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THE CLASSICAL ONE-PHASE STEFAN PROBLEM WITH TEMPERATURE-DEPENDENT THERMAL CONDUCTIVITY AND A CONVECTIVE TERM

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Abstract

We study a one-phase Stefan problem for a semi-infinite material with temperature-dependent thermal conductivity and a convective term with a constant temperature boundary condition or a heat flux boundary condition of the type $-q_0/\sqrt{t}$ $(q_0 > 0)$ at the fixed face $x = 0$. We obtain in both cases sufficient conditions for data in order to have a parametric representation of the solution of the similarity type for $t \geq t_0 > 0$ with t_0 an arbitrary positive time. We improve the results given in Rogers-Broadbridge, ZAMP, 39 (1988), 122-129 and Natale-Tarzia, Int. J. Eng. Sci., 41 (2003), 1685-1698 obtaining explicit solutions through the unique solution of a Cauchy problem with the time as a parameter and we also give an algorithm in order to compute the explicit solutions.

Key words : Stefan problem, free boundary problem, moving boundary problem, phase-change process, nonlinear thermal conductivity, fusion, solidification, similarity solution.

2000 AMS Subject Classification: 35R35, 80A22, 35C05

I. Introduction.

We consider Stefan problems for a semi-infinite region $x > 0$ with temperature-dependent thermal conductivity and a convective term with phase change temperature $\theta_f = 0$ [16]. In all of them is required to determine the evolution of the moving phase separation $x = s(t)$ and the temperature distribution $\theta(x, t)$. The modeling of this kind of systems is a problem with a great mathematical and industrial significance. Phase-change

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problems appear frequently in industrial processes an other problems of technological interest [1],[2],[8] - [14],[17]. A large bibliography on the subject was given in [27]. We consider one-phase Stefan problems in fusion process with nonlinear heat conduction equations.

Owing to [19], [24] we consider the following free boundary (fusion process) problem

$$\rho c\frac{\partial \theta}{\partial t}=\frac{\partial}{\partial x}\left(k(\theta,x)\frac{\partial \theta}{\partial x}\right)-v(\theta)\frac{\partial \theta}{\partial x}\ ,\ 0<x<s(t)\ ,\ t>0 \tag{1}$$

$$k(\theta(0,t),0)\frac{\partial \theta}{\partial x}(0,t)=-\frac{q_0}{\sqrt{t}}\ ,\ q_0>0\ ,\ t>0 \tag{2}$$

$$k\left(\theta\left(s(t),t\right),s(t)\right)\frac{\partial \theta}{\partial x}\left(s(t),t\right)=-\rho l\dot{s}(t)\ ,\ t>0 \tag{3}$$

$$\theta\left(s(t),t\right)=0\ ,\ t>0\ ,\ s(0)=0 \tag{4}$$

where the thermal conductivity $k(\theta,x)$ and the velocity $v(\theta)$ are given by

$$v(\theta)=\rho c\frac{d}{2\left(a+b\theta\right)^2}\quad ,\quad k(\theta,x)=\rho c\frac{1+dx}{\left(a+b\theta\right)^2} \tag{5}$$

and c,ρ and l are the specific heat, the density and the latent heat of fusion of the medium respectively, all of them are assumed to be constant with positive parameters a,d and real parameter b. This kind of nonlinear thermal conductivity or diffusion coefficients was considered in numerous papers, e.g. [3, 6, 7, 15, 20, 21, 25]. The nonlinear transport Eq.(1) arises in connection with unsaturated flow in heterogeneous porous media. If we set $d=0$ and $b=0$ in the free boundary problem $(1)-(5)$ then we retrieve the classical one-phase Lamé-Clapeyron-Stefan problem. The first explicit solution for the one-phase Stefan problem was given in [16]. Here $-q_0/\sqrt{t}$ denotes the prescribed flux on the boundary $x=0$ which is of the type imposed in [26]. We will determine which conditions on the parameters of the problem must be satisfied in order to have an instantaneous phase-change process.

In Section II we consider the free boundary problem $(1)-(4)$ with the nonlinear heat coefficients (5) under the hypotheses $b>0$ and $a>bl/c$, or $b<0$. We follow [19, 24] and we improve [19] in the sense that the existence of the explicit solution of the problem $(1)-(5)$ is obtained through the unique solution of the Cauchy problem $(55)-(56)$ in the spatial variable and the time t is a parameter for $t\geqslant t_0>0$ where t_0 is an arbitrary positive time. This explicit solution can be obtained as a function of a parameter δ which is given as the unique solution of the transcendental Eq. (38). We also give an algorithm in order to compute the explicit solution for the temperature $\theta=\theta(x,t)$ and free boundary $x=s(t)$.

In Section III we consider the free boundary problem $(1),(3)-(4)$ with the nonlinear heat coefficients (5) and the temperature boundary condition (62) (with θ_0 is a constant temperature given on the fixed boundary $x=0$) instead of the heat flux condition (2) under the hypotheses $a > 0, d > 0, b < 0$ and $a + b\theta_0 > 0$. We improve [19] obtaining the explicit solution through the unique solution of the Cauchy problem $(92) - (93)$ in the spatial variable and the time t is a parameter for $t \geqslant t_0 > 0$ where t_0 is an arbitrary positive time. In this case, the explicit solution can be also obtained as a function of a parameter β which is the unique solution of the Eq. (85). We also give an algorithm to compute the temperature $\theta = \theta(x,t)$ and free boundary $x = s(t)$. Other problems with nonlinear thermal coefficients in this subject are also given in [4, 5, 18, 22, 23].

