MAT

Serie A

Conferencias, Seminarios y Trabajos de Matemática.

ISSN(Print)1515-4904
ISSN(Online)2468-9734

21

An Introduction to Fluid Mechanics

Lorenzo Fusi

Departamento de Matemática,
Rosario,
Argentina
Marzo 2016

UNIVERSIDAD AUSTRAL
FACULTAD DE CIENCIAS EMPRESARIALES

MAT

Serie A: CONFERENCIAS, SEMINARIOS Y TRABAJOS DE MATEMATICA

ISSN (Print): 1515-4904
ISSN (Online): 2468-9734
Propiedad de ACES

MAT es una publicación del Departamento de Matemática de la Facultad de Ciencias Empresariales de la Universidad Austral (FCE-UA) cuyo objetivo es contribuir a la difusión de conocimientos y resultados matemáticos. Se compone de dos series:

- Serie A: CONFERENCIAS, SEMINARIOS Y TRABAJOS DE MATEMATICA.
- Serie B: CURSOS Y SEMINARIOS PARA EDUCACION MATEMATICA.

La Serie A contiene trabajos originales de investigación y/o recapitulación que presenten una exposición interesante y actualizada de algunos aspectos de la Matemática, además de cursos, conferencias, seminarios y congresos realizados en el Departamento de Matemática. El Director, los miembros del Comité Editorial y Científico y/o los árbitros que ellos designen serán los encargados de dictaminar sobre los merecimientos de los artículos que se publiquen. La Serie B se compone de cursos especialmente diseñados para profesores de Matemática de cada uno de los niveles de educación: Primaria, Secundaria, Terciaria y Universitaria.
Los trabajos publicados en MAT-Serie A están indexados en las bases de datos del Mathematical Reviews, MathSciNet y Zentralblatt für Mathematik. Todos los trabajos, en formato pdf y en acceso abierto, pueden ser bajados de:
http://web.austral.edu.ar/cienciasEmpresariales-investigacion-mat-A.asp

ISSN (Print) 1515-4904
ISSN (Online) 2468-9734

MAT

SERIE A: CONFERENCIAS, SEMINARIOS Y TRABAJOS DE MATEMATICA

No. 21

AN INTRODUCTION TO FLUID MECHANICS

LORENZO FUSI

Dipartimento di Matematica e Informatica "U. Dini"
Università di Firenze
Viale Morgagni 67A, 50134 Firenze, ITALY
E-mail: fusi@math.unifi.it

Rosario - August - September 2015

Preface

These notes have been devised as a basic introduction course on fluid mechanics. They are primarily addressed to Ph.D. students but they can be employed also by students in postgraduate diploma or master's degree. They are based on a series of lectures I gave at the University Austral (Rosario, Argentina) between August and September 2015.

The main purpose is to give the student the basic knowledge on the kinematics and dynamics of a continuum and to provide a substantial knowledge of the continuum approach adopted in fluid mechanics. The notes have been purposely structured to maintain a nice balance between the rigorous mathematics and the physical laws and principles that govern fluid mechanics. This approach has two aims: avoid to intimidate the students with "too rigorous" mathematics and present the fundamental concepts of fluid mechanics focussing on the basic principles without getting lost in peripheral material. Despite these notes constitute a genuinely theoretical introduction to fluid mechanics, in the sense that the main results are obtained within the theoretical framework of continuum mechanics, they can also be addressed to applied mathematicians and engineers that are mainly interested in practical applications.

The first Chapter has been devoted to the presentation of standard results in tensor algebra and calculus, providing the fundamental mathematical tools that will be used in the subsequent chapters. Chapters 2 and 3 are dedicated to kinematics and dynamics of a generic continuum. Chapter 4 is devoted to ideal fluids, while Chapter 5 deals with Newtonian fluids. Finally, the last chapter is devoted to the study of generalized Newtonian fluids.

Keywords: continuum mechanics; kinematics; Dynamics; inviscid fluids; newtonian fluids; generalized Newtonian fluids;

AMS Mathematics Subject Classification: 74A05; 74A10; 76A02; 76A05; 76B15; 76D05; 76D07; 76D10;

These notes are the enlarged content of a course given by Prof. Lorenzo Fusi at the Universidad Austral between August and September 2015. The visit of Prof. Fusi to Argentina was carried out through the joint project IT 1310 of an international scientific cooperation between MAE (Dipartimento di Matematica e Informatica "U. Dini", Univ. di Firenze, Florence, Italy) and MINCyT (Departamento de Matemática, FCE, Univ. Austral, Rosario, Argentina).

The manuscript has been received on January 2016 and accepted on March 2016.

Contents

Chapter 1

Mathematical preliminaries

This introductory chapter is devoted to the presentation of basic results on vector and tensor algebra and calculus. The reader not familiar with the material presented here should read this chapter very carefully for it provides the mathematical background required in the subsequent sections. Reference books for this section are [1], [5], [6], [12], [13].

1.1 Vector and tensors algebra

The *inner product of two vectors* $\mathbf{u} = (u_1, u_2, u_3)$, $\mathbf{v} = (v_1, v_2, v_3)$ is

$$\mathbf{u} \cdot \mathbf{v} = \sum_{i=1}^{3} u_i v_i.$$

The norm of a vector $\mathbf{u}$ is

$$|\mathbf{u}| = (\mathbf{u} \cdot \mathbf{u})^{1/2}.$$

The inner product can be expressed also as

$$\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}||\mathbf{v}| \cos \theta,$$

where θ is the angle between $\mathbf{u}$ and $\mathbf{v}$. Two vectors are *orthogonal* if $\mathbf{u} \cdot \mathbf{v} = 0$. A set of vectors $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, ...\}$ is called *orthonormal* if

$$\mathbf{u}_i \cdot \mathbf{u}_j = \delta_{ij} = \begin{cases} 0 & i \neq j, \\ \\ 1 & i = j, \end{cases}$$

δ_{ij} being the Kronecker's delta. The *cross* (or *vector*) *product* of two vectors $\mathbf{u}$, $\mathbf{v}$ is denoted by $\mathbf{u} \times \mathbf{v}$. The cross product provides a vector normal to $\mathbf{u}$ and $\mathbf{v}$ with direction specified by the right-hand rule and a magnitude

$$|\mathbf{u} \times \mathbf{v}| = |\mathbf{u}||\mathbf{v}| \sin \theta,$$

where $\theta \in (0, \pi)$ is the angle between $\mathbf{u}$ and $\mathbf{v}$. In a right handed basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ the cross product can be evaluated through the determinant

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ \\ u_1 & u_2 & u_3 \\ \\ v_1 & v_2 & v_3 \end{vmatrix},$$

that is

$$\mathbf{u} \times \mathbf{v} = (u_2 v_3 - u_3 v_2)\,\mathbf{e}_1 - (u_1 v_3 - u_3 v_1)\,\mathbf{e}_2 + (u_1 v_2 - u_2 v_1)\,\mathbf{e}_3.$$

The following properties hold

$$\mathbf{u} \times \mathbf{u} = 0,$$

$$\mathbf{u} \times \mathbf{v} = -\mathbf{v} \times \mathbf{u},$$

$$\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = \mathbf{w} \cdot (\mathbf{u} \times \mathbf{v}) = \mathbf{v} \cdot (\mathbf{w} \times \mathbf{u}). \tag{1.1}$$

A second order *tensor* $\mathbf{S}$ is a linear transformation from $\mathbb{R}^3$ to $\mathbb{R}^3$ that maps the vector $\mathbf{u}$ in a vector $\mathbf{v}$

$$\mathbf{v} = \mathbf{S}\mathbf{u}.$$

The tensor $\mathbf{S}$ is a 3×3 matrix whose components depend on the selected reference basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$. Hence

$$v_i = \sum_{k=1}^{3} S_{ik} u_k, \qquad i = 1, 2, 3.$$

The tensor

$$\mathbf{I} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

is called the *identity tensor* and it is such that $\mathbf{u} = \mathbf{I}\mathbf{u}$. The product of two tensors $\mathbf{S}$ and $\mathbf{T}$ is a tensor $\mathbf{R}$ such that

$$\mathbf{R} = \mathbf{S}\mathbf{T} \qquad R_{ij} = \sum_{k=1}^{3} S_{ik} T_{kj}.$$

It is easy to check that

$$(\mathbf{S}\mathbf{T})\mathbf{v} = \mathbf{S}(\mathbf{T}\mathbf{v}).$$

In general $\mathbf{S}\mathbf{T} \neq \mathbf{T}\mathbf{S}$. The *transpose* of $\mathbf{S}$ is a tensor $\mathbf{S}^T$ such that $S^T_{ij} = S_{ji}$. The following relations hold

$$\mathbf{S}\mathbf{u} \cdot \mathbf{v} = \mathbf{u} \cdot (\mathbf{S}^T \mathbf{v}), \tag{1.2}$$

$$(\mathbf{S} + \mathbf{T})^T = \mathbf{S}^T + \mathbf{T}^T,$$

$$(\mathbf{S}\mathbf{T})^T = \mathbf{T}^T \mathbf{S}^T,$$

$$(\mathbf{S}^T)^T = \mathbf{S}.$$

A tensor is called *symmetric* if $\mathbf{S}^T = \mathbf{S}$ and *skew* if $\mathbf{S}^T = -\mathbf{S}$. Each tensor $\mathbf{S}$ can be decomposed in its symmetric and skew part

$$\mathbf{S}_{sym} = \frac{1}{2}\left[\mathbf{S} + \mathbf{S}^T\right],$$

$$\mathbf{S}_{skew} = \frac{1}{2}\left[\mathbf{S} - \mathbf{S}^T\right],$$

so that

$$\mathbf{S} = \mathbf{S}_{sym} + \mathbf{S}_{skew}.$$

The *dyadic product* of two vectors $\mathbf{u}$, $\mathbf{v}$ is a second order tensor

$$\mathbf{u} \otimes \mathbf{v} = \begin{pmatrix} u_1v_1 & u_1v_2 & u_1v_3 \\ u_2v_1 & u_2v_2 & u_2v_3 \\ u_3v_1 & u_3v_2 & u_3v_3 \end{pmatrix}. \tag{1.3}$$

A tensor $\mathbf{S}$ can be expressed as

$$\mathbf{S} = \sum_{i,j=1}^{3} S_{ij}(\mathbf{e}_i \otimes \mathbf{e}_j).$$

The following properties hold

$$(\mathbf{u} \otimes \mathbf{v})\mathbf{w} = (\mathbf{v} \cdot \mathbf{w})\mathbf{u},$$

$$(\mathbf{u} \otimes \mathbf{v})^T = (\mathbf{v} \otimes \mathbf{u}),$$

$$\sum_{i=1}^{3} \mathbf{e}_i \otimes \mathbf{e}_i = \mathbf{I}.$$

If $\mathbf{n}$ is a unit vector the *projection* of $\mathbf{u}$ in the direction of $\mathbf{n}$ is

$$(\mathbf{u} \cdot \mathbf{n})\mathbf{n} = (\mathbf{n} \otimes \mathbf{n})\mathbf{u}, \tag{1.4}$$

while the projection of $\mathbf{u}$ onto the plane perpendicular to $\mathbf{n}$ is

$$\mathbf{u} - (\mathbf{u} \cdot \mathbf{n})\mathbf{n} = (\mathbf{I} - \mathbf{n} \otimes \mathbf{n})\mathbf{u}. \tag{1.5}$$

The *inner product* of two tensors is

$$\mathbf{S} \cdot \mathbf{T} = \sum_{i,j=1}^{3} S_{ij}T_{ij}.$$

The *trace* of a tensor is

$$\text{tr } \mathbf{S} = \sum_{i=1}^{3} S_{ii} = \mathbf{S} \cdot \mathbf{I}.$$

The following properties hold

$$\text{tr } (\mathbf{ST}) = \text{tr } (\mathbf{TS}),$$

$$\text{tr } (\mathbf{S}^T) = \text{tr } (\mathbf{S}),$$

$$\mathbf{S} \cdot \mathbf{T} = \text{tr } (\mathbf{S}^T\mathbf{T}) = \text{tr } (\mathbf{S}\mathbf{T}^T).$$

If $\mathbf{S}$ is a symmetric tensor and $\mathbf{W}$ is skew

$$\mathbf{S} \cdot \mathbf{W} = \text{tr } (\mathbf{SW}) = \text{tr } (\mathbf{S}\mathbf{W}^T) = -\text{tr } (\mathbf{SW}),$$

so that $\mathbf{S} \cdot \mathbf{W} = 0$. If

$$\mathbf{S} \cdot \mathbf{W} = 0 \qquad \forall \ \mathbf{W} \ \text{ skew}, \tag{1.6}$$

then $\mathbf{S}$ is symmetric. If $\mathbf{W}$ is skew then

$$\mathbf{u} \cdot \mathbf{W}\mathbf{u} = 0, \tag{1.7}$$

for all vectors $\mathbf{u}$. For any skew tensor

$$\mathbf{W} = \begin{bmatrix} 0 & -\gamma & \beta \\ \gamma & 0 & -\alpha \\ -\beta & \alpha & 0 \end{bmatrix}, \tag{1.8}$$

there exists a unique vector $\boldsymbol{\omega} = (\alpha, \beta, \gamma)$ such that $\mathbf{Wu} = \boldsymbol{\omega} \times \mathbf{u}$ for each vector $\mathbf{u}$. The *determinant* of a tensor $\mathbf{S}$ is the determinant of the matrix associated with the tensor. The determinant is independent of the basis $\{\mathbf{e}_1, \mathbf{e}_3, \mathbf{e}_3\}$. A tensor $\mathbf{S}$ is called *invertible* if there exists a tensor $\mathbf{S}^{-1}$ such that

$$\mathbf{S}\mathbf{S}^{-1} = \mathbf{S}^{-1}\mathbf{S} = \mathbf{I}.$$

The following identities hold

$$\det(\mathbf{ST}) = \det(\mathbf{S})\det(\mathbf{T}),$$

$$\det(\mathbf{S}^T) = \det(\mathbf{S}),$$

$$(\mathbf{ST})^{-1} = \mathbf{T}^{-1}\mathbf{S}^{-1},$$

$$(\mathbf{S}^{-1})^T = (\mathbf{S}^T)^{-1} =: \mathbf{S}^{-T}.$$

If a tensor $\mathbf{S}$ is invertible then $\det \mathbf{S} \neq 0$. A tensor $\mathbf{S}$ is *orthogonal* if

$$\mathbf{S}\mathbf{S}^T = \mathbf{S}^T\mathbf{S} = \mathbf{I},$$

or analogously

$$\mathbf{S}^T = \mathbf{S}^{-1}.$$

An orthogonal tensor is sometimes called a *rotation.* A tensor $\mathbf{S}$ is *positive-definite* if

$$\mathbf{u} \cdot \mathbf{Su} > 0 \qquad \forall\, \mathbf{u} \neq 0.$$

A scalar λ is called an *eigenvalue* of a tensor $\mathbf{S}$ if there exists a vector $\mathbf{u} \neq 0$ such that

$$\mathbf{Su} = \lambda\mathbf{u}. \tag{1.9}$$

In that case $\mathbf{u}$ is called an *eigenvector.* If $\mathbf{u}$ is an eigenvector then $\theta\mathbf{u}$ is also an eigenvector. Equation (1.9) can be rewritten as

$$(\mathbf{S} - \lambda\mathbf{I})\mathbf{u} = 0.$$

Hence the eigenvalues of $\mathbf{S}$ are found solving

$$\det(\mathbf{S} - \lambda\mathbf{I}) = 0.$$

If $\mathbf{S}$ is positive definite then the eigenvalues are strictly positive. Indeed, let λ be an eigenvalue of the positive definite tensor $\mathbf{S}$ and let $\mathbf{u}$ be the relative eigenvector with $|\mathbf{u}| = 1$. From (1.9) we have

$$0 < \mathbf{u} \cdot \mathbf{Su} = \lambda|\mathbf{u}|^2 = \lambda.$$

It can be shown that

$$\det(\mathbf{S} - \lambda\mathbf{I}) = -\lambda^3 + i_1(\mathbf{S})\lambda^2 - i_2(\mathbf{S})\lambda + i_3(\mathbf{S}), \tag{1.10}$$

where

$$
\left\{
\begin{aligned}
& i_1(\mathbf{S}) = \mathrm{tr}\,\mathbf{S}, \\
& i_2(\mathbf{S}) = \frac{1}{2}\left[(\mathrm{tr}\,\mathbf{S})^2 - \mathrm{tr}\,(\mathbf{S}^2)\right], \\
& i_3(\mathbf{S}) = \det \mathbf{S},
\end{aligned}
\right.
\tag{1.11}
$$

are the *principal invariants* of $\mathbf{S}$. When $\mathbf{S}$ is symmetric the eigenvalues λ_1, λ_2, $\lambda_3 \in \mathbb{R}$ and

$$
\left\{
\begin{aligned}
& i_1(\mathbf{S}) = \lambda_1 + \lambda_2 + \lambda_3, \\
& i_2(\mathbf{S}) = \lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_1\lambda_3, \\
& i_3(\mathbf{S}) = \lambda_1\lambda_2\lambda_3.
\end{aligned}
\right.
$$

The following theorems hold

Theorem 1 (Cayley-Hamilton) *Every second order tensor* $\mathbf{S}$ *satisfies*

$$-\mathbf{S}^3 + i_1(\mathbf{S})\mathbf{S}^2 - i_2(\mathbf{S})\mathbf{S} + i_3(\mathbf{S})\mathbf{I} = 0.$$

Theorem 2 (Spectral) *If* $\mathbf{S}$ *is symmetric then there exists an orthonormal basis* $\{\mathbf{e}_1, \mathbf{e}_3, \mathbf{e}_3\}$ *formed by eigenvectors of* $\mathbf{S}$*. The matrix representation of* $\mathbf{S}$ *in such a basis is diagonal*

$$\mathbf{S} = \sum_{i=1}^{3} \lambda_i \mathbf{e}_i \otimes \mathbf{e}_i,$$

where λ_i *are the eigenvalues of* $\mathbf{S}$ *(see [5] for the proof of the theorem).*

Theorem 3 (Polar decomposition) *Let* $\mathbf{S}$ *be a symmetric tensor with* $\det \mathbf{S} > 0$*. Then there exists an orthogonal tensor* $\mathbf{R}$ *and two symmetric tensors* $\mathbf{U}$, $\mathbf{V}$ *such that*

$$\mathbf{S} = \mathbf{RU} = \mathbf{VR}.$$

(see [12] for a proof of the theorem).

1.2 Vector and tensor calculus

Suppose $\mathbf{u} : \mathbb{R}^3 \to \mathbb{R}^3$ is a smooth vector field. We write

$$\mathbf{u}(\mathbf{x}) = o(\mathbf{x}), \qquad \mathbf{x} \to 0,$$

when

$$\lim_{\mathbf{x}\to 0} \frac{\mathbf{u}(\mathbf{x})}{|\mathbf{x}|} = 0.$$

The function $\mathbf{u}$ is said to be *differentiable* at $\mathbf{x}$ if there exists a linear transformation

$$D\mathbf{u}(\mathbf{x}) : \mathbb{R}^3 \to \mathbb{R}^3,$$

such that

$$\mathbf{u}(\mathbf{x}+\mathbf{y}) = \mathbf{u}(\mathbf{x}) + [D\mathbf{u}(\mathbf{x})]\mathbf{y} + o(|\mathbf{y}|).$$

Supposing that $\mathbf{u}$ is sufficiently smooth we have

$$[D\mathbf{u}(\mathbf{x})]\mathbf{y} = [\nabla\mathbf{u}(\mathbf{x})]\mathbf{y}. \tag{1.12}$$

where

$$\nabla\mathbf{u}(\mathbf{x}) = \begin{pmatrix} \dfrac{\partial u_1}{\partial x_1} & \dfrac{\partial u_1}{\partial x_2} & \dfrac{\partial u_1}{\partial x_3} \\ \dfrac{\partial u_2}{\partial x_1} & \dfrac{\partial u_2}{\partial x_2} & \dfrac{\partial u_2}{\partial x_3} \\ \dfrac{\partial u_3}{\partial x_1} & \dfrac{\partial u_3}{\partial x_2} & \dfrac{\partial u_3}{\partial x_3} \end{pmatrix},$$

is the *gradient* of $\mathbf{u}$ (sometimes grad $\mathbf{u}$ is used instead of $\nabla\mathbf{u}$). If we consider a scalar field φ relation (1.12) reduces to

$$[D\varphi(\mathbf{x})]\mathbf{y} = \nabla\varphi(\mathbf{x}) \cdot \mathbf{y},$$

where

$$\nabla\varphi(\mathbf{x}) = \left(\frac{\partial\varphi}{\partial x_1}, \frac{\partial\varphi}{\partial x_2}, \frac{\partial\varphi}{\partial x_3}\right).$$

The *divergence* of a vector field $\mathbf{u}$ is

$$\text{div } \mathbf{u}(\mathbf{x}) = \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3}.$$

Sometimes $\nabla \cdot \mathbf{u}$ is used instead of div $\mathbf{u}$. We have

$$\text{div } \mathbf{u} = \text{tr } (\nabla\mathbf{u}).$$

We can also define the divergence of a *tensor field* in the following way

$$(\text{div } \mathbf{S}) \cdot \mathbf{u} = \text{div } (\mathbf{S}^T\mathbf{u}), \qquad \forall \ \mathbf{u}.$$

In particular div $\mathbf{S}$ is a vector with components

$$(\text{div } \mathbf{S})_i = \sum_{j=1}^{3} \frac{\partial S_{ij}}{\partial x_j}.$$

Let $\varphi : \mathbb{R}^3 \to \mathbb{R}$ be a smooth scalar field and let $\mathbf{u}, \ \mathbf{v} : \mathbb{R}^3 \to \mathbb{R}^3$ be smooth vector fields. Further let $\mathbf{S} : \mathbb{R}^3 \to \mathbb{R}^9$ be a smooth tensor field. The following relations hold

$$\nabla(\varphi\mathbf{u}) = \varphi\nabla\mathbf{u} + \mathbf{u} \otimes \nabla\varphi, \tag{1.13}$$

$$\nabla(\mathbf{u} \cdot \mathbf{v}) = (\nabla\mathbf{v})^T\mathbf{u} + (\nabla\mathbf{u})^T\mathbf{v}, \tag{1.14}$$

$$\text{div } (\varphi\mathbf{u}) = \varphi \text{ div } \mathbf{u} + \mathbf{u} \cdot \nabla\varphi, \tag{1.15}$$

$$\text{div } (\mathbf{u} \otimes \mathbf{v}) = \mathbf{u} \text{ div } \mathbf{v} + (\nabla\mathbf{u})\mathbf{v},$$

$$\text{div } (\mathbf{S}^T\mathbf{u}) = \mathbf{S} \cdot \nabla\mathbf{u} + \mathbf{u} \cdot \text{ div } (\mathbf{S}), \tag{1.16}$$

$$\text{div } (\varphi\mathbf{S}) = \varphi \text{ div } \mathbf{S} + \mathbf{S}\nabla\varphi,$$

$$\text{div } (\nabla\mathbf{u}) = \nabla(\text{ div } \mathbf{u}).$$

The proof of these identities can be found in [12]. The *curl* of a vector $\mathbf{u}$ is defined as

$$\text{curl}\,\mathbf{u} = \left(\frac{\partial u_3}{\partial x_2} - \frac{\partial u_2}{\partial x_3}\right)\mathbf{e}_1 + \left(\frac{\partial u_1}{\partial x_3} - \frac{\partial u_3}{\partial x_1}\right)\mathbf{e}_2 + \left(\frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2}\right)\mathbf{e}_3. \tag{1.17}$$

The curl can be evaluated by means of

$$\text{curl}\,\mathbf{u} = \det\begin{bmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ \dfrac{\partial}{\partial x_1} & \dfrac{\partial}{\partial x_2} & \dfrac{\partial}{\partial x_3} \\ u_1 & u_2 & u_3 \end{bmatrix}.$$

The vector field curl$\mathbf{v}$ is also denoted by $\nabla \times \mathbf{v}$. The *Laplacian* of a scalar field φ is

$$\Delta\varphi = \text{ div }(\nabla\varphi).$$

When $\Delta\varphi = 0$ the function φ is called *harmonic*. The *Laplacian* of a vector field $\mathbf{u}$ is given by

$$\Delta\mathbf{u} = \Delta u_1\mathbf{e}_1 + \Delta u_2\mathbf{e}_2 + \Delta u_3\mathbf{e}_3. \tag{1.18}$$

When $\Delta\mathbf{u} = 0$ the vector field $\mathbf{u}$ is called *harmonic*. A vector field $\mathbf{u}$ for which

$$\text{div }\mathbf{u} = 0, \qquad \text{curl }\mathbf{u} = 0$$

is harmonic. If φ is a smooth scalar field and $\mathbf{u}$, $\mathbf{v}$ are smooth vector fields the following identities hold

$$\text{curl }(\varphi\mathbf{u}) = \varphi \text{ curl }\mathbf{u} + \nabla\varphi \times \mathbf{u}. \tag{1.19}$$

$$\text{div }(\mathbf{u} \times \mathbf{v}) = \mathbf{v}\cdot \text{ curl }\mathbf{u} - \mathbf{u}\cdot \text{ curl }\mathbf{v},$$

$$\text{curl }(\mathbf{u} \times \mathbf{v}) = (\nabla\mathbf{u})\mathbf{v} - (\nabla\mathbf{v})\mathbf{u} + \mathbf{u} \text{ div }\mathbf{v} - \mathbf{v} \text{ div }\mathbf{u}, \tag{1.20}$$

$$\Delta\mathbf{u} = \nabla\text{ (div }\mathbf{u}) - \text{curl (curl }\mathbf{u}). \tag{1.21}$$

Moreover

$$\text{div (curl }\mathbf{u}) = 0, \tag{1.22}$$

$$\text{curl }(\nabla\varphi) = 0. \tag{1.23}$$

A vector field $\mathbf{u}$ is said to be *conservative* if $\mathbf{u} = \nabla\varphi$ for some scalar field φ. A curl free vector field defined on a *simply connected domain* is conservative. Proofs of identities (1.19)-(1.23) can be found in [1].

1.3 The divergence theorem and the Stokes' theorem

In this section we present two important theorems widely used in continuum mechanics: the *divergence theorem* and the *Stokes' theorem*. These results are classical so we do not provide any proof, referring the interested reader to [13], [1].

Theorem 4 (divergence) *Let P be a bounded smooth region in $\mathbb{R}^3$ with smooth boundary ∂P with outward unit normal $\mathbf{n}$. Let $\varphi : P \to \mathbb{R}$, $\mathbf{u} : P \to \mathbb{R}^3$, $\mathbf{S} : P \to \mathbb{R}^9$ be a scalar, vector and tensor field respectively. Then*

$$\int_{\partial P} \varphi \mathbf{n} \, d\sigma = \int_P \nabla \varphi \, d\mathbf{x},$$

$$\int_{\partial P} \mathbf{u} \cdot \mathbf{n} \, d\sigma = \int_P \text{div } \mathbf{u} \, d\mathbf{x}, \tag{1.24}$$

$$\int_{\partial P} \mathbf{S}\mathbf{n} \, d\sigma = \int_P \text{div } \mathbf{S} \, d\mathbf{x}, \tag{1.25}$$

From (1.16) we have

$$\int_P \mathbf{u} \cdot \text{div } \mathbf{S} \, d\mathbf{x} = \int_P \text{div } (\mathbf{S}^T \mathbf{u}) \, d\mathbf{x} - \int_P \mathbf{S} \cdot \nabla \mathbf{u} \, d\mathbf{x}$$

Applying the divergence theorem (1.24) and recalling (1.2) we get

$$\int_P \mathbf{u} \cdot \text{div } \mathbf{S} \, d\mathbf{x} = \int_{\partial P} \mathbf{S}\mathbf{n} \cdot \mathbf{u} \, d\sigma - \int_P \mathbf{S} \cdot \nabla \mathbf{u} \, d\mathbf{x}. \tag{1.26}$$

Relation (1.26) is called *Gauss-Green* formula.

Theorem 5 (Stokes) *Let S be a smooth surface in $\mathbb{R}^3$ with normal $\mathbf{n}$ and smooth boundary ∂S. Let $\mathbf{u}$ be a C^1 vector field on S. Then*

$$\int_S \text{curl } \mathbf{u} \cdot \mathbf{n} \, dS = \int_{\partial S} \mathbf{u} \cdot \, d\boldsymbol{\ell} \tag{1.27}$$

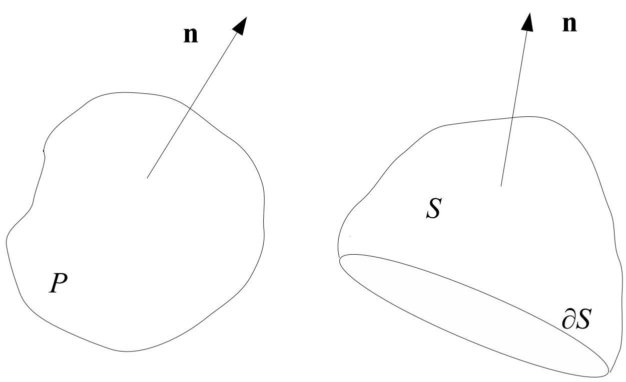


Figure 1.1: Divergence and Stokes' theorems.

