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An Explicit Solution for a Two-Phase Unidimensional Stefan Problem with a Convective Boundary Condition at the Fixed Face*

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Abstract

In this paper we do the mathematical analysis of the problem which was analysed in S.M. Zubair - M.A. Chaudhry, Wärme- und Stoffübertragung, 30 (1994), 77-81. We consider the solidification of a semi-infinite material which is initially at its liquid phase at a uniform temperature T_i. Suddenly at time $t > 0$ the fixed face $x = 0$ is submitted to a convective cooling condition with a time-dependent heat transfer coefficient of the type $h(t) = h_0 t^{-1/2}$ $(h_0 > 0)$. The bulk temperature of the liquid at a large distance from the solid-liquid interface is T_∞, a constant temperature such that $T_\infty < T_f < T_i$ where T_f is the freezing temperature. The density jump between the two phases are neglected.

We obtain that the corresponding phase-change process has an explicit solution of a similarity type for the solid-liquid interface and the temperature of both phases if an only if the coefficient h_0 is large enough, that is $h_0 > \frac{k_l}{\sqrt{\pi\alpha_l}}\frac{T_i - T_f}{T_i - T_\infty}$ where k_l and α_l are the conductivity and diffusion coefficients of the initial liquid phase.

Key words : Stefan problem, free boundary problem, Neumann solution, phase-change process, solidification process, similarity solution.

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I. Introduction.

Heat transfer problems involving a change of phase due to melting or freezing processes are very important in science and technology [5], [6], [9], [13], [14]. This kind of problems are generally refered as moving-free boundary problems which have been the subject of numerous theoretical, numerical and experimental investigations, e.g. we can see the large bibliography on the subject given in [18].

We consider the solidification of a semi-infinite material which is initially at its liquid phase at a uniform temperature T_i. Suddenly at time $t > 0$ the fixed face $x = 0$ is submitted to a convective cooling condition due to a sudden drop in the ambient temperature. The bulk temperature of the liquid at a large distance from the solid- liquid interface is

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T_∞, a constant such that $T_\infty < T_f < T_i$ where T_f is the freezing temperature. The density jump between the two phases are neglected.

In order to solve the phase-change process with a convective condition at the fixed face $x = 0$, approximate method were used, for example in [2], [8], [10], [12]. In [3], [4] a convective condition is considered after a transformation in order to solve a free boundary problem for a nonlinear absorption model of mixed saturated-unsaturated flow with a nonlinear soil water diffusivity.

In [19] the problem was analyzed and a closed-form expression for the solid-liquid interface and both temperatures were found when the heat transfer coefficient h is time-dependent and proportional to $t^{-\frac{1}{2}}$. The solution is obtain graphically.

The goal of this paper is to give the mathematical analysis of this problem, that is the solidification of a semi-infinite material which is initially at the constant temperature T_i and a convective cooling condition is impossed at the fixed boundary $x = 0$ for a time-dependent heat transfer coefficient of the type

$$h(t) = \frac{h_0}{\sqrt{t}}\ , \ h_0 > 0,\ t > 0. \tag{1}$$

We prove that there exists an instantaneous phase-change process if and only if the coefficient h_0 is large enough, that is

$$h_0 > \frac{k_l}{\sqrt{\pi\alpha_l}}\frac{T_i - T_f}{T_i - T_\infty} \tag{2}$$

where k_l and α_l are the conductivity and diffusion coefficients of the initial liquid phase. Moreover we can obtain the explicit expression for the solid-liquid interface $s(t)$ and the temperatures of the solid $T_s(x,t)$ and liquid $T_l(x,t)$ phases respectively.

The plan is the following: in Section II we solve the heat conduction problem for a semi-infinite material which is initially at a constant temperature T_i and a convective cooling condition of the type (1) is imposed at $x = 0$. The solution can be obtained explicitly and we can conclude that inequality (2) must hold if an instantaneous solidification process occurs.