II. Solution of the free boundary problem with heat flux condition on the fixed face.

We consider the problem $(1) - (4)$. Taking into account (5) we can put our problem as

$$\frac{\partial\theta}{\partial t} = \frac{\partial}{\partial x}\left(\frac{1+dx}{(a+b\theta)^2}\frac{\partial\theta}{\partial x} + \frac{d}{2b\,(a+b\theta)}\right), \quad 0 < x < s(t)\ ,\ t > 0 \tag{6}$$

$$\frac{1}{(a+b\,\theta\,(0,t))^2}\frac{\partial\theta}{\partial x}(0,t) = -\frac{q_0^*}{\sqrt{t}}\ ,\ t > 0 \tag{7}$$

$$\frac{1+ds(t)}{a^2}\frac{\partial\theta}{\partial x}\left(s(t),t\right) = -\alpha\dot{s}(t)\ ,\ t > 0 \tag{8}$$

$$\theta\left(s(t),t\right) = 0\ ,\ t > 0\ ,\ s(0) = 0 \tag{9}$$

where $\alpha = \dfrac{l}{c}$, $q_0^* = \dfrac{q_0}{\rho c}$ and $a, d \in \mathbb{R}^+, b \in \mathbb{R}$.

If we define the transformations in the same way as in [20], [24]

$$\begin{cases} y = \dfrac{2}{d}\left[(1+dx)^{\frac{1}{2}} - 1\right] \quad , \quad \overline{S}(t) = \dfrac{2}{d}\left[(1+ds(t))^{\frac{1}{2}} - 1\right] \\ \overline{\theta}(y,t) = \theta(x,t) \end{cases} \tag{10}$$

we obtain the following free boundary problem

$$\frac{\partial\overline{\theta}}{\partial t} = \frac{\partial}{\partial y}\left(\frac{1}{\left(a+b\overline{\theta}\right)^2}\frac{\partial\overline{\theta}}{\partial y}\right), \quad 0 < y < \overline{S}(t)\ ,\ t > 0 \tag{11}$$

$$\frac{1}{\left(a+b\,\overline{\theta}\,(0,t)\right)^2}\frac{\partial\overline{\theta}}{\partial y}(0,t) = -\frac{q_0^*}{\sqrt{t}}\ ,\ t > 0 \tag{12}$$

$$\frac{1}{a^2}\frac{\partial\overline{\theta}}{\partial y}\left(\overline{S}(t),t\right) = -\alpha\dot{\overline{S}}(t)\ ,\ t > 0 \tag{13}$$

$$\overline{\theta}\left(\overline{S}(t),t\right)=0\ ,\ t>0\quad ,\quad \overline{S}(0)=0 \tag{14}$$

Then, we define the new transformation

$$\begin{cases} y^{*}=y^{*}(y,t)=\int\limits_{S(t)}^{y}\left(a+b\overline{\theta}\left(\sigma,t\right)\right)d\sigma+\left(-\alpha b+a\right)\overline{S}(t) \\ \theta^{*}(y^{*},t^{*})=\dfrac{1}{a+b\overline{\theta}(y,t)}\quad ,\quad t^{*}=t \\ S^{*}(t^{*})=y^{*}\Big|_{y=\overline{S}(t)}=\left(-\alpha b+a\right)\overline{S}(t). \end{cases} \tag{15}$$

In order to obtain an alternative expression for y^{*} we compute

$$\begin{aligned} \frac{\partial y^{*}}{\partial t} &= -\left(a+b\overline{\theta}\left(\overline{S}(t),t\right)\right)\overset{\bullet}{\overline{S}}(t)+\int\limits_{S(t)}^{y}b\frac{\partial\overline{\theta}}{\partial t}\left(\sigma,t\right)d\sigma+\left(-\alpha b+a\right)\overset{\bullet}{\overline{S}}(t)= \\ &= -\alpha b\overset{\bullet}{\overline{S}}(t)+\int\limits_{S(t)}^{y}b\frac{\partial}{\partial\sigma}\left(\frac{1}{\left(a+b\overline{\theta}\left(\sigma,t\right)\right)^{2}}\frac{\partial\overline{\theta}}{\partial\sigma}\right)d\sigma= \\ &= -\alpha b\overset{\bullet}{\overline{S}}(t)+b\left(\frac{1}{\left(a+b\overline{\theta}\left(y,t\right)\right)^{2}}\frac{\partial\overline{\theta}}{\partial y}(y,t)-\frac{1}{a^{2}}\frac{\partial\overline{\theta}}{\partial y}(\overline{S}(t),t)\right)= \\ &= \frac{b}{\left(a+b\overline{\theta}\left(y,t\right)\right)^{2}}\frac{\partial\overline{\theta}}{\partial y}(y,t)= \end{aligned}$$

$$=\int\limits_{0}^{y}\frac{\partial}{\partial\sigma}\left(\frac{b}{\left(a+b\overline{\theta}\left(\sigma,t\right)\right)^{2}}\frac{\partial\overline{\theta}}{\partial\sigma}(\sigma,t)\right)d\sigma+\frac{b}{\left(a+b\overline{\theta}\left(0,t\right)\right)^{2}}\frac{\partial\overline{\theta}}{\partial y}(0,t) \tag{16}$$

