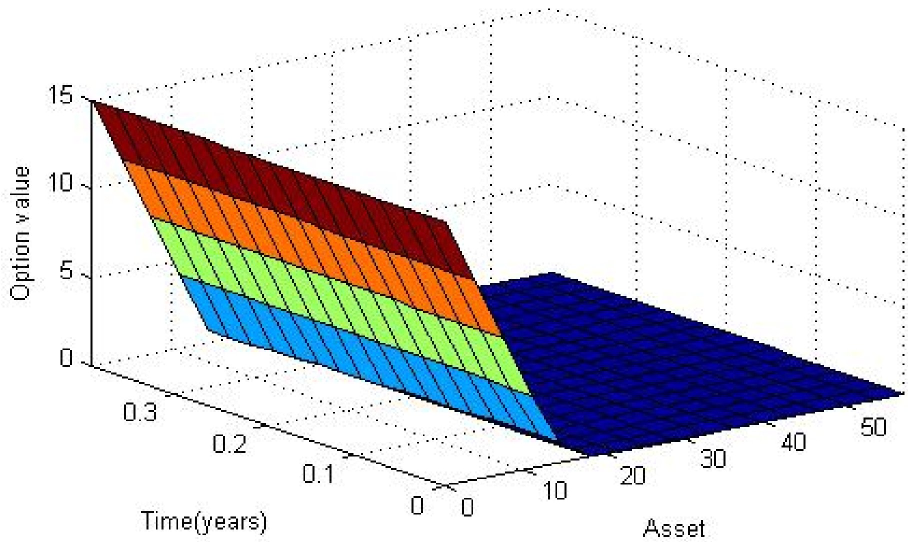

FIGURE 9. American put option price as a function of time and asset.

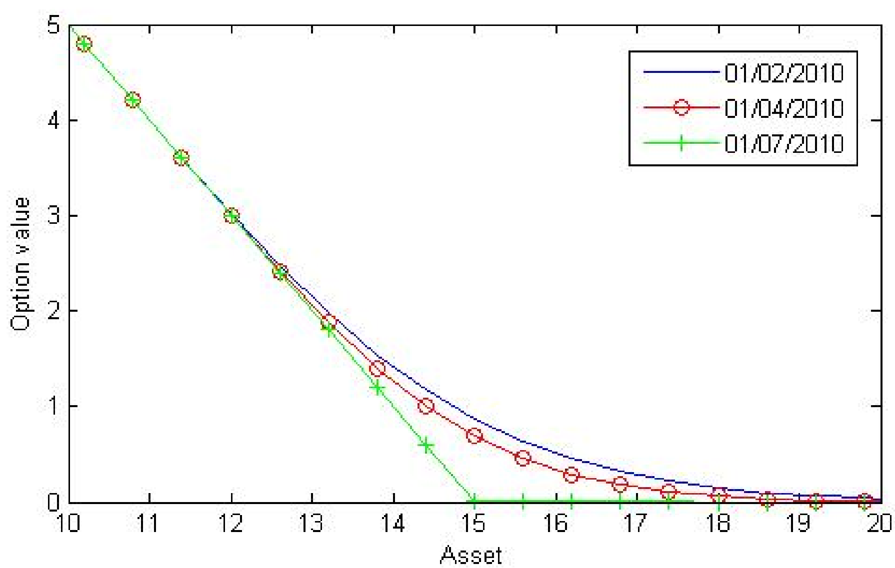

FIGURE 10. American put option price as a function of asset for different times.

during the year. In this case, the underlying is the commodity. Asian options are also used in currency exchange markets, by companies that have continuous sales in a currency, although they must buy the raw materials in different currencies at a fixed date. In this case, the underlying is the rate of change between currencies. In general, Asian options allow investors to protect themselves against losses due to adverse movements in the prices of the assets, without needing to hedge continuously their portfolios. This is the reason why its volume of negotiation grew up quickl in over the counter (OTC) markets.
There are different types of Asian options depending on the payoff and on the averaging procedure. In the following we present some examples, for a detailed classification see [24], for example.

A first classification refers to the way to compute the average, either on continuous or discrete versions. Thus, we have

- *Discrete arithmetic averaging:*

$$M_t = \frac{1}{n(t)} \sum_{i=1}^{n(t)} S(\tau_i).$$

- *Continuous arithmetic averaging:*

$$M_t = \frac{1}{t - T_i} \int_{T_i}^{t} S_\tau \, d\tau.$$

- *Discrete geometric averaging:*

$$M_t = \left[\prod_{i=1}^{n(t)} S(\tau_i) \right]^{\frac{1}{n(t)}}.$$

- *Continuous geometric averaging:*

$$M_t = \exp\left(\frac{1}{t - T_i} \int_{T_i}^{t} \log S_\tau d\tau \right).$$

where T_i denotes the initial averaging date. For the different previous definitions, a second classifications arises from the different possibilities to include the average in the option payoff $V(T, S, M) = G(S, M)$.

- *Fixed-strike Asian options:*

$$G_C(S, M) = \max(M - E, 0), \quad G_P(S, M) = \max(E - M, 0)$$

- *Floating-strike Asian options:*

$$G_C(S, M) = \max(S - M, 0), \quad G_P(S, M) = \max(M - S, 0)$$

where $M = M_T$. The subindex refers to the case of call or put options and E denotes a fixed strike price.

Moreover, the absence or presence of early exercise rights for the owner gives rise to Asian options with European or American style, respectively.

3.8.1. *Black-Scholes modeling for European path-dependent options.* We briefly describe a general framework for pricing path-dependent options following the Black-Scholes methodology. These ideas can be found in [22], for example. The main difference from vanilla options valuation is that a new process, M_t, to cope with the path dependency is introduced. More precisely, we define this new process as

$$M_t = \frac{1}{t - T_i} \int_{T_i}^{t} f(S_\tau, \tau) \, d\tau,$$

where f is appropriately chosen depending on the kind of averaging method. In order to obtain the stochastic differential equation for M_t, we consider a small time step dt and we have that M_t satisfies the deterministic equation

$$dM_t = \frac{f(S_t, t) - M_t}{t - T_i} \, dt. \tag{3.42}$$

Now, we assume that the option price V depends on the three independent variables, namely, $V = V(t,S,M)$. Then, following the standard dynamic hedging methodology already described for vanilla options, first a new version of Ito's lemma for two stochastic factors is applied to function $V(t,S,M)$ (see [15], for example). Thus, we have

$$dV = \sigma S\frac{\partial V}{\partial S}dX + \left(\frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2} + \mu S\frac{\partial V}{\partial S} + \frac{\partial V}{\partial t} + \frac{f(t,S)-M}{t-T_i}\frac{\partial V}{\partial M}\right)dt. \tag{3.43}$$

Next, following delta-hedging methodology, a riskless portfolio can be designed with a long position on an Asian option, and a short position with -Δ units on the underlying. Similarly to the vanilla case, this quantity is found to be equal to $\Delta = \dfrac{\partial V}{\partial S}$. Now, using the no arbitrage arguments we get the following equation:

$$\frac{\partial V}{\partial t} + \frac{f(t,S)-M}{t-T_i}\frac{\partial V}{\partial M} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0. \tag{3.44}$$

Equation (3.44) is similar to Black-Scholes equation, but with an additional first order derivative term associated to the new variable M. Notice the absence of second order derivative terms with respect to the new variable M. This is related to the deterministic equation verified by the associated process M_t. In order to complete (3.44) the final condition provided by the payoff function

$$V(T,S,M) = G(S,M). \tag{3.45}$$

is added.

3.8.2. *Fixed-strike Asian options with continuous arithmetic averaging.* As we have noted before, there exist different types of Asian options. In some cases, an analytical solution of the pricing problem can be obtained, as for the case of European-style Asian options with geometric averaging (see Zhang [24], for example). In other cases, the dimensionality of the problem can be reduced by finding a self-similar solution, like for floating strike Asian options (see [22], for example). Moreover, in [18] a two-dimensional PDE that models the price of both floating and fixed strike Asian options is formulated but it only applies to European style ones.

A fixed-strike Asian call option of European style with continuous arithmetic averaging (hereafter, in short quoted as fixed-strike Eurasian option), is a contract which gives the holder the right to exercise only at expiry date T and get the payoff

$$\max\left(\frac{1}{T-T_i}\int_{T_i}^{T} S_\tau d\tau - E, 0\right). \tag{3.46}$$

In expression (3.46), E denotes the fixed strike price and (T_i,T) is the averaging time interval. Using the notation introduced in the previous section, we define the continuous arithmetic averaging process M_t in the form

$$M_t = \frac{1}{t-T_i}\int_{T_i}^{t} S_\tau d\tau,\ T_i < t < T. \tag{3.47}$$

So, after adding the dividend yield, the final-value problem for pricing the fixed-strike Eurasian is posed as:

Find $C = C(t,S,M)$ defined for $S > 0,\ M > 0,\ T_i < t < T$, such that

$$\frac{\partial C}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} + (r - D_0) S \frac{\partial C}{\partial S} + \frac{S - M}{t - T_i}\frac{\partial C}{\partial M} - rC = 0, \tag{3.48}$$

$$C(T,S,M) = \max(M - E, 0). \tag{3.49}$$

For this model, existence and regularity of the solution is proven in [2] for the case $D_0 = 0$ and in [17] for the case $D_0 \neq 0$.

As in the case of vanilla options, there exist Asian options with American style. In the case of fixed-strike with continuous arithmetic averaging, hereafter we refer to them as fixed-strike Amerasian options. These contracts give to the holder the right to exercise the option at any time t before expiry date T and get the payoff

$$\max\left(\frac{1}{t - T_i}\int_{T_i}^{t} S(\tau)d\tau - E, 0\right), \tag{3.50}$$

for the case of a call, the put payoff being easy to imagine.

