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DIRECT METHODS IN THE CALCULUS OF VARIATIONS

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Abstract. The aim of the present notes is to give a very brief introduction to the main ideas on the direct methods of the calculus of variations and some classical applications to integral functionals.

Resumen. El objetivo de las presentes notas es dar una breve introducción a las ideas principales sobre los métodos directos del cálculo de variaciones y algunas aplicaciones clásicas a los funcionales integrales.

Keywords: Calculus of variations, lower semicontinuity, relaxation.

Parabras claves: Cálculo de variaciones, semicontinuidad inferior, relajación.

AMS Subject Classification: 49-01.

These Notes are the enlarged content of a course given by Prof. Diego Pallara at the Department of Mathematics of FCE-UA, Rosario during September 2003 through a scientific cooperation project with Ministero degli Affari Esteri (Italy). They contain the main ideas on the direct methods of the calculus of variations.

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Direct Methods in the Calculus of Variations

G. G. Garguichevich - C. M. Gariboldi - P. R. Marangunic - D. Pallara

Contents

Introduction

These notes originate from a short series of lectures delivered by the last named author at the Departamento de Matemática, FCE, of the Universidad Austral de Rosario in September, 2003. The aim was to give a quick introduction to the main ideas underlying the direct methods of the Calculus of Variations and to present some classical results on the lower semicontinuity of scalar and first order integral functionals, for which we discuss also existence of minimisers under suitable side conditions. Our aim is to discuss some general ideas and methods, without looking for the most general results. Our presentation is neither original nor exhaustive, and we refer to the treatises [6], [9], [10], [13], [22], [26] and to the lecture notes [8], [14], [11], [16] for further information. More references will be given in each section, but the bibliography is very far from being complete.

The mean theme of the Calculus of Variation is the search for extreme values (maxima and minima) of real functions defined on *a priori* abstract sets of admissible competitors. Probably the most ancient (a formulation of it is referred to the foundation of Carthage by the queen Dido as is mentioned also in Virgil's latin poem *Aeneid*) and famous variational problem is the *isoperimetric problem*:

> among all the closed plane curves of given length, find that which includes the largest area

which admits the following equivalent dual formulation

> among all the plane figures of given area, find that whose boundary has minimal length.

As easy to state it seems to be, a closer look shows that all the terms used to formulate the problem must be defined in a precise way, in order to proceed to a rigorous study. For instance, the terms "area included in a closed curve", "length of a curve" or of the boundary of a plane region (how regular?), "area of a generic plane figure", all require a definition that allows to identify the class of competitors. In any case, historically this problem has been regarded as a well-posed one for a long time, and several proofs of the fact that the optimal set is a circle have been found (see e.g. [28]). Even though more difficult, proofs of the isoperimetric property of the sphere in three dimensions are available as well since a long time, but all these arguments in fact prove that, *if* a solution of the problem exists, *then* it is a circle or a sphere, respectively, whereas the proof of the existence of an optimal set was overlooked for a long time. We shall come back to the isoperimetric problem in the last section, and, for the present, we confine ourselves to notice that showing that the solution of a variational problem exists, and identifying it, or describing its properties, are often different steps and could require different tools. In fact, what is usually referred to as the *classical methods of the calculus of variations* is based on the derivation of *necessary extremality condition*, whereas the *direct method of the calculus of variations*, which is the topic of these notes, focuses rather on the study of the *existence* of solutions of variational problems. In particular, we consider *integral functionals* like

$$F(u) = \int_\Omega f(x, u, Du, \dots, D^k u)dx, \tag{1}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded open set in the whole paper, and $u : \Omega \to \mathbb{R}^n$; more precisely, we deal almost exclusively with *scalar* and *first order* problems. This means that we deal with the case where the admissible functions are real valued and under the integral sign only the first order derivatives of u appear (i.e., with the above notation, $n = 1$ and $k = 1$). We shall briefly discuss the vectorial case $n \geqslant 2$ in Sections 3 and 4 in order to highlight the main differences with respect to the scalar case. Dealing with the general case in (1) still for a while, let us mention that the classical approach to the minimisation of the functional F above under appropriate constraints has been the derivation of necessary conditions for extremality based on the computation of the *first variation* or *Gateaux derivative*: this amounts to writing a system of partial differential equations of order $2k$ with suitable boundary conditions. In the classical approach, the main peculiarity of the scalar case is that the first variation yields an equation and not a system, and in the scalar and first order case the equation thus obtained is of second order, a rather special class.

Furthermore, let us notice that direct methods of the Calculus of variations are by now a quite standard way to prove the existence of weak solutions of elliptic boundary value problems. The very first remark in this direction goes back to Gauss (1839), who pointed out that the harmonic function in an open bounded set $\Omega \subset \mathbb{R}^N$ taking a prescribed boundary datum $g : \partial\Omega \to \mathbb{R}$ can be found by looking at the minimisers of the functional $\mathcal{D}$ (see (2) below) among all functions u such that $u|_{\partial\Omega} = g$. The situation is much more delicate, and took a long time before being completely understood. Nevertheless, Gauss' remark, which was extensively used by Riemann under the name of *Dirichlet principle*, is usually referred as the birth of the direct methods.

The *Dirichlet functional*

$$\mathcal{D}(u) = \int_\Omega |Du|^2 dx \tag{2}$$

is in fact one of leading examples we have in mind, the other being the *non-parametric* *area functional*

$$\mathcal{A}(u) = \int_{\Omega} \sqrt{1 + |Du|^2} dx. \tag{3}$$

There is a big difference between these two examples, because in $\mathcal{D}$ the integrand $f(\xi) = |\xi|^2$ has a growth of order 2 when $|\xi|$ goes to ∞, whereas the integrand $f(\xi) = \sqrt{1 + |\xi|^2}$ in $\mathcal{A}$ has a first order growth. This leads to study the minimisation problems for $\mathcal{D}$ in reflexive Sobolev spaces, and for $\mathcal{A}$ in BV, a completely different situation. We deal with the former case in Section 2, and present some results concerning the latter in Section 4. Of course, the above mentioned examples are rather special because there is no explicit dependence on x and u. The case of irregular dependence of the integrand on x and u are delicate, and will be only mentioned in the sequel.

The plan of the notes closely follows the content of the lectures: after giving some topological preliminaries in Section 1 (lower semicontinuity, coerciveness, relaxation), in Section 2 we present the main classical results on 1^{st}-order integral functionals with scalar-valued variables, in the case of growth $p > 1$, whose model is the Dirichlet functional (2). These problems are well-posed in Sobolev spaces $W^{1,p}$. Section 3 is devoted to a very sketchy discussion of the vectorial case, with the aim to give a flavour of the main differences between the scalar and the vectorial case. In Section 4 we define the space of functions of bounded variation and discuss some results for functional with growth $p = 1$ like (3), which are typically well settled in BV.

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1 Topological preliminaries

The direct methods are concerned with the existence of extreme values, and in this sense are the analogue of the classical Weierstrass Theorem of elementary calculus.

As we have said in the Introduction, we state some topological conditions on the functional that ensure the existence of the solution of a variational problem.

Let us begin by recalling some general definitions that can be given in a topological space X.

Definition 1.1 *Let X be a topological space. The function $F : X \to \overline{\mathbb{R}} = [-\infty, +\infty]$ is lower semicontinuous (shortly l.s.c.) if*

$$\forall t \in \mathbb{R} \quad \textit{the set} \quad \{x \in X : F(x) \leqslant t\} \qquad \textit{is closed in } X. \tag{1.1}$$

Remark 1.2 Condition (1.1) is equivalent to any of the following ones:

$$\forall t \in \mathbb{R} \quad \text{the set} \quad \{x \in X : F(x) > t\} \text{ is open in } X \tag{1.2}$$

$$\big\{(x,t) \in X \times \overline{\mathbb{R}} : F(x) \leqslant t\big\} \text{ is closed in } X \times \overline{\mathbb{R}} \tag{1.3}$$

$$\big\{(x,t) \in X \times \overline{\mathbb{R}} : F(x) > t\big\} \text{ is open in } X \times \overline{\mathbb{R}}. \tag{1.4}$$

Since we are dealing with (possibly extended) real valued functions, and in particular with the search for minimum points, our arguments will often rely upon the consideration of sequences in X (see e.g. the proof of Weierstrass-Fréchet theorem below and the subsequent comments). For this reason, it is convenient to introduce a notion of lower semicontinuity based on sequences of points rather than the underlying topology. Of course, if X has a countable neighbourhood topology (as is the case for metric spaces) the two notions coincide.

Definition 1.3 *The function $F : X \to \overline{\mathbb{R}}$ is sequentially lower semicontinuous (seq. l.s.c.) if*

$$\{x_n\} \subset X, \quad x_n \to x \in X \qquad \Longrightarrow \qquad F(x) \leqslant \liminf_{n\to\infty} F(x_n).$$

Remark 1.4 Characterisations analogous to those in Remark 1.2 hold for sequential lower semicontinuity. Moreover, if X is a metric space, Definitions 1.1 and 1.3 are equivalent.

Let us mention a few properties we will need in the sequel, which are easily obtained from the previous definitions.

(i) If F is l.s.c. and G is continuous, then $F + G$ is l.s.c.

(ii) If $(F_\alpha)_{\alpha\in A}$ is a family of l.s.c. functions (or seq. l.s.c.), then $F = \sup_{\alpha\in A} F_\alpha$ is l.s.c. (resp. seq. l.s.c.).

This is only one half of what we need to deal with minimisation problems. The other side is coerciveness.

Definition 1.5 *The funtion $F : X \to \overline{\mathbb{R}}$ is coercive (or seq. coercive) if $\{x \in X : F(x) \leqslant t\}$ is relatively compact (resp. rel. seq. compact) for all $t \in \mathbb{R}$, that is*

$$\forall t \in \mathbb{R} \quad \exists\, K \textit{ compact (seq. compact) such that} \quad \{x \in X : F(x) \leqslant t\} \subset K.$$

The following classical result is the main tool in the applications.

Theorem 1.6 (Weierstrass-Fréchet Theorem) *If $F : X \to \,]-\infty, +\infty]$ is sequential l.s.c. and sequential coercive then $\inf_X F$ is attained.*

Proof. Let $\{x_n\} \subset X$ be such that $F(x_n) \to \inf_X F$.

Since F is sequentially coercive, there exist $N_0 \in \mathbb{N}$ and K sequentially compact, such that $\{x_n, n \geqslant N_0\} \subset K$, hence there is a subsequence $\{x_{n_k}\}$ and a point $\overline{x}$ such that $\lim_{k\to\infty} x_{n_k} = \overline{x}$.

From the sequential lower semicontinuity of F we infer that

$$F(\overline{x}) \leqslant \liminf_{k\to\infty} F(x_k) = \inf_X F,$$

that is, $F(\overline{x}) = \min_X F$. Notice that the above equality shows, in particular, that $\inf_X F > -\infty$. $\square$

In the above proof, our starting point has been a sequence $\{x_n\} \subset X$ such that $F(x_n) \to \inf_X F$. Such a sequence is called a *minimising sequence*, and we shall meet this construction quite often in the sequel.

In our applications, we shall be mainly concerned with spaces endowed with a richer structure, like Banach spaces (reflexives or not), for instance, $W_0^{1,2}$ in the case of the Dirichlet functional and $W^{1,1}$ or BV in the case of non parametric area functional.

In these spaces, we have two natural topologies: the norm and the weak (or weak*) topology. We have also the notion of convexity, which plays an important role.

In the sequel, we simply write: open, closed, l.s.c., etc, when using the norm topology, while we add the prefix w: w-open, w-closed, w-l.s.c., etc. when using the weak topology.

For Banach spaces, we have other useful characterisations for the previous concepts.

Theorem 1.7 *Let X be a Banach space and $F : X \to]-\infty, +\infty]$*

(i) *If F is convex then:*

$$F \text{ is l.s.c.} \quad \Longleftrightarrow \quad F \text{ is } w\text{-l.s.c.}$$

(ii) *If X is reflexive then:*

$$F \text{ is } w\text{-coercive} \quad \Longleftrightarrow \quad \lim_{\|x\| \to +\infty} F(x) = +\infty.$$

Proof. (i) If F is convex then $\{x \in X : F(x) \leqslant t\}$ is a convex set $\forall t \in \mathbb{R}$.

On the other hand, by the Hahn-Banach theorem, a convex set is closed if and only if is w-closed.

(ii) If $\lim_{\|x\| \to +\infty} F(x) = +\infty$, then for any t the set $\{x \in X : F(x) \leqslant t\}$ is bounded, and therefore, it is relatively w-compact because closed balls are w-compact in reflexive Banach spaces. Conversely, if $\lim_{\|x\| \to +\infty} F(x) = +\infty$ does not hold then, for some $t \in \mathbb{R}$ there exists $\{x_n\} \subset X$ such that $\|x_n\| > n$ and $F(x_n) \leqslant t$, so that $E_t = \{x \in X : F(x) \leqslant t\}$ is not bounded. Since relatively sequentially w-compact sets are norm-bounded, E_t cannot be w-compact and we get a contradiction. $\square$

Both (i) and (ii) can be used for the Dirichlet functional, whereas the situation is more difficult for the area functional, due to the lack of reflexivity of the natural spaces involved. This point is discussed in Section 4.

We wonder what can we say about functionals that do not satisfy the hypothesis of the Weierstrass-Fréchet Theorem and could possibly not have a minimum.

For this purpose, the *relaxation technique* is by now a standard tool. The main ideas go back to Lebesgue thesis at the beginning on the 20^{th} century. In particular, it is convenient to introduce a new functional $\overline{F}(x)$ with the following properties:

(i) $\min_{x \in X} \overline{F}(x) = \inf_{x \in X} F(x)$

(ii) Every $\overline{x} \in X$ such that $\overline{F}(\overline{x}) = \min_{x \in X} \overline{F}(x)$ is the limit of some minimising sequence of F.

(iii) Every minimising sequence of F has a subsequence which converges to a minimum point of $\overline{F}$.

The relaxation technique allows us to handle this situation by studying the behaviour of minimising sequences.

Definition 1.8 *Let X be a topological space and $F : X \to]-\infty, +\infty]$. The functional*

$$\overline{F}(x) = \sup \{G(x) : G \text{ l.s.c.}, \ G \leqslant F \text{ in } X\}$$

is called the relaxed functional of F.