$$=b\int\limits_{0}^{y}\frac{\partial\overline{\theta}}{\partial t}\left(\sigma,t\right)d\sigma-\frac{bq_{0}^{*}}{\sqrt{t}}\ , \tag{17}$$

then the new expression for y^{*} is given by

$$y^{*}(y,t)=\int\limits_{0}^{t}\left(\int\limits_{0}^{y}\frac{\partial}{\partial\sigma}\left(a+b\overline{\theta}\left(\sigma,\tau\right)\right)d\sigma-\frac{bq_{0}^{*}}{\sqrt{\tau}}\right)d\tau+\int\limits_{0}^{y}\left(a+b\overline{\theta}\left(\sigma,0\right)\right)d\sigma=$$

$$=\int\limits_{0}^{y}\left(a+b\overline{\theta}\left(\sigma,t\right)\right)d\sigma-2bq_{0}^{*}\sqrt{t}\ . \tag{18}$$

Now, applying (10) and (15) the problem (6) – (9) is transformed in a Stefan-like problem with a convective boundary condition given by [28]

$$\frac{\partial\theta^{*}}{\partial t^{*}}=\frac{\partial^{2}\theta^{*}}{\partial y^{*2}}\ ,\ -2bq_{0}^{*}\sqrt{t^{*}}<y^{*}<S^{*}(t^{*})\ ,\ t^{*}>0 \tag{19}$$

$$\frac{\partial \theta^*}{\partial y^*}\left(-2bq_0^*\sqrt{t^*}, t^*\right) = \frac{q_0^* b}{\sqrt{t^*}}\,\theta^*\left(-2bq_0^*\sqrt{t^*}, t^*\right)\ ,\ t^* > 0 \tag{20}$$

$$\frac{\partial \theta^*}{\partial y^*}\left(S^*(t^*), t^*\right) = \alpha^* \dot{S}^*(t^*)\ ,\ t^* > 0 \tag{21}$$

$$\theta^*\left(S^*(t^*), t^*\right) = \theta_f^*\ ,\ t^* > 0\quad ,\quad S^*(0) = 0 \tag{22}$$

where

$$\alpha^* = \frac{\alpha b}{a\left(-\alpha b + a\right)}\qquad ,\qquad \theta_f^* = \frac{1}{a}\,. \tag{23}$$

Then, if we introduce the similarity variable:

$$\xi^* = \frac{y^*}{\sqrt{2\gamma^* t^*}} \tag{24}$$

where γ^* is a dimensionless positive constant to be determined, and the solution is sought of the type

$$\theta^*(y^*, t^*) = \Theta^*\left(\xi^*\right)\quad ,\quad S^*(t^*) = \sqrt{2\gamma^* t^*}\,, \tag{25}$$

then, we get that $(19) - (22)$ yields

$$\frac{d^2\Theta^*}{d\xi^{*2}} + \gamma^*\xi^*\frac{d\Theta^*}{d\xi^*} = 0\ ,\ -bq_0^*\sqrt{\tfrac{2}{\gamma^*}} < \xi^* < 1 \tag{26}$$

$$\frac{d\Theta^*}{d\xi^*}\left(-bq_0^*\sqrt{\tfrac{2}{\gamma^*}}\right) = \sqrt{2\gamma^*}q_0^* b\ \Theta^*\left(-bq_0^*\sqrt{\tfrac{2}{\gamma^*}}\right) \tag{27}$$

$$\frac{d\Theta^*}{d\xi^*}\left(1\right) = \alpha^*\gamma^* \tag{28}$$

$$\Theta^*\left(1\right) = \theta_f^*. \tag{29}$$

The solution of the differential equation (26) is given by

$$\Theta^*\left(\xi^*\right) = A\,\mathrm{erf}\left(\sqrt{\tfrac{\gamma^*}{2}}\xi^*\right) + B \tag{30}$$

where A and B are two unknown coefficients to be determined. From (28) and (29) we get

$$A = \sqrt{\pi}\sqrt{\tfrac{\gamma^*}{2}}\alpha^* \exp\left(\tfrac{\gamma^*}{2}\right) \tag{31}$$

$$B = \tfrac{1}{a} - \sqrt{\pi}\sqrt{\tfrac{\gamma^*}{2}}\alpha^* \exp\left(\tfrac{\gamma^*}{2}\right) \operatorname{erf}\left(\sqrt{\tfrac{\gamma^*}{2}}\right) . \tag{32}$$

where

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x \exp\left(-u^2\right) du \ . \tag{33}$$

The unknown constant γ^* is determined by the remaining boundary condition (28) which yields the following equation

$$\alpha^* \sqrt{\gamma^*} = \sqrt{\tfrac{2}{\pi}} \frac{\theta_f^*}{g\left(bq_0^*, \frac{1}{\sqrt{\pi}}\right) + \operatorname{erf}\left(\sqrt{\frac{\gamma^*}{2}}\right)} \exp\left(-\tfrac{\gamma^*}{2}\right) \ , \ \gamma^* > 0, \tag{34}$$

where

$$\begin{aligned} & g(x,p) = \operatorname{erf}(x) + pR(x) \ , \ x > 0 \ , \ p \in \mathbb{R} \\ & R(x) = \frac{\exp(-x^2)}{x} = \frac{1}{P(x)} \ , \ P(x) = x\exp(x^2) \ , \ x > 0 \\ & E(x) = x\exp(x^2)\operatorname{erf}(x) = P(x)\operatorname{erf}(x) \ , \ x > 0. \end{aligned} \tag{35}$$