In order to avoid arbitrage opportunities, the Amerasian option price must be greater or equal than the exercise value (3.50) in options with early exercise opportunity. Therefore, in the present particular Amerasian case we obtain the following unilateral constraint

$$C(t,S,M) \geq G(t,S,M) = \max(M - K, 0). \tag{3.51}$$

Moreover, if we apply the same dynamic hedging methodology and arbitrage free arguments as for American vanilla options, we deduce the following analogous pricing problem for the particular Amerasian call option we are dealing with:

For $S > 0,\ M > 0,\ T_i < t < T$, find the function $C = C(t,S,M)$ such that

$$\mathcal{L}(C) = \frac{\partial C}{\partial t} + \frac{1}{2} S^2 \frac{\partial^2 C}{\partial S^2} + (r - D_0)\, S \frac{\partial C}{\partial S} + \left(\frac{S - M}{t - T_i}\right)\frac{\partial C}{\partial M} - rC \leq 0\,, \tag{3.52}$$

$$C \geq G\,, \tag{3.53}$$

$$\mathcal{L}(C)\cdot(C - G) = 0. \tag{3.54}$$

Equations (3.52) – (3.54) are completed with the final condition (3.49). As typically in American style options, for each time before maturity we can distinguish two regions in the SM-plane. In the first region (coincidence set or exercise region) where it results optimal to exercise the option as $V(t,S,M) = \max(M - K, 0)$. In the second region (non coincidence or hold region), where $V(t,S,M) > \max(M - K, 0)$ and it is optimal to hold the option. For each time, an unknown curve (free boundary or optimal exercise boundary) in the SM-plane separates both regions.

3.9. **Vanilla options on several assets.** A natural generalization of the previous options on one underlying is the consideration of options on a certain number of underlying assets. Many structured products traded by financial institutions include in their contracts the presence of options depending on the price of several assets. As in the case of one asset, the simplest case corresponds to the natural generalization of European vanilla options.

Although it can be easily generalized to the case of d underlying with $d\,(\geq 2)$, for simplicity in notation we briefly describe the case of two underlying assets. For this purpose, first we assume that their prices follow the stochastic differential equations

$$dS_t^k \;=\; \mu_k S_t^k\, dt \;+\; \sigma_k S_t^k\, dX_t^k, \quad k = 1, 2,$$

where μ_k, σ_k and X_t^k denote the drift, the volatility and the Wiener process corresponding to assets $k = 1, 2$. Moreover, in order to account with the possible correlation between prices of both assets, we note this correlation by ρ_{12} and we consider correlated Wiener processes with $dX_t^1 dX_t^2 = \rho_{12} dt$. Let us also denote the generic payoff at expiry date by G, that is

$$V(T, S^1, S^2) = G(S^1, S^2). \tag{3.55}$$

In order to obtain a Black-Scholes equation, we use Ito's lemma for the case $V = V(t, S^1, S^2)$ to obtain

$$\begin{aligned} dV &= \left(\sum_{k=1}^{2} \mu_k S^k \frac{\partial V}{\partial S^k} + \frac{1}{2} \sum_{k,j=1}^{2} \sigma_k \sigma_j \rho_{kj} S^k S^j \frac{\partial^2 V}{\partial S^k \partial S^j} + \frac{\partial V}{\partial t} \right) dt + \\ &\quad + \sum_{k=1}^{2} \sigma_k S^k \frac{\partial V}{\partial S^k}\, dX^k, \qquad (\rho_{11} = \rho_{22} = 1,\ \rho_{21} = \rho_{12}). \end{aligned}$$

Next, we build the portfolio $\pi = V - \Delta_1 S^1 - \Delta_2 S^2$ containing a long position in the option and short positions with $-\Delta_1$ and $-\Delta_2$ units on the first and second assets, respectively. Then, using dynamic hedging methodology to obtain the risk-free condition for portfolio π, we deduce that $\Delta_k = \dfrac{\partial V}{\partial S^k}$ for $k = 1, 2$. Therefore, we get the following Black-Scholes PDE for the case of two assets

$$\frac{\partial V}{\partial t} + \frac{1}{2} \sum_{k,j=1}^{2} \sigma_k \sigma_j \rho_{kj} S^k S^j \frac{\partial^2 V}{\partial S^k \partial S^j} + r \sum_{k=1}^{2} S^k \frac{\partial V}{\partial S^k} - rV = 0, \tag{3.56}$$

which jointly with the final condition (3.55) defines the option pricing problem. Different derivatives give rise to different payoff functions. Some examples are the following:

- Best of : $G(S^1, S^2) = g(max(S^1, S^2))$.
- Worst of : $G(S^1, S^2) = g(min(S^1, S^2))$.
- Call Spread : $G(S^1, S^2) = max(S^1 - S^2 - E, 0)$.
- Put Spread : $G(S^1, S^2) = max(E - (S^1 - S^2), 0)$.

Also American versions can be considered and additional barrier conditions can be added. In most of the cases numerical methods are required.

3.10. **Black-Scholes models for interest rate derivatives.** So far, in option pricing we have assumed a constant risk-free interest rate. Most of the previous methods and formulas can be extended to the case of a deterministic time dependent risk-free interest rate. Nevertheless, in practice several types of interest rates can be distinguished in the markets and their expressions for the future dates are not a priori known. This is specially relevant when pricing derivatives with long periods. In this case, the consideration of an stochastic evolution for the rates is more suitable. A very important part of derivatives markets is related to interest rate derivatives. There exist a lot of instruments where the underlying is an interest rate: bonds, swaps, caps or floors, ratchet caps or ratchet floors, for example.

In this section, we present some examples of application of dynamic hedging methodology to obtain Black-Scholes models to price bonds, as an example of the most common interest rate derivatives.

As in the case of options, the departure point corresponds to the choice of the evolution model for the underlying price. Thus, the more classical models for the short (spot or overnight) interest rate, r, assume that they follow the Ito process

$$dr_t \;=\; u(t, r_t)\, dt \;+\; w(t, r_t)\, dX_t. \tag{3.57}$$

In (3.57), $u(t, r)$ and $w(t, r)$ denote the drift and volatility of the spot rate, and X represents a Wiener process. For simplicity, hereafter we drop subindex "t" in the stochastic processes. The choice of different expressions for the drift and volatility gives rise to different classical models (see [22], for example). Thus, we have

- *Vasicek model (1977):* It is based on the Ornstein-Uhlenbeck process

$$dr \;=\; \alpha(\gamma - r)\, dt \;+\; \rho\, dX,$$

  where $\gamma > 0$ denotes the long term interest rate, $\alpha > 0$ represents the mean reversion velocity rate and ρ is the constant spot rate volatility. Although Vasicek model incorporates the suitable mean reversion property, spot rates can become negative which is not a realistic situation. One advantage is that there exist closed formulas for the mean and the variance of the rates at maturity r_T.
- *Cox-Ingersoll-Ross (CIR) model (1985):* In this case short rates follow the stochastic equation

$$dr \;=\; \alpha(\gamma - r)\, dt \;+\; \rho\sqrt{r}\, dX,$$

  where the difference with respect to Vasicek is that the volatility is no more constant and depends on the square root of short rates, ρ being a parameter. The drift term is the same and there also exist exact expressions for the mean and volatility of r_T. One important advantage with respect to Vasicek is that if the parameters satisfy $2\alpha\gamma \geq \rho^2$ and $r_0 > 0$ then $r_t > 0$, i.e. interest rates are strictly positive.

The previous models involve constant parameters so that it results very difficult to calibrate them for general market data. For this reason, some so called consistent interest rates appeared. They mainly incorporate time dependency on the previous parameters in order to adjust the functions with market data (bonds, spot and future rates, volatility of rates). Three classical examples of consistent models are

- *Ho-Lee model (1985):* The proposed stochastic equation is

$$dr \;=\; \theta(t)\, dt \;+\; \rho\, dX,$$

  where $\theta(t)$ and ρ are the time dependent drift and constant volatility, respectively. The drift function is parameterized and adjusted with present market spot rates.
- *Hull-White model (1990):* In this case, the spot rates evolution is governed by

$$dr \;=\; (\theta(t) - \alpha(t)\, r)\, dt \;+\; \rho(t)\, r^{\beta}\, dX$$

  where $\theta(t)/\alpha(t)$ and $\rho(t)r^{\beta}$ represents the long term rate and volatility, respectively. Notice that Hull-White incorporates the mean reversion property (not present in Ho-Lee). For $\beta = 0$ the model can be understood as an extension of Vasicek model and for $\beta = 1/2$ as an extension of CIR one.
- *Black-Derman-Toy model (1990):* In this case spot rates satisfy

$$d(ln\, r) \;=\; (\theta(t) - \frac{\rho'(t)}{\rho(t)}\, ln\, r)\, dt \;+\; \rho(t)\, dX$$

  where the drift and volatility are calibrated to the market drift and volatility, respectively. Black and Karasinski in 1991 incorporate the uncoupling of mean reversion and volatility with the following proposal:

$$d(ln\, r) \;=\; (\theta(t) - \alpha(t)\, ln\, r)\; dt \;+\; \rho(t)\; dX.$$

Remark 1. *All previous models concern to short rates evolution and can be framed in the Black Scholes setting for bonds we pose in next paragraphs. Nevertheless, more recently the consideration of LIBOR market models and Swap market model has become far more popular. These market models take into account the evolution of forward and swap LIBOR rates. Moreover, the main advantage of these models is that they allow use the same Black formulas handled in the market to price caps and swaps, respectively. We address the reader to the text* [7] *as a very complete reference concerning to interest rates.*

Next, assuming the general stochastic equation (3.57) for short rates dynamics, we apply the dynamic hedging methodology to pose the PDE pricing model for the zero coupon bond, the price of which is denoted by $B = B(t, r)$. The zero coupon bond is an instrument which pays a principal or face value, F, at maturity date and does not include coupon payments during bond life. Thus, applying Ito's lemma to function B, we get

$$dB \;=\; \left(\frac{\partial B}{\partial t} + u\frac{\partial B}{\partial r} + \frac{w^2}{2}\frac{\partial^2 B}{\partial r^2}\right) dt \;+\; w\frac{\partial B}{\partial r}\, dX.$$