In Section III we solve the corresponding phase-change problem; we get that the explicit solution for the solid-liquid interface and the temperature of both phases can be obtained if and only if the inequality (2) is verified for the coefficient h_0 which characterizes the dependent-time heat transfer coefficient $h(t)$ given by (1).

II. Heat conduction problem for a semi-infinite material with a convective condition at x=0.

We consider the heat conduction problem for the liquid phase which is initially at the constant temperature T_i and a convective cooling condition at $x = 0$ is imposed, that is

$$T_t = \alpha T_{xx},\ x > 0,\ t > 0 \tag{3}$$

$$T(x,0) = T(+\infty,t) = T_i,\ x > 0,\ t > 0 \tag{4}$$

$$kT_x(x,0) = \frac{h_0}{\sqrt{t}}\left(T(0,t) - T_\infty\right),\ t > 0 \tag{5}$$

where ρ, c, k and $\alpha = \frac{k}{\rho c}$ are the density mass, heat capacity, heat conductivity and diffusion coefficients of the liquid phase (we must consider the subscript l when it is necessary).

Theorem 1 *(i) The explicit solution to the problem* (3) – (5) *is given by*

$$T(x,t) = \frac{T_\infty + \frac{kT_i}{h_0\sqrt{\pi\alpha}}}{1 + \frac{k}{h_0\sqrt{\pi\alpha}}} + \frac{T_i - T_\infty}{1 + \frac{k}{h_0\sqrt{\pi\alpha}}} \operatorname{erf}\left(\frac{x}{2\sqrt{\alpha t}}\right), \quad x > 0, \; t > 0, \tag{6}$$

where the error function erf *is defined by*

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x \exp\left(-z^2\right) dz. \tag{7}$$

(ii) The temperature at the fixed face $x = 0$ *is constant for all* $t > 0$ *and it is given by*

$$T(0,t) = \frac{T_\infty + \frac{kT_i}{h_0\sqrt{\pi\alpha}}}{1 + \frac{k}{h_0\sqrt{\pi\alpha}}}, \quad \forall t > 0. \tag{8}$$

(iii) The material will undergo an instantaneous phase-change process when the coefficient h_0 *verifies the condition* (2).

Proof.

(i) By using the similarity method [1], [7], [11], [17], we get that a solution of the Eq. (3) is given by

$$T(x,t) = A + B \operatorname{erf}\left(\frac{x}{2\sqrt{\alpha t}}\right)$$

where coefficients A and B must be determined by impossing the two boundary conditions (4) and (5). From these conditions we obtain the following system of equations

$$A + B = T_i \tag{9}$$

$$kB = h_0 (A - T_\infty) \sqrt{\pi\alpha} \tag{10}$$

which solution is given by

$$A = \frac{T_\infty + \frac{kT_i}{h_0\sqrt{\pi\alpha}}}{1 + \frac{k}{h_0\sqrt{\pi\alpha}}}, \qquad B = \frac{T_i - T_\infty}{1 + \frac{k}{h_0\sqrt{\pi\alpha}}} \tag{11}$$

that is the expression (6) holds.

(ii) It follows by taking $x = 0$ in expression (6).

(iii) The material will undergo an instantaneous phase-change process if the constant temperature at the fixed face $T(0,t)$ is less than the freezing temperature T_f, that is

$$\frac{T_\infty + \frac{kT_i}{h_0\sqrt{\pi\alpha}}}{1 + \frac{k}{h_0\sqrt{\pi\alpha}}} < T_f \Longleftrightarrow T_\infty + \frac{kT_i}{h_0\sqrt{\pi\alpha}} < T_f + \frac{T_f k}{h_0\sqrt{\pi\alpha}}$$

$$\Longleftrightarrow \frac{k}{h_0\sqrt{\pi\alpha}} (T_i - T_f) < T_f - T_\infty \Longleftrightarrow \textit{condition } (2). \blacksquare$$

