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OPTIMIZATION OF THE HEAT FLUX IN A MIXED ELLIPTIC PROBLEM WITH TEMPERATURE CONSTRAINTS

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Abstract

For a steady-state heat conduction problem in a poligonal domain $\Omega \subset \mathbb{R}^n$, with heat flux condition in a portion of the boundary, Γ_2, and a Fourier type condition in the rest of the boundary, Γ_1, we obtain the minimum total heat flux on Γ_2, so that the whole material is in the solid phase. For this purpose we use the finite element method in order to convert the optimization problem into a linear programming problem.

Key Words: Mixed Elliptic Problem, Steady-State Stefan Problem, Finite Element Method, Linear Programming Problem.

AMS Subject Classification: 65K10, 49K20, 35J85, 65N30

I. Introduction

We consider a steady-state heat conduction problem in a material Ω which occupies a poligonal bounded domain in $\mathbb{R}^n$, with boundary $\Gamma = \Gamma_1 \cup \Gamma_2$ (with meas$(\Gamma_1) > 0$ and meas$(\Gamma_2) > 0$). We impose a Newton law with a transfer coefficient $\alpha > 0$ and an exterior temperature $b > 0$ on Γ_1, and an outcoming heat flux $q > 0$ on Γ_2. We assume, without loss of generality, that the phase-change temperature of the material is 0°C.

This problem was studied in [TaTa] and it was stablished that if the heat flux q is between a minimum flux q_m and a maximum flux q_M, which are functions of the coefficient α and the temperature b, then there is a steady-state two phase Stefan Problem, that is the temperature is of non-constant sign in Ω.

In [GoTa1] a thermic flux optimization problem was solved: the maximization of the output heat flux on a portion of the boundary domain, Γ_2, while on the other portion, Γ_1, the distribution of the temperature was fixed. The maximization was carried out under the condition that there is no phase change.

In [GoTa2] the maximum heat total flux on Γ_2 was found, such that the temperature is positive in the whole domain Ω considering a boundary Fourier type condition on Γ_1.

Following the ideas of these papers, the goal of the present work is to minimize the total heat flux on the boundary Γ_2 so that all the material is in the solid phase. In order to solve this minimization problem we will use the finite element method and we will obtain a linear programming problem.

We remark that all the results of this work are still valid if we consider that the boundary Γ of the domain Ω is the union of three portions, Γ_1, Γ_2 and Γ_3, such that on Γ_1 and Γ_2 there exist the same conditions stated above, and Γ_3 is a wall impermeable to heat.

In Section II we present the mathematical model of the minimization problem and in Section III we discretize it and a linear programming problem must be solved in order to obtain the solution.

II. Mathematical Model of the Problem

If θ represents the temperature in Ω and we define the function $u = k_l\,\theta^+ - k_s\theta^-$, where k_i is the conductivity of the phase i ($i = l$ for the liquid and $i = s$ for the solid), then the following equations represent the mathematical model of the corresponding steady-state heat conduction problem [Du, Ta]:

$$\begin{aligned} &\Delta \mathbf{u} = 0 \qquad \text{in } \Omega \\ &-\frac{\partial \mathbf{u}}{\partial n}\bigg|_{\Gamma_1} = \alpha(\mathbf{u} - B), \qquad -\frac{\partial \mathbf{u}}{\partial n}\bigg|_{\Gamma_2} = q \end{aligned} \tag{1}$$

where $B = k_l b > 0$ and b is the exterior temperature.

We want to minimize the total heat flux on Γ_2 with the constraint that the whole material is in the solid phase. In other words, the problem is

$$\textit{Find } \mathbf{q}^* \textit{ such that } \mathbf{J}(\mathbf{q}^*) = \mathop{\textit{Inf}}_{u \,\leq\, 0 \text{ in } \Omega} \mathbf{J}(\mathbf{q}) \tag{2}$$

where

$$\mathbf{J}(\mathbf{q}) = \int_{\Gamma_2} q \, d\gamma \ , \ q \in L^2(\Gamma_2)$$

The variational formulation of the problem (1) is given by

$$a_\alpha(u,v) = L_\alpha(v)\,, \qquad \forall\, v \in V\,,\, u \in V \tag{3}$$

where

$$a_\alpha(u,v) = a(u,v) + \alpha\int_{\Gamma_1} uv\, d\gamma \quad , \qquad L_\alpha(v) = L(v) + \alpha B\int_{\Gamma_1} v\, d\gamma$$

$$a(u,v) = \int_\Omega \nabla u\, \nabla v\, dx\ , \qquad L(v) = -\int_{\Gamma_2} q\, v\, d\gamma\ ,\ V = H^1(\Omega).$$

III. The Discrete Problem and the Linear Programming Problem

We construct a regular triangulation τ_h, of the poligonal domain Ω with Lagrange triangles of type 1, with afin equivalent finit elements of class C^o, and we approach the space V by [Ci1]:

$$V_h = \left\{ v_h \in C^0(\bar{\Omega}) \middle/ v_h|_T \in P_1(T),\, \forall T \in \tau_h \right\},$$

where P_1 is the set of the polynomials of degree ≤ 1.