In order to simplify notation in forthcoming calculus, we represent the drift and volatility of the relative return of the bond by

$$\mu_B \;=\; \frac{1}{B}\left(\frac{\partial B}{\partial t} + u\frac{\partial B}{\partial r} + \frac{w^2}{2}\frac{\partial^2 B}{\partial r^2}\right),$$

$$\rho_B \;=\; \frac{w}{B}\frac{\partial B}{\partial r},$$

so that

$$\frac{dB}{B} \;=\; \mu_B(t,r)\, dt \;+\; \rho_B(t,r)\, dX.$$

At this point, we notice that, unlike underlying assets in options, underlying interest rates in bonds cannot be traded (bought or sold). This difference is relevant for the way we argue in dynamic hedging methodology. In the case of bonds, we build a riskless portfolio by using two bonds of different maturities. Thus, for $i = 1, 2$, let B_i denote the bond with maturity T_i. We consider the portfolio $\pi = B_1 - B_2$, so that the portfolio variation in the interval $[t, t + dt]$ is given by

$$d\pi = (B_1\, \mu_{B_1} - B_2\, \mu_{B_2})\; dt \;+\; (B_1\, \rho_{B_1} - B_2\, \rho_{B_2})\; dX.$$

Therefore, if we choose the following proportions of bonds:

$$B_1 = \frac{\rho_{B_2}}{\rho_{B_2} - \rho_{B_1}}\,\pi, \quad B_2 = \frac{\rho_{B_1}}{\rho_{B_2} - \rho_{B_1}}\,\pi,$$

then the portfolio results to be risk free and its variation is

$$d\pi \;=\; \frac{\mu_{B_1}\,\rho_{B_2} - \mu_{B_2}\,\rho_{B_1}}{\rho_{B_2} - \rho_{B_1}}\,\pi\; dt.$$

Next, using the arbitrage free hypothesis, $d\pi \;=\; r\,\pi\, dt$, and then

$$r\,\pi\, dt \;=\; \frac{\mu_{B_1}\,\rho_{B_2} - \mu_{B_2}\,\rho_{B_1}}{\rho_{B_2} - \rho_{B_1}}\,\pi\, dt,$$

so that

$$r \;=\; \frac{\mu_{B_1}\,\rho_{B_2} - \mu_{B_2}\,\rho_{B_1}}{\rho_{B_2} - \rho_{B_1}},$$

or equivalently

$$\frac{\mu_{B_1} - r}{\rho_{B_1}}(t,r) \;=\; \frac{\mu_{B_2} - r}{\rho_{B_2}}(t,r).$$

Taking int account the arbitrary choice of the maturities T_1 and T_2, we get the same following quantity for any bond that matures at time T:

$$\lambda(t,r) \;=\; \frac{\mu_B - r}{\rho_B}(r,t) \;=\; \frac{\dfrac{1}{B}\left(\dfrac{\partial B}{\partial t} + u\dfrac{\partial B}{\partial r} + \dfrac{w^2}{2}\dfrac{\partial^2 B}{\partial r^2}\right) - r}{\dfrac{w}{B}\dfrac{\partial B}{\partial r}}. \tag{3.58}$$

Therefore, Black-Scholes equation for zero coupon bonds can be written as

$$\frac{\partial B}{\partial t} + (u - \lambda w)\frac{\partial B}{\partial r} + \frac{w^2}{2}\frac{\partial^2 B}{\partial r^2} - rB = 0, \;\; 0 < t < T, \; r \in I, \tag{3.59}$$

which is completed with the final condition

$$B(T,r) = F, \;\; r \in I. \tag{3.60}$$

Notice that equation (3.59) depends on the quantity $\lambda(r,t)$ defined by expression (3.58). The function λ can be interpreted as the market price of risk associated to uncertainty of interest rates. This interpretation arises from the identity

$$dB \;=\; \left(w\lambda\frac{\partial B}{\partial r} + rB\right) dt \;+\; w\frac{\partial B}{\partial r}\, dX,$$

which in turn implies that the difference between the return of a risk free bond and a risky one is given by the stochastic quantity

$$dB - rB\, dt \;=\; w\,\frac{\partial B}{\partial r}\,(dX + \lambda dt).$$

Therefore, the extra reward associated to each assumed risk unit is $\lambda\, dt$, which motivates the interpretation of λ.
The interval I depends on the particular choice of the interest rate model. For example, in the case of Vasicek model we choose $I = (-\infty, +\infty)$ while for CIR model $I = (0, +\infty)$ is taken when the involved parameters guarantee positive interest rates. In the both models, exact solutions for zero coupon bond prices can be obtained. More precisely, in the case of Vasicek we have

$$B(t,r) = F\,\exp\left(\alpha^{-1}(1 - e^{-\alpha(T-t)})(R_\infty - r) - R_\infty(T-t) - \frac{\rho^2}{4\alpha^3}(1 - e^{-\alpha(T-t)})^2\right),$$

where $R_\infty = \gamma - \rho\lambda\alpha^{-1} - \frac{\rho^2}{2\,\alpha^2}$.
In the case of CIR model, the solution is

$$B(t,r) = FA(t)e^{-rE(t)}$$

with

$$A(t) = \left(\frac{2\theta e^{(\theta+\phi)(T-t)/2}}{(\theta+\phi)(e^{\theta(T-t)} - 1) + 2\theta}\right)^{\frac{2\alpha\gamma}{\rho^2}}, \;\; E(t) = \frac{2(e^{\theta(T-t)} - 1)}{(\theta+\phi)(e^{\theta(T-t)} - 1) + 2\theta},$$

where $\phi = \alpha + \lambda\rho$ and $\theta = \sqrt{\phi^2 + 2\rho^2}$.
Most bonds include regular coupon payments before maturity. The case of fixed coupon payments at prescribed dates can be treated in Black-Scholes framework in a similar way than discrete dividend payments associated to assets in option pricing. More precisely, let us assume that at times t_i the bond holder receives a payment C_i, for $i = 1, \ldots, M$. So, the following jump discontinuity

$$B(t_i^-, r) = B(t_i^+, r) + C_i, \;\; i = 1, \ldots, M \tag{3.61}$$

in bond prices at payment dates holds. Moreover, as the last coupon, C_M, is usually paid at maturity jointly with the face value, F, the payoff in a coupon bearing bond is given by

$$B(T,r) = F + C_M. \tag{3.62}$$

Therefore, the pricing problem is defined by equation (3.59) jointly with final condition (3.62) and jump conditions (3.61). In the general case, numerical methods are required.
The callable bonds include an additional call option for the bond issuer that gives him/her the right to buy back the bond for a fixed price, $\overline{B}$, at prescribed dates or at any time before maturity. In this case, some analogies to American options or early exercise products can be found. Thus, for a zero coupon callable bond the dynamic hedging methodology combined with arbitrage free arguments leads to the following complementarity problem:

$$\mathcal{L}(B) = \frac{\partial B}{\partial t} + (u - \lambda w)\frac{\partial B}{\partial r} + \frac{w^2}{2}\frac{\partial^2 B}{\partial r^2} - rB \leq 0, \quad 0 < t < T,\ r \in I, \tag{3.63}$$

$$B \leq \overline{B} \tag{3.64}$$

$$\mathcal{L}(B) \cdot (B - \overline{B}) = 0 \tag{3.65}$$

Notice that the previous equations define a free boundary problem, the free boundary being the optimal call boundary at each time t in an analogous way as the optimal exercise boundary in American vanilla options. Numerical methods similar to those ones in American options are required.
Also there exist *puttable* bonds, in which the bond holder has the right to resell the bond to the issuer at a given price $\underline{B}$. In this case, condition (3.64) is replaced by

$$B \geq \underline{B} \tag{3.66}$$

and the complementarity condition (3.65) is replaced by

$$\mathcal{L}(B) \cdot (\underline{B} - B) = 0 \tag{3.67}$$

to define jointly with (3.63) the puttable bond pricing problem. Obviously, the puttable bond pricing problem is another example of free boundary problem and numerical methods are required. Also notice that a bond can have both call and put options included in the contract so that both conditions (3.64) and (3.66) must hold. In this case the complementarity condition can be appropriately written to indicate that when the bond price is strictly between the put and call prices, the identity holds in equation (3.63) instead of the inequality.
From a financial point of view, call options in bonds are designed to protect the issuer in the case of a decay in interest rates during bond life. In this case the issuer calls the bond in order to get a cheaper financial resource at lower rates. Conversely, the put option protects the bond holder when interest rates increase in the market, thus allowing him/her to resell the bond and obtain greater interest rates from the banks.
Also notice that early exercise conditions can coexist with the existence of coupon payments.
Other popular interest rate derivatives, such as swaps, caps and floors admit the use of dynamic hedging methodology to obtain their corresponding Black-Scholes. In order to not extend too much this document we address the interested reader to reference [22], for example.

4. Numerical methods

In a previous section we deduced Black-Scholes formulas for European call and put options when constant volatility and interest rates are assumed. These formulas can be extended to the case of deterministic time dependent volatilities and rates (see [23], for example). Nevertheless, there are no analytical formulas for American options or more complex options (as exotic options) or under more realistic assumptions on parameters (such as stochastic volatility or interest rates). The same lack of exact solution occurs when the underlying asset pays discrete dividends. In all these cases the PDE models are more complex and numerical methods are required. Analogous comments can be done for interest rate derivatives, such as bonds.

The existing numerical methods for derivatives pricing problems can be mainly classified into binomial (or trinomial) trees, Monte Carlo simulation techniques and numerical methods for PDE models. As this lecture notes mainly concern with the PDE approach to derivatives pricing, we start mentioning that finite differences, finite elements or finite volumes can be used. In this document we will focus on finite difference and finite element methods for the most basic derivatives.