Remark 1 *The method utilized in the previous proof follows [15], [16]; it is useful in order to give us the necessary condition* (2) *for the coefficient* h_0 *but it does not give us the explicit solution for the solid-liquid interface and temperatures for the liquid and solid phases which will be the goal of the following Section III.*

III. Instantaneous phase-change process and its corresponding explicit solution.

We consider the following free boundary problem: find the solid-liquid interface $x = s(t)$ and the temperature $T(x,t)$ defined by

$$T(x,t) = \begin{cases} T_s(x,t) & \text{if } 0 < x < s(t), \quad t > 0 \\ T_f & \text{if } x = s(t), \quad t > 0 \\ T_l(x,t) & \text{if } x > s(t), \quad t > 0 \end{cases}$$

which satisfy the following equations and boundary conditions

$$T_{s_t} = \alpha_s T_{s_{xx}}, \ 0 < x < s(t), \ t > 0 \tag{12}$$

$$T_{l_t} = \alpha_l T_{l_{xx}}, \ x > s(t), \ t > 0 \tag{13}$$

$$T_s(s(t),t) = T_l(s(t),t) = T_f, \ x = s(t), \ t > 0 \tag{14}$$

$$T_l(x,0) = T_l(+\infty,t) = T_i, \ x > 0, \ t > 0 \tag{15}$$

$$k_s T_{s_x}(0,t) = \frac{h_0}{\sqrt{t}} (T_s(0,t) - T_\infty), \ t > 0 \tag{16}$$

$$k_s T_{s_x}(s(t),t) - k_l T_{l_x}(s(t),t) = \rho l \, \dot{s}(t), \ t > 0 \tag{17}$$

$$s(0) = 0 \tag{18}$$

where the subscripts s and l represent the solid and liquid phases respectively, ρ is the common density of mass and l is the latent heat of fusion, and $T_\infty < T_f < T_i$.

We obtain the following results:

Theorem 2 *(i) If the coefficient h_0 verifies the inequality* (2) *then the free boundary problem* (12)—(18) *has the explicit solution of a similarity type given by*

$$s(t) = 2\lambda\sqrt{\alpha_l t} \tag{19}$$

$$T_s(x,t) = T_\infty + \frac{(T_f - T_\infty)\left[1 + \frac{h_0\sqrt{\pi\alpha_s}}{k_s}\operatorname{erf}\left(\frac{x}{2\sqrt{\alpha_s t}}\right)\right]}{1 + \frac{h_0\sqrt{\pi\alpha_s}}{k_s}\operatorname{erf}\left(\lambda\sqrt{\frac{\alpha_l}{\alpha_s}}\right)} \tag{20}$$

$$T_l(x,t) = T_i - (T_i - T_f)\frac{\operatorname{erf}c\left(\frac{x}{2\sqrt{\alpha_l t}}\right)}{\operatorname{erf}c(\lambda)} \tag{21}$$

where $\operatorname{erf}c$ *is the complementary error function defined by* $\operatorname{erf}c(z) = 1 - \operatorname{erf}(z)$, $\forall z \geqslant 0$; *and the dimensionless parameter* $\lambda > 0$ *satisfies the following equation*

$$F(x) = x, \ x > 0 \tag{22}$$

where function F *and the* $b's$ *coefficients are given by*

$$F(x) = b_1\frac{\exp(-bx^2)}{1 + b_2\operatorname{erf}\left(x\sqrt{b}\right)} - b_3\frac{\exp(-x^2)}{\operatorname{erf}c(x)} \tag{23}$$

$$b = \frac{\alpha_l}{\alpha_s} > 0; \qquad b_1 = \frac{h_0\left(T_f - T_\infty\right)}{\rho l \sqrt{\alpha_l}} > 0 \tag{24}$$

$$b_2 = \frac{h_0}{h_s}\sqrt{\pi \alpha_s} > 0; \qquad b_3 = \frac{c_l\left(T_i - T_f\right)}{l\sqrt{\pi}} > 0. \tag{25}$$

(ii) The Eq. (22) has a unique solution if and only if the coefficient h_0 satisfies the inequality (2). In this case, there exists an instantaneous solidification process.