The approximate variational problem consists in finding $u_{h\alpha} \in V_h$ so that

$$a_\alpha(u_{h\alpha}, v_h) = L_\alpha(v_h)\ ,\ \forall\, v_h \in V_h\,. \tag{4}$$

We call ***N*** the total number of nodes of the triangulation, r is the number of nodes on the portion of the boundary Γ_1 and p is the number of nodes on Γ_2.

Let $\{\omega_i\}_{i=1}^N$ a basis of the space V_h. We can think the basis as

$$\{\omega_i\}_{i=1}^N = \{\omega_i^1\}_{i=1}^r \cup \{\omega_i^\Omega\}_{i=r+1}^{N-p} \cup \{\omega_i^2\}_{i=N-p+1}^N$$

where we denote ω_i^j the function whose value is 1 on the node N_i, of the boundary Γ_1 if j = 1, of the boundary Γ_2 if j = 2 or of the interior of the domain Ω if j = Ω, and whose values are zero in the other nodes. Then we have

$$u_{h\alpha} = \sum_{i=1}^r u_i^1 \omega_i^1 + \sum_{i=r+1}^{N-p} u_i^\Omega \omega_i^\Omega + \sum_{i=N-p+1}^N u_i^2 \omega_i^2 \tag{5}$$

where u_i^1 (i = 1, . . ., r), u_i^Ω (i = r + 1, . . ., $N-$ p) and u_i^2 (i = $N-$ p + 1, . . ., N) are the real unknown values at the corresponding nodes.

With the expresion (5) and considering $v_h = \omega_i^j$ in (4), we get the following system of linear equations

$$\begin{aligned} \boldsymbol{A}_1\mathbf{u}^1 + \boldsymbol{A}_2\mathbf{u}^\Omega + \boldsymbol{A}_3\mathbf{u}^2 &= \mathbf{b}^{\boldsymbol{\alpha}} \\ \boldsymbol{A}_4\mathbf{u}^1 + \boldsymbol{A}_5\mathbf{u}^\Omega + \boldsymbol{A}_6\mathbf{u}^2 &= \mathbf{0} \\ \boldsymbol{A}_7\mathbf{u}^1 + \boldsymbol{A}_8\mathbf{u}^\Omega + \boldsymbol{A}_9\mathbf{u}^2 &= \mathbf{b}(\boldsymbol{q}) \end{aligned} \qquad (6)$$

where the matrices $\boldsymbol{A}_1, ..., \boldsymbol{A}_9$ are given by:

$\boldsymbol{A}_1 = (a^1_{ij}) \in \boldsymbol{R}^{r\,x\,r}$, $a^1_{ij} = a_\alpha(\omega^1_j, \omega^1_i)$, $\boldsymbol{A}_2 = (a^2_{ij}) \in \boldsymbol{R}^{r\,x\,N-(p+r)}$, $a^2_{ij} = a(\omega^\Omega_j, \omega^1_i)$,

$\boldsymbol{A}_3 = (a^3_{ij}) \in \boldsymbol{R}^{r\,x\,p}$, $a^3_{ij} = a(\omega^2_j, \omega^1_i)$, $\boldsymbol{A}_4 = (a^4_{ij}) \in \boldsymbol{R}^{N-(p+r)\,x\,r}$, $a^4_{ij} = a(\omega^1_j, \omega^\Omega_i)$,

$\boldsymbol{A}_5 = (a^5_{ij}) \in \boldsymbol{R}^{N-(p+r)\,x\,N-(p+r)}$, $a^5_{ij} = a(\omega^\Omega_j, \omega^\Omega_i)$,

$\boldsymbol{A}_6 = (a^6_{ij}) \in \boldsymbol{R}^{N-(p+r)\,x\,p}$, $a^6_{ij} = a(\omega^2_j, \omega^\Omega_i)$, $\boldsymbol{A}_7 = (a^7_{ij}) \in \boldsymbol{R}^{p\,x\,r}$, $a^7_{ij} = a(\omega^1_j, \omega^2_i)$,