Remark 1.9 By Remark 1.4(ii), $\overline{F}$ is the greatest l.s.c. functional such that $\overline{F}(x) \leqslant F(x)$.

In order to deal with $\overline{F}$, it is convenient to look for a constructive characterisation of it. In particular, we highlight its local character, which is hidden in Definition 1.8.

Theorem 1.10 *For all $x \in X$, let $N(x)$ be the set of all neighbourhoods of x. Then,*

$$\overline{F}(x) = \sup_{U \in N(x)} \left(\inf_{y \in U} F(y) \right).$$

Proof. Set $F'(x) = \sup_{U \in N(x)} (\inf_{y \in U} F(y))$. Since $F'(x) \leqslant F(x)$ for all x, it is sufficient to prove that F' is l.s.c. to conclude $F' \leqslant \overline{F}$. Let $x \in X$, $t \in \mathbb{R}$ such that $F'(x) > t$. By definition of F', there exists $U \in N(x)$ such that $\inf_{y \in U} F(y) > t$. As $U \in N(y)$ for all $y \in U$, we have

$$F'(y) = \sup_{V \in N(y)} \left(\inf_{z \in V} F(z) \right) \geqslant \inf_{z \in U} F(z) > t$$

so that $\{F' > t\}$ is open, and then F' is l.s.c.

In order to prove that $\overline{F} \leqslant F'$, let us consider $G \leqslant F$, G l.s.c. and see that $G \leqslant F'$. If we assume that $G(x) > F'(x)$ for some $x \in X$, we can fix $t \in]F'(x), G(x)[$. As G is l.s.c., there is $U \in N(x)$ such that $G(y) > t$ for all $y \in U$. Therefore $F(y) \geqslant G(y) > t$ for all $y \in U$ and $F'(x) = \sup_{U \in N(x)} (\inf_{y \in U} F(y)) \geqslant t$ and this yields a contradiction. □

Relaxed functionals can be characterised in an even more explicit way, using sequences, in the important case of metrisable topologies.

Theorem 1.11 *Let X be a metric space. Then,*

$$\overline{F}(x) = \inf \left\{ \liminf_{n \to \infty} F(x_n) : \ x_n \in X, \ x_n \to x \right\}.$$

Proof. Set $G(x) = \inf \{\liminf_{n \to \infty} F(x_n) : \ x_n \in X, \ x_n \to x\}$.

It is sufficient to consider the constant sequence $x_n = x$ for all n to conclude that

$$G(x) \leqslant F(x). \tag{1.5}$$

Let us now prove that G is l.s.c. Let be $x_n \to x$. Then for each n there exists $\{x_n^k\} \subset X$ such that $x_n^k \to x$ as $k \to \infty$ and $\liminf_{k \to \infty} F(x_n^k) < G(x_n) + \frac{1}{n}$. We can always choose a subsequence of $\{x_n^k\}$, also called $\{x_n^k\}$ for the sake of simplicity, such that

$$d(x_n^k, x_n) < \frac{1}{n} \quad \forall k \quad \text{and} \quad F(x_n^n) < G(x_n) + \frac{1}{n}$$

Hence we have $x_n^n \to x$ and

$$\begin{aligned}\liminf_{n\to\infty} G(x_n) &\geqslant \liminf_{n\to\infty}\left(F(x_n^n)-\frac{1}{n}\right)=\liminf_{n\to\infty}F(x_n^n)\\ &\geqslant \inf\left\{\liminf_{n\to\infty}F(y_n):\ y_n\in X, y_n\to x\right\}=G(x)\end{aligned}$$

i.e.,

$$G \text{ is l.s.c.} \tag{1.6}$$

and (1.5), (1.6) imply $G \leqslant \overline{F}$.

To attain the thesis we now suppose $G(x) < \overline{F}(x)$ for some x, and show that this leads to a contradiction. Indeed, for some $\varepsilon > 0$, we would have $\overline{F}(x) > G(x) + \varepsilon$ and by Theorem 1.10 we could find $U \in N(x)$ such that

$$F(y) > G(x) + \varepsilon \qquad \forall y \in U.$$

This implies that for all $x_n \to x$ the inequality $F(x_n) > G(x) + \varepsilon$ holds for n large enough. Then,

$$G(x) = \inf\left\{\liminf_{n\to\infty} F(x_n):\ x_n \in X,\ x_n \to x\right\} \geqslant G(x) + \varepsilon,$$

which is a contradiction. □

Remark 1.12 We can modify our reasoning in Theorem 1.11 and see that for every $x \in X$, there exists a sequence $y_n \to x$ such that $\lim_{n\to\infty} F(y_n) = G(x)$.

In fact, for each n we can find a sequence $x_n^k \to x$ as $k \to \infty$ such that for all k the inequalities

$$d(x_n^k, x) < \frac{1}{n} \qquad \text{and} \qquad F\left(x_n^k\right) < G(x) + \frac{1}{n}$$

hold. Then, $x_n^n \to x$, $\liminf_{n\to\infty} F(x_n^n) \leqslant G(x)$ and for a suitable subsequence $\{y_n\}$ of $\{x_n^n\}$ we have

$$\lim_{n\to\infty} F(y_n) \leqslant G(x), \qquad \text{i.e.} \qquad \lim_{n\to\infty} F(y_n) = G(x).$$

In other words, we have the following result:

Proposition 1.13 *If X is a metric space then $H = \overline{F}$ if and only if for every $x \in X$ we have*

(i) $x_n \to x \Rightarrow H(x) \leqslant \liminf_{n\to\infty} F(x_n)$

(ii) $\exists \{x_n\} \subset X$ *such that* $x_n \to x$ *and* $\lim_{n\to\infty} F(x_n) = H(x)$.

Let us now show that the infimum of F and $\overline{F}$ coincide. As stated at the beginning of this discussion, this was one of the main requests leading to the relaxation procedure.

Theorem 1.14 *Let X be a topological space and $F : X \to]-\infty, +\infty]$. If $\overline{F}$ is the relaxed functional of F, then*

$$\inf_{x\in X} \overline{F}(x) = \inf_{x\in X} F(x).$$

Proof. Obviously, $\inf_{x\in X} F(x) \geqslant \inf_{x\in X} \overline{F}(x)$. If the strict inequality holds, then there exists $x_0 \in X$ such that

$$\inf_{x\in X} \overline{F}(x) \leqslant \overline{F}(x_0) < \inf_{x\in X} F(x)$$

and then for every neighbourhood $U \in N(x_0)$ we would have

$$\inf_{y\in U} \overline{F}(x) \leqslant \overline{F}(x_0) < \inf_{x\in X} F(x),$$

and this contradicts the definition of $\overline{F}$. □

Remark 1.15 It is interesting to see how Theorem 1.14 can be used to get a characterization of $\overline{F}$, in terms of infimum values, by means of a perturbation argument. First, note that if $G : X \to \mathbb{R}$ is a continuous functional, then $\overline{F+G} = \overline{F} + \overline{G}$ and therefore

$$\inf_{x\in X} (F(x) + G(x)) = \inf_{x\in X} \left(\overline{F}(x) + G(x)\right)$$

Now, if F and ϕ are non negative functionals such that ϕ is l.s.c. and

$$\inf_{x\in X} (F(x) + G(x)) = \inf_{x\in X} (\phi(x) + G(x))$$

for all continuous non negative G, it is possible to prove that $\phi = \overline{F}$ (see [14, Chapter 9]).

We note that $\left\{\overline{F}(x) \leqslant t\right\} \subset \overline{\{F(x) \leqslant t\}}$ and then we have the following

Proposition 1.16 *If F coercive then $\overline{F}$ is coercive as well.*

As a consequence, we obtain:

Theorem 1.17 *If F is coercive then there exists $\min_{x\in X} \overline{F}(x) = \inf_{x\in X} F(x)$. Moreover, if F is coercive and G is continuous and nonnegative then*

$$\min_{x\in X} \left(\overline{F}(x) + G(x)\right) = \inf_{x\in X} (F(x) + G(x)).$$

The goals of this section are fulfilled. The relaxation technique has provided us with a "good" description of the behavior of the minimising sequences of a coercive functional F, even if it does not have a minimum.

Indeed, in a metric space X, to study the relaxed functional $\overline{F}$ is equivalent to study the behavior of the minimising sequences of F, since if F is not l.s.c. and $\{x_n\}$ is such that $F(x_n) \to \inf_{x\in X} F(x)$ it can happen that $x_n \to x$ but $F(x) > \liminf_{n\to\infty} F(x_n)$. In the next sections we will solve this problem for some particular cases. Moreover, for some suitable F we shall find an explicit form for $\overline{F}$.

2 Functionals defined in Sobolev spaces

A classical problem of the Calculus of Variations is to find

$$\min_{\mathcal{A}} \left\{ \int_{\Omega} f(x, u, Du)\, dx \right\} \qquad \text{where } \ \mathcal{A} \subset C^1(\Omega).$$

There were two main ways to approach the solution of this problem.

Initially, L. Tonelli looked for a strong solution u as the limit of a minimising sequence $\{u_n\}$ such that $\{u_n\}$ and $\{Du_n\}$ converge uniformly, that is $\{u_n\}$ and $\{Du_n\}$ had to satisfy a Cauchy condition in the uniform norm. Although he obtained some results, he could not achieve his goal and, as G. Fichera has said in 1994, that became "a lost struggle" of Italian Mathematicians.

Taking into account that the problem deals with integral functionals, the second trend considered integral norms instead of the uniform one, in order to obtain weak solutions. The following necessary and difficult step, that is to analyse the regularity of these weak solutions, has his main result in the De Giorgi-Nash-Moser Theorem.

In this section we present some by now classical results on the lower semicontinuity and relaxation of integral functionals like

$$F(u) = \int_{\Omega} f(x, u, Du)\, dx$$

were $\Omega \subset \mathbb{R}^N$ is an open bounded set and u belongs to the Sobolev Space $W^{1,p}(\Omega)$ for some $p \geqslant 1$ depending upon the shape of the integrand f. Existence theorems for a suitable minimum problem follow at once in several cases, but the cases $p = 1$ or $p > 1$ are very different. In this section we discuss the case $p > 1$, while the next one is devoted to the case $p = 1$ which leads to the considerations of BV-functions.

We assume that the reader has a previous knowledge of Sobolev Spaces, for which we refer to [2], [32] and [6].

We first consider the integral functional

$$F(u) = \int_{\Omega} f(x, u)\, dx, \qquad u \in L^p(\Omega)$$

where u is either a scalar or a vectorial function.

Theorem 2.1 *Assume that $f(\cdot, s)$ is measurable $\forall s \in \mathbb{R}^n$, $n \geqslant 1$, and $f(x, \cdot)$ is Borel measurable for all $x \in \Omega$, with $1 \leqslant p < \infty$. If:*

(i) *There exists $a \in L^1(\Omega)$, $b \in \mathbb{R}$ such that*

$$f(x, s) \geqslant a(x) + b\,|s|^p\,.$$

(ii) *$f(x, .)$ is l.s.c. for almost everywhere $x \in \Omega$.*

Then

(a) *$F(u) = \int_\Omega f(x,u)\,dx$ is strongly l.s.c. in L^p.*

(b) *If, in addition, $f(x,.)$ is convex then F is weakly l.s.c. in L^p.*

PROOF. (a) First, notice that by hypothesis (i), for every $u \in L^p(\Omega)$ the function $x \mapsto f(x,u(x))$ is measurable, and moreover $\int f^-(x,u)\,dx < +\infty$, hence,

$$F(u) = \int\limits_\Omega f^+(x,u(x))\,dx - \int\limits_\Omega f^-(x,u(x))\,dx$$

is well defined and $F(u) \in \left]-\infty,+\infty\right]$. We consider $(u_h) \subset L^p$ such that $u_h \to u$ in $L^p(\Omega)$, and let $F(u)$ be finite. Then, by hypothesis (ii),

$$\begin{aligned} 0 &\leqslant f(x,u(x)) - a(x) - b\,|u(x)|^p \\ &\leqslant \liminf_{h\to+\infty} \left(f(x,u_h(x)) - a(x) - b\,|u_h(x)|^p \right) \text{ a.e. } x \in \Omega. \end{aligned}$$

We integrate and use the Fatou Lemma in order to obtain

$$\begin{aligned} &\int\limits_\Omega f(x,u(x)) - a(x) - b\,|u(x)|^p\,dx \\ &\leqslant \liminf_{h\to+\infty} \left[\int\limits_\Omega f(x,u_h(x))\,dx - \int\limits_\Omega a(x)\,dx - b\int\limits_\Omega |u_h(x)|^p\,dx \right] \end{aligned}$$

and by the strong convergence of u_h in $L^p(\Omega)$, we conclude that

$$\int\limits_\Omega f(x,u(x))\,dx \leqslant \liminf_{h\to+\infty} \int\limits_\Omega f(x,u_h(x))\,dx$$

i.e., the strong l.s.c. of F in $L^p(\Omega)$.
(b) If $f(x,.)$ is convex, then $\int_\Omega f(x,u)\,dx = F(u)$ is convex. Now, since F is strong l.s.c. and convex, from Theorem 1.7(i) we infer that F is w-l.s.c. in $L^p(\Omega)$. □

An easy consequence is the following:

Theorem 2.2 *Let $1 < p < \infty$, $\varphi \in W^{1,p}(\Omega)$, $f : \Omega \times \mathbb{R}^N \to \mathbb{R}$ satisfy hypotheses (i) and (ii) of the preceding theorem with $b > 0$ and consider the functional $F(u) = \int_\Omega f(x,Du)\,dx$ for $u : \Omega \to \mathbb{R}$ in the Sobolev space $W^{1,p}(\Omega)$.*

If $f(x,\cdot)$ is convex for a.e. $x \in \Omega$, then there exists $u_0 \in W^{1,p}(\Omega)$ such that

$$F(u_0) = \min\left\{ F(u) : \; u - \varphi \in W_0^{1,p}(\Omega) \right\}.$$

PROOF. By the preceding theorem $G(v) = \int_\Omega f(x,v)\,dx$ is w-l.s.c. in $L^p(\Omega)$. Since the differentiation operator $D : W^{1,p} \to L^p$ is weakly continuous, if $u_h \to u$ weakly in $W^{1,p}$ then $Du_h \to Du$ weakly in L^p. Thus

$$G(Du) \leqslant \liminf_{h\to+\infty} G(Du_h),$$

i.e.,

$$F(u) \leqslant \liminf_{h\to+\infty} F(u_h),$$

that is F is w-l.s.c. in $W^{1,p}$.