Proof.

Following the Neumann's method [6], [7], [17], the solution of the free boundary problem (12)-(18) is given by

$$T_s\left(x,t\right) = A + B\,\mathrm{erf}\left(\frac{x}{2\sqrt{\alpha_s t}}\right) \tag{26}$$

$$T_l\left(x,t\right) = C + D\,\mathrm{erf}\left(\frac{x}{2\sqrt{\alpha_l t}}\right) \tag{27}$$

$$s\left(t\right) = 2\lambda\sqrt{\alpha_l t} \tag{28}$$

where the coefficients A, B, C, D and λ must be determinated by imposing conditions $(14)-(17)$. We obtain

$$A = \frac{T_f + T_\infty \frac{h_0\sqrt{\pi\alpha_s}}{k_s}\,\mathrm{erf}\left(\lambda\sqrt{\frac{\alpha_l}{\alpha_s}}\right)}{1 + \frac{h_0\sqrt{\pi\alpha_s}}{k_s}\,\mathrm{erf}\left(\lambda\sqrt{\frac{\alpha_l}{\alpha_s}}\right)} \tag{29}$$

$$B = \frac{h_0\sqrt{\pi\alpha_s}}{k_s}\,\frac{T_f - T_\infty}{1 + \frac{h_0\sqrt{\pi\alpha_s}}{k_s}\,\mathrm{erf}\left(\lambda\sqrt{\frac{\alpha_l}{\alpha_s}}\right)} \tag{30}$$

$$C = \frac{T_f - T_i\,\mathrm{erf}\left(\lambda\right)}{\mathrm{erf}\,c\left(\lambda\right)}, \qquad D = \frac{T_i - T_f}{\mathrm{erf}\,c\left(\lambda\right)} \tag{31}$$

and coefficient λ must satisfy the Eq. (22).

Function F has the following properties:

$$F\left(0^+\right) = b_1 - b_3 = \frac{h_0\left(T_f - T_\infty\right)}{\rho_l\sqrt{\alpha_l}} - \frac{c_l\left(T_i - T_f\right)}{l\sqrt{\pi}} \tag{32}$$

$$F\left(+\infty\right) = -\infty, \qquad F'\left(x\right) < 0,\ \forall x > 0. \tag{33}$$

Therefore, there exists a unique solution $\lambda > 0$ of the Eq. (22) if and only if $F\left(0^+\right) > 0$, that is inequality (2) holds.■

Remark 2 *(i) We note that the temperature at the fixed face $x = 0$ is given by $T_s\left(0,t\right) = A < T_f$ because*

$$T_f - A = T_f - \frac{T_f + T_\infty \frac{h_0\sqrt{\pi\alpha_s}}{k_s}\,\mathrm{erf}\left(\lambda\sqrt{\frac{\alpha_l}{\alpha_s}}\right)}{1 + \frac{h_0\sqrt{\pi\alpha_s}}{k_s}\,\mathrm{erf}\left(\lambda\sqrt{\frac{\alpha_l}{\alpha_s}}\right)}$$

$$= \frac{\left(T_f - T_\infty\right)\frac{h_0\sqrt{\pi\alpha_s}}{k_s}\,\mathrm{erf}\left(\lambda\sqrt{\frac{\alpha_l}{\alpha_s}}\right)}{1 + \frac{h_0\sqrt{\pi\alpha_s}}{k_s}\,\mathrm{erf}\left(\lambda\sqrt{\frac{\alpha_l}{\alpha_s}}\right)} > 0.$$

(ii) We note that the inequality (2) *for the coefficient* h_0 *which characterizes the time-dependent heat transfer is of the type that it was obtained in [16] when a time-dependent heat flux condition on the fixed face is impossed.*

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