1.4 Cylindrical polar coordinates

In a Cartesian coordinate system a point in space is specified by its position

$$\mathbf{x} = x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + x_3\mathbf{e}_3.$$

When using a Cylindrical polar coordinate system (see [14]) the position is specified by the *polar coordinates* (r, θ, z) such that

$$\begin{cases} x_1 = r\cos\theta, \\ x_2 = r\sin\theta, \\ x_3 = z \end{cases} \qquad \begin{cases} r = \sqrt{x_1^2 + x_2^2}, \\ \theta = \arctan\left(\dfrac{x_1}{x_2}\right), \\ z = x_3 \end{cases}$$

In cylindrical coordinates vectors are specified as components in the $\{\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_z\}$ basis where

$$\begin{cases} \mathbf{e}_r = \cos\theta\mathbf{e}_1 + \sin\theta\mathbf{e}_2 = \dfrac{\partial\mathbf{x}}{\partial r}\left|\dfrac{\partial\mathbf{x}}{\partial r}\right|^{-1} \\ \mathbf{e}_\theta = -\sin\theta\mathbf{e}_1 + \cos\theta\mathbf{e}_2 = \dfrac{\partial\mathbf{x}}{\partial\theta}\left|\dfrac{\partial\mathbf{x}}{\partial\theta}\right|^{-1} \\ \mathbf{e}_z = \mathbf{e}_3 = \dfrac{\partial\mathbf{x}}{\partial z}\left|\dfrac{\partial\mathbf{x}}{\partial z}\right|^{-1} \end{cases}$$

Given a vector in Cartesian coordinates

$$\mathbf{a} = a_1\mathbf{e}_1 + a_2\mathbf{e}_2 + a_3\mathbf{e}_3,$$

we can convert it into cylindrical coordinates

$$\mathbf{a} = a_r\mathbf{e}_r + a_\theta\mathbf{e}_\theta + a_z\mathbf{e}_z,$$

by means of

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a_r \\ a_\theta \\ a_z \end{bmatrix},$$

or conversely

$$\begin{bmatrix} a_r \\ a_\theta \\ a_z \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}.$$

Notice that the matrices above are orthogonal, i.e. $\mathbf{Q}\mathbf{Q}^T = \mathbf{I}$. The basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ can be thus transformed into $\{\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_z\}$ by means of

$$\begin{cases} \mathbf{e}_r = \cos\theta\mathbf{e}_1 + \sin\theta\mathbf{e}_2, \\ \mathbf{e}_\theta = -\sin\theta\mathbf{e}_1 + \cos\theta\mathbf{e}_2, \\ \mathbf{e}_z = \mathbf{e}_3, \end{cases}$$

and conversely

$$\begin{cases} \mathbf{e}_1 = \cos\theta \mathbf{e}_1 - \sin\theta \mathbf{e}_2, \\ \\ \mathbf{e}_2 = \sin\theta \mathbf{e}_1 + \cos\theta \mathbf{e}_2, \\ \\ \mathbf{e}_3 = \mathbf{e}_z. \end{cases}$$

The basis $\{\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_z\}$ is orthonormal and tensors can be expressed in this basis. Hence

$$\mathbf{S} = \begin{bmatrix} S_{rr} & S_{r\theta} & S_{rz} \\ \\ S_{\theta r} & S_{\theta\theta} & S_{\theta z} \\ \\ S_{zr} & S_{z\theta} & S_{zz} \end{bmatrix}.$$

Tensors can be converted from the Cartesian system to the cylindrical system by means of

$$\begin{bmatrix} S_{11} & S_{12} & S_{13} \\ \\ S_{21} & S_{22} & S_{23} \\ \\ S_{31} & S_{32} & S_{33} \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \\ \sin\theta & \cos\theta & 0 \\ \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} S_{rr} & S_{r\theta} & S_{rz} \\ \\ S_{\theta r} & S_{\theta\theta} & S_{\theta z} \\ \\ S_{zr} & S_{z\theta} & S_{zz} \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ \\ -\sin\theta & \cos\theta & 0 \\ \\ 0 & 0 & 1 \end{bmatrix}, \tag{1.28}$$

$$\begin{bmatrix} S_{rr} & S_{r\theta} & S_{rz} \\ \\ S_{\theta r} & S_{\theta\theta} & S_{\theta z} \\ \\ S_{zr} & S_{z\theta} & S_{zz} \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ \\ -\sin\theta & \cos\theta & 0 \\ \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} S_{11} & S_{12} & S_{13} \\ \\ S_{21} & S_{22} & S_{23} \\ \\ S_{31} & S_{32} & S_{33} \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \\ \sin\theta & \cos\theta & 0 \\ \\ 0 & 0 & 1 \end{bmatrix}. \tag{1.29}$$

Calculating the derivatives of scalar, vector and tensor functions in cylindrical coordinate can be complicated, since the vectors forming the basis are function of position. Without giving the detailed derivation we introduce some basic differential operators in cylindrical coordinates. Suppose $\varphi = \varphi(r, \theta, z)$ is a scalar function. Then

$$\nabla\varphi = \frac{\partial\varphi}{\partial r}\mathbf{e}_r + \left(\frac{1}{r}\frac{\partial\varphi}{\partial\theta}\right)\mathbf{e}_r + \frac{\partial\varphi}{\partial z}\mathbf{e}_z.$$

while

$$\Delta\varphi = \frac{\partial^2\varphi}{\partial r^2} + \frac{1}{r}\frac{\partial\varphi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\varphi}{\partial\theta^2} + \frac{\partial^2\varphi}{\partial z^2}.$$

If $\mathbf{u} = (u_r, u_\theta, u_z)$ is a vector function

$$\nabla\mathbf{u} = \begin{bmatrix} \dfrac{\partial u_r}{\partial r} & \left(\dfrac{1}{r}\dfrac{\partial u_r}{\partial\theta} - \dfrac{u_\theta}{r}\right) & \dfrac{\partial u_r}{\partial z} \\ \\ \dfrac{\partial u_\theta}{\partial r} & \left(\dfrac{1}{r}\dfrac{\partial u_\theta}{\partial\theta} + \dfrac{u_r}{r}\right) & \dfrac{\partial u_\theta}{\partial z} \\ \\ \dfrac{\partial u_z}{\partial r} & \left(\dfrac{1}{r}\dfrac{\partial u_z}{\partial\theta}\right) & \dfrac{\partial u_z}{\partial z} \end{bmatrix}.$$

The divergence of $\mathbf{u}$ is

$$\operatorname{div}\mathbf{u} = \frac{\partial u_r}{\partial r} + \frac{u_r}{r} + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{\partial u_z}{\partial z}. \tag{1.30}$$

The Laplacian of $\mathbf{u}$ is

$$\Delta\mathbf{u} = \begin{bmatrix} \dfrac{\partial^2 u_r}{\partial r^2} + \dfrac{1}{r^2}\dfrac{\partial^2 u_r}{\partial \theta^2} + \dfrac{\partial^2 u_r}{\partial z^2} + \dfrac{1}{r}\dfrac{\partial u_r}{\partial r} - \dfrac{2}{r^2}\dfrac{\partial u_\theta}{\partial \theta} - \dfrac{u_r}{r^2} \\[2ex] \dfrac{\partial^2 u_\theta}{\partial r^2} + \dfrac{1}{r^2}\dfrac{\partial^2 u_\theta}{\partial \theta^2} + \dfrac{\partial^2 u_\theta}{\partial z^2} + \dfrac{1}{r}\dfrac{\partial u_\theta}{\partial r} + \dfrac{2}{r^2}\dfrac{\partial u_r}{\partial \theta} - \dfrac{u_\theta}{r^2} \\[2ex] \dfrac{\partial^2 u_z}{\partial r^2} + \dfrac{1}{r^2}\dfrac{\partial^2 u_z}{\partial \theta^2} + \dfrac{1}{r}\dfrac{\partial u_z}{\partial r} + \dfrac{\partial^2 u_z}{\partial z^2} \end{bmatrix}$$

where the three components are the components of the $\{\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_z\}$ basis. If $\mathbf{S}$ is a tensor in cylindrical coordinates, the divergence of $\mathbf{S}$ is

$$\operatorname{div}\mathbf{S} = \begin{bmatrix} \dfrac{\partial S_{rr}}{\partial r} + \dfrac{S_{rr}}{r} + \dfrac{1}{r}\dfrac{\partial S_{\theta r}}{\partial \theta} + \dfrac{\partial S_{zr}}{\partial z} - \dfrac{S_{\theta\theta}}{r} \\[2ex] \dfrac{1}{r}\dfrac{\partial S_{\theta\theta}}{\partial \theta} + \dfrac{\partial S_{r\theta}}{\partial r} + \dfrac{S_{r\theta}}{r} + \dfrac{S_{\theta r}}{r} + \dfrac{\partial S_{z\theta}}{\partial z} \\[2ex] \dfrac{\partial S_{zz}}{\partial z} + \dfrac{\partial S_{rz}}{\partial r} + \dfrac{S_{rz}}{r} + \dfrac{1}{r}\dfrac{\partial S_{\theta z}}{\partial \theta} \end{bmatrix}.$$

1.5 Spherical polar coordinates

In spherical coordinates the position of vector

$$\mathbf{x} = x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + x_3\mathbf{e}_3,$$

is specified by (r, θ, ϕ) such that (see [14])

$$\left\{\begin{aligned} x_1 &= r\sin\theta\cos\phi, \\ x_2 &= r\sin\theta\sin\phi, \\ x_3 &= r\cos\theta \end{aligned}\right. \qquad \left\{\begin{aligned} r &= \sqrt{x_1^2 + x_2^2 + x_3^2}, \\ \theta &= \arccos\left(\frac{x_3}{r}\right), \\ z &= \arctan\left(\frac{x_2}{x_1}\right). \end{aligned}\right. \tag{1.31}$$

Vectors are specified as components in the basis $\{\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_\phi\}$

$$\left\{\begin{aligned} \mathbf{e}_r &= \sin\theta\cos\phi\,\mathbf{e}_1 + \sin\theta\sin\phi\,\mathbf{e}_2 + \cos\theta\,\mathbf{e}_3 = \frac{\partial\mathbf{x}}{\partial r}\left|\frac{\partial\mathbf{x}}{\partial r}\right|^{-1} \\ \mathbf{e}_\theta &= \cos\theta\cos\phi\,\mathbf{e}_1 + \cos\theta\sin\phi\,\mathbf{e}_2 - \sin\theta\,\mathbf{e}_3 = \frac{\partial\mathbf{x}}{\partial \theta}\left|\frac{\partial\mathbf{x}}{\partial \theta}\right|^{-1} \\ \mathbf{e}_\phi &= -\sin\phi\,\mathbf{e}_1 + \cos\phi\,\mathbf{e}_2 = \frac{\partial\mathbf{x}}{\partial \phi}\left|\frac{\partial\mathbf{x}}{\partial \phi}\right|^{-1} \end{aligned}\right.$$

Given a vector in Cartesian coordinates

$$\mathbf{a} = a_1\mathbf{e}_1 + a_2\mathbf{e}_2 + a_3\mathbf{e}_3,$$

we can convert it into spherical coordinates

$$\mathbf{a} = a_r\mathbf{e}_r + a_\theta\mathbf{e}_\theta + a_\phi\mathbf{e}_\phi,$$

by means of

$$\begin{bmatrix} a_r \\ a_\theta \\ a_z \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ -\sin\phi & \cos\phi & 0 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix},$$

or conversely

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\phi & \cos\theta\cos\phi & -\sin\phi \\ \sin\theta\sin\phi & \cos\theta\sin\phi & \cos\phi \\ \cos\theta & -\sin\theta & 0 \end{bmatrix} \begin{bmatrix} a_r \\ a_\theta \\ a_\phi \end{bmatrix}.$$

Notice that the matrices above are orthogonal, i.e. $\mathbf{Q}\mathbf{Q}^T = \mathbf{I}$. The basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ can be thus transformed into $\{\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_\phi\}$ by means of

$$\begin{cases} \mathbf{e}_r = \sin\theta\cos\phi\mathbf{e}_1 + \sin\theta\sin\phi\mathbf{e}_2 + \cos\theta\mathbf{e}_3 \\ \mathbf{e}_\theta = \cos\theta\cos\phi\mathbf{e}_1 + \cos\theta\sin\phi\mathbf{e}_2 - \sin\theta\mathbf{e}_3 \\ \mathbf{e}_\phi = -\sin\phi\mathbf{e}_1 + \cos\phi\mathbf{e}_2, \end{cases}$$

and conversely

$$\begin{cases} \mathbf{e}_1 = \sin\theta\cos\phi\mathbf{e}_r + \cos\theta\cos\phi\mathbf{e}_\theta - \sin\phi\mathbf{e}_\phi \\ \mathbf{e}_2 = \sin\theta\sin\phi\mathbf{e}_r + \cos\theta\sin\phi\mathbf{e}_\theta + \cos\phi\mathbf{e}_\phi \\ \mathbf{e}_3 = \cos\theta\mathbf{e}_r - \sin\theta\mathbf{e}_\theta. \end{cases} \tag{1.32}$$

The basis $\{\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_\phi\}$ is orthonormal and tensors can be expressed in this basis.

$$\mathbf{S} = \begin{bmatrix} S_{rr} & S_{r\theta} & S_{r\phi} \\ S_{\theta r} & S_{\theta\theta} & S_{\theta\phi} \\ S_{\phi r} & S_{\phi\theta} & S_{\phi\phi} \end{bmatrix}.$$

Tensors can be converted from the Cartesian system to the cylindrical system by means of

$$\begin{bmatrix} S_{11} & S_{12} & S_{13} \\ S_{21} & S_{22} & S_{23} \\ S_{31} & S_{32} & S_{33} \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\phi & \cos\theta\cos\phi & -\sin\phi \\ \sin\theta\sin\phi & \cos\theta\sin\phi & \cos\phi \\ \cos\theta & -\sin\theta & 0 \end{bmatrix}.$$

$$\cdot\begin{bmatrix} S_{rr} & S_{r\theta} & S_{r\phi} \\ S_{\theta r} & S_{\theta\theta} & S_{\theta\phi} \\ S_{\phi r} & S_{\phi\theta} & S_{\phi\phi} \end{bmatrix}\begin{bmatrix} \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ -\sin\phi & \cos\phi & 0 \end{bmatrix},$$

and

$$\begin{bmatrix} S_{rr} & S_{r\theta} & S_{r\phi} \\ S_{\theta r} & S_{\theta\theta} & S_{\theta\phi} \\ S_{\phi r} & S_{\phi\theta} & S_{\phi\phi} \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ -\sin\phi & \cos\phi & 0 \end{bmatrix}\cdot$$

$$\cdot\begin{bmatrix} S_{11} & S_{12} & S_{13} \\ S_{21} & S_{22} & S_{23} \\ S_{31} & S_{32} & S_{33} \end{bmatrix}\begin{bmatrix} \sin\theta\cos\phi & \cos\theta\cos\phi & -\sin\phi \\ \sin\theta\sin\phi & \cos\theta\sin\phi & \cos\phi \\ \cos\theta & -\sin\theta & 0 \end{bmatrix}.$$

We also define the following differential operators in spherical coordinates. Let $\zeta = \zeta(r, \theta, \phi)$ be a scalar function. Then the gradient of ζ is

$$\nabla\zeta = \frac{\partial\zeta}{\partial r}\mathbf{e}_r + \frac{1}{r}\frac{\partial\zeta}{\partial\theta}\mathbf{e}_\theta + \frac{1}{r\sin\theta}\frac{\partial\zeta}{\partial\phi}\mathbf{e}_\phi. \tag{1.33}$$

The Laplacian of ζ is

$$\Delta\zeta = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\zeta}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\zeta}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\zeta}{\partial\phi^2}.$$

Given a vector function $\mathbf{u} = (u_r, u_\theta, u_\phi)$ the gradient of $\mathbf{u}$ is

$$\nabla\mathbf{u} = \begin{bmatrix} \dfrac{\partial u_r}{\partial r} & \left(\dfrac{1}{r}\dfrac{\partial u_r}{\partial\theta} - \dfrac{u_\theta}{r}\right) & \left(\dfrac{1}{r\sin\theta}\dfrac{\partial u_r}{\partial\phi} - \dfrac{u_\phi}{r}\right) \\ \dfrac{\partial u_\theta}{\partial r} & \left(\dfrac{1}{r}\dfrac{\partial u_\theta}{\partial\theta} + \dfrac{u_r}{r}\right) & \left(\dfrac{1}{r\sin\theta}\dfrac{\partial u_\theta}{\partial\phi} - \cot\theta\dfrac{u_\phi}{r}\right) \\ \dfrac{\partial u_\phi}{\partial r} & \left(\dfrac{1}{r}\dfrac{\partial u_\phi}{\partial\theta}\right) & \left(\dfrac{1}{r\sin\theta}\dfrac{\partial u_\phi}{\partial\phi} + \cot\theta\dfrac{u_\theta}{r} + \dfrac{u_r}{r}\right) \end{bmatrix}.$$

The divergence of $\mathbf{u}$ is

$$\operatorname{div}\mathbf{u} = \frac{\partial u_r}{\partial r} + 2\frac{u_r}{r} + \frac{1}{r}\frac{\partial u_\theta}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial u_\phi}{\partial\phi} + \cot\theta\frac{u_\theta}{r}. \tag{1.34}$$

Finally we write the divergence of a tensor

$$\operatorname{div}\mathbf{S} = \begin{bmatrix} \dfrac{\partial S_{rr}}{\partial r} + 2\dfrac{S_{rr}}{r} + \dfrac{1}{r}\dfrac{\partial S_{\theta r}}{\partial\theta} + \cot\theta\dfrac{S_{\theta r}}{r} + \dfrac{1}{r\sin\theta}\dfrac{\partial S_{\phi r}}{\partial\phi} - \dfrac{S_{\theta\theta} + S_{\phi\phi}}{r} \\ \dfrac{\partial S_{r\theta}}{\partial r} + 2\dfrac{S_{r\theta}}{r} + \dfrac{1}{r}\dfrac{\partial S_{\theta\theta}}{\partial\theta} + \cot\theta\dfrac{S_{\theta\theta}}{r} + \dfrac{1}{r\sin\theta}\dfrac{\partial S_{\phi\theta}}{\partial\phi} + \dfrac{S_{\theta r}}{r} - \cot\theta\dfrac{\partial S_{\phi\phi}}{\partial r} \\ \dfrac{\partial S_{r\phi}}{\partial r} + 2\dfrac{S_{r\phi}}{r} + \dfrac{\sin\theta}{r}\dfrac{\partial S_{\theta\phi}}{\partial\theta} + \cos\theta\dfrac{S_{\theta\phi}}{r} + \dfrac{1}{r\sin\theta}\dfrac{\partial S_{\phi\phi}}{\partial\phi} + \dfrac{S_{\phi r} + S_{\phi\theta}}{r} \end{bmatrix}.$$

1.6 Isotropic functions

Suppose that φ, $\mathbf{u}$ and $\mathbf{S}$ are a scalar, vector and tensor valued functions of a tensor $\mathbf{T}$. Then φ, $\mathbf{u}$ and $\mathbf{S}$ are *isotropic* if

$$\varphi(\mathbf{QTQ}^T) = \varphi(\mathbf{T}),$$

$$\mathbf{u}(\mathbf{QTQ}^T) = \mathbf{Qu}(\mathbf{T}),$$

$$\mathbf{S}(\mathbf{QTQ}^T) = \mathbf{QS}(\mathbf{T})\mathbf{Q}^T,$$

where $\mathbf{Q}$ is any othogonal tensor. It is easy to check that the determinant and the trace of a tensor are isotropic functions. More in general, the invariants of a tensor are all isotropic functions. We state (the interest reader can find the proof in [12]) the following theorems:

Theorem 6 (Isotropic functions) *Any isotropic tensor-valued function* $\mathbf{S}(\mathbf{T})$ *can be written as*

$$\mathbf{S}(\mathbf{T}) = \alpha\mathbf{I} + \beta\mathbf{T} + \gamma\mathbf{T}^2, \tag{1.35}$$

where α*,* β*,* γ *are functions of the invariants of* $\mathbf{T}$*.*

Theorem 7 (Isotropic linear functions) *Any linear isotropic tensor-valued function* $\mathbf{S}(\mathbf{T})$ *can be written as*

$$\mathbf{S}(\mathbf{T}) = 2\mu\mathbf{T} + \lambda(\mathrm{tr}\,\mathbf{T})\mathbf{I}, \tag{1.36}$$

for some scalar λ*,* μ*.*

Chapter 2

Kinematics

Continuum mechanics can be seen as a combination of mathematical and physical laws that describe the macroscopic behavior of a body subjected to mechanical load. For those familiar with particle dynamics, we can say that continuum mechanics is a generalization of that theory based on the same physical assumptions. The evolution of a continuum is described in a three dimensional Euclidean space endowed with a metric

$$d = \sqrt{\sum_{i=0}^{3}(x_i - y_i)^2},$$

which establishes the distance between points. Vectors are expressed as components in a basis of mutually perpendicular unit vectors, while physical quantities such as force, displacement, velocity, acceleration are expressed as vectors in this space. A Cartesian coordinate system is composed by a fixed point together with a basis $\{O, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$.

One of the principal assumption of continuum mechanics is that matter can be idealized as a continuum that is infinitely divisible and locally homogeneous. In practice we can subdivide the medium as many times as we wish and find identical properties (e.g. mass density) at each subdivision. A continuum body can be thought of as an infinite set of sufficiently small particles connected together. In a continuum system mass distribution is described through a *density function* $\varrho(\mathbf{x}, t)$ (whose dimensions are mass over a volume $[m\ell^{-3}]$) such that the mass of each measurable portion D of the system is given by

$$M(D) = \int_D \varrho(\mathbf{x}, t) d\mathbf{x}. \tag{2.1}$$

The euristic procedure leading to the definition of density originates from the definition of *average mass density* of a representative volume ΔV centered at some point $\mathbf{x}$. We consider

$$\varrho_m = \frac{\Delta m}{\Delta V},$$

as the ratio between the mass Δm contained in the cell and the volume ΔV of the cell. Reducing the diameter of the cell, the quantity ϱ_m hovers around a value which can be representative of the macroscopic density (meaning it contains a sufficiently high number of molecules). This is what we call density at point $\mathbf{x}$ at time t. A continuum body can be 3D, 2D or 1D depending on the particular system considered (e.g. a membrane is a 2D continuum, a string is a 1D continuum). Accordingly the dimensions of the density will be $[m\ell^{-3}]$, $[m\ell^{-2}]$ or $[m\ell^{-1}]$.

The material presented in this chapter and in the next one is mainly based on the approach developed in the monumental work [25] by Truesdell and Noll on which the books [12], [24] are also based. The interested reader can refer to these books for a more rigorous and detailed treatment on the kinematics and dynamics of continua.

2.1 Bodies, configurations and motion

The aim of continuum mechanics is to describe the evolution of a body with respect to a reference frame $\{O, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$. Consider a body B that at time $t = 0$ occupies a *reference configuration* $\kappa_R \subset \mathbf{R}^3$ and let $\mathbf{X}$ represent a point of κ_R, as depicted in Fig. 2.1. Every possible configuration that can be attained by the body is defined through the map

$$\mathbf{x} = \boldsymbol{\chi}(\mathbf{X}, t), \tag{2.2}$$

which is supposed continuous in time, globally invertible (ensuring the body does not intersect itself) and orientation-preserving, so that reflections are not possible. In particular we assume that the map $\boldsymbol{\chi}$ is twice continuously differentiable to ensure that the differential equations describing the dynamics may actually be formulated. The motion of the body B is described

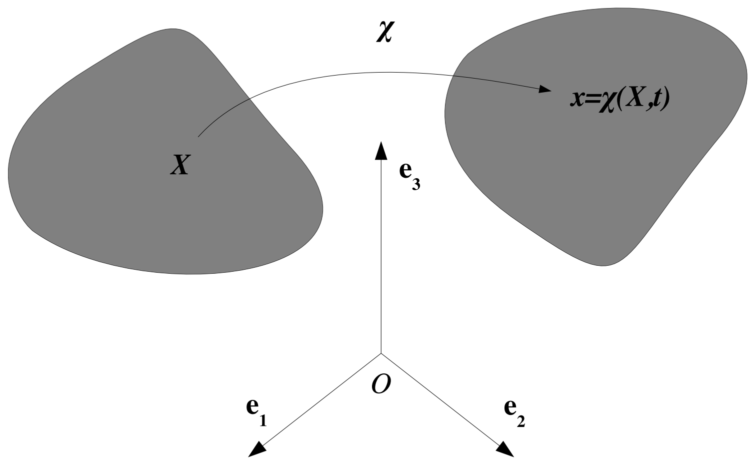


Figure 2.1: Reference and actual configuration.

by the differentiable map (2.2) where $\mathbf{x}$ is the position occupied at time t by the particle that at time $t = 0$ was in $\mathbf{X}$. The coordinates $\mathbf{X}$ and $\mathbf{x}$ are called the *Lagrangian* and *Eulerian* coordinates respectively. The coordinates are intended with respect to the orthogonal reference frame $\{O, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$. The map $\boldsymbol{\chi}$ represents the *material* description of the motion, since the domain of $\boldsymbol{\chi}$ consists of all the material particles that form the body. We denote by κ_t the spatial domain occupied by B at time t, i.e. the *actual configuration.*

A *material volume* P is any subset of particles of B that evolves according to (2.2). Hence at time t the volume P occupies the position $P_t = \boldsymbol{\chi}(P, t)$ and the boundary ∂P is mapped in ∂P_t. A material volume is formed by the same set of particles at any time. A non material volume Q is a set of particles whose evolution is not governed by (2.2), so that in general $Q_t \neq \boldsymbol{\chi}(Q, t)$. A non material volume may contain different particles at different times.