In order to solve the boundary value problem, let us first consider the case $\varphi = 0$. Since by Poincaré inequality the norms $\|u\|_{W_0^{1,p}}$ and $\|Du\|_{L^p}$ are equivalent, hypothesis (i) of Theorem 2.1 implies $\lim\limits_{\|u\|_{W^{1,p}}\to+\infty} F(u) = +\infty$ and by Theorem 1.7(ii), F is coercive. Hence, there exists u_0 such that $F(u_0) = \min\{F(u) : \; u \in W_0^{1,p}(\Omega)\}$.

For a general φ, if we set $v = u - \varphi$, arguing in a similar way with the functional $G(v) = \int_\Omega f(x, Du + D\varphi)\, dx$, we obtain the thesis. □

Remark 2.3 The above theorem is false for $p = 1$. Indeed, we can consider $N = 1$, $\Omega = (-1, 1)$, $\varphi(x) = x$ and $f(x, u') = (1 + |x|)\,|u'|$. The functional

$$F(u) = \int_{-1}^{1} (1 + |x|)\,|u'|\, dx$$

satisfies the required hypotheses but does not have a minimum in the class $\mathcal{A} = \left\{u \in W^{1,1}(\Omega) : \; u - \varphi \in W_0^{1,1}\right\}$ because for all $u \in \mathcal{A}$ the estimate

$$\begin{aligned} F(u) &= \int_{-1}^{1} (1 + |x|)\,|u'(x)|\, dx > \int_{-1}^{1} |u'(x)|\, dx \geqslant \left|\int_{-1}^{1} u'(x)\, dx\right| \\ &= |u(1) - u(-1)| = 2 \end{aligned}$$

holds, whereas for

$$u_h(x) = \begin{cases} -1 & -1 \leqslant x \leqslant -\frac{1}{h} \\ hx & -\frac{1}{h} \leqslant x \leqslant \frac{1}{h} \\ 1 & \frac{1}{h} \leqslant x \leqslant 1 \end{cases}$$

we have

$$\begin{aligned} F(u_h) &= \int_{-1}^{1} (1 + |x|)\,|u_h'|\, dx \\ &= \int_{-1/h}^{1/h} (1 + |x|)\, h dx = 2 + \frac{1}{h} \to 2 \quad \text{as } h \to +\infty. \end{aligned}$$

This phenomenon is related with the non-reflexivity of $W^{1,1}$ and the "bad" behavior of minimising sequences in $W^{1,1}$ which converge to a function of bounded variation rather than to a Sobolev one. In fact, the "natural" minimiser of F is the function

$$u(x) = \begin{cases} -1 & -1 \leqslant x < 0 \\ 1 & 0 < x \leqslant 1 \end{cases}$$

which is BV but does not belong to $\mathcal{A}$.

Let us slightly restrict the class of integrands f, introducing the class of *Carathéodory integrands*, and show that for the corresponding integral functionals a sort of converse of Theorem 2.1 holds.

Definition 2.4 *A function $f : \Omega \times \mathbb{R}^N \to \mathbb{R}$ is a* Carathéodory function *if $f(., \xi)$ is measurable and $f(x, .)$ is continuous.*

For Carathéodory integrands, the convexity is necessary for the lower semicontinuity.

Theorem 2.5 *Let f be a Carathéodory function on an open bounded set $\Omega \subset \mathbb{R}^N$ and assume that for all $R > 0$ there exists $g_R \in L^1(\Omega)$ such that $|f(x, \xi)| \leqslant g_R(x)$ for all $x \in \Omega$, $|\xi| \leqslant R$.*

If $F(u) = \int_\Omega f(x, Du)\, dx$ is w-l.s.c. on $W^{1,\infty}(\Omega)$, then $f(x, .)$ is convex for a.e. $x \in \Omega$.*

Proof. We must prove that for every $\xi_1, \xi_2 \in \mathbb{R}^N$ and $\lambda \in (0, 1)$, setting $\xi = \lambda\xi_1 + (1 - \lambda)\xi_2$, the inequality

$$f(x, \xi) \leqslant \lambda f(x, \xi_1) + (1 - \lambda) f(x, \xi_2) \tag{2.1}$$

holds a.e. in Ω. For, we first prove that

$$\int_\Omega f(x, \xi)\, dx \leqslant \lambda \int_\Omega f(x, \xi_1)\, dx + (1 - \lambda) \int_\Omega f(x, \xi_2)\, dx. \tag{2.2}$$

To obtain it, we exploit the w*-l.s.c. functional $F(u) = \int_\Omega f(x, Du)\, dx$ at a function u_ξ such that $Du_\xi = \xi$ (specifically $u_\xi(x) = \xi \cdot x$). The idea is to approach u_ξ by a sequence u_h such that $Du_h \in \{\xi_1, \xi_2\}$ and the two values are assumed in convenient subdomains.

Let us first consider the case $N = 1$. For each $h \in \mathbb{N}$, we consider the following covering of $\mathbb{R}$:

$$\mathbb{R} = \bigcup_{k\in\mathbb{Z}} \left(\left[\frac{k}{h}, \frac{k+\lambda}{h}\right] \cup \left[\frac{k+\lambda}{h}, \frac{k+1}{h}\right] \right).$$

Let us define $u_\xi(x) = \xi x$, and

$$u_h(x) = \begin{cases} \dfrac{k}{h}\xi + \xi_1 \left(x - \dfrac{k}{h}\right) & \dfrac{k}{h} \leqslant x \leqslant \dfrac{k+\lambda}{h} \\[2ex] \dfrac{k+1}{h}\xi + \xi_2 \left(x - \dfrac{k+1}{h}\right) & \dfrac{k+\lambda}{h} \leqslant x \leqslant \dfrac{k+1}{h} \end{cases}$$

Then, $u_h'(x) = \xi_1$ in $\Omega_{1,h}$ and $u_h'(x) = \xi_2$ in $\Omega_{2,h}$ where

$$\begin{aligned} \Omega_{1,h} &= \Omega \cap \left[\bigcup_{k\in\mathbb{Z}} \left(\frac{k}{h}, \frac{k+\lambda}{h}\right) \right] \\ \Omega_{2,h} &= \Omega \cap \left[\bigcup_{k\in\mathbb{Z}} \left(\frac{k+\lambda}{h}, \frac{k+1}{h}\right) \right] \end{aligned}$$

Moreover, $u_h \to u_\xi$ w^* in $W^{1,\infty}(\Omega)$ as $h \to \infty$, and for every $g \in L^1(\Omega)$ we have

$$\int_{\Omega_{1,h}} g(x)\,dx \longrightarrow \lambda \int_\Omega g(x)\,dx$$

and

$$\int_{\Omega_{2,h}} g(x)\,dx \longrightarrow (1-\lambda) \int_\Omega g(x)\,dx,$$

since this is true for characteristic functions and they are dense in $L^1(\Omega)$.

Therefore,

$$\chi_{\Omega_{1,h}} \longrightarrow \lambda \text{ and } \chi_{\Omega_{2,h}} \longrightarrow (1-\lambda) \quad w^* \text{ in } L^\infty$$

and, taking into account that F is w*-$W^{1,\infty}$ lower semicontinuous,

$$\begin{aligned} \int_\Omega f(x,\xi)\,dx &= F(u_\xi) \leqslant \liminf_{h\to\infty} F(u_h) \\ &= \liminf_{h\to\infty} \left[\int_\Omega f(x,\xi_1)\,\chi_{\Omega_{1,h}} dx + \int_\Omega f(x,\xi_2)\,\chi_{\Omega_{2,h}} dx \right] \\ &= \lambda \int_\Omega f(x,\xi_1)\,dx + (1-\lambda) \int_\Omega f(x,\xi_2)\,dx. \end{aligned}$$

In the case $N \geqslant 2$ it is possible to proceed in an analogous way, taking into account a technical difficulty in building a $W^{1,\infty}$ function whose gradient is ξ_1 in a region Ω_1 and ξ_2 in the complementary region Ω_2. This is possible if and only if the common boundary of Ω_1 and Ω_2 lies on hyperplanes orthogonal to $\xi_1 - \xi_2$. Let us now sketch how the procedure described in $\mathbb{R}$ can be used in the higher dimensional case. Let us fix $h \in \mathbb{N}$, and cover $\mathbb{R}^N$ with a family of strips $S^k_{h,\lambda}$ and $S^k_{h,1-\lambda}$ of width $\frac{\lambda}{h}$ and $\frac{1-\lambda}{h}$ respectively, defined between hyperplanes orthogonal to the vector $\xi_1 - \xi_2$. We define

$$\begin{aligned} \Omega_{1,h} &= \Omega \cap \left(\bigcup_{k\in\mathbb{N}} S^k_{h,\lambda} \right), \\ \Omega_{2,h} &= \Omega \cap \left(\bigcup_{k\in\mathbb{N}} S^k_{h,1-\lambda} \right). \end{aligned}$$

As the reader can verify, this allows us to construct a sequence u_h such that

$$Du_h = \chi_{\Omega_{1,h}}\xi_1 + \chi_{\Omega_{2,h}}\xi_2 \qquad \forall h \in \mathbb{N}$$

and $u_h \to u_\xi$ uniformly in Ω and weakly* in $W^{1,\infty}(\Omega)$. At this point, one can proceed exactly as in the case $N = 1$.

The final step is to deduce (2.1) from (2.2). The idea is to show that (2.2) holds for an arbitrary open set $A \subset \Omega$ instead of Ω. Obviously, if (2.2) is true for every such A, then the pointwise convexity estimate (2.1) holds too. On the other hand, to prove (2.2) for all A is not a trivial task because we do not know if F is w^*-l.s.c. in $W^{1,\infty}(A)$; thus we shall use a joining argument.

Let $B \subset\subset A \subset\subset \Omega$, $\varphi \in C_0^\infty(\Omega)$ with $0 \leqslant \varphi \leqslant 1$ and $\varphi = 1$ in B, $\varphi = 0$ in $\Omega \setminus A$. We take $v_h = \varphi u_h + (1-\varphi)\, u_\xi$, where u_h and u_ξ are as before.

Since $v_h \to u_\xi$ in $L^\infty(\Omega)$ and

$$Dv_h = \varphi Du_h + (1-\varphi)\,\xi + (u_h - u_\xi)\, D\varphi \longrightarrow Du_\xi \text{ in } w^*\text{-}L^\infty(\Omega)$$

we obtain

$$v_h \to u_\xi \text{ in } w^*\text{-}W^{1,\infty}(\Omega).$$

Furthermore, $v_h = u_h$ in B and $v_h = u_\xi$ in $\Omega \setminus A$, then

$$\begin{aligned} F(u_\xi) &= \int_A f(x,\xi)\, dx + \int_{\Omega\setminus A} f(x,\xi)\, dx \leqslant \liminf_{h\to\infty} \int_\Omega f(x, Dv_h)\, dx \\ &= \liminf_{h\to\infty} \left[\int_B f(x, Du_h)\, dx + \int_{A\setminus B} f(x, Dv_h)\, dx \right] + \int_{\Omega\setminus A} f(x,\xi)\, dx. \end{aligned}$$

If we consider $R = 1 + \max\{|\xi_1|, |\xi_2|\}$ we have $|Dv_h| \leqslant R$ for h large enough. By hypothesis

$$|f(x, Dv_h)| \leqslant g_R(x) \qquad \forall x \in \Omega.$$

Then

$$\int_A f(x,\xi)\, dx \leqslant \lambda \int_B f(x,\xi_1)\, dx + (1-\lambda) \int_B f(x,\xi_2)\, dx + \int_{A\setminus B} g_R(x)\, dx$$

for all $B \subset\subset A$. As $g_R \in L^1(\Omega)$, $\sup_{B\subset\subset A} \int_{A\setminus B} g_R(x)\, dx = 0$ and we have

$$\int_A f(x,\xi)\, dx \leqslant \lambda \int_A f(x,\xi_1)\, dx + (1-\lambda) \int_A f(x,\xi_2)\, dx \qquad \forall A \subset\subset \Omega.$$

This allows us to conclude that for every $\xi = \lambda\xi_1 + (1-\lambda)\xi_2$ the inequality

$$f(x) \leqslant \lambda f(x,\xi_1) + (1-\lambda)\, f(x,\xi_2)$$

holds for every $x \in \Omega \setminus M_{\xi_1,\xi_2,\lambda}$, where $\left|M_{\xi_1,\xi_2,\lambda}\right| = 0$. To obtain the thesis we must prove that the last inequality is true for $x \in \Omega \setminus M$, where M is a set of measure 0 which depends only on ξ but not on ξ_1, ξ_2, λ.

For this purpose, given $\xi \in \mathbb{Q}^N$, we define

$$M = \bigcup \left\{ M_{\xi_1,\xi_2,\lambda} :\ \xi_1, \xi_2 \in \mathbb{Q}^N,\ \lambda \in \mathbb{Q} \cap [0,1],\ \xi = \lambda\xi_1 + (1-\lambda)\,\xi_2 \right\}$$

then, $\forall \xi_1, \xi_2 \in \mathbb{Q}^N$, $\lambda \in \mathbb{Q} \cap [0,1]$ such that $\xi = \lambda\xi_1 + (1-\lambda)\,\xi_2$ the inequality

$$f(x,\xi) \leqslant \lambda f(x,\xi_1) + (1-\lambda)\, f(x,\xi_2)$$

holds for all $x \in \Omega \setminus M$, with $|M| = 0$. Therefore, for all $x \in \Omega \setminus M$ the function $\xi \to f(x,\xi)$ is convex in $\mathbb{Q}^N$.

Finally we deduce the convexity in $\mathbb{R}^N$ by the continuity of $f(x,\cdot)$ and a density argument. □

Remark 2.6 The hypothesis of w^*-$W^{1,\infty}(\Omega)$ l.s.c. can be replaced by the w-$W^{1,p}(\Omega)$ l.s.c., $1 \leqslant p < \infty$, provided that $|f(x,\xi)| \leqslant a(x) + b|\xi|^p$.