By defining the new unknown

$$\delta = \sqrt{\frac{\gamma^*}{2}} \tag{36}$$

the equation (34) is equivalent to the equation

$$g(bq_0^*, \tfrac{1}{\sqrt{\pi}})P(\delta) = \frac{\frac{ac}{bl} - 1}{\sqrt{\pi}} - E(\delta) \quad , \quad \delta > 0 \quad , \tag{37}$$

that is

$$g\left(\delta, \tfrac{1}{\sqrt{\pi}}(1 - \tfrac{a}{\alpha b})\right) = -g\left(bq_0^*, \tfrac{1}{\sqrt{\pi}}\right) \quad , \quad \delta > 0 \ . \tag{38}$$

Theorem 1.- If $b > 0$ and $a > \frac{bl}{c}$, or $b < 0$, then the free boundary problem $(1) - (4)$ has a unique solution of the similarity type which is given by

$$\begin{aligned} \theta(\xi) &= \frac{1}{b}\left[\frac{1}{A\operatorname{erf}\left(\sqrt{\frac{\gamma^*}{2}}\xi^*\right) + B} - a\right] \\ \xi &= \tfrac{y}{\sqrt{2\gamma t}} = \frac{\frac{2}{d}\left[(1+dx)^{\frac{1}{2}} - 1\right]}{\sqrt{2\gamma t}} \\ s(t) &= \frac{1}{d}\left[\left(1 + \frac{d}{2}\sqrt{2\gamma t}\right)^2 - 1\right] \end{aligned} \tag{39}$$

where

$$\xi = (-\alpha b + a) \int_{-\sqrt{\frac{2}{\gamma^*}}bq_0^*}^{\xi^*} \left[A \operatorname{erf}\left(\sqrt{\tfrac{\gamma^*}{2}}\sigma\right) + B \right] d\sigma \tag{40}$$

and A, B and γ are given by $(31), (32)$ and (43) respectively.

Proof. First, we have to study the existence and uniqueness of the equation (34) or (38). It's easy to see that $g(0,p) = -\infty$, $g(+\infty,p) = 1$ and $\frac{\partial g}{\partial x}(x,p) > 0$, $\forall x > 0$, $\forall p < 0$ [4]. Therefore if we have $b > 0$ and

$$1 - \frac{a}{\alpha b} < 0 \text{ (that is } a > \alpha b = \tfrac{bl}{c}) \tag{41}$$

and taking into account that $g\left(bq_0^*, \frac{1}{\sqrt{\pi}}\right) > 1$, the equation (38) has a unique solution $\delta > 0$. From (25) we obtain the expression of $S(t)$ given by

$$S(t) = \sqrt{2\gamma t} \tag{42}$$

where γ is given by

$$\gamma = \frac{\gamma^*}{(-\alpha b + a)^2} \quad . \tag{43}$$

From $(24), (25)$ and (30) we can obtain the parametric solution of the problem $(1)-(4)$ given by (39) and (40). Note that $\xi^* \mid_{\xi=0} = \frac{-2bq_0^*\sqrt{t^*}}{\sqrt{2\gamma^* t^*}} = -\sqrt{\frac{2}{\gamma^*}}bq_0^* = -\frac{bq_0^*}{\delta}$.

If we have now $b < 0$ then the equation (38) for δ is given by

$$g(\delta, \alpha_0) = -g\left(-bq_0^*, \tfrac{1}{\sqrt{\pi}}\right) \quad , \quad \delta > 0 \quad . \tag{44}$$

which has a unique solution $\delta > 0$ taking into account that

$$\alpha_0 = \tfrac{1}{\sqrt{\pi}}(1 - \tfrac{a}{\alpha b}) > 0 \quad , \quad g\left(-bq_0^*, \tfrac{1}{\sqrt{\pi}}\right) > 1 \quad , \quad \forall b < 0 \; . \tag{45}$$

Remark 1. There does not exist any solution to the free boundary problem $(1)-(4)$ if $b > 0$ and $0 < a < \frac{bl}{c}$ because the non existence of a real solution of the Eq.(38).

In order to obtain the explicit solution for the cases $b > 0$ and $a > \frac{bl}{c}$, or $b < 0$ we have:

(i) There exists a unique $\delta > 0$ solution of the Eq. (38).

(ii) We have

$$\gamma^* = 2\delta^2, A = \alpha^*\sqrt{\pi}P(\delta) \; , \; B = \theta_f^* - \alpha^*\sqrt{\pi}E(\delta) \tag{46}$$

where

$$\alpha^* = \tfrac{\alpha b}{a(a-\alpha b)} \; , \; \theta_f^* = \tfrac{1}{a} \; , \; \alpha = l/c. \tag{47}$$

(iii) We have

$$\theta^*(y^*,t^*) = \Theta^*\left(\xi^*\right) = A\,\mathrm{erf}\left(\sqrt{\tfrac{\gamma^*}{2}}\xi^*\right) + B = B + A\,\mathrm{erf}\left(\delta\xi^*\right) \tag{48}$$

with $\xi^* = \frac{y^*}{2\delta\sqrt{t}}$ and

$$S^*(t) = \sqrt{2\gamma^* t} = 2\delta\sqrt{t}\,. \tag{49}$$

(iv) We have

$$y^*(y,t) = \int_0^y \left(a + b\overline{\theta}\left(\sigma,t\right)\right)d\sigma - 2bq_0^*\sqrt{t}\ ,\ q_0^* = q_0/\rho c. \tag{50}$$