The notion of Lagrangian and Eulerian coordinates illustrates two equivalent ways for investigating the kinematics of a continuum. The first is based on the description of physical quantities along the motion of a single particle (*Lagrangian* or *material description*). The second simply takes note of the variation of such quantities as they change in time at a fixed point (*Eulerian* or *local description*). For a given quantity G we have the Lagrangian expression

$$G = G_\ell(\mathbf{X}, t), \tag{2.3}$$

and the Eluerian expression

$$G = G_e(\mathbf{x}, t), \tag{2.4}$$

To switch from (2.3) to (2.4) and vice-versa we make use of (2.2), i.e.

$$G_\ell(\mathbf{X}, t) = G_e(\boldsymbol{\chi}(\mathbf{X}, t), t).$$

As a consequence

$$\frac{\partial G_\ell}{\partial t} = \frac{\partial G_e}{\partial t} + \nabla G_e \cdot \frac{\partial \boldsymbol{\chi}}{\partial t}, \tag{2.5}$$

where ∇ indicates spatial gradient w.r.t. Eulerian coordinates. The first term on the r.h.s. of (2.5) is called *local time derivative* and corresponds to the rate of change of G at a fixed $\mathbf{x}$. The second term on the r.h.s. is called the *convective rate of change* of G. The velocity and the acceleration of a particle $\mathbf{X}$ are

$$\mathbf{v}(\mathbf{x}, t) = \left.\frac{\partial \boldsymbol{\chi}(\mathbf{X}, t)}{\partial t}\right|_{\mathbf{X}=\boldsymbol{\chi}^{-1}(\mathbf{x},t)}, \tag{2.6}$$

$$\mathbf{a}(\mathbf{x}, t) = \left.\frac{\partial^2 \boldsymbol{\chi}(\mathbf{X}, t)}{\partial t^2}\right|_{\mathbf{X}=\boldsymbol{\chi}^{-1}(\mathbf{x},t)}.$$

The *material time derivative* (time derivative along a particle path) for a scalar quantity $G_e(\mathbf{x}, t)$ is

$$\dot{G}_e = \frac{dG_e}{dt} = \frac{\partial G_e}{\partial t} + \nabla G_e \cdot \mathbf{v}. \tag{2.7}$$

The material time derivative for a vector field $\mathbf{w}(\mathbf{x}, t)$ is

$$\dot{\mathbf{w}} = \frac{d\mathbf{w}}{dt} = \frac{\partial \mathbf{w}}{\partial t} + (\nabla \mathbf{w})\mathbf{v}, \tag{2.8}$$

where

$$[(\nabla \mathbf{w})\mathbf{v}]_i = \sum_{j=1}^{3} \frac{\partial w_i}{\partial x_j} v_j, \qquad i = 1, 2, 3. \tag{2.9}$$

Sometimes the term $(\nabla \mathbf{w})\mathbf{v}$ is denoted by $(\mathbf{w} \cdot \nabla)\mathbf{w}$. It is useful to consider the so-called *streamlines*, i.e. the curves that are instantaneously tangent to the velocity field. When the velocity field is locally constant the motion is called *steady* and

$$\frac{\partial \mathbf{v}}{\partial t}(\mathbf{x}, t) = 0. \tag{2.10}$$

Steady flows are characterized by streamlines that do not depend on time. If $G_e(\mathbf{x}, t)$ is a scalar quantity in a steady motion, then

$$\frac{\partial G_e}{\partial t}(\mathbf{x}, t) = 0. \tag{2.11}$$

For a given flow we can define the *vorticity*

$$\boldsymbol{\omega} = \text{curl } \mathbf{v}$$

When curl $\mathbf{v} = 0$ the flow is called *irrotational* (or *curl-free*). In this case we speak of *potential flow* since $\mathbf{v}$ can be derived (in a simply connected domain, see [1]) from a potential $\varphi(\mathbf{x}, t)$

$$\nabla \varphi(\mathbf{x}, t) = \mathbf{v}(\mathbf{x}, t).$$

2.2 Deformation and measures of deformation

For simplicity here we investigate the deformation of a body from a reference configuration κ_R to a final configuration κ_F without specifying how κ_f is obtained. In practice we neglect the time dependence in (2.2), so that $\mathbf{x} = \boldsymbol{\chi}(\mathbf{X})$. Minor changes allow one to consider also the dependence on time. The *displacement* of a particle at time t is

$$\mathbf{u} = \boldsymbol{\chi}(\mathbf{X}) - \mathbf{X}. \tag{2.12}$$

The gradient of the map $\boldsymbol{\chi}$ is a second order tensor called the *deformation gradient*[1]

$$\mathbf{F}(\mathbf{X}) = \mathrm{grad}\boldsymbol{\chi}(\mathbf{X}), \qquad F_{ij} = \frac{\partial \chi_i}{\partial X_j} \tag{2.13}$$

The deformation gradient quantifies the change in shape of infinitesimal line elements

$$d\mathbf{x} = \mathbf{F}(\mathbf{X})d\mathbf{X},$$

as shown in Fig.2.2. The tensor $\mathbf{F}$ provides local information on the deformation $\mathbf{x} = \boldsymbol{\chi}(\mathbf{X})$.

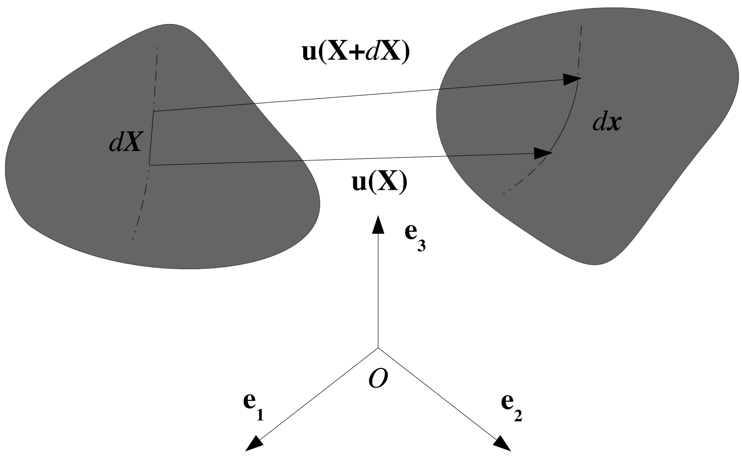


Figure 2.2: Deformation gradient

Indeed, given $\mathbf{Y} \in B$ we can use Taylor expansion and write

$$\boldsymbol{\chi}(\mathbf{X}) = \boldsymbol{\chi}(\mathbf{Y}) + \mathbf{F}(\mathbf{Y})(\mathbf{X} - \mathbf{Y}) + O(|\mathbf{X} - \mathbf{Y}|^2). \tag{2.14}$$

When $\mathbf{F}$ is constant the deformation is called *homogeneous* and (2.14) reduces to

$$\boldsymbol{\chi}(\mathbf{X}) = \boldsymbol{\chi}(\mathbf{Y}) + \mathbf{F}(\mathbf{X} - \mathbf{Y}).$$

A homogeneous deformation is called a *translation* if

$$\boldsymbol{\chi}(\mathbf{X}) = \mathbf{X} + \mathbf{c},$$

where $\mathbf{c}$ is a constant vector. A homogeneous deformation has a *fixed point* $\mathbf{Y}$ if

$$\boldsymbol{\chi}(\mathbf{X}) = \mathbf{Y} + \mathbf{F}(\mathbf{X} - \mathbf{Y}).$$

[1] "grad" is intended w.r.t. Lagrangian coordinates.

A homogeneous deformation is called a *rotation* around a fixed point $\mathbf{Y}$ if

$$\boldsymbol{\chi}(\mathbf{X}) = \mathbf{Y} + \mathbf{R}(\mathbf{X} - \mathbf{Y}),$$

where $\mathbf{R}\mathbf{R}^T = \mathbf{I}$. A homogeneous deformation is called a *stretch* with a fixed point $\mathbf{Y}$ if

$$\boldsymbol{\chi}(\mathbf{X}) = \mathbf{Y} + \mathbf{U}(\mathbf{X} - \mathbf{Y}),$$

with $\mathbf{U}$ symmetric and positive definite tensor. Recalling the *polar decomposition theorem* we know that the tensor $\mathbf{F}$ can be decomposed in $\mathbf{F} = \mathbf{R}\mathbf{U}$, $\mathbf{F} = \mathbf{V}\mathbf{R}$, where $\mathbf{U}$, $\mathbf{V}$ are positive definite and symmetric tensors called the *right* and *left stretch tensors* respectively and where $\mathbf{R}$ is a rotation. A particular type of homogeneous deformations is the one in which

$$\det \mathbf{F} = 1. \tag{2.15}$$

Such a deformation is called *isochoric* or *volume-preserving*. When (2.15) holds the volume of a part P of the body does not change from the reference to the final state. A *rigid deformation* is a homogeneous deformation in which

$$|\boldsymbol{\chi}(\mathbf{X}) - \boldsymbol{\chi}(\mathbf{Y})| = |\mathbf{X} - \mathbf{Y}|.$$

We can prove that a rigid transformation is given by

$$\boldsymbol{\chi}(\mathbf{X}) = \boldsymbol{\chi}(\mathbf{Y}) + \mathbf{R}(\mathbf{X} - \mathbf{Y}),$$

where $\mathbf{R}$ is a rotation. The tensors

$$\mathbf{C} = \mathbf{U}^2 = \mathbf{F}^T\mathbf{F}, \qquad \mathbf{B} = \mathbf{V}^2 = \mathbf{F}\mathbf{F}^T, \tag{2.16}$$

are called the *left Cauchy-Green* and *right Cauchy-Green strain tensors* respectively.

Let P be a part of the body B and let $\boldsymbol{\chi}(P)$ be its deformed state. If φ is a continuous scalar field defined on $\boldsymbol{\chi}(P)$ then

$$\int_{\boldsymbol{\chi}(P)} \varphi(\mathbf{x}) d\mathbf{x} = \int_P \varphi(\boldsymbol{\chi}(\mathbf{X})) \det \mathbf{F}(\mathbf{X}) d\mathbf{X}.$$

As a consequence, when $\det \mathbf{F} = 1$ we have

$$\text{Vol}\,(\boldsymbol{\chi}(P)) = \int_{\boldsymbol{\chi}(P)} d\mathbf{x} = \int_P d\mathbf{X} = \text{Vol}\,(P).$$

proving that deformations for which $\det \mathbf{F} = 1$ are volume preserving.

2.3 Infinitesimal strain theory

From (2.12) we see that

$$\mathbf{F} = \mathbf{I} + \text{grad}\mathbf{u}.$$

Hence from (2.13), (2.16) it is trivial to show that

$$\mathbf{C} = \mathbf{I} + 2\mathbf{E} + (\text{grad}\mathbf{u})^T(\text{grad}\mathbf{u}),$$

$$\mathbf{B} = \mathbf{I} + 2\mathbf{E} + (\text{grad}\mathbf{u})(\text{grad}\mathbf{u})^T,$$

where

$$\mathbf{E} = \frac{1}{2}\left[(\text{grad}\mathbf{u}) + (\text{grad}\mathbf{u})^T\right], \tag{2.17}$$

is the *infinitesimal strain*. It is easy to show that a deformation is rigid when $\mathbf{C} = \mathbf{B} = \mathbf{I}$. When $|\text{grad}\mathbf{u}| = \mathcal{O}(\varepsilon)$ we speak of *small deformations* and

$$2\mathbf{E} = \mathbf{C} - \mathbf{I} + o(\varepsilon) = \mathbf{B} - \mathbf{I} + o(\varepsilon),$$

meaning that, within an error of $o(\varepsilon)$, the tensors $\mathbf{C}-\mathbf{I}$, $\mathbf{B}-\mathbf{I}$ and $2\mathbf{E}$ coincide. Moreover, when the infinitesimal deformation is rigid

$$(\text{grad}\mathbf{u}) = -(\text{grad}\mathbf{u})^T + o(\varepsilon),$$

so that, within an error of $o(\varepsilon)$, the displacement gradient corresponding to a rigid deformation is skew. We recall that every tensor can be decomposed in a symmetric and anti-symmetric part

$$(\text{grad}\mathbf{u})_{sym} = \frac{1}{2}\ \left[(\text{grad}\mathbf{u}) + (\text{grad}\mathbf{u})^T\right],$$

$$(\text{grad}\mathbf{u})_{skew} = \frac{1}{2}\ \left[(\text{grad}\mathbf{u}) - (\text{grad}\mathbf{u})^T\right].$$

Hence, since

$$\mathbf{u}(\mathbf{X},t) = \mathbf{u}(\mathbf{Y},t) + \text{grad }\mathbf{u}(\mathbf{Y})(\mathbf{X}-\mathbf{Y}),$$

we have that an infinitesimal rigid displacement is a vector field $\mathbf{u}$ admitting the representation

$$\mathbf{u}(\mathbf{X},t) = \mathbf{u}(\mathbf{Y},t) + (\text{grad}\mathbf{u})_{skew}(\mathbf{X}-\mathbf{Y}),$$

Recalling (1.8) one can easily show that

$$\mathbf{u}(\mathbf{X},t) = \mathbf{u}(\mathbf{Y},t) + \boldsymbol{\omega} \times (\mathbf{X}-\mathbf{Y}),$$

where

$$\boldsymbol{\omega} = \frac{1}{2}\text{curl }\mathbf{u}.$$

The vector $\boldsymbol{\omega}$ is sometimes called the *axial vector.*

2.4 Local description of the velocity field

Let $\mathbf{v}(\mathbf{x},t)$ be the Eulerian description of the velocity field given in (2.6). From Taylor expansion

$$\mathbf{v}(\mathbf{x},t) = \mathbf{v}(\mathbf{y},t) + \nabla\mathbf{v}(\mathbf{y},t)(\mathbf{x}-\mathbf{y}) + O(|\mathbf{x}-\mathbf{y}|^2). \tag{2.18}$$

The *velocity gradient* is

$$\mathbf{L}(\mathbf{x},t) = \nabla\mathbf{v}(\mathbf{x},t).$$

The symmetric part and the skew part of $\mathbf{L}$ are

$$\mathbf{D} = \frac{1}{2}\left[\mathbf{L} + \mathbf{L}^T\right], \tag{2.19}$$

$$\mathbf{W} = \frac{1}{2}\left[\mathbf{L} - \mathbf{L}^T\right].$$

The tensor $\mathbf{D}$ is called the *rate of deformation tensor*, while $\mathbf{W}$ is called the *vorticity*(or *spin*)*tensor*. We can easily check that

$$\mathrm{tr}\,\mathbf{D} = \mathrm{div}\,\mathbf{v}$$

where div is the divergence operator w.r.t. Eulerian coordinates. Recalling that

$$F_{ij} = \frac{\partial \chi_i}{\partial X_j} \qquad \Longrightarrow \qquad \dot{F}_{ij} = \frac{\partial v_i}{\partial X_j} = \sum_{k=1}^{3} \frac{\partial v_i}{\partial x_k}\frac{\partial x_k}{\partial X_j},$$

we get $\dot{\mathbf{F}} = (\nabla \mathbf{v})\mathbf{F}$ and

$$\mathbf{L}(\mathbf{x}, t) = \dot{\mathbf{F}}\mathbf{F}^{-1}\Big|_{\mathbf{X}=\boldsymbol{\chi}^{-1}(\mathbf{x},t)} \tag{2.20}$$

Given a material point $\mathbf{X}$, the function $\mathbf{s}(t) = \boldsymbol{\chi}(\mathbf{X}, t)$ is called a *path line*. The path line clearly satisfies the differential equation $\dot{\mathbf{s}}(t) = \mathbf{v}(\mathbf{s}(t), t)$. Conversely the *streamlines* at time t are the maximal solutions of $\dot{\mathbf{s}}(\tau) = \mathbf{v}(\mathbf{s}(\tau), t)$. When (2.10) holds and the motion is steady, every path line is a streamline and vice-versa.

Proposition 1 *Let P_t be the configuration occupied by a generic subset of B at time t and let $\mathbf{v}$ be C^1. If $\mathbf{D} \equiv 0$ then there exist vectors $\mathbf{v}_o$ and $\boldsymbol{\omega}$ such that*

$$\mathbf{v} = \mathbf{v}_o + \boldsymbol{\omega} \times \mathbf{x},$$

i.e. a rigid motion.

Proof. Let $\mathbf{D} \equiv 0$, i.e. $\mathbf{W} = \mathbf{L}$. We prove that $\mathbf{W}$ is spatially homogeneous. Suppose P_t is a ball and let $\mathbf{x}, \mathbf{y} \in P_t$. The line connecting $\mathbf{x}, \mathbf{y}$ is

$$\mathbf{c}(s) = \mathbf{x} + s(\mathbf{y} - \mathbf{x}), \qquad s \in [0, 1].$$

We have

$$\mathbf{v}(\mathbf{y}, t) - \mathbf{v}(\mathbf{x}, t) = \int_{\mathbf{c}} \mathbf{L}(\mathbf{z}, t) d\mathbf{z} = \int_0^1 \mathbf{W}(\mathbf{c}(s), t)\mathbf{c}'(s) ds,$$

so that

$$\mathbf{v}(\mathbf{y}, t) - \mathbf{v}(\mathbf{x}, t) = \int_0^1 \mathbf{W}(\mathbf{c}(s), t)(\mathbf{y} - \mathbf{x}) ds. \tag{2.21}$$

Multiplying (2.21) by $(\mathbf{y} - \mathbf{x})$ and recalling (1.7) we get

$$\Big[\mathbf{v}(\mathbf{y}, t) - \mathbf{v}(\mathbf{x}, t)\Big] \cdot (\mathbf{y} - \mathbf{x}) = 0. \tag{2.22}$$

Now recall (1.14) and take the gradient of (2.22) w.r.t. $\mathbf{y}$

$$\mathbf{W}^T(\mathbf{y}, t)(\mathbf{y} - \mathbf{x}) + \mathbf{v}(\mathbf{y}, t) - \mathbf{v}(\mathbf{x}, t) = 0, \tag{2.23}$$

and then take the gradient of (2.23) w.r.t. $\mathbf{x}$

$$-\mathbf{W}^T(\mathbf{y}, t) - \mathbf{W}(\mathbf{x}, t) = 0,$$

so that $\mathbf{W}(\mathbf{y}, t) = \mathbf{W}(\mathbf{x}, t)$, implying $\mathbf{W}$ constant in space. From the arbitrariness of P_t we get the thesis. □

Proposition 1 implies that in a rigid motion the tensor $\mathbf{L}$ is skew and the spatial velocity has the representation

$$\mathbf{v}(\mathbf{x},t) = \mathbf{v}(\mathbf{y},t) + \mathbf{W}(t)(\mathbf{x}-\mathbf{y}),$$

or equivalently

$$\mathbf{v}(\mathbf{x},t) = \mathbf{v}(\mathbf{y},t) + \boldsymbol{\omega}(t)\times(\mathbf{x}-\mathbf{y}).$$

It is easy to see that, in general, the vector $\boldsymbol{\omega}$ associated with the skew part of the velocity gradient $\mathbf{W}$ is

$$\boldsymbol{\omega} = \frac{1}{2}\,\mathrm{curl}\,\mathbf{v},$$

so that

$$\mathbf{W}(\mathbf{y},t)\mathbf{u} = \frac{1}{2}\,\mathrm{curl}\,\mathbf{v}(\mathbf{y},t)\times\mathbf{u} \tag{2.24}$$

for all vectors $\mathbf{u}$. Hence, from (2.18), we find that

$$\mathbf{v}(\mathbf{x},t) = \mathbf{v}(\mathbf{y},t) + \frac{1}{2}\mathrm{curl}\,\mathbf{v}(\mathbf{y},t)\times(\mathbf{x}-\mathbf{y}) + \mathbf{D}(\mathbf{y},t)(\mathbf{x}-\mathbf{y}) + O(|\mathbf{x}-\mathbf{y}|^2) \tag{2.25}$$

In the case of irrotational flow

$$\mathbf{v}(\mathbf{x},t) = \mathbf{v}(\mathbf{y},t) + \mathbf{D}(\mathbf{y},t)(\mathbf{x}-\mathbf{y}) + O(|\mathbf{x}-\mathbf{y}|^2) \tag{2.26}$$

The acceleration is

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = \frac{\partial\mathbf{v}}{\partial t} + (\nabla\mathbf{v})\mathbf{v}$$

or equivalently

$$\mathbf{a} = \frac{\partial\mathbf{v}}{\partial t} + \frac{1}{2}\nabla(|\mathbf{v}|^2) + \mathrm{curl}\,\mathbf{v}\times\mathbf{v}. \tag{2.27}$$

Suppose ℓ is a smooth closed curve. The *circulation* on ℓ is

$$C_\ell = \oint_\ell \mathbf{v}\cdot d\boldsymbol{\ell}. \tag{2.28}$$

From Stokes' theorem (1.27) we notice that irrotational flows are such that $C_\ell = 0$ on every closed curve ℓ.

2.5 Euler's formula

The Jacobian

$$J(\mathbf{X},t) = \det\mathbf{F}(\mathbf{X},t)$$

gives a measure of the volume change produced by the deformation. When the infinitesimal volume dV_o is mapped into dV we have

$$\frac{dV}{dV_o} = \det\mathbf{F} = J.$$

We will prove that

$$\frac{\partial J}{\partial t}(\mathbf{X},t) = J(\mathbf{X},t)\,\mathrm{div}\,\mathbf{v}(\mathbf{x},t)\Big|_{\mathbf{x}=\boldsymbol{\chi}(\mathbf{X},t)}, \tag{2.29}$$

where we recall that the divergence is made w.r.t. Eulerian coordinates $\mathbf{x}$. Formula (2.29) is also known as *Euler's formula.* To prove (2.29) we need the following

Lemma 1 *Given a matrix A*

$$\det(\mathbf{I} + \varepsilon \mathbf{A}) = 1 + \varepsilon \, \text{tr} \, \mathbf{A} + o(\varepsilon). \tag{2.30}$$

Proof. From (1.10) we know that

$$\det(\varepsilon \mathbf{A} - \lambda \mathbf{I}) = -\lambda^3 + i_1(\varepsilon \mathbf{A})\lambda^2 - i_2(\varepsilon \mathbf{A})\lambda + i_3(\varepsilon \mathbf{A}),$$

where i_k are the principal invariants and where $i_k(\varepsilon \mathbf{A}) = \varepsilon^k i_k(\mathbf{A})$. Setting $\lambda = -1$ we obtain (2.30). □

To prove (2.29) we consider the Taylor expansion of $\mathbf{F}$ around t

$$\mathbf{F}(\mathbf{X}, t + \varepsilon) = \mathbf{F}(\mathbf{X}, t) + \varepsilon \frac{\partial \mathbf{F}}{\partial t} + o(\varepsilon).$$

From (2.20) we have

$$\frac{\partial \mathbf{F}}{\partial t} = \mathbf{L}\mathbf{F},$$

where $\mathbf{L} = \mathbf{L}(\boldsymbol{\chi}(\mathbf{X}, t), t)$. Hence

$$\mathbf{F}(\mathbf{X}, t + \varepsilon) = [\mathbf{I} + \varepsilon \mathbf{L} + o(\varepsilon)] \, \mathbf{F}(\mathbf{X}, t). \tag{2.31}$$

Applying the det operator to both sides of (2.31) we get

$$J(\mathbf{X}, t + \varepsilon) = J(\mathbf{X}, t) \det \left[\mathbf{I} + \varepsilon \mathbf{L} + o(\varepsilon)\right],$$

or equivalently

$$J(\mathbf{X}, t + \varepsilon) = J(\mathbf{X}, t) \left[1 + \varepsilon \text{tr} \, \mathbf{L} + o(\varepsilon)\right],$$

Dividing by ε and taking the limit $\varepsilon \to 0$ we get (2.29).

2.6 The Reynolds transport theorem

Suppose to take a part $P \subset B$ and define $P_t = \boldsymbol{\chi}(P, t)$, so that P_t represents the current configuration of P. The volume of P_t is

$$\text{Vol}\,(P_t) = \int_{P_t} d\mathbf{x} = \int_P J(\mathbf{X}, t) \; d\mathbf{X},$$

and

$$\frac{d\text{Vol}\,(P_t)}{dt} = \int_P \frac{\partial J(\mathbf{X}, t)}{\partial t} \; d\mathbf{X}. \tag{2.32}$$

Recalling (2.29) we have

$$\frac{d\text{Vol}\,(P_t)}{dt} = \int_{P_t} \text{div} \, \mathbf{v} \; d\mathbf{x}. \tag{2.33}$$

Hence a motion is volume-preserving (*isochoric*) when $\text{div} \, \mathbf{v} = 0$. Obviously rigid motions are isochoric. Relation (2.33) can be generalized to a generic scalar field $G_e(\mathbf{x}, t)$ providing the so-called *Reynolds transport theorem.*

Theorem 8 (Reynolds) *Suppose that P_t is a part of the actual configuration κ_t and G_e is the Eulerian description of some scalar quantity. Then*

$$\frac{d}{dt}\int_{P_t} G_e \; d\mathbf{x} = \int_{P_t}\left[\frac{\partial G_e}{\partial t} + \operatorname{div}\left(G_e \mathbf{v}\right)\right] \; d\mathbf{x}. \tag{2.34}$$

The theorem holds also for a generic vector field $\mathbf{w}$

$$\frac{d}{dt}\int_{P_t} \mathbf{w} \; d\mathbf{x} = \int_{P_t}\left[\frac{\partial \mathbf{w}}{\partial t} + \operatorname{div}\left(\mathbf{w}\otimes\mathbf{v}\right]\right. \; d\mathbf{x}, \tag{2.35}$$

where $\otimes$ is the dyadic product defined in (1.3).

Proof. We prove only relation (2.34), since (2.35) is a direct consequence. We have

$$\frac{d}{dt}\int_{P_t} G_e \; d\mathbf{x} = \frac{d}{dt}\int_{P} G_\ell J \; d\mathbf{X} = \int_{P}\left[\frac{\partial G_\ell}{\partial t}J + \frac{\partial J}{\partial t}G_\ell\right] \; d\mathbf{X}.$$

From (2.29)

$$\frac{d}{dt}\int_{P_t} G_e \; d\mathbf{x} = \int_{P} J\left[\frac{\partial G_\ell}{\partial t} + G_\ell \operatorname{div}\mathbf{v}\right] \; d\mathbf{X}. \tag{2.36}$$

Recalling (2.5) we get

$$\left[\frac{\partial G_\ell}{\partial t} + G_\ell \operatorname{div}\mathbf{v}\right] = \frac{\partial G_e}{\partial t} + \nabla G_e\cdot\mathbf{v} + G_e \operatorname{div}\mathbf{v},$$

or equivalently, recalling (1.15)

$$\frac{\partial G_\ell}{\partial t} + G_\ell \operatorname{div}\mathbf{v} = \frac{\partial G_e}{\partial t} + \operatorname{div}\left(G_e\mathbf{v}\right). \tag{2.37}$$

Inserting (2.37) into (2.36) we get (2.34). □

Remark 1 *Using the divergence theorem relations (2.34), (2.35) can be rewritten as*

$$\frac{d}{dt}\int_{P_t} G_e \; d\mathbf{x} = \int_{P_t}\frac{\partial G_e}{\partial t}d\mathbf{x} + \int_{\partial P_t} G_e\left(\mathbf{v}\cdot\mathbf{n}\right)d\sigma,$$

$$\frac{d}{dt}\int_{P_t} \mathbf{w} \; d\mathbf{x} = \int_{P_t}\frac{\partial \mathbf{w}}{\partial t}d\mathbf{x} + \int_{\partial P_t} \mathbf{w}\left(\mathbf{v}\cdot\mathbf{n}\right)d\sigma.$$

Remark 2 *The Reynolds theorem can be actually written in a more general form for an* evolving non material volume $V(t)$

$$\frac{d}{dt}\int_{V(t)} \mathbf{w} \; d\mathbf{x} = \int_{V(t)}\frac{\partial \mathbf{w}}{\partial t} + \int_{\partial V(t)} \mathbf{w}(\mathbf{U}\cdot\mathbf{n}) \; d\sigma \tag{2.38}$$

where $\mathbf{U}$ is the velocity of $\partial V(t)$ and $\mathbf{n}$ the outward normal to $\partial V(t)$ (see [20] for the proof of (2.38)).

2.7 Simple shear

A motion commonly encountered in continuum mechanics is the so-called *simple shear.* This motion is steady, with uni-directional fully developed velocity. In a Cartesian reference frame the simple shear is described by

$$\begin{aligned} x_1 &= X_1 + \dot{\gamma} t X_2, \\ x_2 &= X_2, \\ x_3 &= X_3. \end{aligned}$$

As a consequence

$$\mathbf{F} = \begin{bmatrix} 1 & \dot{\gamma} t & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad \mathbf{E} = \begin{bmatrix} 0 & \frac{\dot{\gamma}}{2} t & 0 \\ \frac{\dot{\gamma}}{2} t & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Moreover

$$\begin{aligned} v_1 &= \dot{\gamma} x_2, \\ v_2 &= 0, \\ v_3 &= 0. \end{aligned}$$

so that

$$\mathbf{D} = \begin{bmatrix} 0 & \frac{\dot{\gamma}}{2} & 0 \\ \frac{\dot{\gamma}}{2} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad \mathbf{W} = \begin{bmatrix} 0 & \frac{\dot{\gamma}}{2} & 0 \\ -\frac{\dot{\gamma}}{2} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

We can easily check that the simple shear is an isochoric motion. Simple shear can be generated by parallel plates separated by a distance h, where the upper plate is moving with constant velocity V and the bottom plate is fixed (see Fig. 2.3). In this case $\dot{\gamma} = V/h$. Simple shear is sometimes called *lineal Couette flow.*

2.8 Uniaxial extension

Another important simple motion, which is of particular interest in viscoelasticity, is *uniaxial extension.* This motion has the form

$$\begin{aligned} x_1 &= X_1 e^{\dot{\gamma} t}, \\ x_2 &= X_2 e^{-\frac{\dot{\gamma}}{2} t}, \\ x_3 &= X_3 e^{-\frac{\dot{\gamma}}{2} t}. \end{aligned}$$

The deformation along x_2, x_3 is the same. We get

$$\mathbf{F} = \begin{bmatrix} e^{\dot{\gamma} t} & 0 & 0 \\ 0 & e^{-\frac{\dot{\gamma}}{2} t} & 0 \\ 0 & 0 & e^{-\frac{\dot{\gamma}}{2} t} \end{bmatrix}, \qquad \mathbf{E} = \begin{bmatrix} e^{\dot{\gamma} t} - 1 & 0 & 0 \\ & e^{-\frac{\dot{\gamma}}{2} t} - 1 & 0 \\ 0 & 0 & e^{-\frac{\dot{\gamma}}{2} t} - 1 \end{bmatrix}.$$

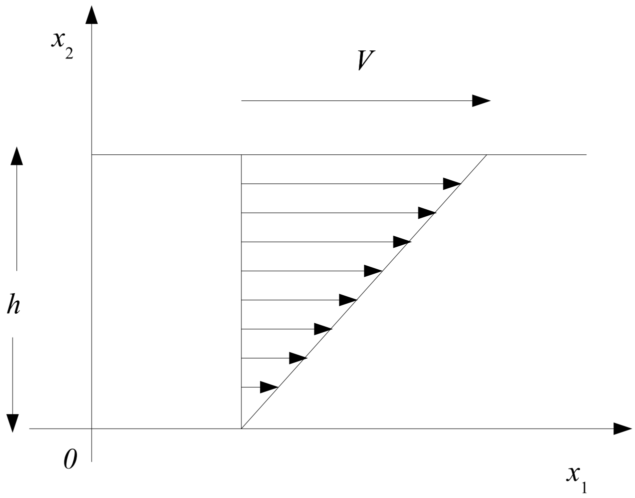


Figure 2.3: Simple shear.