In fact, w-$W^{1,p}(\Omega)$ l.s.c. implies w^*-$W^{1,\infty}(\Omega)$ l.s.c. and the function $g_R(x) = a(x) + bR^p$ can be used in Theorem 2.5.

Remark 2.7 Let us compare the weak $W^{1,p}$ lower semicontinuity with the strong L^p for $p > 1$. Obviously, the weak $W^{1,p}$ topology is stronger than the norm topology in L^p, hence if F is l.s.c. with respect to the strong L^p convergence, it is so with respect to the weak $W^{1,p}$, too. On the other hand, convergences with respect to the two topologies are equivalent on norm-bounded subsets of $W^{1,p}$, hence both notions of lower semicontinuity are equivalent for a coercive functional.

We now turn our attention to functionals which depend explicitly upon u, i.e.,

$$F(u) = \int_\Omega f(x,u,Du)\, dx \tag{2.3}$$

with $u \in W^{1,p}$, $p > 1$.

These functionals are a particular case of the more general class

$$F(u,v) = \int_\Omega f(x,u,v)\, dx \tag{2.4}$$

with $u \in L^p(\Omega)\,, v \in L^q(\Omega,\mathbb{R}^m)$, arising in optimal control problems. In this more general setting, u is the state variable and v is the control variable, and the typical variational problem is as follows:

$$\min\left\{\int_\Omega f(x,u,v)\, dx :\ u \in U,\ v \in V,\ Lu = Bv\right\}$$

where U is the state space and V the control space and the equation $Lu = Bv$, for suitable operators L, B, relates the state and control variables. Of course, we get (2.3) taking as constitutive equation $Du = v$.

For $1 < p < \infty$ we wonder if F in (2.3) is $w - W^{1,p}$ l.s.c., that is, F is l.s.c. when $u_h \to u$ strongly and $Du_h \to Du$ weakly in L^p. For $p = \infty$, we ask if F is $w^* - W^{1,\infty}$ l.s.c. We shall deal directly with the general case (2.4), and in fact it will be sufficient to prove that $F(u,v)$ is l.s.c. in the $s - L^1 \times w - L^1$ topology, because any functional of the type (2.3) which is $w - W^{1,1}$ l.s.c. is also $w - W^{1,p}$ l.s.c.

Theorem 2.8 (De Giorgi) *Let $\Omega \subset \mathbb{R}^N$ be an open bounded set and $f : \Omega \times \mathbb{R}^n \times \mathbb{R}^m \to [0,+\infty]$ be such that:*

(i) *$f(.,s,\xi)$ is measurable $\forall(s,\xi) \in \mathbb{R}^n \times \mathbb{R}^m$,*

(ii) *$f(x,.,.)$ is Borel measurable and l.s.c $\forall x \in \Omega$,*

(iii) *$f(x,s,.)$ is convex $\forall\,(x,s) \in \Omega \times \mathbb{R}^n$.*

Then $F(u,v) = \int_\Omega f(x,u(x),v(x))\, dx$ is $s - L^1(\Omega,\mathbb{R}^n) \times w - L^1(\Omega,\mathbb{R}^m)$ l.s.c.

PROOF. The idea is to write f as the supremum of a countable family of functions for which the property of l.s.c. is easier to verify. In particular, we shall obtain a representation of the form

$$f(x,s,\xi) = \sup_{h\in\mathbb{N}}(a_h(x,s)\cdot\xi + b_h(x,s))$$

with $a_h : \Omega\times\mathbb{R}^n \to \mathbb{R}^m$ and $b_h : \Omega\times\mathbb{R}^n \to \mathbb{R}$ bounded functions such that $a_h(\cdot,s)$ and $b_h(\cdot,s)$ are measurable and $a_h(x,\cdot)$ and $b_h(x,\cdot)$ are Lipschitz continuous.

1stStep: We initially consider f dependent only on the variable ξ andsuperlinear. Let then be $f:\mathbb{R}^m \to [0,\infty]$ be a convex and l.s.c. function such that

$$\lim_{|\xi|\to\infty}\frac{f(\xi)}{|\xi|} = +\infty.$$

Let $A = \{(a,b)\in\mathbb{R}^m\times\mathbb{R} : a\cdot\xi + b \leqslant f(\xi), \forall\xi\in\mathbb{R}^m\}$. Obviously, $A\neq\emptyset$. Let us recall that $epi(f) = \{(\xi,t)\in\mathbb{R}^m\times\mathbb{R} : t \geqslant f(\xi)\}$ is a closed and convex set because f is a convex function. Next, as a consequence of the Hahn-Banach Theorem applied to the $epi(f)$ we obtain the following representation of f

$$f(\xi) = \sup_{(a,b)\in A}(a\cdot\xi + b). \tag{2.5}$$

Now, we consider the set $B = \{(a,b)\in\mathbb{Q}^m\times\mathbb{Q} : a\cdot\xi + b \leqslant f(\xi), \forall\xi\in\mathbb{R}^m\}$. Let us show that A can be replaced by B in (2.5), i.e.,

$$f(\xi) = \sup_{(a,b)\in B}(a\cdot\xi + b).$$

This is possible taking into account the superlinearity of the function f. In fact, we may fix $\xi_0\in\mathbb{R}^m$ and take $t_0\in\mathbb{R}$ and $\epsilon>0$ such that $t_0 < f(\xi_0) - \epsilon$. We apply the Hahn-Banach Theorem to the superlinear function $f-\epsilon$ and we have that

$$(f-\epsilon)(\xi) = \sup_{(a,b)\in A}(a\cdot\xi + b) \tag{2.6}$$

or equivalently

$$f(\xi) - \epsilon = \sup_{(a,b)\in A}(a\cdot\xi + b).$$

Now, we choose $(a_0,b_0)\in\mathbb{R}^m\times\mathbb{R}$ such that $a_0\cdot\xi + b_0 \leqslant f(\xi)-\epsilon$ for all $\xi\in\mathbb{R}^m$ and $t_0 < a_0\cdot\xi_0 + b_0$. Next, because of the superlinearity of f, there exist $R>0$ such that

$$f(\xi) > (|a_0|+1)\,|\xi| + |b_0| + 1 \quad \text{for } |\xi| \geqslant \mathrm{R}.$$

Taking $(a,b)\in\mathbb{Q}^m\times\mathbb{Q}$ sufficiently close to (a_0,b_0) we have

$$a\cdot\xi + b \leqslant f(\xi) \quad \text{for } |\xi| \geqslant \mathrm{R},$$

$$a\cdot\xi + b \leqslant a_0\cdot\xi + b_0 + \epsilon \leqslant f(\xi) \quad \text{for } |\xi| < \mathrm{R},$$

$$a\cdot\xi_0 + b > t_0,$$

and (2.6) follows.

$\mathbf{2}^{nd}$ **Step**: We now consider $f : \mathbb{R}^n \times \mathbb{R}^m \to [0, \infty]$ such that $f = f(s, \xi)$ is a convex function in $\mathbb{R}^m$ and l.s.c. in $\mathbb{R}^n \times \mathbb{R}^m$. Additionally, we assume that there exists a superlinear function ψ such that

$$f(s, \xi) \geqslant \psi(\xi), \quad \forall (s, \xi) \in \mathbb{R}^n \times \mathbb{R}^m.$$

For fixed $a \in \mathbb{R}^m$ and $b \in \mathbb{R}$ we define the (possibly empty) set

$$E_{ab} = \{s \in \mathbb{R}^n : a \cdot \xi + b < f(s, \xi) \quad \forall \xi \in \mathbb{R}^m\}$$

and we prove that E_{ab} is a open set.

Since ψ is a superlinear function, there exist $R > 0$ such that if $|\xi| \geqslant R$

$$\psi(\xi) > |a|\,|\xi| + |b|\,.$$

Consequently

$$a \cdot \xi + b \leqslant |a|\,|\xi| + |b| < \psi(\xi) \leqslant f(s, \xi) \;\; \forall s \in \mathbb{R}^n \tag{2.7}$$

and then

$$\begin{aligned} E_{ab} &= \{s \in \mathbb{R}^n : a \cdot \xi + b < f(s, \xi) \;\; \forall \xi \in \mathbb{R}^m : |\xi| \leqslant R\} \\ &= \left\{ s \in \mathbb{R}^n : \min_{|\xi| \leqslant R} (f(s, \xi) - a \cdot \xi - b) > 0 \right\}. \end{aligned}$$

We now recall a well-known result:

> Let X be a metric space, let K be a compact metric space, $G : X \times K \to \overline{\mathbb{R}}$ a l.s.c. function. Then g defined by $g(x) = \min_{y \in K} G(x, y)$ is a l.s.c. function in X.

In our case $X = \mathbb{R}^n$, $K = \{\xi \in \mathbb{R}^m : |\xi| \leqslant R\}$ is a compact set and $G(s, \xi) = f(s, \xi) - a \cdot \xi - b$ is a l.s.c. function in $X \times K$. Therefore, $\min_{\xi \in K} G(s, \xi)$ is l.s.c., that is $\forall t \in \mathbb{R}$ the set $\left\{ s \in \mathbb{R}^n : \min_{|\xi| \leqslant R} (f(s, \xi) - a \cdot \xi - b) > t \right\}$ is open, in particular for $t = 0$.

We define $\forall k \in \mathbb{N}$

$$\Psi^k_{ab}(s) = \min\{1, k\, dist(s, E^c_{ab})\}$$

where E^c_{ab} denote the complementary set of E_{ab}. Let us notice that Ψ^k_{ab} is a lipschitzian application with constant k.

Now, we prove that

$$f(s, \xi) = \sup\left\{\Psi^k_{ab}(s)(a \cdot \xi + b) : a \in \mathbb{Q}^m, b \in \mathbb{Q},\; k \in \mathbb{N}\right\}.$$

Indeed, if $\Psi^k_{ab}(s) = 0$, by the hypothesis we obtain that $f(s, \xi) \geqslant 0$. If $\Psi^k_{ab}(s) \neq 0$, then $s \in E_{ab}$ and since $\Psi^k_{ab}(s) \leqslant 1$ and $f(s, \xi) \geqslant 0$ we have

$$f(s, \xi) \geqslant \Psi^k_{ab}(s)(a \cdot \xi + b).$$

Then,

$$f(s, \xi) \geqslant \sup\left\{\Psi^k_{ab}(s)(a \cdot \xi + b) : a \in \mathbb{Q}^m, b \in \mathbb{Q}, k \in \mathbb{N}\right\}.$$

On the other hand, if we fix $(s_0, \xi_0) \in \mathbb{R}^n \times \mathbb{R}^m$ and choose $\alpha < f(s_0, \xi_0)$ we can proceed as in the 1st. step. Then there exists $(a, b) \in \mathbb{Q}^m \times \mathbb{Q}$ such that $\alpha < a \cdot \xi_0 + b$ and $a \cdot \xi + b < f(s_0, \xi)$, $\forall \xi \in \mathbb{R}^m$. Therefore $s_0 \in E_{ab}$ and since E_{ab} is an open set there exists $k_0 \in \mathbb{N}$ such that $k_0 dist(s_0, E^c_{ab}) \geqslant 1$. Then $\Psi^k_{ab}(s_0) = 1$, $\alpha < \Psi^k_{ab}(s_0)(a.\xi_0 + b)$ and from the arbitrariness of α we have the other inequality.

3^{rd} Step: Let $f : \Omega \times \mathbb{R}^n \times \mathbb{R}^m \to [0, \infty]$ be a function satisfying the hypothesis of the theorem, and let us assume that there exists a superlinear function φ such that

$$f(x, s, \xi) \geqslant \varphi(\xi) \quad \forall (x, s, \xi) \in \Omega \times \mathbb{R}^n \times \mathbb{R}^m.$$

We consider the set

$$E_{ab}(x) = \{s \in \mathbb{R}^n : a \cdot \xi + b < f(x, s, \xi) \ \forall \xi \in \mathbb{R}^m\}$$

and the function

$$\Psi^k_{ab}(x, s) = \min\{1, k\, dist(s, E^c_{ab}(x))\}\,.$$

We deduce from the 2^{nd} step that $\forall (x, s, \xi) \in \Omega \times \mathbb{R}^n \times \mathbb{R}^m$

$$f(x, s, \xi) = \sup\left\{\Psi^k_{ab}(x, s)(a \cdot \xi + b) : a \in \mathbb{Q}^m, b \in \mathbb{Q},\ k \in \mathbb{N}\right\} \tag{2.8}$$

Now, we note that $\forall k \in \mathbb{N}$, $\Psi^k_{ab}(x, s)$ is a Carathéodory function, that is

(a) $\Psi^k_{ab}(x, \cdot)$ is a continuous function.

(b) $\Psi^k_{ab}(\cdot, s)$ is a measurable function.

In fact, $dist(s, E^c_{ab}(x))$ is a measurable function for any fixed s, because $\forall s \in \mathbb{R}^n, \forall r > 0$ the set

$$G := \{x \in \Omega : dist(s, E^c_{ab}(x)) < r\}$$

is measurable. Indeed, let $B_r(s)$ be the open ball of $\mathbb{R}^n$ with radius r and centre s; then

$$\begin{aligned} G &= \{x \in \Omega : \exists\ z \in B_r(s) \cap E^c_{ab}(x)\} \\ &= \{x \in \Omega : \exists\ z \in B_r(s)\ \exists \xi \in \mathbb{R}^m : f(x, z, \xi) \leqslant a \cdot \xi + b\} \\ &= P_\Omega \{(x, z, \xi) \in \Omega \times B_r(s) \times \mathbb{R}^m : f(x, z, \xi) \leqslant a \cdot \xi + b\}\,, \end{aligned}$$

where $P_\Omega : \Omega \times \mathbb{R}^n \times \mathbb{R}^m \to \Omega$ is the projection on Ω. Taking into account the hypothesis on f, the set

$$\{(x, s, \xi) \in \Omega \times \mathbb{R}^n \times \mathbb{R}^m : f(x, s, \xi) \leqslant a \cdot \xi + b\}$$

is measurable for all $z \in B_r(s)$, for fixed $\xi \in \mathbb{R}^m$ and it is measurable Borel in $\mathbb{R}^N \times \mathbb{R}^m$ and therefore G is measurable as a consequence of the projection theorem.