4.1. **Finite difference methods for European options.** In order to explain the use of finite differences for financial derivatives pricing, we consider the model problem of European call options which is defined by equations (3.12)-(3.13) plus the inclusion of the dividend yield D_0S.

First notice that PDE models for option pricing problems are usually posed on unbounded domains while finite difference or finite elements require the consideration of bounded domains. Therefore, the first step is the localization procedure. Localization mainly consists in the domain truncation to pose an approximated problem in an appropriate bounded domain, so that the solutions to both problems are very close each other in the region of financial interest.

More precisely, for a sufficiently large value S_∞, we find the function C, defined in the domain $D = [0,T] \times [0, S_\infty]$, such that:

$$\begin{aligned} \frac{\partial C}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} + (r - D_0) S \frac{\partial C}{\partial S} - rC &= 0 \text{ in } D, && (4.1)\\ C(T,S) = max(S-E,0), & \qquad S > 0, && (4.2)\\ C(t,S_\infty) = S_\infty\, exp(-D_0(T-t)) - E\, exp(-r(T-t))\,, & \qquad t \in [0,T], && (4.3)\\ C(t,0) = 0, & \qquad t \in [0,T], && (4.4) \end{aligned}$$

Usually in financial literature, either the value $S_\infty = 3E$ or $S_\infty = 4E$ is chosen. In the mathematical literature, the effect the approximation due to domain truncation is analyzed in [13], where also the same financial choice is recommended. Notice that the artificial boundary S_∞ is introduced and the boundary conditions (4.3) and (4.4) can be justified by financial arguments.

Once we fix the bounded domain, we will consider uniform finite differences meshes, although the methodology could easily be extended to variable time and space steps. Thus, for the natural numbers $N > 1$ and $M > 1$, the constant time and space steps are defined as

$$\Delta t = T/(N+1), \quad \Delta S = S_\infty/(M+1)\,,$$

so that the finite differences mesh is defined by the set of nodes

$$(t_j, S_i) = (j\Delta t, i\Delta S), \quad j = 0, \ldots, N+1;\; i = 0, \ldots, M+1.$$

4.1.1. *Euler implicit finite differences scheme.* In the Euler implicit finite differences scheme, the derivatives appearing in equation (4.2) are approximated at the finite differences mesh nodes as follows:

$$\begin{aligned}
\frac{\partial^2 C}{\partial S^2}(t_j,S_i) &\approx \frac{C(t_j,S_{i+1})-2C(t_j,S_i)+C(t_j,S_{i-1})}{\Delta S^2},\\
\frac{\partial C}{\partial S}(t_j,S_i) &\approx \frac{C(t_j,S_{i+1})-C(t_j,S_i)}{\Delta S},\\
\frac{\partial C}{\partial t}(t_j,S_i) &\approx \frac{C(t_{j+1},S_i)-C(t_j,S_i)}{\Delta t}.
\end{aligned} \tag{4.5}$$

Thus, writing equation (4.2) at the point (t_j,S_i) for $j=1,\dots,N$; $i=1,\dots,M$ and replacing the previous derivatives approximations, we get

$$\begin{aligned}
&\frac{C(t_{j+1},S_i)-C(t_j,S_i)}{\Delta t}+\frac{1}{2}\sigma^2 S_i^2\frac{C(t_j,S_{i+1})-2C(t_j,S_i)+V(t_j,S_{i-1})}{\Delta S^2}+\\
&+(r-D_0)S_i\frac{C(t_j,S_{i+1})-C(t_j,S_i)}{\Delta S}-rC(t_j,S_i)\approx 0.
\end{aligned}$$

Next, reordering on the left hand side the terms at time t_j, we obtain

$$\begin{aligned}
&\left(1+\Delta t r+\frac{\Delta t\sigma^2 S_i^2}{\Delta S^2}+\frac{\Delta t(r-D_0)S_i}{\Delta S}\right)C(t_j,S_i)-\frac{\Delta t\sigma^2 S_i^2}{2\Delta S^2}C(t_j,S_{i-1})+\\
+\;&\left(-\frac{\Delta t\sigma^2 S_i^2}{2\Delta S^2}-\frac{\Delta t(r-D_0)S_i}{\Delta S}\right)C(t_j,S_{i+1})\approx C(t_{j+1},S_i).
\end{aligned}$$

Then, we define the approximations $C_{j,i}\approx C(t_j,S_i)$ as the values that verify exactly the previous approximated identities. Therefore, we obtain the equations

$$\begin{aligned}
&-\frac{\Delta t\sigma^2 S_i^2}{2\Delta S^2}C_{j,i-1}+\left(1+\Delta t r+\frac{\Delta t\sigma^2 S_i^2}{\Delta S^2}+\frac{\Delta t(r-D_0)S_i}{\Delta S}\right)C_{j,i}+\\
+\;&\left(-\frac{\Delta t\sigma^2 S_i^2}{2\Delta S^2}-\frac{\Delta t(r-D_0)S_i}{\Delta S}\right)C_{j,i+1}=C_{j+1,i}.
\end{aligned}$$

In the backward in time marching scheme, the system of equations obtained for $i=1,\dots,M$ is solved recursively for $j=N,N-1,\dots,1,0$, starting from the final condition

$$C_{N+1,i}=C(T,S_i)\quad\forall i=1,\dots,\,M+1.$$

Moreover, for each j, by using (4.3) and (4.4) the following boundary conditions are considered:

$$C_{j,M+1}=C(t_j,S_\infty),\quad C_{j,0}=C(t_j,0).$$

Thus, at each time step $j=N,N-1,\dots,1,0$, we have to solve the tridiagonal linear system

$$A\,C_j \;=\; b_j\,, \tag{4.6}$$

where the nonzero coefficients of the tridiagonal matrix A are given by

$$A_{ii}=1+\rho+\gamma S_i^2+\kappa S_i,\quad A_{i,i+1}=-\theta S_i^2-\kappa S_i,\quad A_{i+1,i}=-\theta S_i^2,$$

in terms of the constant parameters

$$\gamma=\frac{\Delta t\sigma^2}{\Delta S^2},\quad \theta=\frac{\Delta t\sigma^2}{2\Delta S^2},\quad \rho=\Delta t r,\quad \kappa=\frac{\Delta t(r-D_0)}{\Delta S}.$$

Moreover, the solution and the right hand side of the system (4.6) are given by

$$\begin{aligned} C_j &= \left(C_{j,1},\, C_{j,2}, \ldots,\, C_{j,M-1},\, C_{j,M})\right)^t, \\ b_j &= \left(C_{j+1,1} + \theta S_1^2 \alpha\left(t_j\right),\, C_{j+1,2}, \ldots,\, C_{j+1,M-1},\, C_{j+1,M} + \left(\theta S_{N+1}^2 + \kappa S_{N+1}\right) \beta\left(t_j\right)\right)^t, \end{aligned}$$

where we use the notation $\alpha(t_j) = C(t_j, 0)$ and $\beta(t_j) = C(t_j, S_\infty)$.

Remark 2. *Notice that system (4.6) is equivalent the following optimization problem without constraints:*

$$\frac{1}{2} C_j^T A C_j - C_j^T b_j = \min_{y \in R^n} \left(\frac{1}{2} y^T A y - y^T b_j\right).$$

In order to solve the system (4.6) either the specific Thomas algorithm or appropriate iterative methods can be used.
The unconditional convergence of the Euler implicit scheme when Δt and ΔS tend to zero is analyzed in [1].

4.1.2. *Euler explicit finite differences scheme.* The Euler explicit scheme avoids the solution of a linear system at each time step, the price to pay being its conditional stability which imposes a small enough time stepsize to guarantee the convergence of the method. After writing the equation (4.2) at the point (t_{j+1}, S_i), the explicit scheme is based on the approximations:

$$\begin{aligned} \frac{\partial^2 C}{\partial S^2}\left(t_{j+1}, S_i\right) &\approx \frac{C\left(t_{j+1}, S_{i+1}\right) - 2C\left(t_{j+1}, S_i\right) + C\left(t_{j+1}, S_{i-1}\right)}{\Delta S^2}, \\ \frac{\partial C}{\partial S}\left(t_{j+1}, S_i\right) &\approx \frac{C\left(t_{j+1}, S_{i+1}\right) - C\left(t_{j+1}, S_i\right)}{\Delta S}, \\ \frac{\partial C}{\partial t}\left(t_{j+1}, S_i\right) &\approx \frac{C\left(t_{j+1}, S_i\right) - C\left(t_j, S_i\right)}{\Delta t}. \end{aligned} \tag{4.7}$$

This allows to deduce an explicit algorithm to obtain the approximations at time t_j from the ones at time t_{j+1}. The conditional convergence properties are analyzed in [1], for example. Summarizing these properties, the time step Δt has to be of order $(\Delta S)^2$.

4.1.3. *General finite differences and Crank-Nicolson schemes.* A general scheme that includes the two previous ones can be obtained in terms of an additional parameter $\omega \in [0,1]$ and it is known as ω-method. The general scheme is based on the following approximations at the point $(t_j^\omega, S_i) = (\omega t_j + (1-\omega) t_{j+1}, S_i)$:

$$\begin{aligned} \frac{\partial^2 C}{\partial S^2}\left(t_j^\omega, S_i\right) &\approx \omega\left(\frac{C\left(t_j, S_{i+1}\right) - 2C\left(t_j, S_i\right) + C\left(t_j, S_{i-1}\right)}{\Delta S^2}\right) + \\ &\quad + (1-\omega)\left(\frac{C\left(t_{j+1}, S_{i+1}\right) - 2C\left(t_{j+1}, S_i\right) + V\left(t_{j+1}, S_{i-1}\right)}{\Delta S^2}\right) \\ \frac{\partial C}{\partial S}\left(t_j^\omega, S_i\right) &\approx \omega\left(\frac{C\left(t_j, S_{i+1}\right) - C\left(t_j, S_i\right)}{\Delta S}\right) + (1-\omega)\left(\frac{C\left(t_{j+1}, S_{i+1}\right) - C\left(t_{j+1}, S_i\right)}{\Delta S}\right), \\ \frac{\partial C}{\partial t}\left(t_j^\omega, S_i\right) &\approx \frac{C\left(t_{j+1}, S_i\right) - C\left(t_j, S_i\right)}{\Delta t}. \end{aligned} \tag{4.8}$$

For the case $\omega = 0$ we recover the explicit Euler scheme while for $\omega = 1$ the previously described implicit one is obtained. For $\omega \geq 0.5$, the schemes are unconditionally stable (and therefore unconditionally convergent as all of them are consistent). Amongst them, Crank-Nicolson scheme corresponds to $\omega = 0.5$ and presents the best consistency error which is of order two in time and space while the other implicit schemes are only of order one in time.