Moreover

$$\begin{aligned} v_1 &= \dot{\gamma} x_1, \\ v_2 &= -\frac{\dot{\gamma}}{2} x_2, \\ v_3 &= -\frac{\dot{\gamma}}{2} x_3. \end{aligned}$$

so that

$$\mathbf{D} = \begin{bmatrix} \dot{\gamma} & 0 & 0 \\ 0 & -\frac{\dot{\gamma}}{2} & 0 \\ 0 & 0 & -\frac{\dot{\gamma}}{2} \end{bmatrix}, \qquad \mathbf{W} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

2.9 Boundary conditions

Let $P_t = \boldsymbol{\chi}(P, t)$ be the actual configuration of a subset P of the body B. As observed earlier $\partial P_t = \boldsymbol{\chi}(\partial P, t)$, that is the particle belonging to the boundary of P will also belong to the evolution of such boundary (∂P is a material surface). We have

Proposition 2 *Let us consider a material volume P with boundary ∂P and let us consider its evolution $P_t = \boldsymbol{\chi}(P, t)$. Suppose also that at each time t the boundary ∂P_t can be written as*

(2.39) $$F(\mathbf{x}, t) = 0,$$

where F is a C^1 function. Then

(2.40) $$\frac{dF}{dt} = 0,$$

for each $\mathbf{x} \in \partial P_t$ and $t > 0$.

Proof. We have $F(\boldsymbol{\chi}(\mathbf{X}, t), t) = 0$, so that

(2.41) $$\frac{dF}{dt} = \frac{\partial F}{\partial t} + \nabla F \cdot \mathbf{v} = 0.$$

□

Suppose that $F(\mathbf{x},t)$ describes the boundary of B and denote by $\mathbf{n}$ the outward normal of B. From (2.41)

$$\mathbf{n} = \frac{\nabla F}{|\nabla F|} \qquad v_n = \mathbf{v}(\mathbf{x},t)\cdot\mathbf{n} = \mathbf{v}(\mathbf{x},t)\cdot\frac{\nabla F}{|\nabla F|} = -\frac{1}{|\nabla F|}\frac{\partial F}{\partial t}.$$

2.10 General balance law

Suppose that $\mathcal{G}$ represents some quantity defined on P_t. The rate of change of $\mathcal{G}$ is given by

$$\frac{d\mathcal{G}}{dt} = -\mathcal{F} + \mathcal{S}, \tag{2.42}$$

where $\mathcal{F}$ represents the outflow of $\mathcal{G}$ through the boundary of P_t and $\mathcal{S}$ represents the supply/removal of $\mathcal{G}$ in P_t. When $\mathcal{G}$, $\mathcal{F}$, $\mathcal{S}$ are defined through density functions we write

$$\mathcal{G} = \int_{P_t} g(\mathbf{x},t)d\mathbf{x}, \qquad \mathcal{F} = \int_{\partial P_t} \mathbf{\Phi}\cdot\mathbf{n}d\sigma, \qquad \mathcal{S} = \int_{P_t} s(\mathbf{x},t)d\mathbf{x},$$

where $\mathbf{\Phi}$ is the flux across ∂P_t. Equation (2.42) can be rewritten as

$$\frac{d}{dt}\left[\int_{P_t} g(\mathbf{x},t)d\mathbf{x}\right] = -\int_{\partial P_t} \mathbf{\Phi}\cdot\mathbf{n}d\sigma + \int_{P_t} s(\mathbf{x},t)d\mathbf{x}. \tag{2.43}$$

Exploiting Reynolds transport theorem and the divergence theorem

$$\int_{P_t}\left[\frac{\partial g}{\partial t} + \text{ div }(g\mathbf{v}+\mathbf{\Phi}) - s\right]d\mathbf{x} = 0,$$

that, because of the arbitrariness of P_t, yields

$$\frac{\partial g}{\partial t} + \text{ div }(g\mathbf{v}+\mathbf{\Phi}) - s = 0. \tag{2.44}$$

Equation (2.44) represents the local form of a generic *balance law* for the density g.

2.11 Jump conditions

When deriving (2.34), (2.35) we are tacitly assuming that the scalar and vector fields under the integral sign are sufficiently smooth on P_t. The latter assumption may be actually relaxed, allowing possible jumps on a singular surface contained in P_t.

Suppose, for instance, that Σ is a non-material surface contained in P_t, with velocity $\mathbf{U}$ and normal $\boldsymbol{\nu}$, as depicted in Fig. 2.4. Denote with $\mathbf{n}$ the outward unit normal to P_t and write

$$P_t = P_t^+ \cup P_t^-, \qquad \partial P_t = \partial P_t^+ \cup \partial P_t^-,$$

where $^+$ stands for the part of P_t that contains $\boldsymbol{\nu}$ and $^-$ for the part containing $-\boldsymbol{\nu}$. Considered a scalar quantity $\mathcal{G}$ which is defined on P_t through a density function g. Suppose also that

$$[\![g]\!]\Big|_{\Sigma} = g^+ - g^- \neq 0,$$

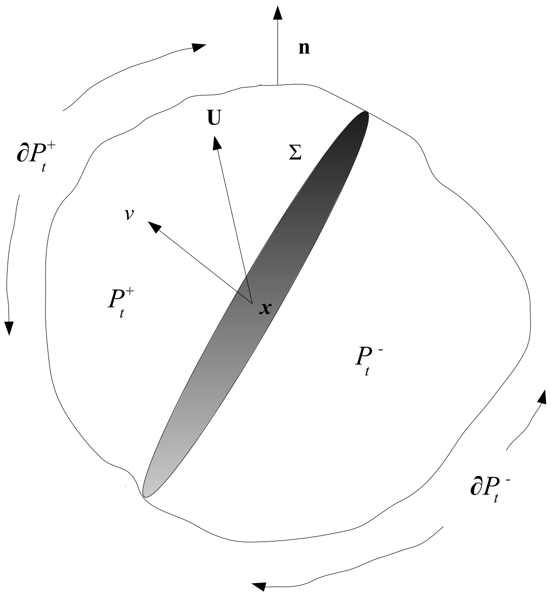


Figure 2.4: Jump conditions and singular surfaces.

that is Σ is a singular surface for g. The Reynolds transport theorem applied to the non-material domains P_t^+, P_t^- yields

$$\frac{d}{dt}\left[\int_{P_t^+} g d\mathbf{x}\right] = \int_{P_t^+} \frac{\partial g}{\partial t} d\mathbf{x} + \int_{\partial P_t^+} g(\mathbf{v}\cdot\mathbf{n})d\sigma - \int_\Sigma g^+(\mathbf{U}\cdot\boldsymbol{\nu})d\sigma,$$

$$\frac{d}{dt}\left[\int_{P_t^-} g d\mathbf{x}\right] = \int_{P_t^-} \frac{\partial g}{\partial t} d\mathbf{x} + \int_{\partial P_t^-} g(\mathbf{v}\cdot\mathbf{n})d\sigma + \int_\Sigma g^-(\mathbf{U}\cdot\boldsymbol{\nu})d\sigma.$$

Summing up we find

$$\frac{d}{dt}\left[\int_{P_t} g d\mathbf{x}\right] = \int_{P_t} \frac{\partial g}{\partial t} d\mathbf{x} + \int_{\partial P_t} g(\mathbf{v}\cdot\mathbf{n})d\sigma - \int_\Sigma [\![g]\!](\mathbf{U}\cdot\boldsymbol{\nu})d\sigma, \tag{2.45}$$

which is the generalization of the Reynolds transport theorem in the presence of a singular surface. If the case of a vector field $\mathbf{w}$ we get

$$\frac{d}{dt}\left[\int_{P_t} \mathbf{w} d\mathbf{x}\right] = \int_{P_t} \frac{\partial \mathbf{w}}{\partial t} d\mathbf{x} + \int_{\partial P_t} \mathbf{w}(\mathbf{v}\cdot\mathbf{n})d\sigma - \int_\Sigma [\![\mathbf{w}]\!](\mathbf{U}\cdot\boldsymbol{\nu})d\sigma.$$

2.12 Rankine-Hugoniot conditions

Let us consider the integral balance law (2.43) and suppose that g experiences a jump across the non material surface Σ. Then, recalling (2.45)

$$\int_{P_t} \frac{\partial g}{\partial t} d\mathbf{x} + \int_{\partial P_t} g(\mathbf{v}\cdot\mathbf{n})d\sigma - \int_\Sigma [\![g]\!](\mathbf{U}\cdot\boldsymbol{\nu})d\sigma = -\int_{\partial P_t} \boldsymbol{\Phi}\cdot\mathbf{n}d\sigma + \int_{P_t} s d\mathbf{x}. \tag{2.46}$$

Looking at Fig. 2.5, we consider the ball $\mathcal{B}$ centered in $\mathbf{x}\in\Sigma$ and define

$$S = \Sigma \cap \mathcal{B}.$$

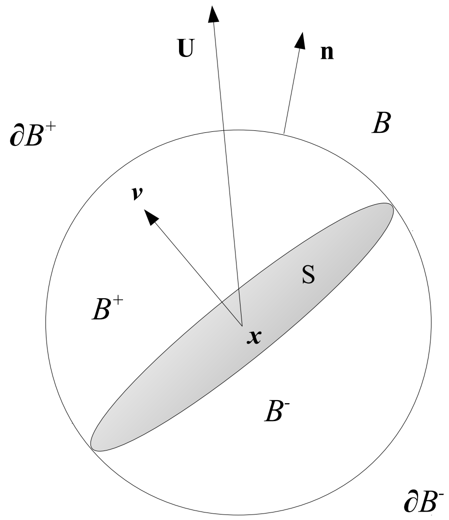


Figure 2.5: Rankine-Hugoniot conditions.

Suppose now to squeeze the two emispheres of $\mathcal{B}$ on S. The volume integrals in (2.46) vanish (provided that the integrands are bounded). Hence we are left with

$$\int_S [\![g]\!](\mathbf{v}\cdot\boldsymbol{\nu})d\sigma - \int_S [\![g]\!](\mathbf{U}\cdot\boldsymbol{\nu})d\sigma = -\int_S [\![\boldsymbol{\Phi}]\!]\cdot\boldsymbol{\nu}d\sigma.$$

Once again, because of the arbitrariness of S

$$[\![g(\mathbf{v}\cdot\boldsymbol{\nu}) + \boldsymbol{\Phi}\cdot\boldsymbol{\nu}]\!] = [\![g]\!](\mathbf{U}\cdot\boldsymbol{\nu}). \tag{2.47}$$

The jump relation (2.47) is called *Rankine-Hugoniot condition.*

2.13 Mass balance

The mass of a part P of the body is

$$M(P) = \int_P \varrho_\ell(\mathbf{X})d\mathbf{X},$$

where we denote by $\varrho_\ell(\mathbf{X})$ the Lagrangian description of density function in the reference configuration. Since the mass cannot be altered by any deformation the quantity $M(P)$ is independent of $\boldsymbol{\chi}(\mathbf{X},t)$. Hence

$$\int_P \varrho_\ell(\mathbf{X})d\mathbf{X} = \int_{\boldsymbol{\chi}(P,t)} \varrho_e(\mathbf{x},t)d\mathbf{x}.$$

We have

$$\int_P \varrho_\ell(\mathbf{X})d\mathbf{X} = \int_P J(\mathbf{X},t)\varrho_e(\boldsymbol{\chi}(\mathbf{X},t),t)d\mathbf{X}.$$

which provides

$$\varrho_\ell(\mathbf{X}) = J(\mathbf{X},t)\varrho_e(\mathbf{x},t)|_{\mathbf{x}=\boldsymbol{\chi}(\mathbf{X},t)}\,.$$

Let us drop the subscript e in the local description of the density. The mass of a part $P_t \subset \kappa_t$ is

$$m(P_t) = \int_{P_t} \varrho(\mathbf{x},t)\ d\mathbf{x}.$$

Conservation of mass yields $\dot{m}(P_t) = 0$. Hence, recalling Reynolds transport theorem (2.34)

$$\frac{\partial \varrho}{\partial t} + \text{div}\,(\varrho\,\mathbf{v}) = 0, \tag{2.48}$$

or equivalently

$$\frac{d\varrho}{dt} + \varrho\,\text{div}\,\mathbf{v} = 0. \tag{2.49}$$

In an isochoric motion

$$\dot{\varrho} = \frac{d\varrho}{dt} = 0.$$

For any given scalar field G

$$\frac{d}{dt}\int_{P_t} \varrho G d\mathbf{x} = \int_{P_t} [\underbrace{\varrho G\,\text{div}\,\mathbf{v} + \dot{\varrho} G}_{=0} + \dot{G}\varrho] d\mathbf{x}.$$

Therefore

$$\frac{d}{dt}\int_{P_t} \varrho G d\mathbf{x} = \int_{P_t} \varrho \frac{dG}{dt} d\,\mathbf{x}. \tag{2.50}$$

Recalling Section 1.4, 1.5 we can write mass balance in cylindrical and in spherical coordinates. In cylindrical coordinates mass balance is expressed by

$$\frac{\partial \varrho}{\partial t} + \frac{1}{r}\frac{\partial(\varrho r v_r)}{\partial r} + \frac{1}{r}\frac{\partial(\varrho v_\theta)}{\partial \theta} + \frac{\partial(\varrho v_z)}{\partial z} = 0, \tag{2.51}$$

where $\mathbf{v} = v_r \mathbf{e}_r + v_\theta \mathbf{e}_\theta + v_z \mathbf{e}_z$. In spherical coordinates mass balance is expressed by

$$\frac{\partial \varrho}{\partial t} + \frac{1}{r^2}\frac{\partial(\varrho r^2 v_r)}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial(\varrho v_\theta \sin\theta)}{\partial \theta} + \frac{1}{r\sin\theta}\frac{\partial(\varrho v_\phi)}{\partial \phi} = 0, \tag{2.52}$$

where $\mathbf{v} = v_r \mathbf{e}_r + v_\theta \mathbf{e}_\theta + v_\phi \mathbf{e}_\phi$.

Chapter 3

Dynamics

In this chapter we investigate the dynamics of a continuum, that is the action of externally applied forces that produce motion. Reference books on the topics presented here are [12], [24], [25].

3.1 Forces in a continuum

The forces exerted on a portion P_t of a continuum are of two types:

- *surface* (or *contact*) *forces*: forces exerted by the portion of the body that surrounds P_t acting on the bounding surface ∂P_t ;
- *body forces*: forces originating outside the body acting on the volume (or mass) of the body (e.g. *gravitational* or *electro-magnetical*).

Body forces are expressed through a density function $\mathbf{b}(\mathbf{x}, t)$ (force per unit mass) so that the total body force acting on P_t is

$$\int_{P_t} \varrho(\mathbf{x}, t)\mathbf{b}(\mathbf{x}, t) d\mathbf{x}.$$

Regarding the contact forces we assume the existence of a force density

$$\mathbf{t}(\mathbf{x}, t; \mathbf{n}) \tag{3.1}$$

representing the force per unit surface acting of any surface S that contains $\mathbf{x}$ with normal $\mathbf{n}$. We assume that $\mathbf{t}$ is a smooth function of its arguments. The existence of (3.1) and the assumption that $\mathbf{t}$ depends on S only through $\mathbf{n}$ is called the *Cauchy's postulate*. Exploiting the *third law* of mechanics we write

$$\mathbf{t}(\mathbf{x}, t; -\mathbf{n}) = -\mathbf{t}(\mathbf{x}, t; \mathbf{n}).$$

The total contact force acting on P is thus

$$\int_{\partial P_t} \mathbf{t}(\mathbf{x}, t; \mathbf{n}) d\sigma,$$

where $\mathbf{n}$ represents the outward unit normal to ∂P_t.

3.2 Balance of linear momentum and Cauchy's theorem

The *balance of linear momentum* (or *Newton's second law*) is given by

$$\underbrace{\frac{d}{dt}\int_{P_t} \varrho \mathbf{v} d\mathbf{x}}_{\text{linear momentum}} = \underbrace{\int_{\partial P_t} \mathbf{t} d\sigma + \int_{P_t} \varrho \mathbf{b} d\mathbf{x}}_{\text{Total force acting on } P_t}.$$

Recalling (2.50) we can rewrite the above as

$$\int_{P_t} \varrho \frac{d\mathbf{v}}{dt} d\mathbf{x} = \int_{\partial P_t} \mathbf{t} d\sigma + \int_{P_t} \varrho \mathbf{b} d\mathbf{x}. \tag{3.2}$$

The equilibrium equation is

$$\int_{\partial P_t} \mathbf{t} d\sigma + \int_{P_t} \varrho \mathbf{b} d\mathbf{x} = 0. \tag{3.3}$$

The dependence of $\mathbf{t}$ on $\mathbf{n}$ is determined by the following

Theorem 9 (Cauchy) *For each* $(\mathbf{x}, t)$ *the vector field* $\mathbf{t}(\mathbf{x}, t; \mathbf{n})$ *is linear in* $\mathbf{n}$, *so that there exists a second order tensor* $\mathbf{T}(\mathbf{x}, t)$ *such that*

$$\mathbf{t}(\mathbf{x}, t; \mathbf{n}) = \mathbf{T}(\mathbf{x}, t)\mathbf{n}.$$

The tensor $\mathbf{T}(\mathbf{x}, t)$ *is called the Cauchy stress tensor.*

Proof. Consider the infinitesimal tetrahedron depicted in Fig. 3.1. The tetrahedron has three faces perpendicular to the coordinate axes and one face normal to a unit vector $\mathbf{n}$. Suppose that

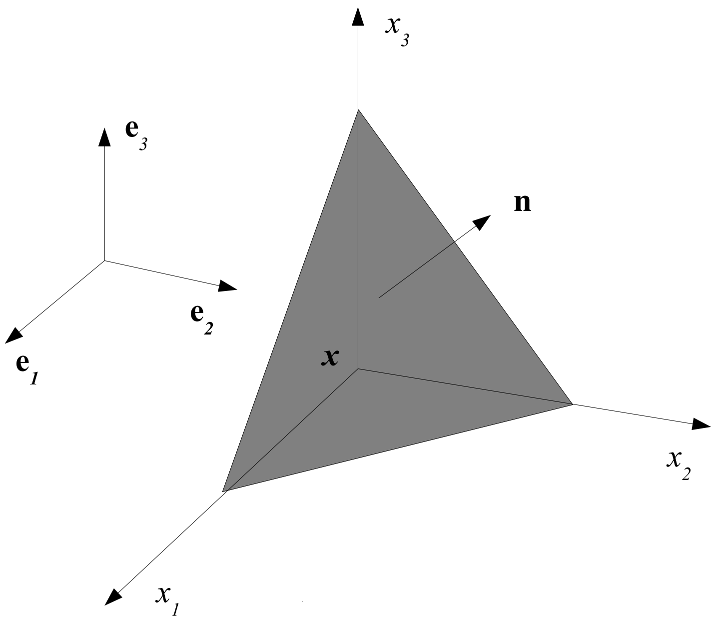


Figure 3.1: Cauchy's tetrahedron.

$d\sigma$ is the area of the surface normal to $\mathbf{n}$ and that $d\sigma_i$ is the area of the surface normal to $\mathbf{e}_i$. The area $d\sigma_i$ can be found projecting $d\sigma$ onto the face perpendicular to $\mathbf{e}_i$

$$d\sigma_i = (\mathbf{n} \cdot \mathbf{e}_i) d\sigma = n_i d\sigma,$$

where

$$\mathbf{n} = n_1\mathbf{e}_1 + n_2\mathbf{e}_2 + n_3\mathbf{e}_3.$$

The equilibrium equation (3.3) for the infinitesimal tetrahedron is

$$\varrho(\mathbf{x},t)\mathbf{b}(\mathbf{x},t)d\mathbf{x} + \sum_{i=1}^{3}\mathbf{t}(\mathbf{x},t;-\mathbf{e}_i)d\sigma_i + \mathbf{t}(\mathbf{x},t;\mathbf{n})d\sigma = 0,$$

or equivalently

$$\varrho(\mathbf{x},t)\mathbf{b}(\mathbf{x},t)d\mathbf{x} + \sum_{i=1}^{3}\mathbf{t}(\mathbf{x},t;-\mathbf{e}_i)n_i d\sigma + \mathbf{t}(\mathbf{x},t;\mathbf{n})d\sigma = 0. \tag{3.4}$$

We have

$$\frac{d\mathbf{x}}{d\sigma} \to 0 \qquad d\sigma \to 0,$$

so that, dividing (3.4) by $d\sigma$ and letting $d\sigma \to 0$, we get

$$\mathbf{t}(\mathbf{x},t;\mathbf{n}) = \sum_{i=1}^{3}\mathbf{t}(\mathbf{x},t;\mathbf{e}_i)n_i.$$

If we define the Cauchy stress tensor $\mathbf{T}(\mathbf{x},t)$ as the tensor whose columns are the vectors $\mathbf{t}(\mathbf{x},t;\mathbf{e}_i)$ we find

$$\mathbf{t}(\mathbf{x},t;\mathbf{n}) = \mathbf{T}(\mathbf{x},t)\mathbf{n}.$$

Hence

$$T_{ij}(\mathbf{x},t) = \mathbf{t}(\mathbf{x},t;\mathbf{e}_i)\cdot\mathbf{e}_j.$$

□

The immediate consequence of Cauchy's theorem is that the momentum balance (3.2) can be rewritten as

$$\int_{P_t}\varrho\frac{d\mathbf{v}}{dt}d\mathbf{x} = \int_{\partial P_t}\mathbf{T}\mathbf{n}d\sigma + \int_{P_t}\varrho\mathbf{b}d\mathbf{x}.$$

Recalling the divergence theorem we get

$$\int_{P_t}\varrho\frac{d\mathbf{v}}{dt}d\mathbf{x} = \int_{P_t}\mathrm{div}\,\mathbf{T}d\mathbf{x} + \int_{P_t}\varrho\mathbf{b}d\mathbf{x}.$$

Because of the arbitrariness of P_t we get the *local form* of the motion equation

$$\varrho\frac{d\mathbf{v}}{dt} = \varrho\mathbf{b} + \mathrm{div}\,\mathbf{T}, \tag{3.5}$$

or

$$\varrho\mathbf{a} = \varrho\mathbf{b} + \mathrm{div}\,\mathbf{T}. \tag{3.6}$$

Recalling (2.8), equation (3.5) can be rewritten as

$$\varrho\left[\frac{\partial\mathbf{v}}{\partial t} + (\nabla\mathbf{v})\mathbf{v}\right] = \varrho\mathbf{b} + \mathrm{div}\,\mathbf{T}. \tag{3.7}$$

3.3 Balance of angular momentum

The balance of *angular momentum* states that the rate of change of angular momentum for any region P_t of the actual configuration equals the moment of all external forces acting on this part. Hence we write

$$\frac{d}{dt} \underbrace{\int_{P_t} \varrho \mathbf{x} \times \mathbf{v}\, d\mathbf{x}}_{\text{angular momentum}} = \underbrace{\int_{\partial P_t} \mathbf{x} \times \mathbf{t}\, d\sigma + \int_{P_t} \varrho \mathbf{x} \times \mathbf{b}\, d\mathbf{x}}_{\text{Total external forces moment acting on } P_t} . \tag{3.8}$$

We prove the following

Theorem 10 *Equation (3.8) implies that the Cauchy stress tensor* $\mathbf{T}$ *is symmetric.*

Proof. Let $\boldsymbol{\omega}$ be a vector and let $\mathbf{W}$ be the skew tensor associated to $\boldsymbol{\omega}$ defined in (1.8). Cross multiply equation (3.5) by $\mathbf{x}$, integrate on P_t and multiply by $\boldsymbol{\omega}$

$$\boldsymbol{\omega} \cdot \int_{P_t} \varrho \mathbf{x} \times \frac{d\mathbf{v}}{dt} d\mathbf{x} = \boldsymbol{\omega} \cdot \int_{P_t} \varrho \mathbf{x} \times \mathbf{b}\, d\mathbf{x} + \boldsymbol{\omega} \cdot \int_{P_t} \mathbf{x} \times \operatorname{div} \mathbf{T}\, d\mathbf{x}. \tag{3.9}$$

From (1.1) we see that

$$\boldsymbol{\omega} \cdot (\mathbf{x} \times \operatorname{div} \mathbf{T}) = (\boldsymbol{\omega} \times \mathbf{x}) \cdot \operatorname{div} \mathbf{T} = \mathbf{W}\mathbf{x} \cdot \operatorname{div} \mathbf{T},$$

so that

$$\boldsymbol{\omega} \cdot \int_{P_t} \mathbf{x} \times \operatorname{div} \mathbf{T}\, d\mathbf{x} = \int_{P_t} \mathbf{W}\mathbf{x} \cdot \operatorname{div} \mathbf{T}\, d\mathbf{x}.$$

Applying Green formula (1.26) we get

$$\int_{P_t} \mathbf{W}\mathbf{x} \cdot \operatorname{div} \mathbf{T}\, d\mathbf{x} = \int_{\partial P_t} \mathbf{T}\mathbf{n} \cdot \mathbf{W}\mathbf{x}\, d\sigma - \int_{P_t} \mathbf{T} \cdot \nabla(\mathbf{W}\mathbf{x})\, d\mathbf{x}.$$

Now we observe that

$$\int_{\partial P_t} \mathbf{T}\mathbf{n} \cdot \mathbf{W}\mathbf{x}\, d\sigma = \int_{\partial P_t} \mathbf{t} \cdot (\boldsymbol{\omega} \times \mathbf{x})\, d\sigma = \omega \cdot \int_{\partial P_t} \mathbf{x} \times \mathbf{t}\, d\sigma.$$

In conclusion

$$\boldsymbol{\omega} \cdot \int_{P_t} \mathbf{x} \times \operatorname{div} \mathbf{T}\, d\mathbf{x} = \boldsymbol{\omega} \cdot \int_{\partial P_t} \mathbf{x} \times \mathbf{t}\, d\sigma - \int_{P_t} \mathbf{T} \cdot \mathbf{W}\, d\mathbf{x},$$

since $\nabla(\mathbf{W}\mathbf{x}) = \mathbf{W}$. Recalling (2.50)

$$\boldsymbol{\omega} \cdot \int_{P_t} \varrho \mathbf{x} \times \frac{d\mathbf{v}}{dt} d\mathbf{x} = \boldsymbol{\omega} \cdot \frac{d}{dt} \int_{P_t} \varrho \mathbf{x} \times \mathbf{v} d\mathbf{x}.$$

In conclusion (3.9) can be rewritten as

$$\boldsymbol{\omega} \cdot \underbrace{\left[\frac{d}{dt} \int_{P_t} \varrho \mathbf{x} \times \mathbf{v} d\mathbf{x} - \int_{P_t} \varrho \mathbf{x} \times \mathbf{b}\, d\mathbf{x} - \int_{\partial P_t} \mathbf{x} \times \mathbf{t}\, d\sigma \right]}_{=0 \text{ because of (3.8)}} = - \int_{P_t} \mathbf{T} \cdot \mathbf{W}\, d\mathbf{x}.$$

Recalling that P_t is an arbitrary portion of the actual configuration and that $\mathbf{W}$ is a generic skew tensor,

$$\mathbf{T} \cdot \mathbf{W} = 0,$$

meaning that $\mathbf{T}$ is symmetric, as shown in (1.6).