Now, the expression (2.8) tells us that there are two sequences of bounded Carathéodory functions (a_h) and (b_h) such that

$$f(x, s, \xi) = \sup_{h \in \mathbb{N}} (a_h(x, s) \cdot \xi + b_h(x, s)). \tag{2.9}$$

4^{th} Step: We cannot deduce from (2.9) a representation of the functional F in the form

$$F(u, v) = \sup_{h \in \mathbb{N}} \int_\Omega (a_h(x, u(x)) \cdot v(x) + b_h(x, u(x)))\, dx$$

(which would give at once the lower semicontinuity as a consequence of Remark 1.4), but we may exploit (2.9) in connection with a localization procedure. Indeed, we shall prove that

$$F(u,v) = \sup_{k\in\mathbb{N}} \sup_{B_h} \sum_{h=1}^{k} \int_{B_h} (a_h(x,u(x))\cdot v(x) + b_h(x,u(x)))\,dx$$

where $\{B_1,\ldots,B_k\}$ are all possible disjoint families of measurable sets such that $\Omega = \bigcup_{h=1}^{k} B_h$ and the functions a_h, b_h are those in (2.9).

Set

$$g_h(x) = a_h(x,u(x))\cdot v(x) + b_h(x,u(x))$$

and define $f_k = \max\{g_1(x), g_2(x),\ldots,g_k(x)\}$. Since (f_k) is a monotone sequence that is a.e. convergent to $g = \sup_h g_h$, we can apply Beppo Levi's monotone convergence theorem, so

$$\int_\Omega g(x)dx = \sup_{k\in N} \int_\Omega f_k(x)dx.$$

On the other hand, if we consider $F_h = \{x : f_k(x) = g_h(x)\}$, $F_0 = \emptyset$ and we define $E_h = F_h \setminus F_{h-1}$, we have

$$\sup_{k\in\mathbb{N}} \int_\Omega f_k(x)dx = \sup_k \sum_{h=1}^{k} \int_{E_h} g_h \leqslant \sup_k \sup_{\{B_1,\ldots,B_k\}} \sum_{h=1}^{k} \int_{B_h} g_h.$$

Therefore

$$F(u,v) \leqslant \sup_{k\in\mathbb{N}} \sup_{\{B_1,\ldots,B_k\}} \sum_{h=1}^{k} \int_{B_h} a_h(x,u(x))\cdot v(x) + b_h(x,u(x))dx.$$

In order to prove the equality, we consider a measurable set $E\subset\Omega$, and we have

$$\int_E f_k \geqslant \int_E g_h, \text{for } h = 1,\ldots,k$$

and

$$\int_E g \geqslant \int_E g_h.$$

Therefore, for all families $\{B_1,\ldots,B_k\}$ we have that

$$\int_\Omega g \geqslant \sum_{h=1}^{k} \int_{B_h} g_h$$

and the equality easily follows.

Now we prove that

$$G(u,v) = \int_E a(x,u(x))\cdot v(x) + b(x,u(x))\,dx$$

is a $s\!-\!L^1(E,\mathbb{R}^N)\times w\!-\!L^1(E,\mathbb{R}^m)$ continuous functional with a and b bounded Carathéodory functions.

We fix $E\subset\Omega$ and we consider two sequences $(u_h)_{h\in\mathbb{N}}$ and $(v_h)_{h\in\mathbb{N}}$, such that they are strongly convergent to u in $L^1(E,\mathbb{R}^n)$ and weakly convergent to v in $L^1(E,\mathbb{R}^m)$, respectively. Since (possibly taking a subsequance) $u_h\to u$ a.e., we have from Lebesgue's dominated convergence theorem

$$\int_E b(x,u(x))dx=\lim_{h\to\infty}\int_E b(x,u_h(x))\,dx.$$

The sequence $g_h(x)=a(x,u_h(x))$ is uniformly bounded in $L^\infty(E,\mathbb{R}^m)$, converges to $g\,(x)=a\,(x,u\,(x))$ a.e., so that

$$\lim_{h\to\infty}\int_E g_h v_h dx=\int_E gv\,dx.$$

In fact,

$$\int_E g_h v_h\,dx-\int_E gv\,dx=\int_E (g_h-g)v_h\,dx-\int_E g(v-v_h)\,dx.$$

As g is a bounded function and v_h is weakly convergent to v in $L^1(E,\mathbb{R}^m)$ we obtain that the last integral goes to zero. Now, from the Severini-Egorov theorem, for all $\varepsilon>0$ there exists a set B_ε with $|B_\varepsilon|<\varepsilon$, such that $g_h\to g$ uniformly in $E\setminus B_\varepsilon$.

Next

$$\int_E (g_h-g)v_h\,dx=\int_{E\setminus B_\varepsilon}(g_h-g)v_h\,dx+\int_{B_\varepsilon}(g_h-g)v_h\,dx$$

from the uniform convergence of the sequence g_h and the equintegrability of the sequence v_h we deduce that

$$\int_{E\setminus B_\varepsilon}|(g_h-g)v_h|\;dx\leqslant\sup_h\|v_h\|_{L^1(E\setminus B_\varepsilon)}\|g_h-g\|_{L^\infty(E\setminus B_\varepsilon)}$$

and

$$\int_{B_\varepsilon}|(g_h-g)v_h|\;dx\leqslant 2\,\|g\|_{L^\infty(B_\varepsilon)}\int_{B_\varepsilon}|v_h|\;dx\leqslant 2\,\|g\|_{L^\infty(B_\varepsilon)}\,.C$$

with $C=const.>0$.

Then, if $h\to\infty$ and afterwards $\varepsilon\to 0$, both integrals go to 0. At this point, we have proved the theorem under the special hypothesis that there exists a convex superlinear function $\Psi:\mathbb{R}^m\to[0,\infty)$ such that

$$f(x,s,\xi)\geqslant\Psi(\xi),\qquad\forall(x,s,\xi)\in\Omega\times\mathbb{R}^n\times\mathbb{R}^m.$$

In the last step we shall remove this restriction.

5^{th} **Step**: Let u_h be strongly convergent to u in $L^1(\Omega,\mathbb{R}^n)$ and v_h weakly convergent to v in $L^1(\Omega,\mathbb{R}^m)$; then v_h is a bounded and equintegrable sequence, i.e.,

(a) for all $\varepsilon > 0$ there exists $A \subset \Omega$ such that

$$\int_{\Omega \setminus A} |g_h(x)| \, dx < \varepsilon$$

for all $h \in \mathbf{N}$.

(b) there exists $\delta > 0$ such that if $|B| < \delta$, then

$$\int_B |g_h(x)| \, dx < \varepsilon$$

for all $h \in \mathbf{N}$.

Therefore, see e.g. [6, Prop. 1.27], there exists a convex and superlinear application Ψ such that $\int_\Omega \Psi(v_h)\, dx \leqslant 1$, $\forall h \in \mathbb{N}$. Next, for each $\varepsilon > 0$ we can consider the function $f(x, s, \xi) + \varepsilon\Psi(\xi)$, which satisfies the previous hypothesis. Then

$$\begin{aligned} &\int_\Omega f(x, u(x), v(x))\, dx + \varepsilon \int_\Omega \Psi(v(x))\, dx \\ &\leqslant \liminf_{h\to\infty} \left(\int_\Omega f(x, u_h(x), v_h(x))\, dx + \varepsilon \int_\Omega \Psi(v_h(x))\, dx \right) \\ &\leqslant \liminf_{h\to\infty} \int_\Omega f(x, u_h(x), v_h(x))\, dx + \varepsilon. \end{aligned}$$

As $\Psi > 0$, we deduce that

$$\int_\Omega f(x, u(x), v(x))\, dx \leqslant \liminf_{h\to\infty} \int_\Omega f(x, u_h(x), v_h(x))\, dx + \varepsilon$$

and the proof of the theorem is now complete. □

The following property is a generalization of the preceding theorem for a non positive function f.

Theorem 2.9 (Ambrosio, [3]) *Let $f : \Omega \times \mathbb{R}^n \times \mathbb{R}^m \to]-\infty, +\infty]$ be such that $f(x, ., .)$ is l.s.c in $\mathbb{R}^{n+m}$ and $f(x, s, .)$ is a convex function. If $u_h \to u$ strongly in $L^1(\Omega)$, $v_h \to v$ weakly in $L^1(\Omega)$ and $\limsup_{h\to\infty} \int_\Omega f(x, u_h(x), v_h(x))\, dx < \infty$, then the sequence $g_h(x) = f^-(x, u_h(x), v_h(x))$ is equintegrable. Moreover, if there exists $(u_0, v_0) \in L^1(\Omega) \times L^1(\Omega)$ such that $f(x, u_0, v_0) \in L^1(\Omega)$ then $F(u, v)$ is l.s.c. in the $s - L^1 \times w - L^1$ topology.*

Remark 2.10 A convenient coerciveness hypothesis enables us to obtain results about the existence of a minimum. For instance, if we consider $\Psi \in W^{1,p}(\Omega)$ for $1 < p < \infty$ and f such that

$$a\, |\xi|^p \leqslant f(x, s, \xi) \leqslant b\, |\xi|^p$$

the existence of

$$\min_{u\in W^{1,p}(\Omega)}\left\{\int_\Omega f(x,u(x),Du(x))\,dx:\ u-\psi\in W_0^{1,p}(\Omega)\right\}$$

can be easily proved.

Remark 2.11 If the integrand f in (2.3) is a convex function of (s,ξ) then the weakly l.s.c. in $W^{1,p}(\Omega)$ would follow immediately from Theorem 2.1. But, from Rellich-Kondrachov compact embedding theorem it follows that the weak convergence of the gradients in $L^p(\Omega)$ implies the strong convergence of the u_h and this allows to disregard the convexity of f with respect to s.

The following result, which we present without proof, extends in some sense Theorem 2.5 to integral functionals whose integrand depends explicitly upon u.

Theorem 2.12 (Olech, [27]) *Let $\Omega\subset\mathbb{R}^N$ be a open bounded set and $f:\Omega\times\mathbb{R}^n\times\mathbb{R}^m\to[0,+\infty]$ such that*

i) *$f(.,s,\xi)$ is measurable for all $(s,\xi)\in\mathbb{R}^N\times\mathbb{R}^m$*

ii) *$f(x,.,.)$ is Borel measurable*

iii) *$F(u,v)=\int_\Omega f(x,u(x),v(x))dx$ is l.s.c. in the $s-L^1(\Omega,\mathbb{R}^n)\times w-L^1(\Omega,\mathbb{R}^m)$ topology.*

Then $f(x,.,.)$ is l.s.c. for a.e. $x\in\Omega$ and $f(x,s,.)$ is a convex function for a.e. $(x,s)\in\Omega\times\mathbb{R}^n$.

Another result in the same direction is the following.

Theorem 2.13 (Marcellini-Sbordone, [25]) *Let $f:\Omega\times\mathbb{R}\times\mathbb{R}^m\to\mathbb{R}$ be a Carathéodory function, and assume that there exists a function $g(x,u,v)$ increasing with respect to u and v and such that $g(\cdot,u,v)\in L^1_{loc}(\Omega)$ for all (u,v) and $0\leqslant tf(x,s,\xi)\leqslant g(x,|s|,|\xi|)$. If the functional*

$$u\mapsto F(u)=\int_\Omega f(x,u(x),Du(x))dx$$

is weakly l.s.c. in $W^{1,p}(\Omega)$, $1lqsp\leqslant\infty$, then $f(x,s,.)$ is a convex function.

Remark 2.14 Notice that if $F(u,v)=\int_\Omega f(x,u(x),v(x))dx$ is l.s.c. in $s-L^1(\Omega)\times w-L^1(\Omega)$, with $u:\Omega\to\mathbb{R}^n$ and $v:\Omega\to\mathbb{R}^m$ then Theorem 2.12 says that $f(x,s,.)$ is a convex function. Analogously, Theorem 2.13 states a similar result in Sobolev spaces, but *only for the case* $n=1$. We shall later see that this result is no longer true in the vectorial case.

As before, an existence result easily follows:

Theorem 2.15 *Let $\Omega \subset \mathbb{R}^N$ be an open bounded set with Lipschitz boundary and $f : \Omega \times \mathbb{R} \times \mathbb{R}^N \to]-\infty, +\infty]$ a Carathéodory function, $f(x, s, .)$ a convex application such that $f(x, s, \xi) \geqslant a(x) + b\,|\xi|^p$ with $a \in L^1(\Omega), b > 0$ and $p > 1$. Let $\varphi \in W^{1,p}(\Omega)$ be such that there exists $u_0 \in W^{1,p}(\Omega)$ with $u_0 - \varphi \in W_0^{1,p}(\Omega)$ and*

$$\int_\Omega f(x, u_0(x), Du_0(x))\, dx < \infty.$$

Then, the minimum

$$\min_{u \in W^{1,p}(\Omega)} \left\{ \int_\Omega f(x, u(x), Du(x))dx : \ u - \varphi \in W_0^{1,p}(\Omega) \right\}$$

is achieved.

Remark 2.16 If we consider f such that

$$|f(x, s, \xi)| \leqslant a(x) + b_1(|s|^q + |\xi|^p)$$

with $q = \frac{Np}{N-p}$ for $p < N$ and no condition on q if $p \geqslant N$, then

$$\inf_{W^{1,p}(\Omega)} \int_\Omega f(x, u(x), Du(x))dx < \infty.$$

Note that, in this case, the condition of the preceding theorem: "there exists $u_0 \in W^{1,p}(\Omega)$ such that $u_0 - \varphi \in W_0^{1,p}(\Omega)$ and $\int_\Omega f(x, u_0(x), Du_0(x))dx < \infty$" can to be omitted.

Remark 2.17 We know that $F(u) = \int_\Omega f(x, Du(x))dx$ is a convex functional if only if $f(x, .)$ is a convex function. This result is not true for $F(u) = \int_\Omega f(x, u(x), Du(x))\, dx$. In fact, if F is defined by

$$F(u) = \int_0^1 \left[|u'(x)|^4 + (u^2(x) - 1)^2 \right] dx$$

we obtain that F is a weakly l.s.c. functional in $W^{1,4}(\Omega)$ but F is not a convex functional. Moreover, note that the integrand is a convex function of u'.