(v) We have

$$\overline{S}(t) = \tfrac{S^*(t)}{a-\alpha b} = \tfrac{2\delta}{a-\alpha b}\sqrt{t}\ , \tag{51}$$

$$\begin{aligned}\tfrac{1}{a+b\overline{\theta}(y,t)} &= \theta^*(y^*,t) = \Theta^*\left(\xi^*\right) = A\,\mathrm{erf}\left(\tfrac{y^*}{2\sqrt{t}}\right) + B = \\ &= B + A\,\mathrm{erf}\left[\tfrac{1}{2\sqrt{t}}\int_0^y \left(a + b\overline{\theta}\left(\sigma,t\right)\right)d\sigma - bq_0^*\right]\end{aligned}$$

which is an integral equation for $\overline{\theta} = \overline{\theta}(y,t)$ where $t > 0$ is a parameter.

(vi) The free boundary $s = s(t)$ is given by:

$$\begin{aligned}s(t) &= \tfrac{1}{d}\left[\left(1 + \tfrac{d}{2}\overline{S}(t)\right)^2 - 1\right] = \tfrac{1}{d}\left[\left(1 + \tfrac{d\delta}{a-\alpha b}\sqrt{t}\right)^2 - 1\right] = \\ &= \tfrac{2\delta}{a-\alpha b}\sqrt{t} + \tfrac{d\delta^2}{(a-\alpha b)^2}t = \tfrac{\delta}{a-\alpha b}\left[2\sqrt{t} + \tfrac{d\delta}{a-\alpha b}t\right]\end{aligned} \tag{52}$$

(vii) If we define

$$Y(y,t) = -bq_0^* = \tfrac{1}{2\sqrt{t}}\int_0^y \left(a + b\overline{\theta}\left(\sigma,t\right)\right)d\sigma \tag{53}$$

then we obtain

$$\frac{\partial Y}{\partial y}(y,t) = \frac{a + b\overline{\theta}\left(y,t\right)}{2\sqrt{t}} \tag{54}$$

that is $Y = Y(y,t)$ satisfies the following Cauchy problem in variable y :

$$\tfrac{\partial Y}{\partial y}(y,t) = \tfrac{1}{2\sqrt{t}}\left(\tfrac{1}{B+A\,\mathrm{erf}(Y(y,t))}\right), 0 < y < \overline{S}(t), t > 0 \tag{55}$$

$$Y(0,t) = -bq_0^*\ , \tag{56}$$

where $t > 0$ is a parameter.

$(viii)$ The temperature $\overline{\theta} = \overline{\theta}\,(y,t)$ is given by:

$$\overline{\theta}\,(y,t) = \frac{1}{b}\left[\tfrac{1}{B+A\,\mathrm{erf}(Y(y,t))} - a\right]\ ,\ 0 < y < \overline{S}(t)\ ,\ t > 0 \tag{57}$$

as a function of Y.

(ix) The temperature $\theta = \theta(x,t)$ is given by:

$$\theta(x,t) = \overline{\theta}\left(\tfrac{2}{d}\left(\sqrt{1+dx}-1\right),t\right) = \tag{58}$$

$$= \frac{1}{b}\left[\tfrac{1}{B+A\,\mathrm{erf}(Y(\frac{2}{d}\left(\sqrt{1+dx}-1\right),t))} - a\right], 0 < x < s(t), t > 0$$

as a function of Y where $s(t)$ as defined in (52).

Theorem 2 Let us consider the hypothesis $b > 0$ and $a > bl/c$, or $b < 0$. Let $\delta > 0$ be the unique solution of the Eq. (38) and A and B the coefficients defined by (31) and (32) respectively or by (46).

(i) There exists a unique solution $Y = Y(y,t)$ of the Cauchy problem $(55) - (56)$ for all $t \geqslant t_0 > 0$ where t_0 is an arbitrary positive time.

(ii) There exists a unique solution $\theta = \theta(x,t)$ and $s = s(t)$ given by (58) and (52) respectively of the free boundary problem $(1) - (4)$ for $t \geqslant t_0 > 0$ where t_0 is an arbitrary positive time.

Proof:

It is sufficient to prove that the Cauchy problem has a unique solution for $t \geqslant t_0 > 0$. The ordinary differential equation, with parameter $t > 0$, can be written as

$$\frac{\partial Y}{\partial y}(y,t) = G(y,t,Y(y,t))) \tag{59}$$

where

$$G(y,t,Y) = \frac{1}{2\sqrt{t}}\left(\frac{1}{B + A\,\mathrm{erf}(Y)}\right) \tag{60}$$

satisfies the condition

$$\left|\frac{\partial G}{\partial Y}(y,t,Y)\right| \leq Const \quad , \quad \forall t \geqslant t_0 > 0 \tag{61}$$

with $t_0 > 0$ an arbitrary positive time.

Remark 2. The existence of a solution for $t \geqslant t_0 > 0$ with t_0 an arbitrary positive time for the cases $b > 0$ and $a > bl/c$, or $b < 0$ is similar to the one obtained in the free boundary problem studied in [20].