$\boldsymbol{A}_8 = (a^8_{ij}) \in \boldsymbol{R}^{p\,x\,r}$, $a^8_{ij} = a(\omega^\Omega_j, \omega^2_i)$, $\boldsymbol{A}_9 = (a^9_{ij}) \in \boldsymbol{R}^{p\,x\,p}$, $a^9_{ij} = a(\omega^2_j, \omega^2_i)$,

$\mathbf{u}^1 \in \boldsymbol{R}^r$, $\mathbf{u}^2 \in \boldsymbol{R}^p$, $\mathbf{u}^\Omega \in \boldsymbol{R}^{N-(p+r)}$

and $\mathbf{b}^{\boldsymbol{\alpha}} = (b^\alpha_i)^r_{i=1} \in \boldsymbol{R}^r$, $\mathbf{b}(\boldsymbol{q}) = (b_i(\boldsymbol{q}))^N_{i=N-p+1} \in \boldsymbol{R}^p$,

with

$$b^\alpha_i = \alpha B \int_{\Gamma_1} \omega^1_i \, d\gamma \,, \qquad b_i(\boldsymbol{q}) = -\int_{\Gamma_2} q\, \omega^2_i \, d\gamma \,.$$

The system (6) can be expressed as follows

$$\boldsymbol{A}\mathbf{u} = \boldsymbol{b}, \qquad (7)$$

where $\boldsymbol{A} \in \boldsymbol{R}^{N\,x\,N}$ is the finite element matrix , $\boldsymbol{A} = \begin{pmatrix} \boldsymbol{A}_1 & \boldsymbol{A}_2 & \boldsymbol{A}_3 \\ \boldsymbol{A}_4 & \boldsymbol{A}_5 & \boldsymbol{A}_6 \\ \boldsymbol{A}_7 & \boldsymbol{A}_8 & \boldsymbol{A}_9 \end{pmatrix}$

and

$$\mathbf{u} = (\mathbf{u}^1, \mathbf{u}^\Omega, \mathbf{u}^2) \in \boldsymbol{R}^N, \qquad \boldsymbol{b} = (\mathbf{b}^{\boldsymbol{\alpha}}, 0, \mathbf{b}(\boldsymbol{q})) \in \boldsymbol{R}^N.$$

We obtain the elements b^α_i and $b_i(\boldsymbol{q})$ through a numerical computation of the corresponding integrals, for example by using the trapezoidal rule. For this purpose we consider that the curve Γ_i $(i = 1, 2)$ can be decomposed by

$$\Gamma_i = \bigcup_{j=1}^{s} \Gamma_i^j, \qquad \text{with } s = \text{p} - 1 \text{ if } \Gamma_i \text{ is open or} \qquad s = \text{p if } \Gamma_i \text{ is closed}$$

and

$$\int_{\Gamma_i} f \, d\gamma = \sum_{j=1}^{s} \int_{\Gamma_i^j} f \, d\gamma \approx \sum_{j=1}^{s} \frac{1}{2} \left|\Gamma_i^j\right| \Big(f(\mathcal{N}_j) + f(\mathcal{N}_{j+1}) \Big),$$

where we have denoted with Γ_i^j the portion of the curve Γ_i whose limit points are the nodes $\mathcal{N}_j$ and $\mathcal{N}_{j+1}$ and we have denoted with $\left|\Gamma_i^j\right|$ the measure of that portion of the curve.

In this way the linear system (7) becomes in

$$\boldsymbol{A}\mathbf{u} = \boldsymbol{M}\widetilde{\boldsymbol{q}} + \widetilde{\boldsymbol{b}}$$

where :

- $\widetilde{\boldsymbol{q}} \in \boldsymbol{R}^{\boldsymbol{p}}$ and $\widetilde{\boldsymbol{q}}_{\boldsymbol{i}}$ is the value of the flux $\boldsymbol{q}$ in the node $\mathcal{N}_{N-\text{p+i}}$,

- $\widetilde{\boldsymbol{b}} \in \boldsymbol{R}^{\boldsymbol{N}}$, $\widetilde{\boldsymbol{b}}_{\boldsymbol{i}}$ approaches the value of b_i^{α} for i = 1 . . .r and $\widetilde{\boldsymbol{b}}_{\boldsymbol{i}} = 0$ for i = r+1, . . ., N,