Example 2.18 We consider $N = 1$, $\Omega = (0, 1)$ and $f(x, s, \xi) = x\xi^2$. We ask if there exists the following

$$\min_{u \in W^{1,2}(\Omega)} \left\{ F(u) = \int_0^1 x(u'(x))^2 dx : \ u(0) = 1, u(1) = 0 \right\}.$$

If we define the sequence u_h by

$$u_h(x) = \begin{cases} 1 & \text{if } 0 \leqslant x \leqslant \dfrac{1}{h}, \\ -\dfrac{\log x}{\log h} & \text{if } \dfrac{1}{h} < x \leqslant 1, \end{cases}$$

we can see that

$$F(u_h) = \int_{\frac{1}{h}}^{1} x \frac{1}{x^2 \log^2 h} dx = -\frac{\log \frac{1}{h}}{\log^2 h} = \frac{1}{\log h} \longrightarrow 0 \qquad \text{as } h \to \infty$$

and $\inf\limits_{W^{1,2}(\Omega)} F(u_h) \geqslant 0$, but $F(u) = 0$, then $u' = 0$ a.e. and this is a contradiction. Note that F is not a coercive functional.

In the case where $F(u,v) = \int_\Omega f(x,u,v)\ dx$ is not l.s.c. we follow the ideas of Chapter I and try to find the relaxed functional $\overline{F}(u,v)$. In applications, it is very important to know if the relaxed functional is again an integral functional. This is not always the case (from a physical point of view, *nonlocal effects* may appear), but we have the following rather general result. We present only a sketch of the proof, because there are many technical difficulties to show that $\overline{F}$ is an integral functional. A complete proof can be found in [8].

Theorem 2.19 *Let* $f : \Omega \times \mathbb{R}^n \times \mathbb{R}^m \to [0,+\infty]$ *be Borel measurable and* $F(u,v) = \int_\Omega f(x,u,v)\ dx$*.*

Then there exists $g : \Omega \times \mathbb{R}^n \times \mathbb{R}^m \to [0,+\infty]$ *such that the* $s - L^1 \times w - L^1$ *relaxed functional is*

$$\overline{F}(u,v) = \int_\Omega g(x,u,v)\ dx.$$

Moreover, g is the l.s.c. convex hull of f, that is

$$g(x,s,\xi) = \underline{co}\, f = \sup\{\psi \leqslant f : \ \psi \text{ l.s.c. in } x,\ s \text{ and } \psi(x,s,\cdot) \ \ \text{convex}\}. \qquad (2.10)$$

Sketch of the Proof. It is clear from Olech's Theorem 2.12 that $g(x,s,\cdot)$ must be a convex function and it is natural to try with $g = \underline{co}\, f$.

1stStep: We first localise F by defining

$$F(u,v,B) = \int_B f(x,u,v)\ dx \text{ for each measurable } B \subset \Omega.$$

Then, we relax with respect to the $s - L^1 \times w - L^1$ sequential topology and we obtain for each B the l.s.c. envelope

$$\overline{F}(u,v,B) = \overline{sc}\, F(u,v,B).$$

This relaxed localised functional has the following properties:

(a) $\overline{F}(\cdot,\cdot,B)$ is sequentially $s - L^1 \times w - L^1$ l.s.c. and hence $s - L^1 \times s - L^1$ l.s.c. (recall Remark 1.4).

(b) $\overline{F}(u,v,B)$ is local, that is, if $u_1 = u_2$ and $v_1 = v_2$ a.e. in B then $\overline{F}(u_1,v_1,B) = \overline{F}(u_2,v_2,B)$.

(c) $\overline{F}(u,v,\cdot)$ is additive, that is, for all pairs $B_1,\ B_2 \subset \Omega$ such that $B_1 \cap B_2 = \emptyset$ the equality $\overline{F}(u,v,B_1 \cup B_2) = \overline{F}(u,v,B_1) + \overline{F}(u,v,B_2)$ holds for all u,v in $L^1(\Omega)$.

Let us notice that (b) and (c) are obvious properties for an integral functional.

2ndStep: It can be proved that $F(u,v,\cdot)$ is a measure, that is, for each (u,v) there exists a measure μ_{uv} such that

$$\overline{F}(u,v,B) = \mu_{uv}(B) = \int\limits_B d\mu_{uv} \ \forall \text{ measurable } B \subset \Omega.$$

This result is a consequence of a De Giorgi-Letta theorem (see e.g.[6, 1.53]) by noting that $\overline{F}(u,v,\cdot)$ is an increasing set function.

3rdStep: Moreover, $\overline{F}(u,v,\cdot)$ is an absolutely continuous measure with respect to dx. This is actually the crucial and hardest step. Then, there exists $g : \Omega\times\mathbb{R}^n\times\mathbb{R}^m \to [0,+\infty)$ such that

- $\overline{sc}\, F(u,v,B) = \int_B g(x,u,v)\, dx \ \forall$ measurable $B \subset \Omega$.
- $g(\cdot,s,\xi)$ is measurable in B.
- $g(x,\cdot,\cdot)$ is l.s.c. and, by Olech's Theorem 2.12.
- $g(x,s,\cdot)$ is a convex function.

4thStep: We prove that $g = \underline{co}\, f$ because of the definition of $\underline{co}\, f$ (2.10) and the properties of g quoted in Step 3.

On the other hand

$$\int\limits_\Omega \underline{co}\, f\, dx \leqslant \int\limits_\Omega f\, dx = F(u,v)$$

and $\int_\Omega \underline{co}\, f\, dx$ is l.s.c.

Hence, $\int_\Omega \underline{co}\, f\, dx \leqslant \overline{F}(u,v) = \int_\Omega g(x,u,v)\, dx$, from which, taking into account that $g \leqslant \underline{co}\, f$, we have $g = \underline{co}\, f$.

$\square$

3 The vectorial case

In the preceding Section we have studied functionals of the following types

$$\begin{aligned} F(u) &= \int\limits_\Omega f(x,u)\, dx, \\ F(u) &= \int\limits_\Omega f(x,Du)\, dx \\ F(u) &= \int\limits_\Omega f(x,u,Du)\, dx \end{aligned}$$

for *scalar-valued* functions $u : \Omega \to \mathbb{R}$. We have also considered functionals like

$$F(u,v) = \int\limits_\Omega f(x,u,v)\, dx$$

for vector-valued u, v, but in the particular case $v = Du$ much more can be said, besides what immediately follows from the general case. In particular, as it was pointed out in Remark 2.14, the hypothesis of convexity with respect to Du is too strong, as the following example suggests.

Example 3.1 Let Ω be the unit square in the plane, and think of it as the rest configuration of an elastic membrane, so that $u : \Omega \to \mathbb{R}^2$ are the deformations. For linear deformations, it is reasonable to assume that the corresponding energy F is proportional (for simplicity, say equal) to the increase of area. If u_1 maps Ω onto $[0, 2] \times [0, 1]$ and u_2 maps Ω onto $[0, 1] \times [0, 2]$ then of course $F(u_1) = F(u_2) = 1$ but

$$F\Big(\frac{u_1 + u_2}{2}\Big) = \Big(\frac{3}{2}\Big)^2 = \frac{5}{4} > \frac{1}{2}F(u_1) + \frac{1}{2}F(u_2)$$

and the energy functional is not convex. On the other hand, one expects that physical energies are lower semicontinuous in some reasonable topology.

Let us then consider the functional

$$F(u) = \int_\Omega f(x, u, Du)\, dx$$

where $f : \Omega \times \mathbb{R}^n \times \mathbb{R}^{Nn} \to \left]-\infty, +\infty\right]$, $u : \Omega \subset \mathbb{R}^N \to \mathbb{R}^n$ and $Du : \Omega \to \mathbb{R}^{Nn}$ is the Jacobian matrix.

If $f(x, s, \cdot)$ is convex $\forall (x, s) \in \Omega \times \mathbb{R}^n$ all the above results hold, but this condition is not necessary. In 1952, J.B. Morrey discovered an useful condition which turns out to be necessary and sufficient for the $w - W^{1,p}$ l.s.c. of F and plays a role analogous to that of convexity in the vectorial case. Unfortunately, it is not easy to handle, and even its definition is not algebraic in nature.

Definition 3.2 *The function* $f : \mathbb{R}^{nN} \to \mathbb{R}$ *is* quasiconvex *if* f *is Borel measurable,* $f \in L^\infty_{loc}\left(\mathbb{R}^{nN}\right)$ *and*

$$\int_\Omega f(\xi + D\varphi)\, dx \geqslant |\Omega|\, f(\xi) \tag{3.1}$$

for all bounded sets $\Omega \subset \mathbb{R}^N$*, for all* $\xi \in \mathbb{R}^{nN}$*, for all* $\varphi \in C^1_0(\Omega, \mathbb{R}^n)$.

Let us stress the variational character of the notion of quasiconvexity.

Remark 3.3 Note that taking $u(x) = \xi x + \varphi(x)$ with $\varphi \in C^1_0(\Omega, \mathbb{R}^n)$ inequality (3.1) becomes

$$\int_\Omega f(Du)\, dx \geqslant \int_\Omega f(\xi)\, dx,$$

for all bounded $\Omega \subset \mathbb{R}^N$, $\forall \xi \in \mathbb{R}^{nN}$, and for all $u \in C^1(\Omega, \mathbb{R}^n)$ with $u|_{\partial\Omega} = \xi x$.

This means that $f \in L^\infty_{loc}\left(\mathbb{R}^{nN}\right)$ and Borel measurable is quasiconvex if and only if for all $\Omega \subset \mathbb{R}^N$ and for all $\xi \in \mathbb{R}^{nN}$

$$\min\left\{\int_\Omega f(Du)\, dx : \ u \in C^1(\Omega, \mathbb{R}^n),\, u|_{\partial\Omega} = \xi x\right\}$$

is achieved at $u_0 = \xi x$.

It is easily seen that convex functions are quasiconvex.

Remark 3.4 Let $f : \mathbb{R}^m \to \mathbb{R}$ be convex. Then, by Jensen's inequality,

$$\frac{1}{|\Omega|}\int_\Omega f(\xi + D\varphi)\,dx \geqslant f\Big(\frac{1}{|\Omega|}\int_\Omega \xi + D\varphi\,dx\Big) = f(\xi)$$

for all $\varphi \in C_0^1(\Omega)$. As every finite convex function f is continuous, f is quasiconvex.

On the other hand, there are quasiconvex functions that are not convex.

Example 3.5 An example of a quasiconvex function which is not a convex one is the function

$$f\,(\xi) = |\det \xi|,$$

where ξ is any $N \times n$ matrix. In fact, for $N = n = 2$

$$\begin{aligned}\int_\Omega \det(Du(x))\,dx &= \int_\Omega \left(D_1u^1(x)D_2u^2(x) - D_2u^1(x)D_1u^2(x)\right)\,dx\\ &= \int_\Omega D_1\left(u^1(x)D_2u^2(x)\right) - D_2\left(u^1(x)D_1u^2(x)\right)\,dx\\ &= \int_{\partial\Omega} u^1\frac{\partial u^2}{\partial\tau}\,ds\end{aligned}$$

where τ is the unit tangent vector to $\partial\Omega$ (positevely oriented). Then, the integral $\int_\Omega \det\left(Du\left(x\right)\right)\,dx$ depends only upon the values of u on $\partial\Omega$ and, if we choose $u\left(x\right) = \xi x + \varphi\left(x\right)$ with $\varphi \in C_0^2\left(\Omega, \mathbb{R}^2\right)$ we get

$$\int_\Omega \det\left(\xi + D\varphi\left(x\right)\right)\,dx = \int_\Omega \det\xi\,dx = |\Omega|\det\xi.$$

This equality is also valid $\forall\varphi \in C_0^1\left(\Omega, \mathbb{R}^2\right)$ by continuity and as

$$\int_\Omega \left|\det\left(\xi + D\varphi\left(x\right)\right)\right|\,dx \geqslant \left|\int_\Omega \det\left(\xi + D\varphi\left(x\right)\right)\,dx\right| = |\Omega|\,|\det\xi|$$

we have the thesis.

The above example is a (very particular!) case of *polyconvex* function, introduced in [7].

Definition 3.6 *A function $f : \mathbb{R}^{nN} \to \mathbb{R}$ is called polyconvex if it is a convex function of the minors of the variable, thought of as a $N \times n$ matrix.*

Remark 3.7 Polyconvex functions are quasiconvex. We only sketch the proof of this statement, which is not difficult but requires some technical preliminaries from linear algebra. Denoting by $M(\xi)$ the vector of all minors of the matrix ξ, any polyconvex function $f : \mathbb{R}^{Nn} \to \mathbb{R}$ can be written in the form $f(\xi) = g\left(M(\xi)\right)$ for some convex function g. Then the proof can be done in two steps:

(i) $M(\xi) = \dfrac{1}{|\Omega|} \displaystyle\int_\Omega M(\xi + D\varphi)\, dx$ for all bounded domains Ω and for all functions $\varphi \in C_0^1(\Omega)$. This property hes been proved in Example 3.5 for the determinant, but in fact holds true for all minors. As for the determinant, all the integrals of the minors depend only upon the boundary values.

(ii) We may argue as in Remark 3.4. Jensen's inequality gives

$$\begin{aligned}\frac{1}{|\Omega|}\int_\Omega f(\xi + D\varphi)\, dx &= \frac{1}{|\Omega|}\int_\Omega g\left(M(\xi + D\varphi)\right)\, dx \\ &\geqslant g\Big(\frac{1}{|\Omega|}\int_\Omega M(\xi + D\varphi)\, dx\Big) = g\left(M(\xi)\right) = f(\xi)\end{aligned}$$

The function det is an easy example of polyconvex, but nonconvex, function. It has been an open problem for a long time to decide if there are quasiconvex functions that are not polyconvex. This question is very important: on one hand, polyconvexity is an algebraic condition much more treatable than quasiconvexity; on the other hand, polyconvex (and non convex) functions have always a superlinear growth, hence the question was tied with the existence of quasiconvex functions with *linear* growth. Such functions exist (see [30]), and are very useful because they play in quasiconvex analysis the same role that affine functions play in convex analysis (as we have seen in the proof of Theorem 2.8, see (2.5)).

There are some analogies and several differences between the scalar case dealt with in the preceding section and the vectorial case. In this section, we limit ourselves to list some key results.