3.4 The theorem of mechanical energy balance

We define the kinetic energy

$$K = \frac{1}{2}\int_{P_t} \varrho\,(\mathbf{v}\cdot\mathbf{v})\,d\mathbf{x},$$

and prove the following

Theorem 11 (Power expended) *Assuming the validity of balance of mass (2.48), balance of linear momentum (3.5) and balance of angular momentum (3.8), the following formula holds*

$$\frac{dK}{dt} = \int_{P_t} \varrho\,\mathbf{b}\cdot\mathbf{v}\,d\mathbf{x} + \int_{\partial P_t} \mathbf{Tn}\cdot\mathbf{v}\,d\sigma - \int_{P_t} \mathbf{T}\cdot\mathbf{D}\,d\mathbf{x}. \tag{3.10}$$

Proof. From (2.50)

$$\frac{dK}{dt} = \int_{P_t} \varrho\mathbf{v}\cdot\frac{d\mathbf{v}}{dt}\,d\mathbf{x}.$$

Hence, recalling (3.5)

$$\frac{dK}{dt} = \int_{P_t} \varrho\mathbf{v}\cdot\mathbf{b}\,d\mathbf{x} + \int_{P_t} \mathbf{v}\cdot(\,\mathrm{div}\,\mathbf{T})\,d\mathbf{x}.$$

From Green fromula (1.26), we get

$$\int_{P_t} \mathbf{v}\cdot(\,\mathrm{div}\,\mathbf{T})\,d\mathbf{x} = \int_{\partial P_t} \mathbf{Tn}\cdot\mathbf{v}d\sigma - \int_{P_t} \mathbf{T}\cdot\nabla\mathbf{v}\,d\mathbf{x}.$$

Recalling that

$$\mathbf{T}\cdot\mathbf{L} = \mathbf{T}\cdot(\mathbf{D}+\mathbf{W}) = \mathbf{T}\cdot\mathbf{D},$$

we get the thesis of the theorem. The term

$$\int_{P_t} \mathbf{T}\cdot\mathbf{D}\,d\mathbf{x},$$

is called the *stress power*. □

Formula (3.10) can be interpreted in the following way: the *power expended* by the body forces and surface forces equals the rate of change of the kinetic energy plus the stress power.

3.5 Energy balance: first law of thermodynamics

We define the *internal energy* of a portion P_t as

$$E = \int_{P_t} \varrho\varepsilon d\mathbf{x},$$

where $\varepsilon(\mathbf{x},t)$ is the *density of internal energy.* Suppose that adjacent parts of the continuum may exchange heath through their common boundary and define $\mathbf{q}$ as the *heath flux* flowing from one part to the other. Suppose also that heath may be generated (or absorbed) by means of heat sources (or sinks) whose density is denoted by r. The *first law of thermodynamics* states that the rate of change of total energy equals the sum of mechanical power and heat adsorbed. Therefore we write the integral formulation of the *energy balance*

$$\frac{d}{dt}\Big(K+E\Big) = \int_{P_t} \varrho\,\mathbf{b}\cdot\mathbf{v}\,d\mathbf{x} + \int_{\partial P_t} \mathbf{t}\cdot\mathbf{v}d\sigma + \int_{P_t} \varrho r d\mathbf{x} - \int_{\partial P_t} \mathbf{q}\cdot\mathbf{n}d\sigma. \tag{3.11}$$

Combining (3.11) with (3.10) we get

$$\frac{dE}{dt} = \int_{P_t} \varrho r d\mathbf{x} - \int_{\partial P_t} \mathbf{q} \cdot \mathbf{n} d\sigma + \int_{P_t} \mathbf{T} \cdot \mathbf{D}\, d\mathbf{x}.$$

Applying the divergence's theorem we get the local formulation of the energy balance

$$\varrho \frac{d\varepsilon}{dt} = \varrho r - \operatorname{div} \mathbf{q} + \mathbf{T} \cdot \mathbf{D}. \tag{3.12}$$

3.6 The second principle of thermodynamics

The *second law of thermodynamics* can be expressed by

$$\frac{Q}{\theta} \leqslant \frac{dS}{dt},$$

where Q denotes the *heating rate*, θ the *absolute temperature* and where S is *entropy*. The supply of entropy to a part P_t of the current configuration is due to two sources: i) bulk rate of entropy generated in P_t; ii) rate of entropy flux through ∂P_t. Hence the total rate at which entropy is provided to P_t is

$$\int_{P_t} \frac{\varrho r}{\theta} d\mathbf{x} - \int_{\partial P_t} \frac{\mathbf{q} \cdot \mathbf{n}}{\theta} d\sigma$$

Now suppose entropy S is defined through a density function η so that

$$S = \int_{P_t} \varrho \eta d\mathbf{x}.$$

The second principle of thermodynamics states that the total rate at which entropy is supplied to P_t cannot exceed the rate of increase of entropy of P_t

$$\int_{P_t} \frac{\varrho r}{\theta} d\mathbf{x} - \int_{\partial P_t} \frac{\mathbf{q} \cdot \mathbf{n}}{\theta} d\sigma \leqslant \frac{dS}{dt},$$

or equivalently we can state that the net rate of entropy production

$$\Gamma = \frac{d}{dt} \int_{P_t} \varrho \eta d\mathbf{x} - \int_{P_t} \frac{\varrho r}{\theta} d\mathbf{x} + \int_{\partial P_t} \frac{\mathbf{q} \cdot \mathbf{n}}{\theta} d\sigma \geqslant 0.$$

Introducing the density γ of net entropy production rate we get

$$\Gamma = \int_{P_t} \varrho \gamma d\mathbf{x},$$

so that

$$\varrho \gamma = \varrho \frac{d\eta}{dt} - \frac{\varrho r}{\theta} + \operatorname{div} \left(\frac{\mathbf{q}}{\theta} \right) \geqslant 0. \tag{3.13}$$

Inequality (3.13) is called the *Clausius-Duhem* inequality and represents the local form of the second principle of thermodynamics.

3.7 Helmholtz free energy and dissipation function

From (3.12) we have

$$-\frac{\varrho r}{\theta}+\frac{1}{\theta}\mathrm{div}\,\mathbf{q}=-\frac{\varrho}{\theta}\frac{d\varepsilon}{dt}+\frac{1}{\theta}\mathbf{T}\cdot\mathbf{D}, \tag{3.14}$$

while (3.13) can be rewritten as

$$\varrho\frac{d\eta}{dt}-\frac{\varrho r}{\theta}+\frac{1}{\theta}\mathrm{div}\,\mathbf{q}-\frac{1}{\theta^2}\nabla\theta\cdot\mathbf{q}\geqslant 0. \tag{3.15}$$

Coupling (3.14 and (3.15) we find

$$\varrho\theta\frac{d\eta}{dt}-\varrho\frac{d\varepsilon}{dt}+\mathbf{T}\cdot\mathbf{D}-\frac{1}{\theta}\nabla\theta\cdot\mathbf{q}\geqslant 0. \tag{3.16}$$

Let introduce the *Helmholtz free energy*

$$\Psi=\varepsilon-\theta\eta.$$

We have

$$\varrho\theta\frac{d\eta}{dt}-\varrho\frac{d\varepsilon}{dt}=-\varrho\frac{d\Psi}{dt}-\varrho\eta\frac{d\theta}{dt}. \tag{3.17}$$

Substitution into (3.16) yields

$$\xi=\mathbf{T}\cdot\mathbf{D}-\varrho\frac{d\Psi}{dt}-\varrho\eta\frac{d\theta}{dt}-\frac{1}{\theta}\nabla\theta\cdot\mathbf{q}\geqslant 0, \tag{3.18}$$

where ξ is called the *rate of dissipation*. When the process is isothermal we get

$$\xi=\mathbf{T}\cdot\mathbf{D}-\varrho\frac{d\Psi}{dt}-\geqslant 0.$$

3.8 Constitutive equations

The fundamental equations of continuum mechanics are

$$\begin{cases}\dfrac{d\varrho}{dt}+\varrho\nabla\cdot\mathbf{v}=0,\\[2em] \varrho\dfrac{d\mathbf{v}}{dt}=\varrho\mathbf{b}+\nabla\cdot\mathbf{T},\\[2em] T_{ij}=T_{ji},\end{cases} \tag{3.19}$$

The system (3.19) has to be solved with suitable initial and boundary conditions. We notice that the differential equations in (3.19) are not sufficient to determine the motion (or the equilibrium) of a continuum. Indeed the number of equations is smaller than the number of unknowns.

System (3.19) does not contain any information on the intrinsic *mechanical nature* of the system and holds for any type of material. To distinguish between various kind of continua we must provide supplementary equations (called *constitutive equations*) that correlate the stress state to the deformations of the system. In the next chapters we will limit ourselves to study *fluid systems.*

3.9 Balance of linear momentum in polar coordinates

Equation (3.5) can be expressed in the polar coordinates systems introduced in Sections 1.4, 1.5. In particular, in the case of cylindrical coordinates we have

$$\varrho\left(\frac{\partial v_r}{\partial t}+v_r\frac{\partial v_r}{\partial r}+\frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta}-\frac{v_\theta^2}{r}+v_z\frac{\partial v_r}{\partial z}\right)=$$
$$\left[\frac{1}{r}\frac{\partial(rT_{rr})}{\partial r}+\frac{1}{r}\frac{\partial T_{\theta r}}{\partial \theta}-\frac{T_{\theta\theta}}{r}+\frac{\partial T_{zr}}{\partial z}\right]+\varrho b_r,$$

$$\varrho\left(\frac{\partial v_\theta}{\partial t}+v_r\frac{\partial v_\theta}{\partial r}+\frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta}+\frac{v_\theta v_r}{r}+v_z\frac{\partial v_\theta}{\partial z}\right)=$$
$$\left[\frac{1}{r^2}\frac{\partial(r^2T_{r\theta})}{\partial r}+\frac{1}{r}\frac{\partial T_{\theta\theta}}{\partial \theta}+\frac{\partial T_{z\theta}}{\partial z}\right]+\varrho b_\theta,$$

$$\varrho\left(\frac{\partial v_z}{\partial t}+v_r\frac{\partial v_z}{\partial r}+\frac{v_\theta}{r}\frac{\partial v_z}{\partial \theta}+v_z\frac{\partial v_z}{\partial z}\right)=$$
$$\left[\frac{1}{r}\frac{\partial(rT_{rz})}{\partial r}+\frac{1}{r}\frac{\partial T_{\theta z}}{\partial \theta}+\frac{\partial T_{zz}}{\partial z}\right]+\varrho b_z,$$

where

$$\mathbf{v}=v_r\mathbf{e}_r+v_\theta\mathbf{e}_\theta+v_z\mathbf{e}_z \quad \mathbf{b}=b_r\mathbf{e}_r+b_\theta\mathbf{e}_\theta+b_z\mathbf{e}_z, \qquad \mathbf{T}=\begin{bmatrix} T_{rr} & T_{r\theta} & T_{rz}\\ T_{\theta r} & T_{\theta\theta} & T_{\theta z}\\ T_{zr} & T_{z\theta} & T_{zz}\end{bmatrix}.$$

In the case of spherical coordinates we have

$$\varrho\left(\frac{\partial v_r}{\partial t}+v_r\frac{\partial v_r}{\partial r}+\frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta}+\frac{v_\phi}{r\sin\theta}\frac{\partial v_r}{\partial \phi}-\frac{v_\theta^2+v_\phi^2}{r}\right)=$$
$$\left[\frac{1}{r^2}\frac{\partial(r^2T_{rr})}{\partial r}+\frac{1}{r\sin\theta}\frac{\partial(T_{\theta r}\sin\theta)}{\partial \theta}+\frac{1}{r\sin\theta}\frac{\partial T_{\phi r}}{\partial \phi}-\frac{T_{\theta\theta}+T_{\phi\phi}}{r}\right]+\varrho b_r,$$

$$\varrho\left(\frac{\partial v_\theta}{\partial t}+v_r\frac{\partial v_\theta}{\partial r}+\frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta}+\frac{v_\phi}{r\sin\theta}\frac{\partial v_\theta}{\partial \phi}+\frac{v_r v_\theta}{r}-\frac{v_\phi^2\cot\theta}{r}\right)=$$
$$\left[\frac{1}{r^3}\frac{\partial(r^3T_{r\theta})}{\partial r}+\frac{1}{r\sin\theta}\frac{\partial(T_{\theta\theta}\sin\theta)}{\partial \theta}+\frac{1}{r\sin\theta}\frac{\partial T_{\phi\theta}}{\partial \phi}+\frac{T_{\theta r}-T_{r\theta}}{r}-\frac{T_{\phi\phi}\cot\theta}{r}\right]+\varrho b_\theta,$$

$$\varrho\left(\frac{\partial v_\phi}{\partial t}+v_r\frac{\partial v_\phi}{\partial r}+\frac{v_\theta}{r}\frac{\partial v_\phi}{\partial \theta}+\frac{v_\phi}{r\sin\theta}\frac{\partial v_\phi}{\partial \phi}+\frac{v_r v_\phi}{r}+\frac{v_\phi v_\theta\cot\theta}{r}\right)=$$
$$\left[\frac{1}{r^3}\frac{\partial(r^3T_{r\phi})}{\partial r}+\frac{1}{r\sin\theta}\frac{\partial(T_{\theta\phi}\sin\theta)}{\partial \theta}+\frac{1}{r\sin\theta}\frac{\partial T_{\phi\phi}}{\partial \phi}+\frac{T_{\phi r}-T_{r\phi}}{r}+\frac{T_{\phi\theta}\cot\theta}{r}\right]+\varrho b_\phi,$$

where

$$\mathbf{v}=v_r\mathbf{e}_r+v_\theta\mathbf{e}_\theta+v_\phi\mathbf{e}_\phi, \quad \mathbf{b}=b_r\mathbf{e}_r+b_\theta\mathbf{e}_\theta+b_z\mathbf{e}_\phi, \qquad \mathbf{T}=\begin{bmatrix} T_{rr} & T_{r\theta} & T_{r\phi}\\ T_{\theta r} & T_{\theta\theta} & T_{\theta\phi}\\ T_{\phi r} & T_{\phi\theta} & T_{\phi\phi}\end{bmatrix}.$$

3.10 Frame indifference

The principle of *frame indifference* states that any quantitative description of a physical phenomenon must be invariant under a change of observer. This means that the measured quantity must be *objective* and do non depend on the particular observer that records it. Suppose that $(\mathbf{x}, t)$ and $(\mathbf{z}, t + \tau)$ are two *events* in the Euclidean space $\mathbb{R}^3$, as recorded by an observer O. Suppose that $(\mathbf{x}^*, t^*)$ and $(\mathbf{z}^*, t^* + \tau^*)$ are the same events as recorded by another observer O^*. The events are clearly related by an orthogonal time-dependent map $\mathbf{Q}(t)$ such that

$$\mathbf{x}^* - \mathbf{z}^* = \mathbf{Q}(t)(\mathbf{x} - \mathbf{z}), \tag{3.20}$$

or equivalently

$$\mathbf{x}^* = \mathbf{q}(t) + \mathbf{Q}(t)\mathbf{x}, \tag{3.21}$$

where

$$\mathbf{q}(t) = \mathbf{z}^* - \mathbf{Q}(t)\mathbf{z},$$

and where $t^* - t$ is the time shift. The one to one mapping defined by (3.21) is called an *Euclidean transformation.* Of course the displacement is a physical quantity that must be invariant under Euclidean transformation. Indeed, by (3.20)

$$|\mathbf{z}^* - \mathbf{x}^*| = |\mathbf{z} - \mathbf{x}|,$$

since $\mathbf{Q}$ is orthogonal. If we now consider velocity

$$\mathbf{v} = \frac{d}{dt}\left[\mathbf{Q}^T(\mathbf{x}^* - \mathbf{q})\right] = \dot{\mathbf{Q}}^T(\mathbf{x}^* - \mathbf{q}) + \mathbf{Q}^T(\mathbf{v}^* - \dot{\mathbf{q}}),$$

so that

$$\mathbf{v}^* = \mathbf{Q}\mathbf{v} + \Big[\dot{\mathbf{q}} + \mathbf{\Omega}(\mathbf{x}^* - \mathbf{q})\Big],$$

where

$$\mathbf{\Omega} = -\mathbf{Q}\dot{\mathbf{Q}}^T = \dot{\mathbf{Q}}\mathbf{Q}^T,$$

is a skew tensor representing the *angular velocity* or *spin* of the observer O relative to O^*. As a consequence velocity is not objective unless

$$\dot{\mathbf{q}} - \mathbf{Q}\dot{\mathbf{Q}}^T(\mathbf{x}^* - \mathbf{q}) = 0. \tag{3.22}$$

Time independent transformations

$$\mathbf{x}^* = \mathbf{q}_o + \mathbf{Q}_o\mathbf{x}, \tag{3.23}$$

satisfies (3.22). Acceleration is given by

$$\mathbf{a}^* = \dot{\mathbf{Q}}\mathbf{v} + \mathbf{Q}\mathbf{a} + \ddot{\mathbf{q}} + \dot{\mathbf{\Omega}}(\mathbf{x}^* - \mathbf{q}) + \mathbf{\Omega}(\mathbf{x}^* - \dot{\mathbf{q}}),$$

Therefore $\mathbf{a}$ is non objective as well. When the transformation is (3.23) we have $\mathbf{a}^* = \mathbf{Q}\mathbf{a}$ and acceleration becomes objective. In general we can state that a scalar a, a vector $\mathbf{u}$ and a tensor $\mathbf{S}$ are objective when they meet the following requirements

$$\begin{aligned} a^* &= a, \\ \mathbf{u}^* &= \mathbf{Q}\mathbf{u}, \\ \mathbf{S}^* &= \mathbf{Q}\mathbf{S}\mathbf{Q}^T. \end{aligned}$$

The deformation gradient $\mathbf{F}$ is not an objective tensor. Indeed from (3.21)

$$\mathbf{F}^* = \mathbf{QF}.$$

The Cauchy stress tensor is objective. Indeed the stress is given by $\mathbf{t} = \mathbf{Tn}$, where $\mathbf{t}$, $\mathbf{n}$ are objective vectors. The observer O^* sees the stress as $\mathbf{t}^* = \mathbf{T}^*\mathbf{n}^*$, whence

$$\mathbf{t}^* = \mathbf{Qt} = \mathbf{QTn} = \mathbf{T}^*\mathbf{n}^* = \mathbf{T}^*\mathbf{Qn},$$

implying

$$\mathbf{T}^* = \mathbf{QTQ}^T,$$

that is $\mathbf{T}$ is objective. From (2.16) we see that the left and right Cauchy-Green tensors

$$\mathbf{C}^* = \mathbf{F}^{*^T}\mathbf{F}^* = \mathbf{F}^T\mathbf{Q}^T\mathbf{QF} = \mathbf{C},$$

$$\mathbf{B}^* = \mathbf{F}^*\mathbf{F}^{*^T} = \mathbf{QFF}^T\mathbf{Q}^T = \mathbf{QBQ}^T,$$

are objective.

3.11 Objective time derivatives

We have seen that an objective displacement does not automatically imply an objective velocity. This fact seems to indicate that objective vectors (or objective tensors) do not conserve their objectivity when differentiating with respect to time. *Objective time derivatives* are material time derivatives that allows for objective time differentiation. Recall that

$$\mathbf{L} = \dot{\mathbf{F}}\mathbf{F}^{-1}.$$

Under the Euclidean transformation (3.21)

$$\mathbf{L}^* = \dot{\mathbf{F}}^*\mathbf{F}^{*^{-1}} = (\dot{\mathbf{Q}}\mathbf{F} + \mathbf{Q}\dot{\mathbf{F}})(\mathbf{QF})^{-1}.$$

After some algebra we find

$$\mathbf{L}^* = \mathbf{QLQ}^T + \mathbf{\Omega}.$$

Substituting $\mathbf{L} = \mathbf{D} + \mathbf{W}$ in the above and separating the symmetric and skew part we find

$$\begin{aligned} \mathbf{D}^* &= \mathbf{QDQ}^T, && \text{(objective)} \\ \mathbf{W}^* &= \mathbf{QWQ}^T + \mathbf{\Omega}, && \text{(non objective).} \end{aligned}$$

In general an objective displacement $\mathbf{u}$ is such that the material time derivative is not objective. If we define the *co-rotational time derivative* (also called *Jaumann rate*)

$$\overset{\circ}{\mathbf{u}} = \dot{\mathbf{u}} - \mathbf{Wu},$$

we find that this type of differentiation is objective. Indeed since

$$\mathbf{\Omega} = \dot{\mathbf{Q}}\mathbf{Q}^T = \mathbf{W}^* - \mathbf{QWQ}^T,$$

we find

$$\dot{\mathbf{Q}} = \mathbf{W}^*\mathbf{Q} - \mathbf{QW}. \tag{3.24}$$

Hence $\mathbf{u}^* = \mathbf{Qu}$ yields

$$\dot{\mathbf{u}}^* = \dot{\mathbf{Q}}\mathbf{u} + \mathbf{Q}\dot{\mathbf{u}} = \mathbf{W}^*\mathbf{Q}\mathbf{u} - \mathbf{Q}\mathbf{W}\mathbf{u} + \mathbf{Q}\dot{\mathbf{u}},$$

and

$$\dot{\mathbf{u}}^* = \mathbf{W}^*\mathbf{Q}\mathbf{u} + \mathbf{Q}(\dot{\mathbf{u}} - \mathbf{W}\mathbf{u}) = \mathbf{W}^*\mathbf{u}^* + \mathbf{Q}\,\overset{\circ}{\mathbf{u}}\,.$$

In conclusion

$$\overset{\circ}{\mathbf{u}}{}^* = \dot{\mathbf{u}}^* - \mathbf{W}^*\mathbf{u}^* = \mathbf{Q}\,\overset{\circ}{\mathbf{u}}\,.$$

As for vectors, we may modify rates of spatial tensors to obtain objective time derivatives. Indeed, consider an objective tensor $\mathbf{S}^* = \mathbf{QSQ}^T$ and take the material time derivative

$$\dot{\mathbf{S}}^* = \mathbf{Q}\dot{\mathbf{S}}\mathbf{Q}^T + \dot{\mathbf{Q}}\mathbf{S}\mathbf{Q}^T + \mathbf{Q}\mathbf{S}\dot{\mathbf{Q}}^T, \tag{3.25}$$

which is clearly non objective. Using (3.24) we can rearrange (3.25) to get

$$\dot{\mathbf{S}}^* + (\mathbf{S}^*\mathbf{W}^* - \mathbf{W}^*\mathbf{S}^*) = \mathbf{Q}(\mathbf{SW} - \mathbf{WS})\mathbf{Q}^T,$$

which shows that the Jaumann rate

$$\overset{\circ}{\mathbf{S}} = \dot{\mathbf{S}} - \mathbf{SW} - \mathbf{WS},$$

is objective. Other objective rates can be constructed in a similar manner. For instance the *upper convected time derivative* (also known as *Oldroyd derivative*)

$$\overset{\triangledown}{\mathbf{S}} = \dot{\mathbf{S}} - \mathbf{SL} - \mathbf{L}^T\mathbf{S}$$

is frame indifferent.

3.12 Objective functions

A scalar function φ of a tensor $\mathbf{A}$ is objective if

$$\varphi^*(\mathbf{A}) = \varphi(\mathbf{A}) = \varphi(\mathbf{A}^*).$$

A vector-valued function is objective if

$$\mathbf{v}^*(\mathbf{A}) = \mathbf{Qv}(\mathbf{A}).$$

A tensor-valued function is objective if

$$\mathbf{S}^*(\mathbf{A}) = \mathbf{QS}(\mathbf{A})\mathbf{Q}^T.$$

A function φ of the deformation gradient $\mathbf{F}$ is objective if

$$\varphi(\mathbf{F}) = \varphi(\mathbf{QF}).$$

Recalling the decomposition $\mathbf{F} = \mathbf{RU}$ and choosing $\mathbf{Q} = \mathbf{R}^T$ we find

$$\varphi(\mathbf{F}) = \varphi(\mathbf{RU}) = \varphi(\mathbf{U}).$$

Hence the scalar function φ is objective only if it is independent of the rotational part of $\mathbf{F}$, that is if it depends on the stretching part only. In an analogous fashion a tensor-valued fucntion $\mathbf{f}$ is objective if

$$\mathbf{f}^* = \mathbf{Qf}(\mathbf{F})\mathbf{Q}^T = \mathbf{f}(\mathbf{F}^*) = \mathbf{f}(\mathbf{QF}).$$

Using once again the decomposition $\mathbf{Q} = \mathbf{R}^T$ we find

$$\mathbf{f}(\mathbf{U}) = \mathbf{R}^T \mathbf{f}(\mathbf{R}\mathbf{U})\mathbf{R},$$

or equivalently

$$\mathbf{f}(\mathbf{F}) = \mathbf{R}\mathbf{f}(\mathbf{U})\mathbf{R}^T.$$

If, for instance, we consider $\mathbf{f}(\mathbf{F}) = \alpha(\mathbf{F}\mathbf{F}^T)^2 = \alpha\mathbf{B}^2$, with α scalar

$$\mathbf{f}(\mathbf{Q}\mathbf{F}) = \alpha\left[(\mathbf{Q}\mathbf{F})(\mathbf{Q}\mathbf{F})^T\right]^2 = \mathbf{Q}\alpha(\mathbf{F}\mathbf{F}^T)^2\mathbf{Q}^T = \mathbf{Q}\mathbf{f}(\mathbf{F})\mathbf{Q}^{\prime}.$$

so that objectivity is satisfied.

Chapter 4

Ideal fluids

In this chapter we study a very simple class of fluids called *ideal* (or *inviscid*) *fluids*. Classical reference books on this topic are [15], [21], [23]. Recalling (1.4), (1.5) we decompose the stress **Tn** in its *normal* and *shear* components, as shown in Fig. 4.1

$$\mathbf{Tn} = \underbrace{(\mathbf{Tn}\cdot\mathbf{n})\mathbf{n}}_{\text{Normal}} + \underbrace{[\mathbf{Tn} - (\mathbf{Tn}\cdot\mathbf{n})\mathbf{n}]}_{\text{Shear}}.$$

Equivalently we may write

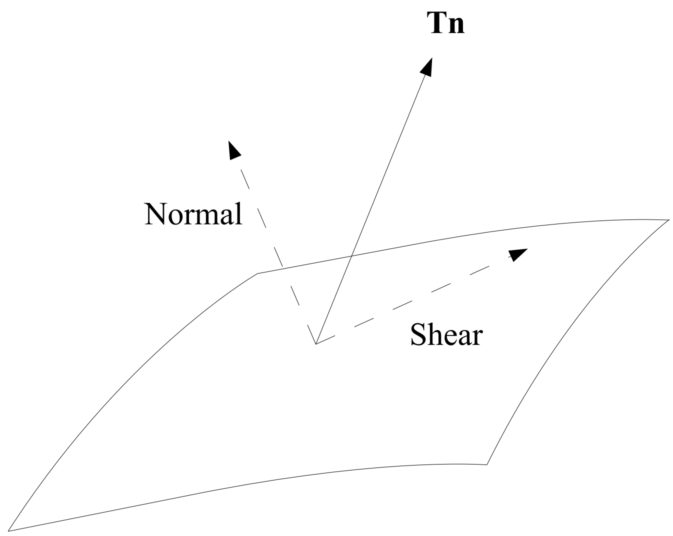


Figure 4.1: Tensor **Tn** decomposition.

$$\mathbf{Tn} = \underbrace{(\mathbf{n}\otimes\mathbf{n})\mathbf{Tn}}_{\text{Normal}} + \underbrace{[\mathbf{I} - (\mathbf{n}\otimes\mathbf{n})]\mathbf{Tn}}_{\text{Shear}}.$$

Ideal fluids are characterized by a null shear component even in dynamical conditions, so that the vector **Tn** is always oriented along the direction of **n**. As a consequence

$$\mathbf{Tn} = (\mathbf{Tn}\cdot\mathbf{n})\mathbf{n},$$

meaning that each vector **n** is an eigenvector of **T**. This implies $\mathbf{T}\mathbf{e}_i = \lambda_i \mathbf{e}_i$ and **T** is diagonal in the canonical base $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$. Moreover

$$\mathbf{T}(\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3) = \lambda(\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3) = \lambda_1\mathbf{e}_1 + \lambda_2\mathbf{e}_2 + \lambda_3\mathbf{e}_3.$$

Therefore $\lambda = \lambda_i$ and we can write

$$\mathbf{T}(\mathbf{x}, t) = -p(\mathbf{x}, t)\mathbf{I}, \tag{4.1}$$

where p is called *pressure* and it is assumed to be positive. Substituting (4.1) into (3.5) we get the *Euler's equation* for inviscid fluids

$$\varrho\frac{d\mathbf{v}}{dt} = \varrho\mathbf{b} - \nabla p. \tag{4.2}$$

Equation (4.2) must be coupled with mass balance

$$\frac{d\varrho}{dt} + \varrho \operatorname{div} \mathbf{v} = 0. \tag{4.3}$$

4.1 Barotropic fluids

A fluid is called *barotropic* if density can be expressed as a function of the pressure

$$\varrho = f(p) \tag{4.4}$$

with $f(p)$ positive and C^1. Introducing

$$P(p) = \int \frac{dp}{\varrho},$$

we get

$$\nabla P = \frac{\nabla p}{\varrho},$$

and Euler's equation becomes

$$\frac{d\mathbf{v}}{dt} = \mathbf{b} - \nabla P. \tag{4.5}$$

In case body forces are conservative $\mathbf{b} = \nabla B$ and

$$\mathbf{a} = \nabla(B - P), \tag{4.6}$$

so that curl $\mathbf{a} = 0$. Therefore barotropic fluids under the action of conservative body forces are irrotational. Recalling (2.27) and setting $\boldsymbol{\omega} = \operatorname{curl} \mathbf{v}$ we get

$$\frac{\partial\boldsymbol{\omega}}{\partial t} = \operatorname{curl}\,(\mathbf{v} \times \boldsymbol{\omega}),$$

in which neither pressure nor density appears.