Remark 3:

The particular case $b > 0$ and $a = bl/c$ can not be studied through a similar method developed for the case $b > 0$ and $a > bl/c$ because the transformation (15) is not useful due to the definition of the free boundary $S^*(t^*)$ as a function of the $\overline{S}(t)$.

Remark 4:

For the cases $b > 0$ and $a > bl/c$, or $b < 0$ we can obtain the explicit solution $\theta = \theta(x,t)$ and $s = s(t)$ of the free boundary problem $(1) - (4)$ by the following process:

(i) Compute $\delta > 0$ as the unique solution of the Eq. (38).

(ii) Compute

$$A = \alpha^* \sqrt{\pi} P(\delta) \quad , \quad B = \theta_f^* - \alpha^* \sqrt{\pi} E(\delta)$$

where α^*, θ_f^* and α are defined in (47).

(iii) Fix t_0 as an arbitrary positive time and compute $Y = Y(y,t)$ as the unique solution of the Cauchy problem $(55) - (56)$ for $t \geqslant t_0 > 0$.

(iv) Compute the free boundary $s = s(t)$ by the explicit expression (52)

(v) Compute the temperature $\theta = \theta(x,t)$ by the explicit expression (58).

III. Solution of the free boundary problem with temperature boundary condition on the fixed face.

Now, we consider the problem $(1) - (4)$ with $a, d \in \mathbb{R}^+$ and $b < 0$ but the heat flux boundary condition (2) will be replaced by the following temperature boundary condition $(\theta_0 > 0)$ given by

$$\theta\,(0,t) = \theta_0 \ , \ t > 0 \ , \ \text{with } a + b\theta_0 > 0. \tag{62}$$

We can define the same transformations (10) and (15) as were done for the previous Section but now we get

$$\begin{aligned}\frac{\partial y^*}{\partial t} &= b \int_{S(t)}^{y} \frac{\partial \overline{\theta}}{\partial t}(\sigma,t)d\sigma + \frac{b}{\left(a + b\overline{\theta}\,(0,t)\right)^2} \frac{\partial \overline{\theta}}{\partial y}(0,t) = \\ &= b \int_0^y \frac{\partial \overline{\theta}}{\partial t}(\sigma,t)d\sigma + \frac{b}{(a + b\theta_0)^2} \frac{\partial \overline{\theta}}{\partial y}(0,t) \ .\end{aligned}$$

Then

$$y^*(y,t) = \int_0^t \left(\int_0^y \frac{\partial}{\partial \tau} \left(a + b\overline{\theta}\,(\sigma,\tau)\right) d\sigma + \frac{b}{(a + b\theta_0)^2} \frac{\partial \overline{\theta}}{\partial y}(0,\tau) \right) d\tau + \int_0^y \left(a + b\overline{\theta}\,(\sigma,0)\right) d\sigma =$$

$$= \int_0^y \left(a + b\overline{\theta}\,(\sigma,t)\right) d\sigma + \frac{b}{(a + b\theta_0)^2} \int_0^t \frac{\partial \overline{\theta}}{\partial y}(0,\tau)d\tau. \tag{63}$$

Therefore our free boundary problem becomes $(21) - (22)$ and

$$\frac{\partial \theta^*}{\partial t} = \frac{\partial^2 \theta^*}{\partial y^*} \ , \ b\theta_0^{*2} \int_0^{t^*} \frac{\partial \overline{\theta}}{\partial y}(0,\tau)d\tau < y^* < S^*(t^*) \ , \ t^* > 0 \tag{64}$$

$$\theta^* \left(b\theta_0^{*2} \int_0^{t^*} \frac{\partial \overline{\theta}}{\partial y}(0,\tau)d\tau, t^* \right) = \theta_0^* \tag{65}$$

where θ_f^* and α^* are given by (23) and

$$\theta_0^* = \tfrac{1}{a+b\overline{\theta}(0,t)} = \tfrac{1}{a+b\theta(0,t)} = \tfrac{1}{a+b\theta_0} \quad . \tag{66}$$

It is easy to see that we have a classical Stefan problem so that the free boundary must be of the type

$$S^*(t^*) = \sqrt{2\gamma^* t^*} \qquad \left(\overline{S}(t) = \sqrt{2\gamma t} \ \ , \ \ \gamma^* = \gamma\,(a-\alpha b)^2\right) \tag{67}$$

where γ^* (i.e. γ) is a dimensionless constant to be determined.

If we introduce the similarity variable (24) and we propose the solution of the type (25) then the problem $(21)-(22)$ and $(64)-(65)$ yields $(28)\,,(29)$ and

$$\frac{d^2\Theta^*}{d\xi^{*2}} + \gamma^*\xi^*\frac{d\Theta^*}{d\xi^*} = 0\ ,\ \frac{b\theta_0^{*2}}{\sqrt{2\gamma^* t^*}}\int\limits_0^{t^*}\frac{\partial\overline{\theta}}{\partial y}(0,\tau)d\tau < \xi^* < 1\ ,\ t^* > 0 \tag{68}$$

$$\Theta^*\left(\tfrac{b\theta_0^{*2}}{\sqrt{2\gamma^* t^*}}\int_0^{t^*}\frac{\partial\overline{\theta}}{\partial y}(0,\tau)d\tau\right) = \theta_0^*\ ,\ t^* > 0 \tag{69}$$