Theorem 3.8 (Morrey) *Let $f : \mathbb{R}^{nN} \to [0, +\infty]$ be Borel measurable and $f \in L^\infty_{loc}$. Then, the functional*

$$F(u) = \int_\Omega f(Du)\, dx$$

is $w^* - W^{1,\infty}$ if and only if f is quasiconvex.

Notice that in order to ensure $w - W^{1,p}$ lower semicontinuity for some $p < \infty$ additional growth conditions must be added to quasiconvexity. In fact, the following result holds:

Theorem 3.9 *Let $f : \mathbb{R}^{Nn} \to \mathbb{R}$ be quasiconvex and assume that there is $c > 0$ such that*

$$c(1 + |\xi|^q) \leqslant f(\xi) \leqslant c(1 + |\xi|^p)$$

for all $\xi \in \mathbb{R}^{Nn}$, $1 \leqslant q < p$. Then for every open bounded set $\Omega \subset \mathbb{R}^N$ the functional

$$F(u) = \int_\Omega f(Du(x))\, dx$$

is $w - W^{1,p}(\Omega, \mathbb{R}^n)$ l.s.c.

The following theorem is in some sense an optimal generalization of the results valid in the scalar case. We refer to [1], [24], [22].

Theorem 3.10 *Let Ω be an open bounded set in $\mathbb{R}^N$ with Lipschitz boundary and $f : \Omega \times \mathbb{R}^n \times \mathbb{R}^{nN} \to \mathbb{R}$ a Carathéodory function such that*

(i) $-c\left(|\xi|^q + |s|^r\right) - h(x) \leqslant f(x,s,\xi) \leqslant g(x,s)\left(1 + |\xi|^p\right)$, *where* $c > 0$, $p > 1$, $1 \leqslant q < p$, *and* $1 \leqslant r < p^* = \frac{Np}{N-p}$ *if* $p < N$, $r \geqslant 1$ *if* $p \geqslant N$, *and* $h \in L^1(\Omega)$, $g \geqslant 0$, *with g Carathéodory.*

(ii) $f(x,s,\cdot)$ *is quasiconvex.*

Then the functional

$$F(u) = \int_\Omega f(x,u,Du)\, dx$$

is $w - W^{1,p}$ *l.s.c.*

Let us turn our attention to the relaxation problem in the vectorial case. It seems reasonable to expect that the relaxed functional will admit an integral representation involving the quasiconvex hull. Indeed, the following result holds.

Theorem 3.11 (Acerbi-Fusco, [1]) *Let $f : \Omega \times \mathbb{R}^n \times \mathbb{R}^{nN} \to \mathbb{R}$ be such that $f(\cdot, s, \xi)$ is measurable and $f(x,\cdot,\cdot)$ is continuous, and consider the functional*

$$F(u) = \int_\Omega f(x,u,Du)\, dx.$$

If $1 < p < \infty$ *and there are* $a \in L^1_{loc}(\Omega)$, $b \geqslant 0$ *such that*

$$0 \leqslant f(x,\xi) \leqslant a(x) + b\left(|s|^p + |\xi|^p\right),$$

then the relaxed functional with respect to the $w - W^{1,p}$ *topology is*

$$\overline{F}(u) = \int_\Omega g(x,u,Du)\, dx$$

where

$$g(x,u,\xi) = \sup\left\{\psi(x,s,\xi) : \psi \leqslant f,\ \psi \text{ quasiconvex with respect to } \xi\right\}$$

is the quasiconvex hull of f.

If $p = \infty$ *and there are* $a \in L^1_{loc}(\Omega)$, $b \in L^\infty_{loc}$ *such that* $0 \leqslant f(x,u,\xi) \leqslant a(x) + b(s,\xi)$, *then the same result holds with respect to the* $w^* - W^{1,\infty}$ *topology.*

4 Functionals with first order growth and BV functions

As we have seen in Remark 2.3, functionals of first order growth are not coercive on $W^{1,1}$. This is the reason functions with bounded variation play a crucial role.

Definition 4.1 *A function $u : \Omega \to \mathbb{R}$ is said of* bouded variation *if $u \in L^1(\Omega)$ and for all $i = 1, 2, \ldots, N$ there exists a real measure μ_i such that*

$$\int_\Omega u\, D_i\varphi\, dx = \int_\Omega \varphi\, d\mu_i$$

for all $\varphi \in C_0^\infty(\Omega)$. That is, in the sense of distributions,

$$D_i u := -\mu_i.$$

The *total variation of* u is defined as the total variation of the (vector valued) measure $Du = (D_1 u, \ldots, D_N u)$:

$$|Du|(\Omega) = \sup\left\{ \int_\Omega u\, (div\, \psi)\, dx, \psi \in C_0^1\left(\Omega, \mathbb{R}^N\right), \|\psi\|_{L^\infty} \leqslant 1 \right\},$$

and $BV(\Omega)$ is a Banach Space with the norm

$$\|u\|_{BV} = \|u\|_{L^1} + |Du|(\Omega).$$

Remark 4.2 If $u \in W^{1,1}(\Omega)$ then $u \in BV(\Omega)$. Moreover, $\mu_i = -D_i u\, dx$ with $D_i u$ considered in the Sobolev sense. Conversely, if u has bounded variation and $\mu_i << dx$ for all i, then u belongs to $W^{1,1}(\Omega)$. Of course, for general BV functions we have the decomposition

$$Du = \nabla u dx + D^s u$$

by the Lebesgue decomposition theorem, where $D^s u$ is singular with respect to dx, and ∇u denotes the Radon-Nikodým derivative of Du with respect to dx.

In the sequel we only list some properties and results related to $BV(\Omega)$, and we refer to [6], [18], [21], [31], [32] for more information on the subject.

Let us now state the main embedding and compactness properties of BV. Notice that BV is not reflexive, and can be described as a dual space (see e.g. [6, Remark 3.12]).

Theorem 4.3 *If Ω is a bounded set in $\mathbb{R}^N$ with Lipschitz continuous boundary, we have:*

(i) *if $u \in BV(\Omega)$, set $u_\Omega = |\Omega|^{-1} \int_\Omega u dx$, there is $c > 0$ such that the inequality*

$$\int_\Omega |u - u_\Omega|\, dx \leqslant c|Du|(\Omega) \tag{4.1}$$

holds;

(ii) *the space* $BV(\Omega)$ *is continuously embedded in* $L^p(\Omega)$ *for all* $1 \leqslant p < \frac{N}{N-1}$.

(iii) $u_h \to u$ *weakly* in* BV *if* $|Du_h|(\Omega) \leqslant c < \infty$ *and* $u_h \to u$ *in the* $s - L^1(\Omega)$ *topology.*

(iv) *If* (u_h) *is bounded in* $BV(\Omega)$, *then there exists a subsequence* (u_{h_k}) *which is convergent in* $L^1(\Omega)$.

Let us point out why for functionals with linear growth the approach of Section 2, 3, based on Sobolev spaces, fails. If $f(x,s,\xi) : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ is measurable in x, continuous in s and convex in ξ and has linear growth, i.e.,

$$a\,|\xi| \leqslant f(x,s,\xi) \leqslant b\,|\xi| \tag{4.2}$$

for some $a, b > 0$, then the sublevels of the functional

$$F(u) = \int_\Omega f(x,u,Du)\,dx, \qquad u \in W^{1,1}(\Omega),$$

are bounded in $W^{1,1}(\Omega)$. Nevertheless, F is not coercive even in the weak topology, because $W^{1,1}(\Omega)$ is not reflexive and Theorem 1.7(ii) does not apply. On the other hand, to set the variational problem in $BV(\Omega)$ is not an easy task. Let us first consider the special case in which f does not depend on s and is 1-homogeneous with respect to ξ, i.e.,

$$f(x,t\xi) = t f(x,\xi) \qquad \forall\,(x,\xi) \in \Omega \times \mathbb{R}^N,\ \forall\, t > 0.$$

In this case, we can define the functional F on $BV(\Omega)$ setting

$$F(u) = \int_\Omega f\Big(x, \frac{Du}{|Du|}\Big)\,|Du|, \qquad u \in BV(\Omega),$$

where we have used the polar decomposition $Du = \frac{Du}{|Du|}|Du|$ of the measure Du. But, what can we do if f verifies (4.2), but is not 1-homogeneous? The basic idea is to start a relaxation procedure from the functional

$$F(u) = \begin{cases} \displaystyle\int_\Omega f(Du)\;dx & u \in W^{1,1}(\Omega) \\ +\infty & u \in L^1(\Omega) \setminus W^{1,1}(\Omega) \end{cases}$$

and look for an explicit expression of the L^1-relaxed functional $\overline{F}$. The best result would be to have an integral form of $\overline{F}(u)$. This can be done, but not with respect to the Lebesgue measure. The following theorem is proved in [23], [20], [29].

Theorem 4.4 *Let* $f(x,\xi) : \Omega \times \mathbb{R}^N \to \mathbb{R}$ *be measurable in* x *and convex in* ξ *and assume that (4.2) holds. Then*

(i) $\overline{F}(u) < +\infty$ *if and only if* $u \in BV(\Omega)$

(ii) the following representation of $\overline{F}$ holds:

$$\overline{F}(u) = \int_\Omega f(x, \nabla u)\, dx + \int_\Omega f_\infty\left(x, \frac{D^s u}{|D^s u|}\right) |D^s u|$$

where

$$f_\infty(x, \xi) = \lim_{t\to+\infty} \frac{f(x, t\xi)}{t} \tag{4.3}$$

is the recession function of f.

Statement (i) in the above theorem says essentially that the natural space where functionals with linear growth are well-posed is BV.

Let us add a few comments on Theorem 4.4. First of all, we point out the following properties of the recession function, as a function of ξ:

(a) If f is convex then the limit in (4.3) exists, hence f_∞ is well defined.

(b) The recession function f_∞ is 1-homogeneous, that is, $f_\infty(t\xi) = t f_\infty(\xi)$ for all $t > 0$.

Property (a) is essential in order to give a meaning to the integral representation formula, and property (b) allows to exploit the polar decomposition of $D^s u$, in agreement with the preceding discussion. Heuristically, we can say that at points x where Du becomes singular the modulus of the gradient becomes bigger and bigger and Du itself goes to ∞ along a certain direction, so that the value of the integrand at such points is given by the (scaled) asymptotic value of f in the direction along which Du diverges. This value is given precisely by the recession function, evaluated in the mentioned direction.

There is another important qualitative difference between Sobolev and BV functions: BV functions may be discontinuous along $(N-1)$-dimensional surfaces in their domain, whereas Sobolev functions may not. For this reason, BV functions are suitable in geometric variational problems. We limit ourselves to mention the *free discontinuity problems* introduced in [17], for which we refer also to [6], and to remark that *characteristic functions* may belong to BV (see Example 4.8 below). If $u = \chi_E$, and the boundary ∂E of E is smooth, then Du is concentrated on ∂E, but there are non-smooth sets E with $\chi_E \in BV$, and in this case $D\chi_E$ is concentrated on a possibly very small portion of the topological boundary ∂E. We do not discuss such geometric aspects of the theory of BV functions, but come back to discussing some aspects of the theory related to integral functionals. We first present the basic example of relaxation, and after we consider more general classes of functionals.

Example 4.5 Consider the functional

$$F(u) = \int_\Omega \sqrt{1 + |Du|^2}\, dx,$$

that for $u \in C^1(\Omega)$ represents the surface measure (in $\Omega \times \mathbb{R}$) of the graph of u.

It is easy to compute the recession function of $f(\xi) = \sqrt{1 + |\xi|^2}$:

$$f_\infty(\xi) = \lim_{t\to+\infty} \frac{\sqrt{1 + t^2\xi^2}}{t} = |\xi|.$$

Then, for $u \in BV(\Omega)$ we obtain

$$\begin{aligned}\overline{F}(u) &= \int_\Omega f(\nabla u)\, dx + \int_\Omega f_\infty\Big(\frac{D^s u}{|D^s u|}\Big)\, |D^s u| \\ &= \int_\Omega \sqrt{1+|\nabla u|^2}\, dx + |D^s u|\ (\Omega)\,.\end{aligned}$$

As remarked above, BV functions may have some $(N-1)$-dimensional discontinuity surfaces, which correspond to "holes" of the graph. When approximating a BV function which is not $W^{1,1}$ with regular functions, the graphs of the approximating functions must become "vertical" near these holes, and the area of these almost vertical parts gives, in the limit, the second term in the relaxation formula above. Let us see two examples.

Let $\Omega = (0,1)$, so that for $u \in C^1(\Omega)$, $F(u)$ is the length of the graph of u and consider the two functions:

(1) let

$$u_1(x) = \begin{cases} 0 & 0 \leqslant x < \frac{1}{2} \\ 1 & \frac{1}{2} < x \leqslant 1 \end{cases}$$

(2) let u_2 be the Cantor-Vitali function.

$Du_1 = \delta_{\frac{1}{2}} \perp dx$ is singular with respect to dx. Then,

$$\overline{F}(u_1) = \int_\Omega \sqrt{1+|\nabla u_1|^2}\, dx + |D^s u_1|(\Omega) = 1+1 = 2,$$

which is the length of the "stitched" graph of u_1, i.e., the connected curve obtained by adding to the graph of u_1 the vertical segment $\{(1/2, y) : 0 \leqslant y \leqslant 1\}$.

Analogously, $u_2 \in BV(0,1)$, as it is continuous in $[0,1]$ and non decreasing, but the classical derivative $\frac{du_2}{dx} = 0$ a.e., then $\nabla u_2 = 0$ *a.e.* and again $Du_2 \perp dx$. The singular term $D^s u_2$ is concentrated on the Cantor set C, and $|Du_2|(\Omega) = 1$. Then

$$\overline{F}(u_2) = \int_\Omega \sqrt{1+|\nabla u_2|^2}\, dx + |D^s u_2|(\Omega) = 1+1 = 2.$$

This result can be interpreted in a way analogous to u_1; in the present case, the graph of u_2 is a connected curve by itself, but "almost all" the curve is horizontal: in some sense it consists of a broken line containing long horizontal segments that project on $\Omega \setminus C$, and very short vertical segments lying on the points of C, whose total length gives 1, the total variation of u_2.