4.2 Incompressible fluids

A fluid is called *incompressible* if ϱ is constant. Mass balance for an incompressible fluid yields

$$\operatorname{div} \mathbf{v} = 0.$$

A barotropic incompressible fluid is such that

$$\nabla P = \frac{\nabla p}{\varrho},$$

and Euler's equation becomes

$$\mathbf{a} = \mathbf{b} - \nabla\left(\frac{p}{\varrho}\right).$$

4.3 Boundary conditions for ideal fluids

Equations (4.2), (4.3) form a system of first order nonlinear partial differential equations for the unknowns $\varrho(\mathbf{x}, t)$, $p(\mathbf{x}, t)$, $\mathbf{v}(\mathbf{x}, t)$. Hence, to solve the problem, we must add initial and boundary conditions (in the barotropic case the relation between p and ϱ is given). The boundary conditions establish a relation between the fluid and the walls of the medium that contain the fluid.

When a portion of the boundary is *free*, conditions (2.39) and (2.40) hold. When the fluid is in touch with the walls the absence of shear stress entails that the fluid can freely move along the boundary but cannot penetrate it, so that we write

$$\mathbf{v}(\mathbf{x}, t) \cdot \mathbf{n}(\mathbf{x}, t) = \mathbf{V}(\mathbf{x}, t) \cdot \mathbf{n}(\mathbf{x}, t) \tag{4.7}$$

where $\mathbf{V}(\mathbf{x}, t)$ is the velocity of the wall and $\mathbf{n}$ its normal. When the wall is fixed we get

$$\mathbf{v}(\mathbf{x}, t) \cdot \mathbf{n}(\mathbf{x}, t) = 0. \tag{4.8}$$

4.4 Equilibrium of a barotropic fluid

From Euler's equation (4.5) the equilibrium of a barotropic fluid is given by

$$\mathbf{b} = \nabla P.$$

We have

Proposition 3 *Necessary condition for the equilibrium of a barotropic fluid is that* $\mathbf{b}$ *is conservative.*

Proof. Let $\mathbf{b} = \nabla B$. Then

$$P(p(\mathbf{x})) - B(\mathbf{x}) = C, \tag{4.9}$$

where the constant C can be determined from the knowledge of B, $P(p)$ and from the knowledge the value of the pressure at some point $\bar{\mathbf{x}}$. The equilibrium is found solving (4.9) for $p(\mathbf{x})$, which is possible because P is invertible, since $dP/dp = \varrho^{-1} > 0$. □

Example 1 (Incompressible barotropic fluid under the action of gravity) *In this case*

$$P = \frac{p}{\varrho} \qquad\qquad B = -gx_3,$$

where g is gravity and x_3 is the vertical axis. Assume that $p = p_o$ at $x_3 = 0$, so that from (4.9)

$$\frac{p}{\varrho} + gx_3 = \frac{p_o}{\varrho},$$

that is

$$p - p_o = -\varrho g x_3, \tag{4.10}$$

where ϱg is the specific weight. Equation (4.10) is also known as Stevin's law. □

Example 2 (Isothermal gas under the action of gravity) *In this case $\varrho = kp$, k being a constant. We have*

$$P = \frac{\ln p}{k} \qquad\qquad B = -gx_3,$$

Assuming again $p = p_o$ at $x_3 = 0$ we get

$$\frac{\ln p}{k} + gx_3 = \frac{\ln p_o}{k},$$

so that

$$p = p_o \, \exp(-kgx_3). \tag{4.11}$$

□

Theorem 12 (Archimedes' principle) *When an object is immersed in a fluid, there is an upward buoyant force equal to the weight of the volume of fluid displaced by the object.*

Proof. Consider an object occupying a volume Ω. Equilibrium yields

$$\nabla p = -\varrho g \mathbf{e}_3.$$

The resultant force exerted by the fluid on the object is

$$\mathbf{R} = -\int_{\partial\Omega} p\mathbf{n} \, d\sigma = -\int_{\Omega} \nabla p \, d\mathbf{x}$$

Hence

$$\mathbf{R} = \int_{\Omega} \varrho g \mathbf{e}_3 d\mathbf{x}.$$

where the r.h.s. is the weight of the fluid displaced by the object. □

4.5 Dynamics of ideal fluids

Theorem 13 (Bernoulli) *In the steady flow of an ideal barotropic fluid under the action of conservative body forces $\mathbf{b} = \nabla B$ the quantity*

$$\Gamma = \frac{1}{2}|\mathbf{v}|^2 - B + P$$

is constant at each point of the fluid.

Proof. The motion equation for a ideal barotropic fluid is

$$\mathbf{a} = \nabla(B - P). \tag{4.12}$$

Recalling property (2.11) of steady motion we find

$$\frac{d}{dt}\Big[B - P\Big] = \underbrace{\frac{\partial}{\partial t}\Big[B - P\Big]}_{=0} + \nabla(B - P) \cdot \mathbf{v}.$$

Hence, multiplying (4.12) by $\mathbf{v}$, we get

$$\frac{1}{2}\frac{d}{dt}|\mathbf{v}|^2 = \frac{d}{dt}\Big[B - P\Big],$$

that is

$$\frac{d\Gamma}{dt} = 0.$$

□

The quantity Γ can be interpreted as *total energy per unit mass.* For a barotropic fluid in a conservative force field, Euler's equation (4.6) can be rewritten as (recall (2.27))

$$\frac{\partial \mathbf{v}}{\partial t} + \frac{1}{2}\nabla(|\mathbf{v}|^2) + \text{curl}\,\mathbf{v} \times \mathbf{v} = \nabla(B - P).$$

so that

$$\frac{\partial \mathbf{v}}{\partial t} + \text{curl}\,\mathbf{v} \times \mathbf{v} = -\nabla\Gamma.$$

When the flow is irrotational there exists a function φ, called *kinetic potential*, such that

$$\nabla\varphi = \mathbf{v}. \tag{4.13}$$

As a consequence

$$\nabla\left(\Gamma + \frac{\partial\varphi}{\partial t}\right) = 0, \tag{4.14}$$

and

$$\Gamma + \frac{\partial\varphi}{\partial t} = c(t).$$

Theorem 14 *Consider the steady flow of a barotropic ideal fluid under the action of conservative body forces. The surfaces Γ = const are such that* $\mathbf{v}$ *and curl* $\mathbf{v}$ *are tangent at each point of Γ = cosnt, see Fig. 4.2. Moreover, if the flow is irrotational the specific energy Γ is spatially homogeneous.*

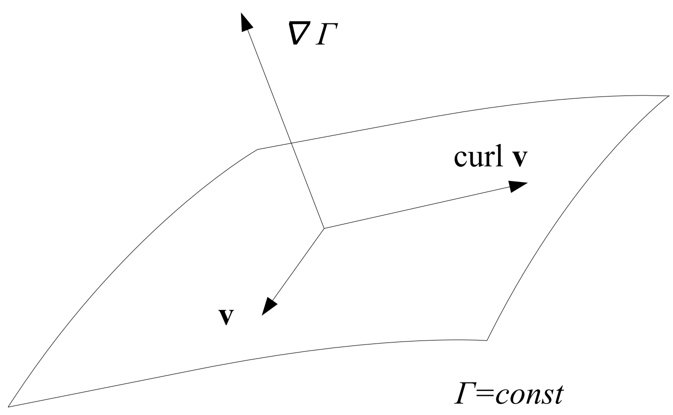


Figure 4.2: Isoenergetic surface $B = const$

Proof. From the hypothesis of steady motion

$$\text{curl}\,\mathbf{v} \times \mathbf{v} = -\nabla\Gamma.$$

Since $\nabla\Gamma$ is normal to $\Gamma = const$, the latter is tangent to both $\mathbf{v}$ and curl $\mathbf{v}$. If curl $\mathbf{v} = 0$ then Γ is spatially homogeneous. □

Example 3 *When $B = -gx_3$ and the fluid is incompressible we have*

$$\Gamma = \frac{1}{2}|\mathbf{v}|^2 + gx_3 + \frac{p}{\varrho},$$

which can be expressed also as

$$\frac{|\mathbf{v}|^2}{2g} + x_3 + \frac{p}{\gamma} = const. \tag{4.15}$$

where $\gamma = \varrho g$ is the specific weight and

$$h = x_3 + \frac{p}{\gamma}$$

is called the piezometric (or hydraulic) head.

Example 4 (Torricelli's law) *This law states that the speed of a fluid at an efflux placed at the bottom of container filled to a depth h of liquid is equal to the speed of a body falling freely from a height h, that is $\sqrt{2gh}$. This is true under the assumption of steady motion and irrotational flow. The result can be proved using Bernoulli's theorem. Indeed, since curl $\mathbf{v} = 0$, the quantity Γ is constant. On $x_3 = 0$ the constant appearing in (4.15) becomes*

$$\frac{p_o}{\gamma} = const,$$

while on $x_3 = -h$ we get

$$\frac{p_o}{\gamma} - h + \frac{v_3^2}{2g} = const.$$

Coupling the two relations we find that the velocity at the efflux is $v_3 = \sqrt{2gh}$.

Example 5 (Venturi effect) *Suppose to have an incompressible ideal fluid is flowing in a pipe with different sections A_1 and A_2. Suppose also that the velocity of the fluid is normal to the cross sections of the pipe, see Fig. 4.3. Balance of mass implies that the discharge is constant*

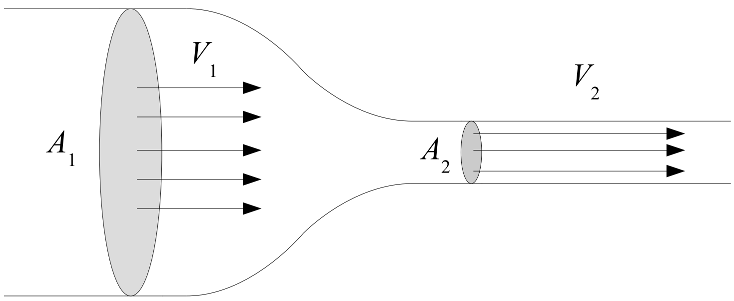


Figure 4.3: Venturi effect

through any cross section, so that

$$v_1 A_1 = v_2 A_2.$$

If we assume that the flow is irrotational and that we may neglect body forces, Bernoulli's theorem yields

$$\frac{p_1}{\varrho} + \frac{v_1^2}{2} = \frac{p_2}{\varrho} + \frac{v_2^2}{2},$$

so that

$$p_2 = p_1 + \frac{\varrho}{2}\left(v_1^2 - v_2^2\right),$$

or equivalently

$$p_2 = p_1 + \frac{\varrho}{2}\left(\frac{A_2^2 - A_1^2}{A_2^2}\right) v_1^2.$$

We observe that pressure is smaller where the tube shrinks, while velocity increases as the cross section reduces. This phenomenon is known as Venturi effect.

4.6 Vorticity

From (2.27), setting $\boldsymbol{\omega} = \operatorname{curl} \mathbf{v}$, we get

$$\operatorname{curl} \mathbf{a} = \frac{\partial \boldsymbol{\omega}}{\partial t} + \operatorname{curl}(\boldsymbol{\omega} \times \mathbf{v}).$$

Recalling relation (1.20) we find

$$\operatorname{curl} \mathbf{a} = \underbrace{\frac{\partial \boldsymbol{\omega}}{\partial t} + (\nabla \boldsymbol{\omega})\mathbf{v}}_{=\frac{d\boldsymbol{\omega}}{dt}} - (\nabla \mathbf{v})\boldsymbol{\omega} + \boldsymbol{\omega} \operatorname{div} \mathbf{v} - \underbrace{\mathbf{v} \operatorname{div} \boldsymbol{\omega}}_{=0}.$$

Therefore

$$\operatorname{curl} \mathbf{a} = \frac{d\boldsymbol{\omega}}{dt} - (\nabla \mathbf{v})\boldsymbol{\omega} + \boldsymbol{\omega} \operatorname{div} \mathbf{v}$$

From mass balance (2.49)

$$\operatorname{div} \mathbf{v} = -\frac{1}{\varrho}\frac{d\varrho}{dt},$$

and

$$\frac{\operatorname{curl} \mathbf{a}}{\varrho} = \frac{1}{\varrho}\frac{d\boldsymbol{\omega}}{dt} - (\nabla \mathbf{v})\frac{\boldsymbol{\omega}}{\varrho} - \frac{\boldsymbol{\omega}}{\varrho^2}\frac{d\varrho}{dt}. \tag{4.16}$$

Recalling that $\nabla \mathbf{v} = \mathbf{D} + \mathbf{W}$ and that $\mathbf{W}\boldsymbol{\omega} = \boldsymbol{\omega} \times \boldsymbol{\omega} = 0$ we find

$$\frac{d}{dt}\left(\frac{\boldsymbol{\omega}}{\varrho}\right) = \frac{\mathbf{D}\boldsymbol{\omega}}{\varrho} + \frac{\operatorname{curl} \mathbf{a}}{\varrho}, \tag{4.17}$$

which is called *Beltrami* or *vorticity equation.* An inviscid barotropic fluid subjected to conservatives forces is such that $\operatorname{curl} \mathbf{a} = 0$ so that (4.17) becomes

$$\frac{d}{dt}\left(\frac{\boldsymbol{\omega}}{\varrho}\right) = \frac{\mathbf{D}\boldsymbol{\omega}}{\varrho}, \tag{4.18}$$

called *Helmoltz equation.* In a *planar flow* $\mathbf{v} = (v_1, v_2, 0)$ and

$$\boldsymbol{\omega} = \operatorname{curl} \mathbf{v} = \left(\frac{\partial v_2}{\partial x_1} - \frac{\partial v_1}{\partial x_2}\right)\mathbf{e}_3.$$

Moreover

$$\mathbf{D} = \begin{bmatrix} \dfrac{\partial v_1}{\partial x_1} & \dfrac{1}{2}\left(\dfrac{\partial v_1}{\partial x_2} + \dfrac{\partial v_2}{\partial x_1}\right) & 0 \\ \dfrac{1}{2}\left(\dfrac{\partial v_1}{\partial x_2} + \dfrac{\partial v_2}{\partial x_1}\right) & \dfrac{\partial v_2}{\partial x_2} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

implying $\mathbf{D}\boldsymbol{\omega} = 0$. Therefore

$$\frac{\boldsymbol{\omega}}{\varrho} = const.$$

When the fluid is incompressible the Heltmoltz equation reduces to

$$\frac{d\boldsymbol{\omega}}{dt} = \mathbf{D}\boldsymbol{\omega}.$$

Hence, in an incompressible planar flow, the vorticity $\boldsymbol{\omega}$ is constant along pathlines.

4.7 Circulation

Suppose that ℓ is a closed smooth curve contained in P_t. The *circulation* along ℓ is (see (2.28))

$$C_\ell = \oint_\ell \mathbf{v} \cdot d\boldsymbol{\ell}. \tag{4.19}$$

Let us parametrize ℓ with

$$\mathbf{x}(t; s) : [0, 1] \to P_t$$

such that $\mathbf{x}(t; 0) = \mathbf{x}(t; 1)$. We have

$$\frac{d}{dt}\left[\oint_\ell \mathbf{v} \cdot d\boldsymbol{\ell}\right] = \frac{d}{dt}\left[\int_0^1 \mathbf{v} \cdot \frac{d\mathbf{x}}{ds} ds\right] = \int_0^1 \left[\mathbf{a} \cdot \frac{d\mathbf{x}}{ds} + \mathbf{v} \cdot \frac{d\mathbf{v}}{ds}\right] ds.$$

Hence

$$\frac{d}{dt}\left[\oint_\ell \mathbf{v} \cdot d\boldsymbol{\ell}\right] = \oint_\ell \mathbf{a} \cdot d\boldsymbol{\ell} + \oint_\ell \mathbf{v} \cdot d\mathbf{v}. \tag{4.20}$$

The last integral of (4.20) is the differential of $|\mathbf{v}|^2/2$ and hence is null on a closed curve. As a consequence we have the following

Theorem 15 (Kelvin) *In an inviscid barotropic fluid subjected to conservatives forces the circulation of* $\mathbf{v}$ *on a closed curve does not change with time.*

Proof. The proof comes from Stokes' theorem (1.27). Indeed

$$\frac{d}{dt}\left[\oint_\ell \mathbf{v} \cdot d\boldsymbol{\ell}\right] = \oint_\ell \mathbf{a} \cdot d\boldsymbol{\ell} = \int_S \operatorname{curl} \mathbf{a} \cdot \mathbf{n} d\sigma = 0,$$

where S is any smooth open surface with boundary ℓ. □

Theorem 16 (Lagrange) *An inviscid barotropic fluid subjected to conservatives forces that is initially irrotational will remain such for all times.*

Proof. Let S be a smooth open surface with boundary ℓ. By Kelvin's theorem the circulation is constant in time along ℓ. Since the motion is initially irrotational, circulation is null and by Stokes' theorem

$$0 = \oint_\ell \mathbf{v} \cdot d\boldsymbol{\ell} = \int_S \operatorname{curl} \mathbf{v} \cdot \mathbf{n}\, d\sigma.$$

From the arbitrariness of S we get the thesis □

4.8 Shallow water gravity waves

Suppose to have an inviscid fluid flowing in a rectangular channel under the action of gravity, as depicted in Fig. 4.4. Suppose that the velocity field is given by

$$\mathbf{v} = v_1(x_1, x_3, t)\mathbf{e}_1 + v_3(x_1, x_3, t)\mathbf{e}_3.$$

and pressure

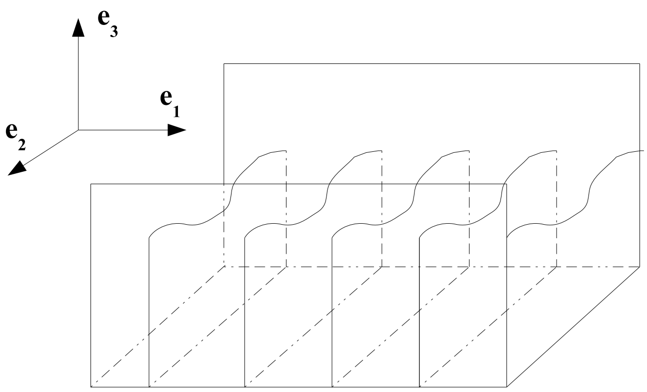


Figure 4.4: Gravity waves in a channel.

$$p = p(x_1, x_3, t).$$

Assume that acceleration in the $\mathbf{e}_3$ direction is negligible and that on the free surface $x_3 = h(x_1, t)$ pressure is null (atmospheric pressure rescaled to zero). Notice that the free surface h is a material surface. Assume that $v_3 = 0$ on the bottom of the channel $x_3 = -H$. Euler's equation (4.2) becomes

$$\begin{cases} \dfrac{dv_1}{dt} = -\dfrac{1}{\varrho}\dfrac{\partial p}{\partial x_1}, \\[2ex] 0 = -g - \dfrac{1}{\varrho}\dfrac{\partial p}{\partial x_3}. \end{cases} \tag{4.21}$$

Pressure is thus given by

$$p = \varrho g\Big[h(x_1, t) - x_3\Big],$$

and $v_1 = v_1(x_1, t)$ does not depend on x_3 since

$$\frac{dv_1}{dt} = -g\frac{\partial h}{\partial x_1},$$

or equivalently

$$\frac{\partial v_1}{\partial t} + v_1\frac{\partial v_1}{\partial x_1} = -g\frac{\partial h}{\partial x_1}. \tag{4.22}$$

Linearizing equation (4.22) (i.e. assuming that the inertial terms are negligible) we get

$$\frac{\partial v_1}{\partial t} = -g\frac{\partial h}{\partial x_1}. \tag{4.23}$$

Balance of mass yields

$$\underbrace{\frac{\partial v_1}{\partial x_1}}_{\text{depending on }(x_1,t)} + \frac{\partial v_3}{\partial x_3} = 0,$$

so that

$$v_3 = -\frac{\partial v_1}{\partial x_1}(H + x_3),$$

where we have exploited the fact that $v_3 = 0$ on $x_3 = -H$. We write the free surface profile as

$$F(x_1, x_3, t) = x_3 - h(x_1, t) = 0.$$

Recalling (2.40)

$$\left.\frac{dF}{dt}\right|_{(x_1,h,t)} = 0.$$

Hence, on $x_3 = h$

$$v_3 - \frac{\partial h}{\partial x_1}v_1 - \frac{\partial h}{\partial t} = 0.$$

so that

$$\frac{\partial h}{\partial t} + \frac{\partial}{\partial x_1}\Big[v_1(H + h)\Big] = 0.$$

If we now suppose $h \ll H$, that is the oscillation of the free surface is much smaller than the depth of the channel, the above reduces to

$$\frac{\partial h}{\partial t} + H\frac{\partial v_1}{\partial x_1} = 0. \tag{4.24}$$

Coupling (4.23) with (4.24) we get

$$\begin{cases} \dfrac{\partial v_1}{\partial t} + g\dfrac{\partial h}{\partial x_1} = 0, \\[2ex] \dfrac{\partial h}{\partial t} + H\dfrac{\partial v_1}{\partial x_1} = 0, \end{cases}$$

so that

$$\frac{\partial^2 v_1}{\partial t^2} - gH\frac{\partial^2 v_1}{\partial x_1^2} = 0,$$

or

$$\frac{\partial^2 h}{\partial t^2} - gH\frac{\partial^2 h}{\partial x_1^2} = 0.$$

In conclusion both h and v_1 satisfy the one dimensional wave equation whose solution can be determined by means of *d'Alembert formula*. Notice that the wave velocity is given by

$$c^2 = gH. \tag{4.25}$$

4.9 Deep water gravity waves

Consider, once again, a channel flow of an incompressible inviscid fluid as the one depicted in Fig. 4.4. Suppose that velocity and pressure do not depend on x_2 and assume that the flow is irrotational so that $\mathbf{v} = \nabla\varphi$ and

$$\Delta\varphi = 0, \tag{4.26}$$

because of incompressibility. From Bernoulli's theorem

$$\frac{\partial \varphi}{\partial t} + \frac{|\mathbf{v}|^2}{2} + \frac{p}{\varrho} + g x_3 = 0,$$

where we have selected an appropriate kinetic potential so $\varphi_t + \Gamma = 0$. Supposing that $|\mathbf{v}|^2$ can be neglected, the above reduce to

$$\frac{\partial \varphi}{\partial t} + \frac{p}{\varrho} + g x_3 = 0,$$

On the free surface $x_3 = h(x_1, t)$ we have (recall that atmospheric pressure is rescaled to zero)

$$\left.\frac{\partial \varphi}{\partial t}\right|_h + gh = 0. \tag{4.27}$$

The free surface equation is still

$$\frac{\partial h}{\partial t} + \frac{\partial h}{\partial x_1} v_1 - v_3 = 0.$$

Linearizing (i.e. neglecting the inertial terms) we find

$$\frac{\partial h}{\partial t} = v_3\Big|_h = \left.\frac{\partial \varphi}{\partial x_3}\right|_h. \tag{4.28}$$

Differentiating (4.27) w.r.t. time and neglecting inertial terms once again, we get

$$\left.\frac{\partial^2 \varphi}{\partial t^2}\right|_h + g\frac{\partial h}{\partial t} = 0,$$

which, recalling (4.28) becomes

$$\left.\frac{\partial^2 \varphi}{\partial t^2}\right|_h + g\left.\frac{\partial \varphi}{\partial x_3}\right|_h = 0, \tag{4.29}$$

Regarding the boundary conditions on the bottom of the channel we consider two different cases: i) the bottom is at a fixed depth $x_3 = -H$ so that

$$\left.\frac{\partial \varphi}{\partial x_3}\right|_{-H} = 0, \tag{4.30}$$

ii) the channel has infinite depth and

$$\lim_{x_3 \to -\infty} \frac{\partial \varphi}{\partial x_3} = 0. \tag{4.31}$$

We look for solutions that can be expressed in the form

$$\varphi = \xi(x_3)\zeta\Big[k(x_1 - ct)\Big].$$

Substitution into $\Delta\varphi = 0$ yields

$$\xi''\zeta + \xi\zeta'' = 0,$$

which, by separation of variables, leads to

$$\varphi = \underbrace{\Big[Ae^{kx_3} + Be^{-kx_3}\Big]}_{=\xi(x_3)} \underbrace{\cos\Big[k(x_1 - ct)\Big]}_{=\zeta\left[k(x_1-ct)\right]},$$

where $k = 2\pi/\lambda$ is the wave number and λ is the wavelength. When considering the boundary condition (4.30), we find

$$\left.\frac{\partial \varphi}{\partial x_3}\right|_{-H} = \Big[Ake^{-kH} - Bke^{kH}\Big] \cos\Big[k(x_1 - ct)\Big] = 0,$$

implying

$$\frac{Q}{2} := Ae^{-kH} = Be^{kH}.$$

Hence

$$\varphi = Q\cosh\Big[k(x_3 + H)\Big] \cos\Big[k(x_1 - ct)\Big]. \tag{4.32}$$

As a consequence

$$\frac{\partial^2 \varphi}{\partial t^2} = -Qc^2k^2 \cosh\Big[k(x_3 + H)\Big] \cos\Big[k(x_1 - ct)\Big],$$

$$\frac{\partial \varphi}{\partial x_3} = Qk\sinh\Big[k(x_3 + H)\Big] \cos\Big[k(x_1 - ct)\Big],$$

Imposing (4.29) we find

$$-c^2k\cosh\Big[k(h + H)\Big] + g\sinh\Big[k(h + H)\Big] = 0.$$

Supposing $h \ll H$ we get

$$c^2 = \frac{g}{k}\tanh\Big[kH\Big],$$

or equivalently

$$c^2 = \frac{g\lambda}{2\pi}\tanh\left[\frac{2\pi H}{\lambda}\right], \tag{4.33}$$

which proves that the wave velocity increases with the wave length. From (4.27) we can also derive the free surface profile. Indeed, recalling that $h + H \sim H$ we get

$$h = -\frac{1}{g}\left.\frac{\partial \varphi}{\partial t}\right|_h = -\frac{Qkc}{g}\cosh\Big[kH\Big] \sin\Big[k(x_1 - ct)\Big].$$

Therefore h has the sinusoidal profile

$$h = \mathcal{F}(k)\sin\Big[k(x_1 - ct)\Big],$$

where

$$\mathcal{F}(k) = -\frac{Qk}{g}\cosh\Big[kH\Big]\sqrt{\frac{g}{k}\tanh\Big[kH\Big]}.$$

When we consider the channel with infinite depth we get

$$\varphi = Ae^{kx_3}\cos\Big[k(x_1 - ct)\Big].$$

Imposing, once again, the free surface condition we find

$$c^2 = \frac{g}{k} = \frac{g\lambda}{2\pi}. \tag{4.34}$$

This result can be found also taking the limit $H \to \infty$ in (4.33). When we consider finite depth and suppose $H \ll 1$, from the relation $\tanh kH \sim kH$, we see that $c^2 = gH$ which is exactly the wave velocity in shallow waters (4.25).