- $\boldsymbol{M} = \begin{pmatrix} \boldsymbol{M}^1 \\ \boldsymbol{M}^2 \end{pmatrix} \in \boldsymbol{R}^{\boldsymbol{N} \times \boldsymbol{p}}$, $\quad \boldsymbol{M}^1 \in \boldsymbol{R}^{(\boldsymbol{N}-\boldsymbol{p}) \times \boldsymbol{p}}$ is a zero matrix,

$\boldsymbol{M}^2 = (m_{ii})_{i=N-p+1}^{N} \in \boldsymbol{R}^{\boldsymbol{p} \times \boldsymbol{p}}$ is a diagonal matrix , with

$$m_{ii} = \begin{cases} -\frac{1}{2}\left|\Gamma_2^i\right| & \text{if } i = N - p + 1 \text{ or } i = N, \\ -\frac{1}{2}\left(\left|\Gamma_2^{i-1}\right| + \left|\Gamma_2^i\right|\right) & \text{if } i = N - p + 2, \ldots, N - 1, \end{cases}$$

if Γ_2 is an open curve, or

$$m_{ii} = -\frac{1}{2}\left(\left|\Gamma_2^{i-1}\right| + \left|\Gamma_2^i\right|\right) \qquad \text{if } i = N - p + 1, \ldots, N,$$

if Γ_2 is a closed curve, (here we have considered $\Gamma_2^{N-p} = \Gamma_2^N$), and the i-th component of the vector $\boldsymbol{M}\widetilde{\boldsymbol{q}}$ approaches the value of $b_i(\boldsymbol{q})$ i.e. $(\boldsymbol{M}\widetilde{\boldsymbol{q}})_i \approx b_i(\boldsymbol{q})$.

After all these considerations, the optimization problem (2) is transformed into the following linear programming problem :

$$\underset{u \in \boldsymbol{U}}{\text{Minimize}} \ \hat{F}(\widetilde{\boldsymbol{q}}) = \left\langle \boldsymbol{T}_{\Gamma_2}, \widetilde{\boldsymbol{q}} \right\rangle_{\boldsymbol{R}^{\boldsymbol{p}}} \tag{8}$$

where

$$\boldsymbol{T}_{\Gamma_2} = \begin{cases} \frac{1}{2}\left(\left|\boldsymbol{\Gamma}_2^1\right|, \left(\left|\Gamma_2^1\right| + \left|\Gamma_2^2\right|\right), \ldots, \left(\left|\Gamma_2^{p-2}\right| + \left|\Gamma_2^{p-1}\right|\right), \left|\Gamma_2^{p-1}\right|\right) & \text{if } \Gamma_2 \text{ is open,} \\ \frac{1}{2}\left(\left(\left|\Gamma_2^p\right| + \left|\boldsymbol{\Gamma}_2^1\right|\right), \left(\left|\Gamma_2^1\right| + \left|\Gamma_2^2\right|\right), \ldots, \left(\left|\Gamma_2^{p-1}\right| + \left|\Gamma_2^p\right|\right)\right) & \text{if } \Gamma_2 \text{ is closed,} \end{cases}$$

$<\,.\,,\,.\,>_{\boldsymbol{R}^p}$ is the usual inner product in $\boldsymbol{R}^p$ and the set $\boldsymbol{U}$ is defined by

$$\boldsymbol{U} = \{\ \tilde{\boldsymbol{q}} \in \boldsymbol{R}^p\ :\ \boldsymbol{C}\tilde{\boldsymbol{q}} \leq \boldsymbol{d}\ ,\ \boldsymbol{C} \in \boldsymbol{R}^{N\times p},\ \boldsymbol{d} \in \boldsymbol{R}^N \},$$

with $\boldsymbol{C} = \boldsymbol{A}^{-1}\boldsymbol{M}$ and $\boldsymbol{d} = -\boldsymbol{A}^{-1}\tilde{\boldsymbol{b}}$.

Taking into account that $\hat{F}(\tilde{\boldsymbol{q}}) \geq 0,\ \forall\ \tilde{\boldsymbol{q}} \geq 0$ it results that the linear programming problem (8) admits at least one solution [Ci2].

We construct a programm in MATLAB and we get the solution of the problem (8) for some different domains. From these numerical results we can guess that the minimum optimum flux is given by $q^* = -\left.\frac{\partial \mathbf{u}^*}{\partial n}\right|_{\Gamma_2}$ where **u*** is the solution of the following elliptic problem

$$\Delta \mathbf{u}^* = 0 \qquad \text{in } \Omega$$

$$\left.\mathbf{u}^*\right|_{\Gamma_1} = 0\ , \qquad \left.\frac{\partial \mathbf{u}^*}{\partial n}\right|_{\Gamma_1} = \alpha B.$$

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