4.1.4. *Postprocessing and computation of greeks.* Once any of the previous finite differences schemes has been applied, the option price approximation is obtained for the set of discrete times t_j and discrete prices S_i of the finite differences mesh. As usually $t = 0$ corresponds to the present time, we are specially interested in the option price corresponding to the spot asset price, that is $C(0, S)$. Nevertheless, maybe the spot price, S, does not match exactly any finite difference node price, S_i. In this case, an appropriate interpolation procedure has to be included in the software to account for asset prices $S \neq S_i$. Moreover, if we are also interested in giving prices for future times $t \neq t_j$ also a time interpolation procedure has to be incorporated. Usually, a piecewise linear product interpolation is carried out to account for both time and underlying variables.
Also it results useful for hedging purposes to compute the approximation of the option Greeks from the computed option prices on the finite differences mesh. At this point we distinguish the Greeks depending on the derivatives with respect to the independent variables t and S (Δ, Γ and Θ) from the ones depending on the derivatives with respect to the parameters σ and ρ (Vega and Rho). In the first case, the following numerical derivation formulas are proposed:

$$\begin{aligned}
\Delta(t_j, S_i) &\approx \frac{C\left(t_j, S_{i+1}\right) - C\left(t_j, S_i\right)}{\Delta S} \\
\Gamma(t_j, S_i) &\approx \frac{C\left(t_j, S_{i+1}\right) - 2C\left(t_j, S_i\right) + C\left(t_j, S_{i-1}\right)}{\Delta S^2} \\
\Theta(t_j, S_i) &\approx \frac{C\left(t_{j+1}, S_i\right) - C\left(t_j, S_i\right)}{\Delta t},
\end{aligned}$$

with some adaption for the cases $i = 0$ and $i = N + 1$. In the case of derivatives with respect to a parameter, such as volatility, the following approximation is proposed:

$$Vega(t_j, S_i) \approx \frac{C_{\sigma+d\sigma}\left(t_j, S_i\right) - C_\sigma\left(t_j, S_i\right)}{d\sigma}. \tag{4.9}$$

In order to compute (4.9), first a small volatility increment $d\sigma$ is considered ($d\sigma = 0.01$, for example) and the problem is numerically solved for the parameters σ and $\sigma + d\sigma$ to approximate the values of Vega at the finite differences mesh nodes with formula (4.9). For the case of the paremeter r, the analogous procedure with the following formula approximates Rho:

$$Rho(t_j, S_i) \approx \frac{C_{r+dr}\left(t_j, S_i\right) - C_\sigma\left(t_j, S_i\right)}{dr}. \tag{4.10}$$

Finally, additional product linear interpolation procedures are used to obtain the approximation of Greeks for points (t, S) not belonging to the finite differences mesh.

4.2. **A finite elements method for European options.** In this section we briefly describe the application of a particular finite element method to the European option pricing problem.
More precisely, first an implicit Euler scheme is applied for the time discretization:

$$\frac{\partial V}{\partial t}(t_j, .) \approx \frac{V(t_{j+1}, .) - V(t_j, .)}{\Delta t} = \frac{V_{j+1} - V_j}{\Delta t}.$$

Next, the diffusion term is written in divergence form by using the identity

$$S^2 \frac{\partial^2 V}{\partial S^2} = \frac{\partial}{\partial S}\left(S^2 \frac{\partial V}{\partial S}\right) - 2S\frac{\partial V}{\partial S}.$$

Thus, if dividend yield is included in the model, we get the equation

$$\frac{\partial V}{\partial t} + (r - D_0 - \sigma^2) S \frac{\partial V}{\partial S} + \frac{\sigma^2}{2} \frac{\partial}{\partial S} \left(S^2 \frac{\partial V}{\partial S} \right) - rV = 0$$

jointly with the appropriate final and boundary conditions depending on the type of option we are dealing with.

In the case of a put option, P, the implicit Euler time discretization leads to the equation

$$\frac{P_{j+1} - P_j}{\Delta t} + (r - D_0 - \sigma^2) S P_j' + \frac{\sigma^2}{2} \left(S^2 P_j' \right)' - r P_j = 0 \,,$$

where $P_j \approx P(t_j, .)$. Reordering the terms in the previous equation, we get

$$(1 + r\Delta t) P_j + \Delta t (\sigma^2 + D_0 - r) S P_j' - \frac{\sigma^2}{2} \left(S^2 P_j' \right)' = P_{j+1}. \tag{4.11}$$

Next, for $j = N, N-1, \ldots, 1, 0$, the following variational formulation is posed at time t_j:

Find $P_j \in W$, such that $P_j(S_\infty) = \beta(t_j)$ and:

$$(1 + r\Delta t) \int_0^{S_\infty} P_j \varphi \, dS \; + \; \Delta t (\sigma^2 + D_0 - r) \int_0^{S_\infty} S P_j' \varphi \, dS +$$

$$+ \frac{\sigma^2 \Delta t}{2} \int_0^{S_\infty} S^2 P_j' \varphi' \, dS \; = \; \int_0^{S_\infty} P_{j+1} \varphi \, dS, \quad \forall \varphi \in W_0,$$

where the following weighed Sobolev spaces are considered:

$$\begin{aligned} W &= \{\varphi \in L^2(0, S_\infty) / S \frac{\partial \varphi}{\partial S} \in L^2(0, S_\infty)\} \,, \\ W_0 &= \{\varphi \in W / \varphi(S_\infty) = 0\}. \end{aligned}$$

Remark 3. *Notice that equation (4.11) degenerates at the boundary $S = 0$ so that no boundary condition is required at this point in the variational formulation.*

For the spatial discretization, we introduce a partition of the interval $[0, S_\infty]$ defined by the subintervals $[S_i, S_{i+1}]$ with $S_i = ih, \ i = 0, \ldots, N+1$ (the stepsize being $h = S_\infty/(N+1)$). So, we can use the piecewise linear finite elements space and subspace defined by

$$\begin{aligned} W_h &= \{\varphi_h \in C[0, S_\infty] \,/\, \varphi_h \,|_{[S_i, S_{i+1}]} \in P_1\} \,, \\ W_{h0} &= \{\varphi_h \in W_h \,/\, \varphi_h(S_\infty) = 0\}, \end{aligned}$$

where $\mathcal{P}_1$ denotes the polinomial of degree less or equal than one (this choice can be easily generalized to $\mathcal{P}_k$, with $k \geq 1$). Then, the following finite elements discretized problem is posed.

Find $P_j^h \in W_h$, such that $P_j^h(S_\infty) = \beta(y_j)$ and

$$(1 + r\Delta t) \int_0^{S_\infty} P_j^h \varphi_h \, dS + \Delta t (\sigma^2 + D_0 - r) \int_0^{S_\infty} S (P_j^h)' \varphi_h \, dS +$$

$$+ \frac{\sigma^2 \Delta t}{2} \int_0^{S_\infty} S^2 (P_j^h)' \varphi_h' \, dS \; = \; \int_0^{S_\infty} P_{j+1}^h \varphi_h \, dS, \; \forall \varphi \in W_{h0}$$

As in the case of finite differences, the fully discretized problem obtained at each time step is equivalent to the linear system

$$B\,\overline{P}_j \;=\; d_j\,, \tag{4.12}$$

where B denotes the usual finite element matrix and d_j represents the usual second member obtained by finite elements (see [1], for further details on the expression of the matrix and second member coefficients).

In order to solve the system (4.12) either direct or iterative methods can be used. In the case of piecewise linear finite elements, a tridiagonal system is obtained so that either the specific Thomas algorithm or the relaxation iterative method can be chosen.

Remark 4. *Having in view to point out the differences between European and American options, we notice that system (4.12) is equivalent the following optimization problem without constraints:*

$$\frac{1}{2}\overline{P}_j^T B\overline{P}_j - \overline{P}_j^T d_j \;=\; \min_{y\in R^n}\left(\frac{1}{2}y^T By - y^T d_j\right) \tag{4.13}$$

Remark 5. *As the piecewise linear finite element method allows to obtain the approximated option prices at the same nodes than the finite differences scheme, the same post-processing procedure described in section* 3.1.4 *provides the approximation of option prices and Greeks at any point* (t, S).

Remark 6. *As in the case of finite differences another implicit time discretization schemes could be used (such as Crank-Nicolson, for example).*

4.2.1. *Some general remarks to finite differences and finite elements.* Clearly, the numerical methods can also be applied to the transformed heat equation we have used to deduce the Black-Scholes formula, although in that case we have to truncate the domain at both a priori unbounded boundaries. The important decision is not the variables but the choice of the grid. Although the methods have been described for a uniform mesh, they can be easily generalized to nonuniform time and space grids and a better choice is a variable step spatial mesh which is refined near the less smooth points of the option payoff function.

The presence of discrete dividend payments can also be treated. For this purpose it is better to handle a time mesh which includes the dividend payment dates. Thus, the backward in time marching schemes start from the final condition and the jump condition (3.33) is applied at each dividend date.