4.10 Flow past obstacles

Consider the steady motion of an incompressible, irrotational, inviscid fluid in a domain in which an undeformable bounded object Ω has been placed. Assume that the domain of the fluid is $\mathbb{R}^3/\Omega$ and that

$$\lim_{|\mathbf{x}|\to\infty} \mathbf{v}(\mathbf{x}) = \mathbf{v}_\infty. \tag{4.35}$$

From (4.13) and from the incompressibility constraint div $\mathbf{v} = 0$ we get

$$\Delta\varphi = 0. \tag{4.36}$$

Condition (4.35) can be rewritten as

$$\lim_{|\mathbf{x}|\to\infty} \varphi(\mathbf{x}) - v_\infty x_1 = 0, \tag{4.37}$$

where we have chosen $\mathbf{e}_1$ parallel to $\mathbf{v}_\infty$. Assuming that the fluid cannot penetrate Ω the velocity $\mathbf{v}$ must be tangent to $\partial\Omega$ and the surfaces $\varphi = \textit{const}$ must be perpendicular to $\partial\Omega$ so that

$$\nabla\varphi \cdot \mathbf{n} = \frac{\partial\varphi}{\partial n} = 0 \qquad \text{on } \partial\Omega, \tag{4.38}$$

where $\mathbf{n}$ is the outward unit normal to $\partial\Omega$. The set of equations (4.36)-(4.38) form the elliptic problem that must be solved to describe the motion of the fluid. In general this problem is quite complex and must be solved numerically. Anyway, we can write a formula for the *drag force* exerted by the fluid on the object. This force is given by the difference between the force exerted on Ω in dynamical condition and the force exerted on Ω in static condition, namely

$$\mathbf{R} = -\int_{\partial\Omega} (p - p_o)\mathbf{n}d\sigma.$$

From (4.14) we have

$$\frac{1}{2}|\mathbf{v}|^2 - B + \frac{p}{\varrho} + \frac{\partial\varphi}{\partial t} = -B + \frac{p_o}{\varrho},$$

so that

$$-(p - p_o) = \varrho\left(\frac{\partial\varphi}{\partial t} + \frac{1}{2}|\mathbf{v}|^2\right).$$

In the stationary case $\varphi_t = 0$ and $\mathbf{R}$ reduces to

$$\mathbf{R} = \frac{1}{2}\varrho\int_{\partial\Omega} |\mathbf{v}|^2\mathbf{n}d\sigma. \tag{4.39}$$

4.11 Steady planar flow

We consider here the dynamics of *planar incompressible fluids*. Planar flows are characterized by the following property: the velocity field is identical in all planes perpendicular to a given vector, so that they can be studied in a representative plane. We assume that this plane is $x_3 = 0$. Considering steady motion the velocity field is given by

$$\mathbf{v} = v_1(x_1, x_2)\mathbf{e}_1 + v_2(x_1, x_2)\mathbf{e}_2,$$

while vorticity

$$\boldsymbol{\omega} = \text{curl}\,\mathbf{v} = \left(\frac{\partial v_2}{\partial x_1} - \frac{\partial v_1}{\partial x_2}\right)\mathbf{e}_3$$

is always normal to $x_3 = 0$. Mass balance yields

$$\frac{\partial v_1}{\partial x_1} + \frac{\partial v_2}{\partial x_2} = 0.$$

The vector field

$$\mathbf{w} = -v_2(x_1, x_2)\mathbf{e}_1 + v_1(x_1, x_2)\mathbf{e}_2,$$

is such that

$$\text{curl}\,\mathbf{w} = \left(\frac{\partial v_1}{\partial x_1} + \frac{\partial v_2}{\partial x_2}\right)\mathbf{e}_3 = \Big(\text{div}\,\mathbf{v}\Big)\mathbf{e}_3 = 0.$$

So, in a simply connected domain there exists a function ψ such that $\nabla\psi = \mathbf{w}$, i.e

$$\frac{\partial \psi}{\partial x_1} = -v_2 \qquad \frac{\partial \psi}{\partial x_2} = v_1. \tag{4.40}$$

Recall that, in irrotational flows, the kinematic potential φ is such that

$$\frac{\partial \varphi}{\partial x_1} = v_1 \qquad \frac{\partial \varphi}{\partial x_2} = v_2. \tag{4.41}$$

Equation (4.40), (4.41) are called *Cauchy-Riemann conditions* (see [17])

$$\begin{cases} \dfrac{\partial \varphi}{\partial x_1} = \dfrac{\partial \psi}{\partial x_2}, \\[2ex] \dfrac{\partial \varphi}{\partial x_2} = -\dfrac{\partial \psi}{\partial x_1}. \end{cases} \tag{4.42}$$

Notice that (4.42) imply $\nabla\varphi \cdot \nabla\psi = 0$, meaning that $\varphi = const$ and $\psi = const$ are perpendicular. Conditions (4.42) are sufficient and necessary conditions for the complex function

$$f(z) = \varphi(x_1, x_2) + \mathbf{i}\psi(x_1, x_2), \qquad z = x_1 + \mathbf{i}x_2$$

to be analytic (*holomorphic).* In practice, if (4.42) holds, there exists the limit

$$f'(z) = \lim_{\delta \to 0} \frac{f(z + \delta) - f(z)}{\delta} \qquad \delta \in \mathbb{C}.$$

When δ is real

$$f'(z) = \frac{\partial \varphi}{\partial x_1} + \mathbf{i}\frac{\partial \psi}{\partial x_1} = v_1 - \mathbf{i}v_2.$$

The function $f(z)$ is called the *complex potential* while

$$V(z) = f'(z) = v_1 - \mathbf{i}v_2,$$

is called *complex velocity.* We have

$$\mathsf{Re}\Big[zV\Big] = \mathsf{Re}\Big[(x_1 + \mathbf{i}x_2)(v_1 - \mathbf{i}v_2)\Big] = \mathbf{v} \cdot \mathbf{x},$$

$$\mathsf{Im}\Big[zV\Big] = \mathsf{Im}\Big[(x_1 + \mathbf{i}x_2)(v_1 - \mathbf{i}v_2)\Big] = \mathbf{w} \cdot \mathbf{x},$$

4.12 Planar flow around obstacles

Consider a planar steady motion past a bounded object Ω with smooth boundary $\partial\Omega$. Suppose that velocity satisfies (4.37). From (4.42)

$$\Delta\psi = 0 \qquad \text{in} \quad \mathbb{R}^2/\Omega, \tag{4.43}$$

with

$$\lim_{|\mathbf{x}|\to\infty} \psi(x_1, x_2) - v_\infty x_2 = 0. \tag{4.44}$$

Since velocity is tangent to $\psi = \mathit{const}$, we conclude that ψ must be constant on the object profile. Therefore we can select a potential such that

$$\psi\Big|_{\partial\Omega} = 0, \tag{4.45}$$

which is a *Dirichlet* boundary condition. The problem for ψ is as complex as the one for φ but when Ω is a circle of radius R, we can easily determine the complex potential. Indeed if Ω is a circle then $\mathbf{v} \cdot \mathbf{x} = 0$ on the circle and hence

$$\mathsf{Re}\Big[zV\Big] = 0,$$

on $|z| = R$. Moreover

$$\lim_{|z|\to\infty} f'(z) = v_\infty.$$

If we take

$$f(z) = v_\infty \left[z + \frac{R^2}{z}\right],$$

we get

$$V(z) = f'(z) = v_\infty \left[1 - \left(\frac{R}{z}\right)^2\right]. \tag{4.46}$$

The function $f(z)$ is called the *Jukowski potential*. We have

$$\varphi(x_1, x_2) = v_\infty x_1 \left(1 + \frac{R^2}{x_1^2 + x_2^2}\right), \tag{4.47}$$

$$\psi(x_1, x_2) = v_\infty x_2 \left(1 - \frac{R^2}{x_1^2 + x_2^2}\right). \tag{4.48}$$

It is easy to verify that (4.48) satisfies (4.43)-(4.45).

4.13 Drag force and D'Alembert paradox

Recall the definition of *drag force* $\mathbf{R}$ for a steady flow given in (4.39) and assume that $\mathbf{v}$ is a planar steady motion around the obstacle Ω. Following Fig. 4.5 we can write

$$\mathbf{n} = (\cos\phi, \sin\phi), \qquad \mathbf{t} = (-\sin\phi, \cos\phi),$$

and

$$d\ell\cos\phi = dx_2, \qquad d\ell\sin\phi = -dx_1.$$

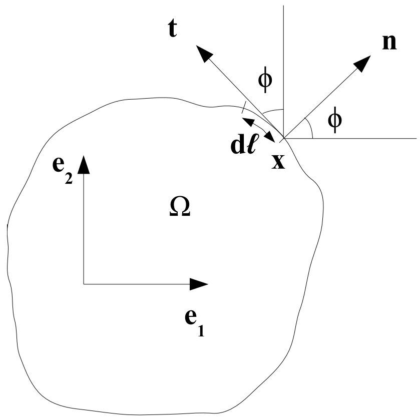


Figure 4.5: Drag force on Ω

The drag force (4.39) can be rewritten as

$$\mathbf{R} = \Big(\underbrace{\frac{1}{2}\varrho \oint_{\partial\Omega} |\mathbf{v}|^2 dx_2}_{R_1}, \underbrace{-\frac{1}{2}\varrho \oint_{\partial\Omega} |\mathbf{v}|^2 dx_1}_{R_2} \Big).$$

We introduce

$$\mathcal{R} = R_2 + \mathbf{i}R_1 = -\frac{1}{2}\varrho \oint_{\partial\Omega} |\mathbf{v}|^2 (dx_1 - \mathbf{i}dx_2).$$

Recalling that the conjugate $\overline{z}$ of a complex number $z = x_1 + \mathbf{i}x_2$ is $\overline{z} = x_1 - \mathbf{i}x_2$ we get

$$|\mathbf{v}|^2 = (v_1 + \mathbf{i}v_2)(v_1 - \mathbf{i}v_2) = \overline{V(z)}V(z),$$

and

$$d\overline{z} = dx_1 - \mathbf{i}dx_2.$$

Hence we write

$$\mathcal{R} = -\frac{1}{2}\varrho \oint_{\partial\Omega} V(z)\overline{V(z)} d\overline{z}.$$

Now

$$\overline{V(z)} d\overline{z} = (v_1 dx_1 + v_2 dx_2) - \mathbf{i} \underbrace{(v_1 dx_2 - v_2 dx_1)}_{= d\mathbf{x}\cdot\mathbf{w} = 0}$$

meaning that $\overline{V(z)}d\overline{z}$ is a real number and $\overline{V(z)}d\overline{z} = V(z)dz$. In conclusion

$$\mathcal{R} = -\frac{1}{2}\varrho \oint_{\partial\Omega} [V(z)]^2 dz, \tag{4.49}$$

called *Blausius formula.* In case the object Ω is a circle, from (4.46) we get

$$V^2 = v_\infty^2 \left[1 + \left(\frac{R}{z}\right)^4 - 2\left(\frac{R}{z}\right)^2 \right].$$

Recalling that for each $n \neq -1$ (see [17])

$$\oint_{|z|=R} z^n dz = 0,$$

we find that $\mathcal{R} = 0$. This result can be extended to the case of an object with a generic profile exploiting Cauchy's integral theorem for holomorphic functions (see [17]), which states that the integral

$$\oint_\gamma g(z)dz$$

is invariant for any closed curve γ defined in the domain in which the function is holomorphic. We have thus proved the following

Theorem 17 (D'Alembert paradox) *The drag force exerted on an object by an inviscid, incompressible fluid in a steady irrotational flow is null.*

It is important to notice that Theorem 17 can be extended to the three dimensional case (see [7]), where not only can there be no drag, there can be no lift either.

4.14 Magnus effect

Following the results of the previous section we want to investigate whether the theory of inviscid fluids can actually predict some kind of dynamical effect on solid objects. Looking at Blausius formula (4.49), we observe that only a z^{-1} term in the expansion of V^2 can lead to a non null dynamical action exerted on the object. If , for instance, the complex velocity $V(z)$ is derived from a logarithmic potential

$$f(z) = \frac{C}{2\pi\mathbf{i}} \ln\left(\frac{z}{R}\right),$$

then

$$V(z) = \frac{C}{2\pi\mathbf{i}z},$$

implying

$$v_1 = -\frac{Cx_2}{2\pi(x_1^2 + x_2^2)}, \qquad v_2 = \frac{Cx_1}{2\pi(x_1^2 + x_2^2)}. \tag{4.50}$$

In this case the streamlines are concentric circles and velocity goes to zero as $|\mathbf{x}| \to \infty$. The constant C is therefore the circulation (2.28) on each streamline. Indeed

$$\frac{1}{2\pi\mathbf{i}} \oint_{|z|=const} \frac{dz}{z} = 1.$$

If we consider the complex velocity

$$V = v_\infty \left[1 - \left(\frac{R}{z}\right)^2\right] + \frac{C}{2\pi\mathbf{i}z},$$

we get

$$\mathcal{R} = -\frac{1}{2}\varrho \oint_{\partial\Omega} \left[v_\infty \left[1 - \left(\frac{R}{z}\right)^2\right] + \frac{C}{2\pi\mathbf{i}z}\right]^2 dz.$$

Recalling that the only term with non null contribution to the integral is z^{-1}, we find

$$\mathcal{R} = -\frac{1}{2}\varrho \oint_{\partial\Omega} \frac{Cv_\infty}{\pi\mathbf{i}z} dz,$$

which leads to

$$\mathcal{R} = -\varrho C v_\infty,$$

or equivalently

$$\mathbf{R} = -\varrho C v_\infty \mathbf{e}_2. \tag{4.51}$$

The dynamical effect (4.51), which is transversal to the asymptotic velocity, is called *Magnus effect*. The velocity (4.50) can be generated by the presence of friction between the fluid and the obstacle when the latter is rotating with angular velocity $\omega = C(2\pi R)^{-1}$.

4.15 Subsonic and supersonic flow

Consider the flow of an inviscid barotropic fluid in which $\varrho = \varrho(p)$ is a strictly increasing function (incompressible fluids are therefore excluded). The function ϱ is invertible and we can define

$$c^2(\varrho) = \frac{dp(\varrho)}{d\varrho} > 0,$$

where $c(\varrho)$ is the *speed of sound*. Differentiating $p(\varrho)$ w.r.t the spatial coordinates we get

$$\nabla p(\varrho) = c^2(\varrho)\nabla\varrho.$$

Euler's equation (4.2) becomes

$$\frac{d\mathbf{v}}{dt} = \mathbf{b} - \frac{c^2(\varrho)}{\varrho}\nabla\varrho, \tag{4.52}$$

while mass balance is expressed by

$$\frac{d\varrho}{dt} + \varrho \operatorname{div} \mathbf{v} = 0. \tag{4.53}$$

We define the *Mach number*

$$M(\mathbf{x}, t) = \frac{|\mathbf{v}(\mathbf{x}, t)|}{c\Big(\varrho(\mathbf{x}, t)\Big)},$$

which characterize *subsonic motion* ($M < 1$), *sonic motions* ($M = 1$) and *supersonic motions* ($M > 1$). We prove the following

Proposition 4 *A steady solution of (4.52), (4.53) with* $\mathbf{b} = 0$ *is such that*

$$\frac{d}{dt}\Big(\varrho|\mathbf{v}|\Big) = \varrho\Big(1 - M^2\Big)\frac{d|\mathbf{v}|}{dt} \tag{4.54}$$

Proof. From the the hypothesis of steady motion we get

$$\frac{d}{dt}\Big(\varrho|\mathbf{v}|\Big) = |\mathbf{v}|\Big(\nabla\varrho \cdot \mathbf{v}\Big) + \varrho\Big(\nabla|\mathbf{v}| \cdot \mathbf{v}\Big), \tag{4.55}$$

and

$$\frac{d|\mathbf{v}|}{dt} = \nabla|\mathbf{v}| \cdot \mathbf{v}. \tag{4.56}$$

Observing that

$$\nabla|\mathbf{v}| = (\nabla\mathbf{v})^T \frac{\mathbf{v}}{|\mathbf{v}|},$$

we find

$$\nabla|\mathbf{v}|\cdot\mathbf{v} = \underbrace{\Big[(\nabla\mathbf{v})\mathbf{v}\Big]}_{=\frac{d\mathbf{v}}{dt}}\cdot\frac{\mathbf{v}}{|\mathbf{v}|} = \frac{d\mathbf{v}}{dt}\cdot\frac{\mathbf{v}}{|\mathbf{v}|} = -\frac{c^2(\varrho)\nabla\varrho}{\varrho}\cdot\frac{\mathbf{v}}{|\mathbf{v}|}.$$

Therefore

$$\nabla\varrho\cdot\mathbf{v} = -\frac{\varrho|\mathbf{v}|}{c^2(\varrho)}\Big[\nabla|\mathbf{v}|\cdot\mathbf{v}\Big]. \tag{4.57}$$

Substituting (4.57) into (4.55) we get

$$\frac{d}{dt}\Big(\varrho|\mathbf{v}|\Big) = \left(-\frac{\varrho|\mathbf{v}|^2}{c^2(\varrho)}+\varrho\right)\Big[\nabla|\mathbf{v}|\cdot\mathbf{v}\Big] = \varrho\Big(1-M^2\Big)\frac{d|\mathbf{v}|}{dt}.$$

□

The proposition above shows a very interesting feature of supersonic/subsonic motions. Consider a streamline s passing trough $\mathbf{x}$ with $\mathbf{v}(\mathbf{x}) \neq 0$ and consider the surface Σ normal to s in $\mathbf{x}$. The unit normal $\mathbf{n}$ to S in $\mathbf{x}$ is

$$\mathbf{n} = \frac{\mathbf{v}}{|\mathbf{v}|},$$

so that

$$\varrho(\mathbf{x})\mathbf{v}(\mathbf{x})\cdot\mathbf{n} = \varrho(\mathbf{x})|\mathbf{v}(\mathbf{x})|.$$

Hence $\varrho|\mathbf{v}|$ represents the flux of linear momentum through the surface Σ. Formula (4.54) shows that such a flux increases or decreases depending on $|\mathbf{v}|$ and M. In the subsonic case ($M < 1$) the flux increases as $|\mathbf{v}|$ increases. In the supersonic case ($M > 1$) the flux decreases as $|\mathbf{v}|$ increases. Therefore in the supersonic regime the decrease of the density is faster than the increase of velocity.

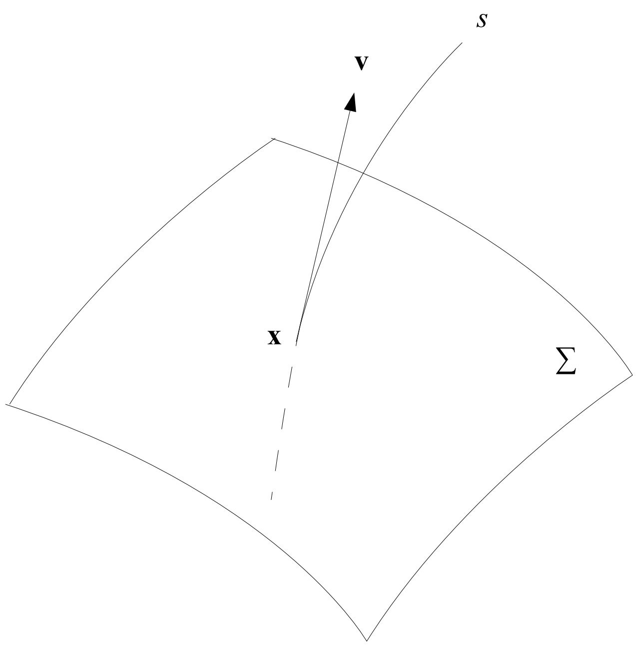


Figure 4.6: Surface Σ.

4.16 Critical velocity in ideal gases

For an ideal gas, an isoentropic flow obeys the equation

$$p = k\varrho^{\gamma},$$

where k, γ are positive constants with $\gamma > 1$. In this case

$$c^2 = \frac{dp}{d\varrho} = k\gamma\varrho^{\gamma-1},$$

while

$$P(p) = \int \frac{dp}{\varrho} = \int k\gamma\varrho^{\gamma-2}d\varrho = \frac{k\gamma\varrho^{\gamma-1}}{\gamma-1} = \frac{c^2}{\gamma-1}.$$

Assuming steady motion and $\mathbf{b} = 0$, from Bernoulli's theorem, we get

$$P + \frac{|\mathbf{v}|^2}{2} = \frac{c^2}{\gamma-1} + \frac{|\mathbf{v}|^2}{2} = const$$

on streamlines. Therefore we can write

$$\frac{2c^2}{\gamma-1} + |\mathbf{v}|^2 = V^2,$$

where V^2 is a constant related to the specific streamline considered. We conclude that

$$|\mathbf{v}| < c \qquad \Longleftrightarrow \qquad |\mathbf{v}| < \sqrt{\frac{\gamma-1}{\gamma+1}}V,$$

$$|\mathbf{v}| > c \qquad \Longleftrightarrow \qquad |\mathbf{v}| > \sqrt{\frac{\gamma-1}{\gamma+1}}V,$$

The quantity

$$c_{crit} = \sqrt{\frac{\gamma-1}{\gamma+1}}V,$$

is called *critical velocity* and it is a constant that depends on the streamline. To determine whether a motion is subsonic or supersonic it is sufficient to compare the modulus $|\mathbf{v}|$ with c_{crit}. The advantage of comparing $|\mathbf{v}|$ with c_{crit} intead of c lies in the fact that c_{crit} does not depend on the position. In particular, if at some point of a fixed streamline the velocity $|\mathbf{v}|$ equals the critical velocity, then the motion is sonic at that specific point.

4.17 Shear flow of an inviscid fluid

Consider an incompressible inviscid fluid flowing between parallel planes $x_2 = 0$ and $x_2 = h$, as shown in Fig. 4.7. Suppose velocity and pressure are of the form

$$\mathbf{v} = v_1(x_1, x_2, t)\mathbf{e}_1,$$

$$p = p(x_1, x_2, t).$$

Suppose that the boundary conditions for the pressure are $p_{in} = p(0, x_2, t)$, $p_{out} = p(L, x_2, t)$ and define

$$\Delta P = p_{out} - p_{in}.$$

Mass balance implies

Figure 4.7: Shear flow of an inviscid fluid.

$$\frac{\partial v_1}{\partial x_1} = 0,$$

so that $v_1 = v_1(x_2, t)$. Euler's equation is

$$\begin{cases} \dfrac{\partial v_1}{\partial t} = -\dfrac{1}{\varrho}\dfrac{\partial p}{\partial x_1}, \\ \\ \dfrac{\partial p}{\partial x_2} = 0. \end{cases} \tag{4.58}$$

and so $p = p(x_1, t)$. Differentiating $(4.58)_1$ w.r.t. x_1 we find

$$\frac{\partial^2 p}{\partial x_1^2} = 0,$$

so that, recalling the boundary conditions for p, we get

$$p(x_1) = \frac{\Delta p}{L} x_1 + p_{in}.$$

As a consequence

$$v_1 = -\left(\frac{\Delta p}{\varrho L}\right) t + v_{o1}(x_2),$$

where $v_{o1}(x_2)$ is the initial profile for v_1. We notice that velocity is an unbounded function of time, which is physically inconsistent. This is another paradox that originates from the assumption of inviscid flow. As we shall see later on, when we take into account *viscosity*, the paradox can be overcome.

4.18 Energy

When considering a compressible fluid the thermodynamical variables η (entropy), θ temperature and ϱ (density) satisfy the following relation

$$\theta \frac{d\eta}{dt} = \frac{d\varepsilon}{dt} + p\frac{d}{dt}\left(\frac{1}{\varrho}\right),$$

or equivalently

$$\theta \frac{d\eta}{dt} = \frac{d\varepsilon}{dt} - \frac{p}{\varrho^2}\frac{d\varrho}{dt} = \frac{d\varepsilon}{dt} + \frac{p}{\varrho} \operatorname{div} \mathbf{v}.$$

Hence, from (3.12)

$$\theta \frac{d\eta}{dt} = r - \frac{1}{\varrho} \operatorname{div} \mathbf{q} + \frac{1}{\varrho} \mathbf{T} \cdot \mathbf{D} + \frac{p}{\varrho} \operatorname{div} \mathbf{v},$$

so that

$$\varrho\theta \frac{d\eta}{dt} = \varrho r - \operatorname{div} \mathbf{q} + (\mathbf{T} + p\mathbf{I}) \cdot \mathbf{D}.$$

As a consequence, from (3.15)

$$\Phi = (\mathbf{T} + p\mathbf{I}) \cdot \mathbf{D} \geqslant \frac{1}{\theta} \nabla\theta \cdot \operatorname{div} \mathbf{v},$$

where the function Φ is called the *dissipation function.* In the case of an isothermal system

$$\Phi = (\mathbf{T} + p\mathbf{I}) \cdot \mathbf{D} \geqslant 0. \tag{4.59}$$

For an ideal fluid $\Phi = 0$ and the entropy production is not due to deformation.

Chapter 5

Newtonian fluids

Even though the assumption of inviscid fluid (4.1), which led to Euler's equation, can be useful in many practical applications, it presents limitations due to the hypothesis of null shear components of the stress. To overcome these limitations - that produce inconsistencies such as the D'Alembert paradox - we need to consider constitutive equations in which shear effects in dynamical conditions are taken into account. Suggested readings on the topics presented here are [3], [16], [18], [22].

5.1 Stokesian fluids

Suppose to modify the constitutive equation (4.1) with

$$\mathbf{T} = -p\mathbf{I} + \mathbf{V}, \tag{5.1}$$

where $\mathbf{V}(\mathbf{D})$ is an isotropic function (see Section 1.6) such that $\mathbf{V}(0) = 0$. Recalling Theorem 6 the function $\mathbf{V}(\mathbf{D})$ can be rewritten in the following form

$$\mathbf{V} = \alpha\mathbf{I} + \beta\mathbf{D} + \gamma\mathbf{D}^2 \tag{5.2}$$

where α, β, γ are functions of the principal invariants of $\mathbf{D}$ only. Fluids with a stress tensor of the form (5.1) are called *Stokesian*. Stokesian fluids present null shear components in static conditions.

5.2 Newtonian fluids

We consider here a particular class of Stokesian fluids where

$$\mathbf{T} = \Big(-p + \lambda \operatorname{div} \mathbf{v} \Big)\mathbf{I} + 2\mu\mathbf{D}. \tag{5.3}$$

Relation (5.3) is called the *Cauchy-Poisson* constitutive equation. The coefficients λ, μ, p do not depend on the kinematical variables but may depend on the thermodynamical variables. The coefficients λ, μ are called the *viscosity coefficients*. Fluids with constitutive equation (5.3) are called *Newtonian fluids*. When the fluid is incompressible, equation (5.3) becomes

$$\mathbf{T} = -p\mathbf{I} + 2\mu\mathbf{D}.$$

Recalling the definition of dissipation function in isothermal condition (4.59), we have

$$\Phi = \Big(\mathbf{T} + p\mathbf{I}\Big) \cdot \mathbf{D} \geqslant 0,$$

so that

$$\Phi = \lambda\,(\operatorname{div}\mathbf{v})^2 + 2\mu\mathbf{D}\cdot\mathbf{D} \geqslant 0.$$

Recalling that $\operatorname{tr}\mathbf{D} = \operatorname{div}\mathbf{v}$ and $\mathbf{D}\cdot\mathbf{D} = \operatorname{tr}(\mathbf{D}^2)$ are invariants of the tensor $\mathbf{D}$ we find

$$\operatorname{tr}\mathbf{D} = d_1 + d_2 + d_3,$$

$$\mathbf{D}\cdot\mathbf{D} = d_1^2 + d_2^2 + d_3^2,$$

where $d_i \in \mathbb{R}$ are the eigenvalues of $\mathbf{D}$. We can easily check that [1]

$$3\Phi = (3\lambda + 2\mu)(d_1 + d_2 + d_3)^2 + 2\mu\Big[(d_1 - d_2)^2 + (d_2 - d_3)^2 + (d_3 - d_1)^2\Big].$$

Since $\mathbf{D}$ is a generic symmetric tensor we may select $d_1 = d_2 = d_3$ so that

$$3\lambda + 2\mu \geqslant 0,$$

or

$$d_1 + d_2 = -d_3,$$

so that

$$\mu \geqslant 0.$$

In particular, when the fluid is incompressible the viscosity coefficient μ must be positive.

5.3 Navier-Stokes equations

If we substitute (5.3) into (3.5) we find

$$\text{(5.4)}\qquad \begin{cases} \varrho\dfrac{d\mathbf{v}}{dt} = \varrho\mathbf{b} - \nabla p + \nabla(\lambda\operatorname{div}\mathbf{v}) + 2\operatorname{div}(\mu\mathbf{D}), \\[2ex] \dfrac{d\varrho}{dt} + \varrho\operatorname{div}\mathbf{v} = 0, \end{cases}$$

to which one may add thermodynamical relations or, in case the fluid is barotropic, equation (4.4). If the fluid is incompressible (5.4) reduces to

$$\text{(5.5)}\qquad \begin{cases} \varrho\dfrac{d\mathbf{v}}{dt} = \varrho\mathbf{b} - \nabla p + 2\operatorname{div}(\mu\mathbf{D}), \\[2ex] \operatorname{div}\mathbf{v} = 0. \end{cases}$$

Recall that

$$\operatorname{div}\mathbf{D} = \frac{1}{2}\operatorname{div}\Big[\nabla\mathbf{v} + \nabla\mathbf{v}^T\Big] = \frac{1}{2}\Big[\Delta\mathbf{v} + \nabla(\operatorname{div}\mathbf{v})\Big].$$

Therefore, when λ and μ are constant (5.4) becomes

$$\text{(5.6)}\qquad \begin{cases} \varrho\dfrac{d\mathbf{v}}{dt} = \varrho\mathbf{b} - \nabla p + (\lambda + \mu)\nabla(\operatorname{div}\mathbf{v}) + \mu\Delta\mathbf{v}. \\[2ex] \dfrac{d\varrho}{dt} + \varrho\operatorname{div}\mathbf{v} = 0. \end{cases}$$

[1] Recall the identity $3(d_1^2 + d_2^2 + d_3^2) = (d_1 + d_2 + d_3)^2 + (d_1 - d_2)^2 + (d_1 - d_3)^2 + (d_2 - d_3)^2$.