From (69) we must necessarily have that there exists a constant ξ_0^* such that

$$\int\limits_0^{t^*}\frac{\partial\overline{\theta}}{\partial y}(0,\tau)d\tau = \xi_0^*\tfrac{1}{b\theta_0^{*2}}\sqrt{2\gamma^* t^*}\ . \tag{70}$$

Therefore (68) and (69) can be written as

$$\frac{d^2\Theta^*}{d\xi^{*2}} + \gamma^*\xi^*\frac{d\Theta^*}{d\xi^*} = 0\ ,\ \xi_0^* < \xi^* < 1 \tag{71}$$

$$\Theta^*\left(\xi_0^*\right) = \theta_0^*\ . \tag{72}$$

The solution of the problem $(28)\,,(29)\,,(71)$ and (72) is given by

$$\Theta^*\left(\xi^*\right) = A'\operatorname{erf}\left(\sqrt{\tfrac{\gamma^*}{2}}\xi^*\right) + B' \qquad , \qquad \xi_0^* < \xi^* < 1 \tag{73}$$

where the unknown coefficients ξ_0^*, A', B' and γ^* must satisfy the following equations

$$A'\operatorname{erf}\left(\xi_0^*\sqrt{\tfrac{\gamma^*}{2}}\right) + B' = \theta_0^* \quad , \quad \sqrt{\tfrac{2}{\pi\gamma^*}}\exp\left(-\tfrac{\gamma^*}{2}\right) = \tfrac{\alpha^*}{A'}, \tag{74}$$

$$A'\operatorname{erf}\left(\sqrt{\tfrac{\gamma^*}{2}}\right) + B' = \theta_f^* \ \text{ and } \ \theta_0^*\sqrt{\tfrac{\gamma^*}{2}}\xi_0^* = -\tfrac{A'}{\sqrt{\pi}}\exp\left(-\tfrac{\gamma^*}{2}\xi_0^{*2}\right) \tag{75}$$

If we define

$$\beta = \sqrt{\tfrac{\gamma^*}{2}} \quad , \quad z = \xi_0^* \sqrt{\tfrac{\gamma^*}{2}} \quad (\beta > z) \tag{76}$$

we have to solve the following system of equations:

$$A' \operatorname{erf}(z) + B' = \theta_0^* \quad , \quad A' \operatorname{erf}(\beta) + B' = \theta_f^* \tag{77}$$

$$\frac{\exp(-\beta^2)}{\beta} = \frac{\alpha^*}{A'}\sqrt{\pi} \quad , \quad \theta_0^* \, z = -\frac{A'}{\sqrt{\pi}} \exp(-z^2) \tag{78}$$

From (77) we get

$$A' = \frac{\theta_f^* - \theta_0^*}{\operatorname{erf}(\beta) - \operatorname{erf}(z)} \tag{79}$$

$$B' = \frac{\theta_0^* \operatorname{erf}(\beta) - \theta_f^* \operatorname{erf}(z)}{\operatorname{erf}(\beta) - \operatorname{erf}(z)} \tag{80}$$

and from (78) we obtain

$$\frac{\exp(-\beta^2)}{\beta} = \delta_1 \left[\operatorname{erf}(\beta) - \operatorname{erf}(z)\right] \tag{81}$$

$$\frac{\exp(-z^2)}{z} = \delta_2 \left[\operatorname{erf}(\beta) - \operatorname{erf}(z)\right] \tag{82}$$

with

$$\delta_1 = \frac{\alpha^* \sqrt{\pi}}{\theta_f^* - \theta_0^*} = \frac{\alpha \sqrt{\pi}\,(a + b\theta_0)}{(a - \alpha b)\,\theta_0} \quad , \quad \delta_2 = -\frac{\sqrt{\pi} a}{b\theta_0}. \tag{83}$$

Taking into account that $\xi_0^* < 1, b < 0$ and $a + b\theta_0 > 0$, we have $[\operatorname{erf}(\beta) - \operatorname{erf}(z)] > 0$, $\delta_1 > 0$ and $\delta_2 > 0$.

Taking into account the properties of the real function $g = g(x, p)$, defined in (35), for $p < 0$, from (81) we have

$$z = \operatorname{erf}^{-1}\left[\operatorname{erf}(\beta) - \frac{1}{\delta_1}\frac{\exp(-\beta^2)}{\beta}\right] \tag{84}$$

if $\beta > x_0$ where $x_0 > 0$ is the unique positive zero of $g(x, -\frac{1}{\delta_1}) = 0$ which is given by $x_0 = E^{-1}(1/\delta_1)$.

Moreover, for β we have the following equation:

$$\frac{\delta_2}{\delta_1}\frac{F(\beta)}{\beta \exp(\beta^2)} = 1 \quad , \quad \beta > x_0 = E^{-1}(1/\delta_1) \tag{85}$$

where the real function F is defined by

$$F(x) = \exp\left[\left(\operatorname{erf}^{-1}\left(g(x, \tfrac{-1}{\delta_1})\right)\right)^2\right] \operatorname{erf}^{-1}\left(g(x, \tfrac{-1}{\delta_1})\right) \; , \; x > 0. \tag{86}$$

If we define the real function:

$$G(x) = \frac{\delta_2}{\delta_1} \frac{F(x)}{x \exp(x^2)} \quad , \quad x > 0 \tag{87}$$

and following [20] we have the following properties:

$$G(x_0^+) = 0, \quad \lim_{x \to +\infty} G(x) = \frac{\delta_2}{\delta_1 + \sqrt{\pi}}. \tag{88}$$

Then there exists a unique solution $\beta > x_0$ of the Eq. (85) if and only if $\frac{\delta_2}{\delta_1 + \sqrt{\pi}} > 1$ if and only if $a + b\theta_0 > 0$ which is our hypothesis.