4.3. **Numerical methods for American options PDE models.** As described in Section 3.7, American put vanilla option price is governed by the complementarity problem (3.36)-(3.41) and numerical methods are required to approximate the solution. In this section we briefly describe the application of a particular finite differences scheme and a finite element method. For both methods we first apply the same implicit Euler finite differences scheme. Thus, for each $j = N, N-1, \ldots, 1, 0$, we consider the function $P_j \approx P(t_j, .)$ that verifies

$$\frac{P_{j+1} - P_j}{\Delta t} + \frac{\sigma^2 S^2}{2} {P_j}'' + rSP_j' - rP_j \;\leq\; 0 \tag{4.14}$$

$$P_j \;\geq\; g_j \tag{4.15}$$

$$\left(\frac{P_{j+1} - P_j}{\Delta t} + \frac{\sigma^2 S^2}{2} {P_j}'' + rSP_j' - rP_j\right)(P_j - g_j) \;=\; 0, \tag{4.16}$$

where $g_j = \max(E - S, 0)$ denotes the exercise price at time t_j which actually does not depend on j in the case of vanilla options.

If we consider the same finite differences discretization in space that has been described for the European option, we obtain the following discrete complementarity problem

$$A\,\overline{P}_j \;\geq\; b_j\,,\; \overline{P}_j \geq \overline{g}_j\,,\; \left(A\,\overline{P}_j - b_j\right)^T \left(\overline{P}_j - \overline{g}_j\right) = 0, \tag{4.17}$$

where the matrix A and the vector b_j are the same as in European options, and $\overline{g}_j$ denotes the vector of function g_j at the finite differences mesh nodes. In order to propose a numerical method for solving the discretized problem (4.17) we notice that (4.17) is equivalent to the following quadratic programming problem with inequality constraints:

$$\frac{1}{2}\overline{P}_j^T A \overline{P}_j - \overline{P}_j^T b_j \;=\; \min_{y \geq \overline{g}_j} \left(\frac{1}{2} y^T A y - y^T b_j \right). \tag{4.18}$$

Thus, a variety of techniques (primal methods, penalization techniques or duality methods, for example) can be applied (see [1, 16], among others). Here, for its simplicity we briefly sketch the application of a projected relaxation algorithm for a generic problem. Alternative algorithms have been used in [20]. Thus, let be the matrix A of order m, the vectors b and $f \in R^m$, then the generic problem is to find the vector $x \in R^m$ such that:

$$\frac{1}{2}x^T A x - x^T b \;=\; \min_{y \geq f} \left(\frac{1}{2} y^T A y - y^T b \right).$$

In the projected relaxation method, the solution x is the limit of the sequence $\{x^k\}$ obtained by the following algorithm:

- Step 0: Initialize $x^0 \geq f$.
- Step k: Given the vector x^k, for $i = 1, \ldots, m$ compute:

$$\begin{aligned} \widehat{x}_i^{k+1} &= (b_i - \sum_{j=1}^{i-1} a_{ij} x_j^{k+1} - \sum_{j=i+1}^{m} a_{ij} x_j^k)/a_{ii} \\ x_i^{k+1} &= \max\left(f_i\,,\; x_i^k + \omega(\widehat{x}_i^{k+1} - x_i^k)\right) \end{aligned}$$

If we use piecewise linear finite elements space for spatial discretization we arrive to problem (4.17) with the finite elements matrix B and second member d_j, instead of the ones corresponding to finite differences discretization. In both cases, the properties of the involved matrices A and B guarantee the convergence of the projected relaxation method.

4.4. **Numerical methods for Asian options PDE models.** Having the idea that the present document should not be very large, in this section we just present brief comments about some existing references in the literature concerning to the numerical solution of Asian options pricing problems. In the Eurasian case, mainly a localization procedure to approximate the problem in a bounded domain, combined with different time and space discretization schemes are required. In the Amerasian case, additional numerical techniques to cope with the complementarity formulation are applied.

First, concerning the Eurasian pricing problem defined by equations (3.48)-(3.49), some changes of variable to reduce it in one dimension have been proposed in [18] and [21]. Nevertheless, these symmetry reduction techniques cannot be applied to the nonlinear problem of Amerasian options. In the early exercise case, an algorithm based on analytical and probabilistic arguments is proposed in [3] to solve problem (3.52)-(3.54). Another numerical approach based on a finite volumes method with a higher order nonlinear flux limiter for the convective terms, combined with a penalization technique to account with the inequality constraint is applied in [25]. Second order Crank-Nicolson characteristics (or semilagrangian) time discretization schemes jointly with piecewise linear and quadratic finite elements and a duality method for the inequality constraint have been recently used in [4], where the computed results are compared with those ones in [3] and [25]. Some of the previous papers include also the Eurasian case. More recently, a very efficient augmented lagrangian active set (ALAS) method has been proposed in [5] as an alternative to duality and penalization techniques. We address the reader to the previous articles and their references.

4.5. **Options on several assets.** The previously described numerical techniques can be generalized to the case of several underlying assets. Classical textbooks on finite differences and finite elements describe the application of these techniques to parabolic PDE problems mainly up to 3 spatial independent variables. Nevertheless, the more classical extensions of finite differences or finite elements result to be inefficient in the case of a high number of assets ($d > 3$), this fact is known as the curse of dimensionality. Recent strategies such as sparse grid techniques try to cope with this drawback (see [1] and the specific references to this topic therein).

4.6. **Numerical methods for bonds pricing with Black-Scholes.** Actually, in view of the equations that govern Black-Scholes models for bond pricing, analogous methods to the ones appearing in vanilla options can be used. Thus, Euler or Crank-Nicolson schemes for time discretization can be applied and combined with finite differences or finite elements for the interest rate variable to solve equation (3.59). In the case of coupon bearing bonds, jump conditions (3.61) have to be additionally implemented and the consideration of payment dates included in the time discretization mesh is recommended. In the case of bonds with call, put or both options the numerical techniques are analogous to the case of American options. Thus, projection, penalization or duality methods are appropriate. Sometimes the call option requires an advanced notice by the issuer indicating that the option will be exercised at the next following call date [11].
In Table 7 the CIR model data for a coupon bearing bond which is callable with notice in advance is presented. The numerical solution of this bond pricing problem has been addressed by using a characteristics method for time discretization combined with piecewise linear finite elements in the spot rate variable proposed in [11]. The computed results are presented in Table 8, where also the values of bond duration and convexity are included. Fig. 11 displays the bond price in terms of spot rates and time. Notice that the representation corresponds to the computational domain, which extends far away from the financially relevant region in the case of interest rates. Although they can be also devised in Fig. 11, the discontinuities of bond prices at coupon payment dates are better illustrated in Fig. 12, where the bond price evolution with time is shown for the fixed rate 3% . Fig. 12 also shows that after the tenth year of the bond life the presence of call options with notice at coupon payment dates smooths the spikes at these dates.

4.7. **Risk neutral probability and Monte Carlo simulation.** In this section we briefly present some arguments that allow to develop a Monte Carlo simulation procedure to price European vanilla options. We address the reader to [12] for the use of Monte Carlo in financial applications.
In order to apply this methodology, let us first consider a market with a riskless product, B, and a risky asset, S, the prices of which follow the dynamics

$$dS_t = \mu S_t\,dt + \sigma S_t\,dX_t, \quad dB_t = rB_t\,dt,$$

respectively, X_t representing a Wienner process under the real probability measure P.
A stochastic process, M_t, adapted to the filtration $\mathcal{F}_t$, is a martingale under measure P if it satisfies the conditions

Bond data	
Issue date:	01-01-2000
Maturity date:	05-03-2020
Principal value:	1
Speed of mean reversion (γ)**:**	0.54958
Long term rate (α)**:**	3.4847 %
Volatility parameter (ρ)**:**	0.38758
Market price of risk (λ)**:**	-1.0492
First annual coupon payment date:	05-03-2001
Last annual coupon payment date:	05-03-2020
Coupon:	0.0425
First annual call date:	05-03-2010
Last annual call date:	05-03-2019
Call price:	1
First annual notice date:	21-01-2010
Last annual notice date:	21-01-2019
Pricing date:	01-11-2010
Spot rate:	3 %

Table 7. Data for a callable with notice coupon bearing bond.

Bond price:	0,8834
Duration:	4,0732
Convexity:	-34,3649

Table 8. Bond price, duration and convexity concerning the data in Table 7.

- $E_P(\mid M_t \mid) < +\infty$.
- For all $\tau \leq t$, $E_P(M_t \mid \mathcal{F}_\tau) = M_\tau$.

The martingale pricing theory asserts that a continuous financial market with trading securities and strategies is arbitrage free if and only if there exists a probability measure under which the discounted asset prices are martingales. We can define this (risk-neutral) probability measure Q such that the process

$$W_t = \frac{(\mu - r)}{\sigma} t + X_t$$

is a Brownian motion under Q. In this new setting, it is easy to prove that the risky asset price verifies the stochastic equation

$$dS_t = rS_t\, dt \,+\, \sigma S_t\, dW_t,$$

where the drift μ has been replaced by the risk-free interest rate r. Notice that the exact solution of the previous stochastic equation is given by

$$S_t = S_0 \, \exp\left((r - \frac{\sigma^2}{2})\, t \,+\, \sigma\, X_t \right). \tag{4.19}$$