There are two more important points missing in our discussion on the relaxation theory of linear growth functionals in BV: integrands depending upon u, which are not covered by Theorem 4.4, and the vectorial case. To deal with these topics, we should enter more deeply the structure of BV functions. In fact, as we have already remarked, BV functions may be discontinuous along $(N-1)$-dimensional surfaces, and then when a BV function u is approximated through regular functions, some energy will concentrate onto the discontinuity surfaces, and, in order to give a quantitative estimate of the amount of energy, both the behavior of u near to the discontinuity set and the geometric properties of this one must be well understood. Concerning the vectorial case, there is the additional serious difficulty to deal with the quasiconvex case. In fact, it is necessary to understand if f_∞ is well-defined when f is not a convex function, but just quasiconvex. The recession function $f_\infty\left(\frac{D^s u}{|D^s u|}\right)$ (as a limsup) can be defined, but the proof of the integral representation formula of the relaxed functional relies on deep properties of BV functions. We recall now the notions necessary to state the main results, but a detailed discussion goes far beyond the aim of these notes. We refer to [6] for a detailed presentation of the results, and quote the original papers. The first notion is that of *Hausdorff measure*. For any $E \subset \mathbb{R}^N$ we set:

$$\mathcal{H}^{N-1}(E) = \frac{\omega_{N-1}}{2^{N-1}} \sup_{\delta>0} \inf \left\{ \sum_{h=0}^{\infty} (\mathrm{diam} E_h)^{N-1}, \ E \subset \bigcup_{h=0}^{\infty} E_h, \ \mathrm{diam} E_h < \delta \right\}.$$

Moreover, for every real function $u \in BV(\Omega)$ we define the *upper* and *lower approximate limits of* u *at* $x \in \Omega$ by

$$\begin{aligned} u^\vee(x) &:= \inf\left\{ t \in [-\infty, +\infty] \colon \lim_{\varrho \downarrow 0} \varrho^{-N} |\{u > t\} \cap B_\varrho(x)| = 0 \right\}, \\ u^\wedge(x) &:= \sup\left\{ t \in [-\infty, +\infty] \colon \lim_{\varrho \downarrow 0} \varrho^{-N} |\{u < t\} \cap B_\varrho(x)| = 0 \right\}, \end{aligned}$$

respectively. If $u^\vee(x) = u^\wedge(x)$ we call their common value, denoted $\tilde{u}(x)$, the *weak approximate limit of* u *at* x. We also set $S_u = \{x \in \Omega : \ u^\wedge(x) < u^\vee(x)\}$. Then, the following results hold

(a) for $\mathcal{H}^{N-1}$ a.a. $x \in S_u$ the values $u^\wedge(x), u^\vee(x)$ can be obtained as one-sided traces $u^+(x)$, $u^-(x)$, i.e., a unit vector $\nu(x) \in \mathbf{S}^{N-1}$ exists such that

$$\lim_{\varrho \to 0} \varrho^{-N} \int\limits_{B_\varrho^+(x)} |u(y) - u^+(x)| \, dy = 0$$

and

$$\lim_{\varrho \to 0} \varrho^{-N} \int\limits_{B_\varrho^-(x)} |u(y) - u^-(x)| \, dy = 0,$$

where $B_\varrho^+(x) = \{y \in B_\varrho(x) : \langle y, \nu_x \rangle > 0\}$, $B_\varrho^-(x) = \{y \in B_\varrho(x) : \langle y, \nu_x \rangle < 0\}$; of course, since ν_x is determined up to the sign, changing ν_x by $-\nu_x$ entails that the traces are in turn exchanged.

(b) $D^s u$ splits as $D^J u + D^C u$, where $D^J u(B) = D^s u(B \cap S_u)$ and $D^C u(B) = D^s u(B \setminus S_u)$ for every Borel set B; the measures $D^J u$, $|D^J u|$ are given by

$$D^J u(B) = \int\limits_{B \cap S_u} \big(u^+(x) - u^-(x)\big) \nu_x \, d\mathcal{H}^{N-1}(x),$$

$$|D^J u|(B) = \int\limits_{B \cap S_u} \big(u^\vee(x) - u^\wedge(x)\big) \, d\mathcal{H}^{N-1}(x).$$

The measure $D^C u$ is also called the "Cantor part" of Du because if u is the classical Cantor-Vitali function then $Du = D^C u$.

We can now present the relaxation theorem for the general scalar case.

Theorem 4.6 (G. Dal Maso, [12]) *Let $f(x,s,\xi) : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ be measurable in x, continuous in s and convex in ξ, and assume that (4.2) holds. Then the following representation of the L^1 relaxed integral $\overline{F}$ holds for every $u \in BV(\Omega)$:*

$$\overline{F}(u) = \int_{\Omega \setminus S_u} f\left(x, \tilde{u}(x), \nabla u(x)\right) dx + \int_{\Omega \setminus S_u} f_\infty\left(x, \tilde{u}(x), \frac{D^C u}{|D^C u|}\right) |D^C u|$$

$$+ \int_{S_u} \int_{u^-(x)}^{u^+(x)} f\left(x, t, \nu(x)\right) dt \, d\mathcal{H}^{N-1}(x),$$

where $\nu(x)$ in chosen is such a way that $u^-(x) < u^+(x)$.

The above formula should be compared with those in Example 4.5. Again, the graph of u is stitched vertically between the points $(x, u^-(x))$ and $(x, u^+(x))$ for $\mathcal{H}^{N-1}$ a.e. $x \in S_u$, and the energy that concentrates onto S_u is measured by the third integral.

Let us come to the vectorial case: now, for $u : \Omega \to \mathbb{R}^n$ of calls BV, if $n > 1$ the definitions of $u^\vee, u^\wedge$ do not make sense anymore, but we can still define S_u as the union of the jump sets of the components of u, and obtain triples $(u^+(x), u^-(x), \nu(x))$ at $\mathcal{H}^{N-1}$-a.a. points x of S_u as above, i.e.,

(a') for $\mathcal{H}^{N-1}$ a.a. $x \in S_u$ there are $u^+(x)$, $u^-(x)$ in $\mathbb{R}^n$ and a unit vector $\nu(x) \in \mathbf{S}^{N-1}$ such that

$$\lim_{\varrho \to 0} \varrho^{-N} \int_{B_\varrho^+(x)} |u(y) - u^+(x)| \, dy = 0$$

and

$$\lim_{\varrho \to 0} \varrho^{-N} \int_{B_\varrho^-(x)} |u(y) - u^-(x)| \, dy = 0.$$

(b') $D^s u$ splits as $D^J u + D^C u$, where $D^J u(B) = D^s u(B \cap S_u)$ and $D^C u(B) = D^s u(B \setminus S_u)$; the measures $D^J u$, $|D^J u|$ are given by

$$D^J u(B) = \int_{B \cap S_u} \big(u^+(x) - u^-(x)\big) \otimes \nu_x \, d\mathcal{H}^{N-1}(x)$$

$$|D^J u|(B) = \int_{B \cap S_u} |u^+(x) - u^-(x)| \, d\mathcal{H}^{N-1}(x)$$

for all Borel sets B.

The problem of the integral representation of relaxed functionals in the vectorial case has been studied first in the convex case [5] and subsequently in the quasiconvex case with f depending only upon ξ in [4], and finally has been settled in as follows in [19]. Notice that in the quasiconvex case the limit in (4.3) does not exist, in general, and the recession function must be defined using the limsup. Nevertheless, f_∞ is still 1-homogeneous and quasiconvex.

Theorem 4.7 *Let $f : \Omega \times \mathbb{R}^n \times \mathbb{R}^{nN} \to [0, \infty)$ be a continuous function satisfying (4.2) and the following assumptions:*

(i) *$z \to f(x, u, z)$ is quasiconvex for all $(x, u) \in \Omega \times \mathbb{R}^n$*

(ii) *for every compact subset K of $\Omega \times \mathbb{R}^n$ there exists a continuous function ω_K, with $\omega_K(0) = 0$, such that*

$$|f(x, u, z) - f(y, v, z)| \leqslant \omega_K(|x - y| + |u - v|)(1 + |z|)$$

for all $(x, u, z), (y, v, z) \in K \times \mathbb{R}^{nN}$ and moreover for every $x_0 \in \Omega$ and any $\epsilon > 0$ there exists $\delta > 0$ such that if $|x - x_0| < \delta$, then

$$f(x, u, z) - f(x_0, u, z) \geqslant -\epsilon(1 + |z|) \qquad \forall (u, z) \in \mathbb{R}^n \times \mathbb{R}^{nN};$$

(iii) *there exist $\alpha \in (0, 1)$ and C, $L > 0$ such that if $t|z| > L$ then*

$$\left| f_\infty(x, u, z) - \frac{f(x, u, tz)}{t} \right| \leqslant C\frac{|z|^{1-\alpha}}{t^\alpha} \qquad \forall (x, u, z) \in \Omega \times \mathbb{R}^n \times \mathbb{R}^{nN}.$$

Then, there is a continuous function $\gamma_f : \Omega \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbf{S}^{N-1} \to [0, \infty)$ such that the following representation formula holds for the relaxed functional $\overline{F}$:

$$\begin{aligned} \overline{F}(u) &= \int_\Omega f\left(x, u(x), \nabla u(x)\right) dx + \int_{S_u} \gamma_f\left(x, u^+(x), u^-(x), \nu(x)\right) d\mathcal{H}^{N-1} \\ &\quad + \int_\Omega f_\infty\left(x, u(x), \frac{D^C u}{|D^C u|}\right) |D^C u|. \end{aligned}$$

Again, the function γ_f measures the energy needed to stitch the graph of u between the asymptotic values $u^-(x)$ and $u^+(x)$ at $x \in S_u$. But now $u^-(x)$ and $u^+(x)$ are points in the space, and there is no analogue of the vertical line appearing in the scalar case, because there are infinitely many curves joining them. Moreover, the function γ_f cannot be computed along any curve, but requires a different procedure.

We end these lectures coming back to the isoperimetric problem quoted in the Introduction.

Example 4.8 (The isoperimetric problem) An interesting special class of BV functions is that of *characteristic functions* $u = \chi_E$ for some measurable set E.

We say that $E \subset \Omega$ is of *finite perimeter in* Ω if its *perimeter*, $P(E) := |D\chi_E|(\Omega)$ is finite.

If Ω is bounded, the condition $|D\chi_E|(\Omega) < \infty$ is equivalent to $\chi_E \in BV(\Omega)$, thanks to the embedding $BV(\Omega) \subset L^{\frac{N}{N-1}}(\Omega)$.

If $\Omega = \mathbb{R}^N$, we may use (4.1) with a cube $Q_\varrho(x)$ in place of Ω and write

$$\int_{Q(x)} |\chi_E - (\chi_E)_{Q_\varrho(x)}|\, dy \leqslant \gamma\varrho |D\chi_E|(Q_\varrho(x))$$

for a suitable $\gamma > 0$ which depends only on N. To see this, let us use simple translation and scaling arguments to check that the constant c there behaves like $c = \gamma\varrho$, γ being the optimal constant relative to any unit cube. Thus,

$$2(\chi_E)_{Q_\varrho(x)}(1 - (\chi_E)_{Q_\varrho(x)}) \leqslant \gamma \frac{P(E, Q_\varrho(x))}{\varrho^{N-1}}.$$

Since $(\chi_E)_{Q_\varrho(x)} \in [0, 1]$ and $\min\{t, 1-t\} \leqslant 2t(1-t)$ for any $t \in [0, 1]$, we obtain

$$\min\left\{(\chi_E)_{Q_\varrho(x)}, 1 - (\chi_E)_{Q_\varrho(x)}\right\} \leqslant \gamma \frac{P(E, Q_\varrho(x))}{\varrho^{N-1}}.$$

Setting $\alpha_\varrho(x) = |Q_\varrho(x) \cap E|/(2\varrho)^N$ and choosing $\varrho = [3\gamma P(E)]^{1/(N-1)}$ we obtain

$$\alpha_\varrho(x) \in [0, 1/2) \cup (1/2, 1].$$

By a continuity argument, either $\alpha_\varrho(x) \in [0, 1/2)$ for any $x \in \mathbb{R}^N$ or $\alpha_\varrho(x) \in (1/2, 1]$ for any $x \in \mathbb{R}^N$. If the first possibility is true, we infer

$$\frac{|E \cap Q_\varrho(x)|}{(2\varrho)^N} = \alpha_\varrho(x) \leqslant \gamma \frac{P(E, Q_\varrho(x))}{\varrho^{N-1}} \qquad \forall x \in \mathbb{R}^N.$$

Covering almost all of $\mathbb{R}^N$ by a disjoint family of cubes $\{Q_\varrho(x_h)\}_{h \in \mathbf{Z}^N}$, eventually we get

$$|E| = \sum_{h \in \mathbf{Z}^N} |Q_\varrho(x_h) \cap E| \leqslant 2^N c\varrho \sum_{h \in \mathbf{Z}^N} P(E, Q_\varrho(x_h)) \leqslant 2^N \gamma \varrho P(E, \mathbb{R}^N).$$

If $\alpha_\varrho(x) \in (1/2, 1]$ for all x, then we may exchange E with $\mathbb{R}^N \setminus E$ and we get an analogous inquality. Summarising, we have proved the *isoperimetric inequality*

$$\min\{|E|, |\mathbb{R}^N \setminus E|\} \leqslant cP(E)^{\frac{N}{N-1}} \tag{4.4}$$

where c is a dimensional constant.

This preparation allows us to formulate in rigorous terms the isoperimetric problem in the class of sets with finite perimeter in $\mathbb{R}^N$:

> among all the sets with given perimeter, find that which includes the maximum volume,

or, equivalently, find the best possible constant c in (4.4) and the set E (if any) for which (4.4) holds as an equality. Notice that now "volume" means Lebesgue measure, the measure of the boundary is given by the perimeter, and the class of competitors is very large, and contains all sets with piecewise smooth boundary.

It is a classical deep result, due in this form to E. De Giorgi (see [15]) that the optimal constant c in the isoperimetric inequality is $\frac{\omega_N^{\frac{N}{N-1}}}{N\,\omega_{N-1}}$ and it is obtained if and only if $E = B_\varrho$. Therefore, the set of minimum perimeter in $\mathbb{R}^N$ among all sets of finite perimeter and fixed area A is any ball of radius $\varrho = \left(\frac{A}{\omega_N}\right)^{\frac{1}{N}}$.

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