Moreover, we have

$$\overline{S}(t) = \frac{2\beta}{a - \alpha b} \sqrt{t} \, , \tag{89}$$

and for $0 < y < \overline{S}(t)$, $t > 0$, we obtain

$$\begin{aligned} \frac{1}{a + b\overline{\theta}(y,t)} &= \theta^*(y^*, t) = \Theta^*\left(\xi^*\right) = A' \operatorname{erf}\left(\beta \xi^*\right) + B' = \\ &= B' + A' \operatorname{erf}\left(\frac{y^*}{2\sqrt{t}}\right) = \\ &= B' + A' \operatorname{erf}\left[z + \frac{1}{2\sqrt{t}} \int_0^y \left(a + b\overline{\theta}\left(\sigma, t\right)\right) d\sigma\right] \end{aligned} \tag{90}$$

which represents an integral equation for $\overline{\theta}$ in variable y with t a parameter.

In order to solve this integral equation we define

$$Y(y,t) = z + \frac{1}{2\sqrt{t}} \int_0^y \left(a + b\overline{\theta}\left(\sigma, t\right)\right) d\sigma, \tag{91}$$

which must satisfy the following Cauchy problem

$$\frac{\partial Y}{\partial y}(y,t) = \frac{1}{2\sqrt{t}} \left(\frac{1}{B' + A' \operatorname{erf}(Y(y,t))}\right), \; 0 < y < \overline{S}(t) \; , \; t > 0 \tag{92}$$

$$Y(0,t) = z \tag{93}$$

Therefore we obtain the following theorem.

Theorem 3. Let us consider the hypothesis $a > 0, d > 0, b < 0$ and $a + b\theta_0 > 0$. Let $\beta > 0$ be the unique solution of the Eq. (85) , and A' and B' the coefficients defined by (79) and (80) respectively.

(i) There exists a unique solution $Y = Y(y,t)$ of the Cauchy problem $(92) - (93)$ for all $t \geqslant t_0 > 0$ where $t_0 > 0$ is an arbitrary positive time.

(ii) There exists a unique solution $\theta = \theta(x,t)$ and $s = s(t)$ given by (95) and (96) respectively of the free boundary problem $(1), (3) - (4)$ and (62).

Proof.

(i) The Cauchy problem $(92) - (93)$ has a unique solution $Y = Y(y,t)$ for all $t \geqslant t_0 > 0$ with $t_0 > 0$ is an arbitrary positive time following a similar method given in Theorem 2.

(ii) From (90) and (91) we get

$$\overline{\theta}\,(y,t) = \frac{1}{b}\left[\frac{1}{B' + A'\,\mathrm{erf}(Y(y,t))} - a\right] \tag{94}$$

that is

$$\theta(x,t) = \overline{\theta}(y,t) = \overline{\theta}\left(\frac{2}{d}\left(\sqrt{1+dx}-1\right),t\right) = \tag{95}$$
$$= \frac{1}{b}\left[\frac{1}{B' + A'\,\mathrm{erf}(Y(\frac{2}{d}\left(\sqrt{1+dx}-1\right),t))} - a\right]$$

and

$$s(t) = \frac{1}{d}\left[\left(1+\frac{d}{2}\overline{S}(t)\right)^2 - 1\right] = \frac{1}{d}\left[\left(1+\frac{d\beta}{a-\alpha b}\sqrt{t}\right)^2 - 1\right] = \tag{96}$$
$$= \frac{\beta}{a-\alpha b}\left[2\sqrt{t} + \frac{d\beta}{a-\alpha b}t\right]$$

Remark 5 For the case $a > 0, d > 0, b < 0$ and $a + b\theta_0 > 0$ we can obtain the explicit solution $\theta = \theta(x,t)$ and $s = s(t)$ of the free boundary $(1), (3) - (4)$ and (62) by the following process:

(i) Compute the positive parameters δ_1 and δ_2 given by (83).

(ii) Compute $\beta > 0$ as the unique solution of the Eq. (85)

(iii) Compute the coefficients

$$\begin{aligned} z &= \mathrm{erf}^{-1}\left[\mathrm{erf}(\beta) - \frac{1}{\delta_1}\frac{\exp(-\beta^2)}{\beta}\right] \\ \gamma^* &= 2\beta^2 \quad , \quad \gamma = \frac{\gamma^*}{(a-\alpha b)^2} = 2\left(\frac{\beta}{a-\alpha b}\right)^2 \\ \xi_0^* &= \frac{z}{\beta} \end{aligned}$$

(iv) Compute the coefficients $A' < 0$ and $B' > 0$ given by (79) and (80) respectively.

(v) Fix t_0 as an arbitrary positive time and compute $Y = Y(y,t)$ as the unique solution of the Cauchy problem $(92) - (93)$.

(vi) Compute the free boundary $s = s(t)$ by the explicit expression (96)

(vi) Compute the temperature $\theta = \theta(x,t)$ by the explicit expression (95).

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