Next, by using Ito's lemma it is easy to prove that the asset discounted value $\overline{S}_t = e^{-rt} S_t$ satisfies

$$d\overline{S}_t = \sigma \overline{S}_t\, dW_t,$$

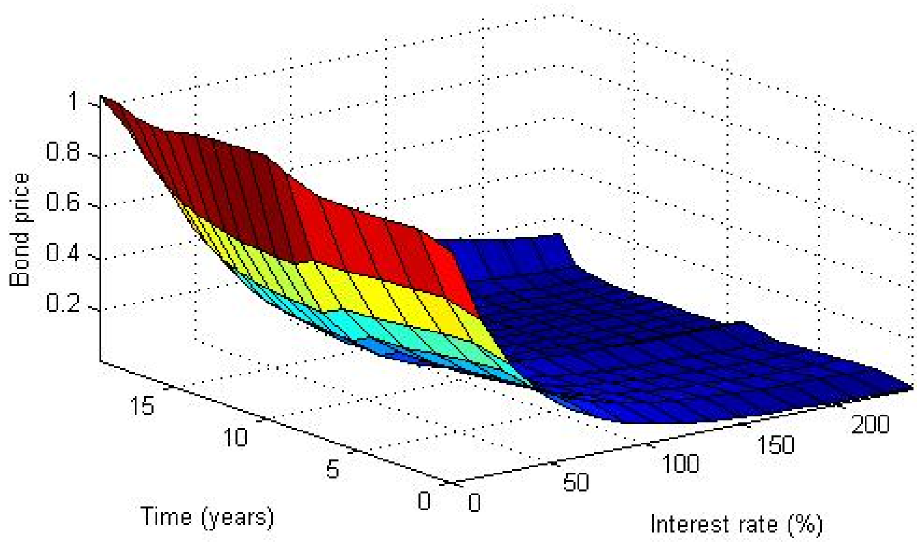

FIGURE 11. Bond price as a function of time and interest rate

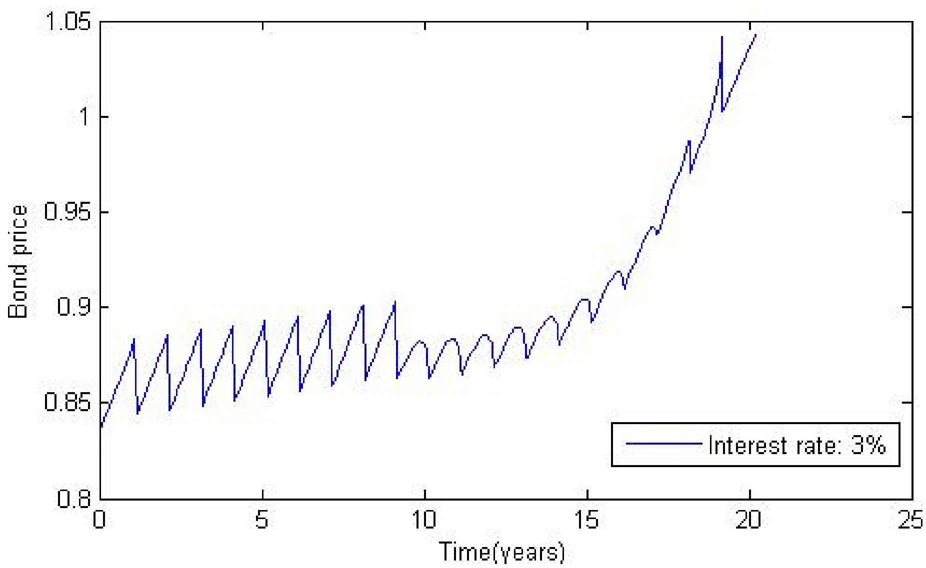

FIGURE 12. Bond price as a function of time for $r = 3\%$

so that $\overline{S}_t$ is a martingale under Q. Next, using the no arbitrage hypothesis, it can be shown that a European vanilla option with payoff given by $G(S)$ can be replicated by a self financed porfolio including risky and riskless assets, so that it is a martingale under Q. Therefore, by using the martingale property, the value of the option can be obtained as

$$V(S,t) = \overline{E}_Q(e^{-r(T-t)}V(S_T,T) \mid \mathcal{F}_t) = \overline{E}_Q(e^{-r(T-t)}G(S_T) \mid \mathcal{F}_t)\,, \tag{4.20}$$

where $\overline{E}_Q$ denotes the expectation under the risk neutral probability Q. This change of measure allows an easy way to compute option prices by simulation. Monte Carlo algorithm contains the following steps:

(1) As the payoff only depends on the final asset price, simulate the risk-neutral random walk to obtain prices at expiry date with expression (4.19) for $t = T$.
(2) Compute the payoffs, $G(S_T)$, for the different simulated prices at expiry date.
(3) Compute the expectation of the discounted prices according to formula (4.20).

As the we consider constant interest rates, actually it is easier to compute the discounted expectation of the prices.

Remark 7. *Monte Carlo simulation can be also applied to path-dependent options. In this case, in the first step a time discretization has to be carried out to simulate intermediate prices as the payoff depends on these intermediate values of the underlying. Typical examples of this case are discrete barrier, where the payoff depends on hitting or not a barrier price, or Asian options in which the payoff depends on certain time average of the asset prices. In the case of continuous barrier options, the time discretization has to be combined with the Brownian bridge technique which allows to incorporate the probabilities of touching the barriers between to successive discrete times considered in the simulation and use these probabilities in the pricing formula. On the other hand, the most popular Monte Carlo technique for American options has been proposed in* [14].

In the case of options depending on two assets, we can simulate the prices of different assets with

$$S^k_{t_{i+1}} = S^k_{t_i}\ \exp\left((r - \frac{\sigma_k^2}{2})\, dt\ +\ \sigma_k \sqrt{dt}\, X^k_{t_i}\right)$$

where we use the correlated Wiener processes X^1 and X^2. In order to correlate them, we can use auxiliar independent Wiener processes Y^1 and Y^2 to obtain

$$X^1 = Y^1, \quad X^2 = \rho_{12}, \quad Y^1 + (1 - \rho_{12})\, Y^2\,.$$

As in the case of one asset, martingale theory allows to obtain the option price with expression

$$V(t, S_0^1, S_0^2) = e^{-r(T-t)}\, \overline{E}_Q(G(S_T^1, S_T^2)),$$

where S_0^1 and S_0^2 denote the spot prices of the assets.
In the case of a greater number of underlying assets, there exist different techniques to take into account the correlation among them (correlation matrix), Choleski factorization being one possibility.

The advantage of Monte Carlo with respect to PDE based methods mainly concentrates in the case of derivatives depending on a high number of assets. Moreover, Monte Carlo is easier to understand than PDE methods and can be applied with small modifications to a large number of derivatives. In the case of one or two asset, finite differences or finite elements are quicker and for American options Monte Carlo results to be rather complex. In order to speed up Monte Carlo or to reduce the error, methods for variance reduction are required, such as antithetic variables or control variate techniques (see [12, 19], for example).

References

[1] I. Achdou, O. Pironneau,*Computational methods for option pricing,* SIAM, Philadelphia, 2005.

[2] E. Barucci, S. Polidoro, V. Vespri, *Some results on partial differential equation and Asian options,* Mathematical Models and Methods in Applied Sciences, 11 (2001) 475-497.

[3] J. Barraquand, T. Pudet, *Pricing American path-dependent contingent claims*, Mathematical Finance, 6 (1996) 17-51.

[4] A. Bermúdez, M.R. Nogueiras, C. Vázquez, *Numerical solution of variational inequalities for pricing Asian options by higher order Lagrange-Galerkin methods*, Applied Numerical Mathematics, 56 (2006), 1256-1270.

[5] A. Bermúdez, M.R. Nogueiras, C. Vázquez, *Comparison of two algorithms to solve the fixed-strike Amerasian options pricing problem*, in Free Boundary Problems. Theory and Applications, Birkhauser, Basel, 2007, 95-106.

[6] F. Black, M. Scholes, *The price of options and corporate liabilities*, J. Pol. Econ. 81 (1973) 637-659.

[7] D. Brigo, F. Mercurio, *Interest Rate Models: Theory and Practice (with Smile, Inflation and Credit)*, Springer Finance, 2007.

[8] R. Cont, P. Tankov, *Financial modeling with jump processes*, Chapman and Hall, Boca Raton, 2003.

[9] J.C. Cox, M. Rubinstein, *Options markets,* Pearson Education, 1985.

[10] C.M. Elliot, J.R. Ockendon, *Weak and variational methods for moving boundary problems,* Research Notes Math. N.59, Pitman, London 1982.

[11] J. Farto, C. Vázquez, *Numerical methods for pricing callable bonds with notice*, Applied Mathematics and Computation, 161 (2005), 989-1013.

[12] P. Glasserman, Monte Carlo methods in financial engineering, Springer, New York, 2004.

[13] R. Kangro, R. Nicolaides, *Far field boundary conditions for Black-Scholes equations*, SIAM Journal of Numerical Analysis, 38 (2000), 1357-1368.

[14] F.A. Longstaff, E.S. Schwartz, *Valuing American options by simulation: a simple least-squares approach,* Review of Financial Studies, 14 (2001), 113-147.

[15] T. Mikosh, *Elementary stochastic calculus with finance in view,* World Scientific Press, Singapur, 1998.

[16] M. Minoux, *Programmation mathématique: théorie et algorithmes,* Vol. 1, Dunod, Paris, 1983.

[17] M.R. Nogueiras, Numerical analysis of second order Lagrange-Galerkin schemes. Application to option pricing problems, Ph. D. Thesis, University of Santiago, 2005.

[18] L.C.G. Rogers, Z. Shi, *The value of an Asian option*, Joournal of Applied Probability, 32 (1995) 1077-1088.

[19] R. Seydel, *Tools for computational finance*, Springer-Verlag, Berlin, 2006.

[20] C. Vázquez, *An upwind numerical approach for an American and European option pricing model*, Applied Mathematics and Computation, 97 (1998), 273-286.

[21] J. Vecer, *A new PDE approach for pricing arithmetic average Asian options*, Journal of Computational Finance (2001), 105-113.

[22] P. Wilmott, S. Howison, J. Dewynne, *Option pricing. Mathematical models and computation* Oxford Financial Press, Oxford, 1993.

[23] P. Wilmott, S. Howison, J. Dewynne, *The mathematics of financial derivatives: a students introduction,* Cambridge University Press, Cambridge, 1995.

[24] P. Zhang, *Exotic options*, World Scientific Press, Singapore, 1998.

[25] R. Zvan, P. Forsyth, K. Vetzal, *Robust numerical methods for PDE for Asian options*, Journal of Computational Finance, 1 (1998), 39-78.

MAT - INDICE GENERAL

Serie A: CONFERENCIAS, SEMINARIOS Y TRABAJOS DE MATEMÁTICA (ISSN 1515-4904)

- Rubén D. Spies, "Differentiability of the solutions of a semilinear abstract Cauchy problem with respect to parameters", 1-10.
- Adriana C. Briozzo – María F. Natale – Domingo A. Tarzia, "An explicit solution for a two-phase Stefan problem with a similarity exponencial heat sources", 11-19.
- Domingo A. Tarzia, "An explicit solution for a two-phase unidimensional Stefan problem with a convective boundary condition at the fixed face", 21-27.

9 (2005): Micol Amar – Roberto Gianni, "A Brief Survey on Homogenization with a Physical Application".

10 (2005): Domingo A. Tarzia – Cristina V. Turner (Eds.), "Segundas Jornadas sobre Ecuaciones Diferenciales, Optimización y Análisis Numérico":

- Marcos Gaudiano – Cristina Turner, "Difusión de un solvente en un polímero vidrioso con una condición de contorno del tipo creciente en el tiempo", 1-9.
- Adriana C. Briozzo – María F. Natale – Domingo A. Tarzia, "A one-phase Lamé-Clapeyron-Stefan problem with nonlinear thermal coefficients", 11-16.
- Eduardo A. Santillan Marcus - Domingo A. Tarzia, "Un caso de determinación de coeficientes térmicos desconocidos de un material semiinfinito poroso a través de un problema de desublimación con acoplamiento de temperatura y humedad", 17-22.

11 (2005): Antonio Fasano, "Mathematical Models of Some Diffusive Processes with Free Boundaries".

12 (2006): Arturo De Pablo, "An Introduction to the Problem of Blow-up for Semilinear and Quasilinear Parabolic Equations".

13 (2006): Graciela G. Garguichevich – Claudia M. Gariboldi – Pedro R. Marangunic – Diego Pallara, "Direct methods in the calculus of variations".

14 (2007): María C. Maciel – Domingo A. Tarzia (Eds.), "Terceras Jornadas sobre Ecuaciones Diferenciales, Optimización y Análisis Numérico":

- Tatiana I. Gibelli – María C. Maciel, "Large-scale algorithms for minimizing a linear function with a strictly convex quadratic constraint", 1-12.
- María C. Maciel – Elvio A. Pilotta – Graciela N. Sottosanto, "Thickness optimization of an elastic beam", 13-23.
- María F. Natale – Eduardo A. Santillan Marcus – Domingo A. Tarzia, "Determinación de dos coeficientes térmicos a través de un problema de desublimación con acoplamiento de temperatura y humedad", 25-30.
- Rubén D. Spies – Karina G. Temperini, "Sobre la no convergencia del método de mínimos cuadrados en dimension infinita", 31-34.
- Juan C. Reginato – Domingo A. Tarzia, "An alternative method to compute Michaelis-Menten parameters from nutrient uptake data", 35-40.

15 (2008): D.A. Tarzia – R.H. Mascheroni (Eds.), "Workshop on Mathematical Modelling of Energy and Mass Transfer Processes, and Applications":

- María F. Natale – Domingo A. Tarzia, "The classical one-phase Stefan problem with temperature-dependent termal conductivity and a convective term", 1-16.
- Carmen Cortazar – Manuel Elgueta, "Non-local diffusion", 17-24.
- Luis T. Villa, "Consideraciones sobre el buen planteo de un modelo de frontera libre-móvil descriptivo de un proceso de freído por inmersión", 25-36.
- Ricardo Simpson – Isabel Figueroa – Arthur Teixeira, "Simple, practical, and efficient on-line correction of process deviations in batch retort through simulation", 37-53.

16 (2009): Mahdi Boukrouche, "A brief survey on lubrication problems with nonlinear boundary conditions".

17 (2010): Carlos Vázquez, "An introduction to Black-Scholes modeling and numerical methods in derivatives pricing".

Serie B: CURSOS Y SEMINARIOS PARA EDUCACIÓN MATEMÁTICA (ISSN 1515-4912)

1 (2000): Domingo A. Tarzia, "Cómo pensar, entender, razonar, demostrar y crear en Matemática".

2 (2003): Domingo A. Tarzia, "Matemática: Operaciones numéricas y geometría del plano"

INFORMACION PARA LOS AUTORES

Los trabajos han de estar escritos en español o inglés. Excepcionalmente el Director y el Comité Editorial podrán admitir trabajos escritos en otros idiomas ampliamente utilizados. Deberá presentarse el texto mecanografiado o elaborado mediante un procesador de textos, con caracteres de 12 puntos, en un rectángulo de 16cm×24cm y en una sola cara del papel. Trabajos escritos en LATEX o en MS-WORD serán bienvenidos y en ese caso el autor deberá adjuntar un diskette con los archivos correspondientes, o bien enviarlos por correo electrónico.

En cada trabajo deberá constar, en la primera página, a partir de la quinta línea, el título en letras mayúsculas y sin punto final, el nombre del o de los autores, su identificación institucional y su correspondiente dirección postal y electrónica. Se acompañará un resumen que no exceda las 200 palabras en español y otro en inglés, añadiendo en ambos las palabras claves. También se solicita la inclusión de la corresponiente AMS-Mathematics Subject Classification. Al final de la última página deberán dejarse al menos dos líneas en blanco para incluir los datos de recepción. Las tablas y gráficos deberán insertarse en el texto y estar numeradas en forma correlativa.

Las referencias bibliográficas se compondrán sólo de los trabajos mencionados en el texto y se incluirán al final, por orden alfabético de autores y en orden cronológico, si existieran varios trabajos del mismo autor; cada una precedida por el correspondiente número de orden, entre corchetes. Las citas en el texto se efectuarán según los siguientes modelos: [1]; Caffarelli & Vazquez [1]; Caffarelli & Vazquez (1995, [1]). Y en la referencia final:

[1] L. A. CAFFARELLI & J.L. VAZQUEZ, *A free-boundary problem for the heat equation arising inflame propagation*, Trans. Amer. Math. Soc., 347 (1995), pp. 411-441.

[2] A. FASANO & M. PRIMICERIO, *Blow-up and regularization for the Hele-Shaw problem,* in *Variational and free boundary problems*, Friedman A. & Spruck J. (Eds.), IMA Math. Appl. Vol. 53, Springer Verlag, New York (1993), pp. 73-85.

[3] J.F. RODRIGUES, *Obstacle problems in mathematical physics*, North-Holland Mathematics Studies N. 134, North-Holland, Amsterdam (1987).

INTERCAMBIOS

Departamento de Matemática – Biblioteca, Servicio de Canje
Facultad de Ciencias Empresariales -Universidad Austral
Paraguay 1950, S2000FZF ROSARIO, ARGENTINA

NUMEROS APARECIDOS

Serie A:

- #1 (2000): E. Mascolo – F. Siepe, “Functionals of the Calculus of Variations with non Standard Growth Conditions”.
- #2 (2000): D.A. Tarzia, “A Bibliography on Moving-Free Boundary Problems for the Heat-Diffusion Equation. The Stefan and Related Problems”.
- # 3 (2001): D.A. Tarzia (Ed.), “VI Seminario sobre Problemas de Frontera Libre y sus Aplicaciones”, Primera Parte.
- #4 (2001): D.A. Tarzia (Ed.), “VI Seminario sobre Problemas de Frontera Libre y sus Aplicaciones”, Segunda Parte.
- #5 (2001): D.A. Tarzia (Ed.), “VI Seminario sobre Problemas de Frontera Libre y sus Aplicaciones”, Tercera Parte.
- #6 (2002): F. Talamucci,“Some Problems Concerning with Mass and Heat Transfer in a Multi-Component System”.
- #7 (2004): D.A. Tarzia (Ed.), “Primeras Jornadas sobre Ecuaciones Diferenciales, Optimización y Análisis Numérico”, Primera Parte.
- #8 (2004): D.A. Tarzia (Ed.), “Primeras Jornadas sobre Ecuaciones Diferenciales, Optimización y Análisis Numérico”, Segunda Parte.
- #9 (2005): M. Amar – R. Gianni, “A Brief Survey on Homogenization with a Physical Application”.
- #10 (2005): D.A. Tarzia – C.V. Turner (Eds.), “Segundas Jornadas sobre Ecuaciones Diferenciales, Optimización y Análisis Numérico”.
- #11 (2005): A. Fasano, “Mathematical Models of Some Diffusive Processes with Free Boundaries”.
- #12 (2006): A. De Pablo, “An Introduction to the Problem of Blow-up for Semilinear and Quasilinear Parabolic Equations”.
- #13 (2006): G.G. Garguichevich – C.M. Gariboldi – P.R. Marangunic – D. Pallara, “Direct methods in the calculus of variations”.
- #14 (2007): M.C. Maciel – D.A. Tarzia (Eds.), “Terceras Jornadas sobre Ecuaciones Diferenciales, Optimización y Análisis Numérico”.
- #15 (2008): D.A. Tarzia – R.H. Mascheroni (Eds.), “Workshop on Mathematical Modelling of Energy and Mass Transfer Processes, and Applications”.
- #16 (2009): M. Boukrouche, “A brief survey on lubrication problems with nonlinear boundary conditions”.
- #17 (2010): C. Vázquez, “An introduction to Black-Scholes modeling and numerical methods in derivatives pricing”.

Serie B:

- #1(2000): D.A. Tarzia, “Cómo pensar, entender, razonar, demostrar y crear en Matemática”.
- #2(2003): D.A. Tarzia, “Matemática: Operaciones numéricas y geometría del plano”.

DEPARTAMENTO DE MATEMÁTICA
FACULTAD DE CIENCIAS EMPRESARIALES
UNIVERSIDAD AUSTRAL
Paraguay 1950 - S2000FZF ROSARIO - ARGENTINA
TEL: (54) 341-522-3000   FAX: (54) 341-522-3001
E-mail: dtarzia@austral.edu.ar