Equation (5.6) is the famous *Navier-Stokes* system for compressible Newtonian fluids. When the fluid is incompressible (5.6) reduces to

$$\begin{cases} \dfrac{d\mathbf{v}}{dt} = \mathbf{b} - \dfrac{\nabla p}{\varrho} + \nu \Delta \mathbf{v}. \\ \\ \text{div}\, \mathbf{v} = 0, \end{cases} \tag{5.7}$$

where

$$\nu = \frac{\mu}{\varrho}$$

is the so-called *kinematic viscosity.*

5.4 Navier Stokes equation in cylindrical coordinates

Recalling the results of Section 1.4 we can write the Navier-Stokes equation (5.7) in cylindrical coordinates. The stress tensor of a viscous incompressible Newtonian fluid expressed in cylindrical coordinates is given by the following

$$\mathbf{T} = \begin{bmatrix} T_{rr} & T_{r\theta} & T_{rz} \\ \\ T_{r\theta} & T_{\theta\theta} & T_{\theta z} \\ \\ T_{rz} & T_{\theta z} & T_{zz} \end{bmatrix}.$$

where

$$\begin{aligned}
T_{rr} &= -p + 2\mu \frac{\partial v_r}{\partial r} \\
T_{r\theta} &= \mu \left(\frac{1}{r} \frac{\partial v_r}{\partial \theta} + \frac{\partial v_\theta}{\partial r} - \frac{v_\theta}{r} \right) \\
T_{rz} &= \mu \left(\frac{\partial v_z}{\partial r} + \frac{\partial v_r}{\partial z} \right) \\
T_{\theta\theta} &= -p + 2\mu \left(\frac{1}{r} \frac{\partial v_\theta}{\partial \theta} + \frac{v_r}{r} \right) \\
T_{\theta z} &= \mu \left(\frac{\partial v_\theta}{\partial z} + \frac{1}{r} \frac{\partial v_z}{\partial \theta} \right) \\
T_{zz} &= -p + 2\mu \frac{\partial v_z}{\partial z}.
\end{aligned}$$

Therefore the motion equation becomes

$$\varrho \left[\frac{\partial v_r}{\partial t} + v_r \frac{\partial v_r}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_r}{\partial \theta} - \frac{v_\theta^2}{r} + v_z \frac{\partial v_r}{\partial z} \right] =$$

$$= b_r - \frac{\partial p}{\partial r} + \mu \left[\frac{\partial}{\partial r} \left(\frac{1}{r} \frac{\partial (r v_r)}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 v_r}{\partial \theta^2} - \frac{2}{r^2} \frac{\partial v_\theta}{\partial \theta} + \frac{\partial^2 v_r}{\partial z^2} \right], \tag{5.8}$$

$$\varrho \left[\frac{\partial v_\theta}{\partial t} + v_r \frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_\theta}{\partial \theta} + \frac{v_r v_\theta}{r} + v_z \frac{\partial v_\theta}{\partial z} \right] =$$

$$= b_\theta - \frac{1}{r}\frac{\partial p}{\partial \theta} + \mu\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial (r v_\theta)}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 v_\theta}{\partial \theta^2} + \frac{2}{r^2}\frac{\partial v_r}{\partial \theta} + \frac{\partial^2 v_\theta}{\partial z^2}\right], \tag{5.9}$$

$$\varrho\left[\frac{\partial v_z}{\partial t} + v_r\frac{\partial v_z}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_z}{\partial \theta} + v_z\frac{\partial v_z}{\partial z}\right] =$$

$$= b_z - \frac{\partial p}{\partial z} + \mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 v_z}{\partial \theta^2} + \frac{\partial^2 v_z}{\partial z^2}\right], \tag{5.10}$$

where $\mathbf{b} = (b_r, b_\theta, b_z)$ is the body force vector in cylindrical coordinates. We recall that mass balance for incompressible fluids in cylindrical coordinates is given by (2.51).

5.5 Navier Stokes equation in spherical coordinates

The stress tensor of a viscous incompressible Newtonian fluid expressed in spherical polar coordinates is given by the following

$$\mathbf{T} = \begin{bmatrix} T_{rr} & T_{r\theta} & T_{r\phi} \\ T_{r\theta} & T_{\theta\theta} & T_{\theta\phi} \\ T_{r\phi} & T_{\theta\phi} & T_{\phi\phi} \end{bmatrix}. \tag{5.11}$$

where

$$\begin{aligned}
T_{rr} &= \left(-p + 2\mu\frac{\partial v_r}{\partial r}\right) \\
T_{r\theta} &= \mu\left(\frac{1}{r}\frac{\partial v_r}{\partial \theta} + \frac{\partial v_\theta}{\partial r} - \frac{v_\theta}{r}\right) \\
T_{r\phi} &= \mu\left(\frac{\partial v_\phi}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial v_r}{\partial \phi} - \frac{v_\phi}{r}\right) \\
T_{\theta\theta} &= -p + 2\mu\left(\frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{v_r}{r}\right) \\
T_{\theta\phi} &= \mu\left(\frac{1}{r\sin\theta}\frac{\partial v_\theta}{\partial \phi} + \frac{1}{r}\frac{\partial v_\phi}{\partial \theta} - \frac{v_\phi}{r}\frac{\cos\theta}{\sin\theta}\right) \\
T_{\phi\phi} &= -p + 2\mu\left(\frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial \phi} + \frac{v_r}{r} + \frac{v_\theta}{r}\frac{\cos\theta}{\sin\theta}\right)
\end{aligned}$$

Therefore the Navier Stokes equation in spherical coordinates is given by

$$\varrho\left(\frac{\partial v_r}{\partial t} + v_r\frac{\partial v_r}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta} + \frac{v_\phi}{r\sin\theta}\frac{\partial v_r}{\partial \phi} - \frac{v_\theta^2 + v_\phi^2}{r}\right) = -\frac{\partial p}{\partial r} + \mu\left[\frac{\partial}{\partial r}\left(\frac{1}{r^2}\frac{\partial (r^2 v_r)}{\partial r}\right) +\right.$$

$$\left. + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial v_r}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 v_r}{\partial \phi^2} - \frac{2}{r^2\sin\theta}\frac{\partial (v_\theta\sin\theta)}{\partial \theta} - \frac{2}{r^2\sin\theta}\frac{\partial v_\phi}{\partial \phi}\right] + \varrho b_r,$$

$$\varrho\left(\frac{\partial v_\theta}{\partial t} + v_r\frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{v_\phi}{r\sin\theta}\frac{\partial v_\theta}{\partial \phi} + \frac{v_r v_\theta}{r} - \frac{v_\phi^2\cot\theta}{r}\right) = -\frac{1}{r}\frac{\partial p}{\partial \theta} + \mu\left[\frac{\partial}{\partial r}\left(\frac{1}{r^2}\frac{\partial (r^2 v_\theta)}{\partial r}\right) +\right.$$

$$+\frac{1}{r^2}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial(v_\theta\sin\theta)}{\partial\theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2 v_\theta}{\partial\phi^2}+\frac{2}{r^2}\frac{\partial v_r}{\partial\theta}-\frac{2\cot\theta}{r^2\sin\theta}\frac{\partial v_\phi}{\partial\phi}\Bigg]+\varrho b_\theta,$$

$$\varrho\left(\frac{\partial v_\phi}{\partial t}+v_r\frac{\partial v_\phi}{\partial r}+\frac{v_\theta}{r}\frac{\partial v_\phi}{\partial\theta}+\frac{v_\phi}{r\sin\theta}\frac{\partial v_\phi}{\partial\phi}+\frac{v_r v_\phi}{r}+\frac{v_\phi v_\theta\cot\theta}{r}\right)=-\frac{1}{r\sin\theta}\frac{\partial p}{\partial\phi}+$$

$$\mu\left[\frac{\partial}{\partial r}\left(\frac{1}{r^2}\frac{\partial(r^2 v_\phi)}{\partial r}\right)++\frac{1}{r^2}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial(v_\phi\sin\theta)}{\partial\theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2 v_\phi}{\partial\phi^2}+\right.$$

$$\left.+\frac{2}{r^2\sin\theta}\frac{\partial v_r}{\partial\phi}+\frac{2\cot\theta}{r^2\sin\theta}\frac{\partial v_\theta}{\partial\phi}\right]+\varrho b_\phi,$$

where $\mathbf{b}=(b_r,b_\theta,b_\phi)$ is the body force vector in spherical coordinates. We recall that mass balance for incompressible fluids in cylindrical coordinates is given by (2.52).

5.6 Boundary conditions

The boundary conditions for ideal fluids (4.7), (4.8) represent impenetrability of the boundary. For Newtonian fluids these conditions are no longer sufficient, because the Navier-Stokes equation is of second order. The presence of viscosity, and hence of shear stresses, suggests to impose the so-called *no-slip* conditions

$$\mathbf{v}(\mathbf{x},t)=\mathbf{V}(\mathbf{x},t), \tag{5.12}$$

where we recall that $\mathbf{V}(\mathbf{x},t)$ represents the velocity of the wall. When the wall does not move

$$\mathbf{v}(\mathbf{x},t)=0. \tag{5.13}$$

Conditions (5.12), (5.13) are stronger than (4.7), (4.8), since they impose a constraint on both the normal and tangent component of the velocity.

5.7 Circulation and vorticity in Newtonian incmpressible fluids

We begin by recalling, from (1.21), that

$$\Delta\mathbf{v}=\nabla(\mathrm{div}\,\mathbf{v})-\mathrm{curl}\,(\mathrm{curl}\,\mathbf{v}).$$

Hence in the incompressible case $(5.7)_1$ can be rewritten as

$$\frac{d\mathbf{v}}{dt}=\mathbf{b}-\frac{\nabla p}{\varrho}-\nu\,\mathrm{curl}\,\boldsymbol{\omega},$$

where we recall that $\boldsymbol{\omega}=\mathrm{curl}\,\mathbf{v}$. Therefore, when the flow is irrotational, the Navier-Stokes equation coincides with Euler's equation. This does not mean that viscous irrotational flows reduce to ideal flows, since the boundary conditions are different. Moreover, irrotational viscous flows do not maintain their irrotational character for all times. Indeed consider an incompressible Newtonian fluid. The curl of the acceleration is

$$\mathrm{curl}\,\mathbf{a}=\mathrm{curl}\,\mathbf{b}+\nu\,\mathrm{curl}\,(\Delta\mathbf{v}).$$

Since $\mathrm{curl}\,(\Delta\mathbf{v})=\Delta(\,\mathrm{curl}\,\mathbf{v})$, in the case of conservative body force

$$\mathrm{curl}\,\mathbf{a}=\nu\Delta\boldsymbol{\omega}. \tag{5.14}$$

so that acceleration is not irrotational, as for inviscid fluids. In particular Beltrami's equation (4.17) (with ϱ constant) becomes

$$\frac{d\boldsymbol{\omega}}{dt} = \mathbf{D}\omega + \nu\Delta\boldsymbol{\omega}.$$

In the planar case $\mathbf{D}\boldsymbol{\omega} = 0$, so that

$$\frac{d\boldsymbol{\omega}}{dt} = \nu\Delta\boldsymbol{\omega},$$

and vorticity is no longer constant on streamlines. From Stokes' theorem and (5.14) we find that

$$\oint_\ell \mathbf{a}\cdot d\boldsymbol{\ell} = \int_S \operatorname{curl}\mathbf{a}\cdot\mathbf{n}d\sigma = \int_S \nu\Delta\boldsymbol{\omega}\cdot\mathbf{n}d\sigma = \nu\oint_\ell \Delta\mathbf{v}\cdot d\boldsymbol{\ell}.$$

Hence, recalling (4.20)

$$\frac{d}{dt}\left[\oint_\ell \mathbf{v}\cdot d\boldsymbol{\ell}\right] = \nu\oint_\ell \Delta\mathbf{v}\cdot d\boldsymbol{\ell},$$

implying that circulation is not conserved.

5.8 Transport and energy dissipation

Consider the Navier-Stokes system (5.7) and neglect body forces. Recalling (2.27), equation $(5.7)_1$ can be rewritten as

$$\frac{\partial\mathbf{v}}{\partial t} + \operatorname{curl}\mathbf{v}\times\mathbf{v} = -\nabla\left(\frac{1}{2}|\mathbf{v}|^2 + \frac{p}{\varrho}\right) + \nu\Delta\mathbf{v}. \tag{5.15}$$

Multiply (5.15) by $\mathbf{v}$ so that

$$\mathbf{v}\cdot\frac{\partial\mathbf{v}}{\partial t} = -\mathbf{v}\cdot\nabla\left(\frac{1}{2}|\mathbf{v}|^2 + \frac{p}{\varrho}\right) + \nu\mathbf{v}\cdot\Delta\mathbf{v}. \tag{5.16}$$

Recall, from (1.15), that

$$\mathbf{v}\cdot\nabla\left(\frac{1}{2}|\mathbf{v}|^2 + \frac{p}{\varrho}\right) = \operatorname{div}\left[\mathbf{v}\left(\frac{1}{2}|\mathbf{v}|^2 + \frac{p}{\varrho}\right)\right] - \underbrace{\left(\frac{1}{2}|\mathbf{v}|^2 + \frac{p}{\varrho}\right)\operatorname{div}\mathbf{v}}_{=0},$$

so that (5.16) becomes

$$\frac{\partial}{\partial t}\left(\frac{1}{2}|\mathbf{v}|^2\right) = -\operatorname{div}\left[\mathbf{v}\left(\frac{1}{2}|\mathbf{v}|^2 + \frac{p}{\varrho}\right)\right] + \nu\mathbf{v}\cdot\Delta\mathbf{v}. \tag{5.17}$$

We notice that

$$\operatorname{div}\Big(2\mathbf{D}\Big) = \operatorname{div}\Big(\nabla\mathbf{v}\Big) + \operatorname{div}\Big(\nabla\mathbf{v}^T\Big),$$

and that the i^{th} component of $\operatorname{div}(\nabla\mathbf{v}^T)$ is

$$\Big[\operatorname{div}(\nabla\mathbf{v}^T)\Big]_i = \sum_{j=1}^{3}\frac{\partial}{\partial x_j}\left(\frac{\partial v_j}{\partial x_i}\right) = \frac{\partial}{\partial x_i}\underbrace{\Big(\operatorname{div}\mathbf{v}\Big)}_{=0}.$$

Therefore

$$\operatorname{div}\Big(\nabla\mathbf{v}\Big) = \operatorname{div}\Big(2\mathbf{D}\Big).$$

From (1.16)

$$\mathbf{v}\cdot\Delta\mathbf{v} = \mathbf{v}\cdot\ \mathrm{div}\left(\nabla\mathbf{v}\right) = \mathbf{v}\cdot\ \mathrm{div}\left(2\mathbf{D}\right) = \mathrm{div}\left(2\mathbf{D}\mathbf{v}\right) - 2\mathbf{D}\cdot\nabla\mathbf{v},$$

so that (5.17) becomes

$$\frac{\partial}{\partial t}\left(\frac{1}{2}|\mathbf{v}|^2\right) + \mathrm{div}\left[\mathbf{v}\left(\frac{1}{2}|\mathbf{v}|^2 + \frac{p}{\varrho}\right) - 2\nu\mathbf{D}\mathbf{v}\right] = -2\nu\mathbf{D}\cdot\nabla\mathbf{v}. \tag{5.18}$$

Recalling that $\mathbf{T} = -p\mathbf{I} + 2\mu\mathbf{D}$ we find

$$2\nu\mathbf{D} = \left(\frac{1}{\varrho}\mathbf{T} + \frac{p}{\varrho}\mathbf{I}\right),$$

so that

$$2\nu\mathbf{D}\mathbf{v} = \frac{1}{\varrho}\mathbf{T}\mathbf{v} + \frac{p}{\varrho}\mathbf{v},$$

and

$$2\nu\mathbf{D}\cdot\nabla\mathbf{v} = \frac{1}{\varrho}\mathbf{T}\cdot\mathbf{L} = \frac{1}{\varrho}\mathbf{T}\cdot\mathbf{D}.$$

Hence (5.18) can be rewritten as

$$\frac{\partial}{\partial t}\left(\frac{\varrho}{2}|\mathbf{v}|^2\right) + \mathrm{div}\left[\varrho\mathbf{v}\left(\frac{1}{2}|\mathbf{v}|^2\right) - \mathbf{T}\mathbf{v}\right] = -\mathbf{T}\cdot\mathbf{D}, \tag{5.19}$$

where the term on the r.h.s. is the *stress power*. Notice that when the fluid is inviscid ($\nu = 0$) the stress reduces to $\mathbf{T} = -p\mathbf{I}$ and (5.19) becomes

$$\frac{\partial}{\partial t}\left(\frac{1}{2}|\mathbf{v}|^2\right) + \mathrm{div}\left[\mathbf{v}\left(\frac{1}{2}|\mathbf{v}|^2 + \frac{p}{\varrho}\right)\right] = 0, \tag{5.20}$$

meaning that ideal fluids do not dissipate energy.

5.9 Scaling of the Navier-Stokes equation

Let us consider system (5.7) and suppose that the body force is zero. We introduce the following non dimensional variables

$$\mathbf{x} = \tilde{\mathbf{x}}L, \qquad \mathbf{v} = \tilde{\mathbf{v}}V, \qquad t = \tilde{t}\left(\frac{L}{V}\right), \qquad p = \tilde{p}P.$$

Substituting into (5.7) we find

$$\frac{d\tilde{\mathbf{v}}}{d\tilde{t}} = -\left(\frac{P}{\varrho V^2}\right)\nabla\tilde{p} + \left(\frac{\mu}{\varrho V L}\right)\Delta\tilde{\mathbf{v}}, \tag{5.21}$$

where the differential operators are intended with respect to the non dimensional variables. We introduce the *Reynolds* number

$$\mathsf{Re} = \frac{\varrho V L}{\mu},$$

so that (5.21) becomes

$$\frac{d\tilde{\mathbf{v}}}{d\tilde{t}} = -\left(\frac{P}{\varrho V^2}\right)\nabla\tilde{p} + \frac{1}{\mathsf{Re}}\Delta\tilde{\mathbf{v}}. \tag{5.22}$$

There is no natural selection for the characteristic pressure and we have to distinguish between two different situations:

(A) (dynamic effects are dominant) $P = \varrho V^2$,

(B) (viscous effects are dominant) $P = \dfrac{\mu V}{L}$.

Case (A) occurs, for instance, when the velocity of the flow is quite large and viscous effects are negligible. In this case $\mathsf{Re} \gg 1$ and the motion equation can be approximated with

$$\frac{\partial \tilde{\mathbf{v}}}{\partial \tilde{t}} + (\nabla \tilde{\mathbf{v}})\tilde{\mathbf{v}} = -\nabla \tilde{p}, \tag{5.23}$$

which is nothing but Euler's equation. On the other hand, in case (B), viscous effects are dominant so that velocities are not large. Hence $\mathsf{Re} \ll 1$ and equation (5.21) becomes the well-known *Stokes equation*

$$-\nabla \tilde{p} + \Delta \tilde{\mathbf{v}} = 0. \tag{5.24}$$

5.10 Plane Poiseuille flow

We investigate here the shear flow of an incompressible Newtonian fluid. In practice we study the problem introduced in Section 4.17 for an incompressible Newtonian fluid. Here we assume also that the upper surface moves with uniform velocity $\mathbf{u} = U\mathbf{e}_1$ and we limit ourselves to the stationary case, see Fig. 5.1. The velocity field is $\mathbf{v} = v_1(x_2)\mathbf{e}_1$ and satisfies the continuity

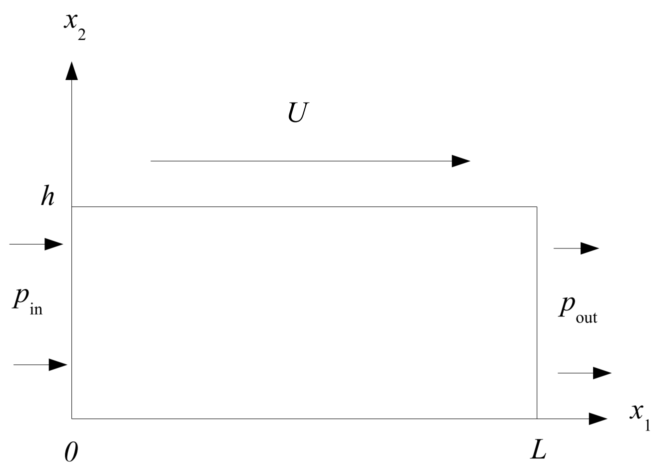


Figure 5.1: Shear flow of a viscous fluid.

equation

$$\frac{\partial v_1}{\partial x_1} = 0.$$

Navier-Stokes equation becomes

$$\begin{cases} -\dfrac{\partial p}{\partial x_1} + \mu \dfrac{\partial^2 v_1}{\partial x_2^2} = 0, \\[2ex] \dfrac{\partial p}{\partial x_2} = 0, \end{cases} \tag{5.25}$$

entailing

$$p(x_1) = \frac{\Delta P}{L} x_1 + p_{in},$$

where ΔP, L, p_{in}, p_out, are as in Section 4.17. Assuming no-slip on the upper and lower planes, from $(5.25)_1$

$$v_1(x_2) = \frac{1}{2\mu}\left(\frac{\Delta P}{L}\right)\left(x_2^2 - hx_2\right) + \frac{Ux_2}{h}.$$

When the upper plane is fixed

$$v_1(x_2) = \frac{1}{2\mu}\left(\frac{\Delta P}{L}\right)\left(x_2^2 - hx_2\right),$$

and the velocity profile is symmetric w.r.t. $x_2 = h/2$. When the upper plane is moving, regions where velocity is null may exist. Indeed, setting $v_1(x_2) = 0$ we find

$$x_2 = h - \frac{2\mu UL}{h\Delta P},$$

which belongs to $(0, h)$ if

$$\frac{U\Delta P}{L} > 0,$$

and

$$-\frac{h^2}{2\mu}\left|\frac{\Delta P}{L}\right| < U < \frac{h^2}{2\mu}\left|\frac{\Delta P}{L}\right|.$$

The mass flow rate is given by

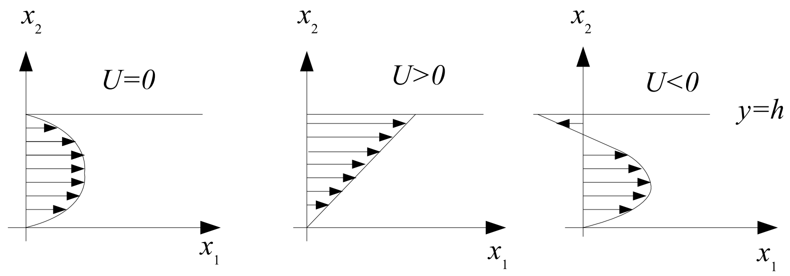


Figure 5.2: Shear flow of an inviscid fluid.

$$Q = \varrho \int_0^h v_1 dx_2 = \varrho \left[\frac{Uh}{2} - \frac{\Delta P h^3}{12\mu L}\right].$$

5.11 Hagen-Poiseuille flow in a cylindrical duct

Consider the steady motion of an incompressible Newtonian fluid in a cylindrical duct of radius R and length L, as shown in Fig. 5.3. We make use of cylindrical polar coordinates described in Section 1.4. We suppose that the velocity field is of the form

$$\mathbf{v} = v_z(r)\mathbf{e}_z.$$

Recalling (2.51) we find that mass balance is satisfied, since

$$\frac{\partial v_z}{\partial z} = 0.$$

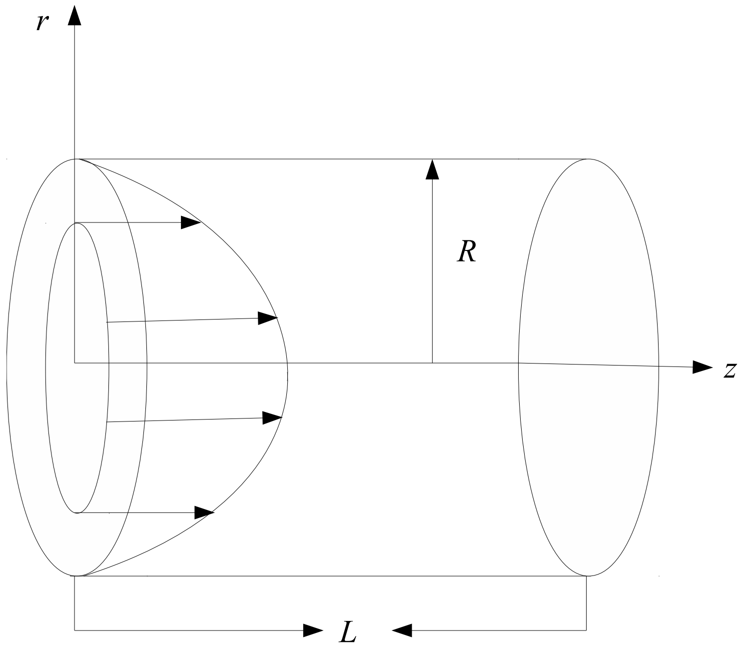


Figure 5.3: Hagen-Poiseuille flow.

Neglecting body forces, Navier-Stokes equation (5.8)-(5.10) reduces to

$$\nabla p = \frac{\mu}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right)\mathbf{e}_z.$$

As a consequence $p = Az + B$, with A and B specified by the boundary conditions

$$p(0) = p_{in} \qquad p(L) = p_{out}.$$

We find

$$p(z) = \frac{\Delta P}{L}z + p_{in},$$

where $\Delta P = p_{out} - p_{in}$. Hence

$$\frac{\Delta P}{\mu L} = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right)$$

If we now impose no-slip on $r = R$

$$v_z(R) = 0,$$

and symmetry on $r = 0$

$$\left.\frac{\partial v_z}{\partial r}\right|_{r=0} = 0,$$

we find

$$v_z(r) = \frac{\Delta P}{4\mu L}(r^2 - R^2).$$

In this case the mass flow rate is

$$Q = 2\pi\varrho\int_0^R v_z r dr = -\frac{\pi \Delta P R^4}{8\nu L}. \tag{5.26}$$

Formula (5.26) is called *Poiseuille formula.* It allows us to calculate the viscosity of the fluid through mass flow rate measurements. Recalling that

$$v_r = v_\theta = \frac{\partial}{\partial z} = \frac{\partial}{\partial \theta} = 0,$$

we can easily find the expression of the Cauchy tensor in cylindrical coordinates

$$\mathbf{T} = \begin{bmatrix} -p & 0 & \dfrac{\Delta P r}{2L} \\ 0 & -p & 0 \\ \dfrac{\Delta P r}{2L} & 0 & -p \end{bmatrix}.$$

5.12 Couette flow between co-axial cylinders

Let us consider the steady motion of an incompressible Newtonian fluid placed between two co-axial cylinders rotating with uniform velocity. Denote by $R_1 < R_2$ the radii of the cylinders and by ω_1, ω_2 the respective angular velocities. Suppose that the velocity field and pressure depend only on the radial coordinate r, so that

$$\mathbf{v} = v_r(r)\mathbf{e}_r + v_\theta(r)\mathbf{e}_\theta + v_z(r)\mathbf{e}_z,$$

and $p = p(r)$. On the rotating cylinders we have

(5.27) $$\mathbf{v}\Big|_{R_1} = \omega_1 R_1 \mathbf{e}_\theta, \qquad \mathbf{v}\Big|_{R_2} = \omega_2 R_2 \mathbf{e}_\theta.$$

Mass balance (2.51) yields

$$\frac{\partial}{\partial r}\Big(r v_r\Big) = 0,$$

so that $v_r \equiv 0$ because of (5.27). The Navier-Stokes equations (5.8)-(5.10) reduces to

(5.28) $$\begin{cases} \dfrac{\partial p}{\partial r} = \dfrac{\varrho v_\theta^2}{r}, \\[2em] \dfrac{\partial}{\partial r}\left[\dfrac{1}{r}\dfrac{\partial}{\partial r}\Big(r v_\theta\Big)\right] = 0, \\[2em] \dfrac{\partial}{\partial r}\left[r\dfrac{\partial v_z}{\partial r}\right] = 0. \end{cases}$$

From $(5.28)_3$

$$\frac{\partial v_z}{\partial r} = \frac{A}{r} \qquad v_z = A \ln r + B.$$

Recalling that $v_z(R_1) = v_z(R_2) = 0$ we get $A = B = 0$, so that $v_z \equiv 0$. From $(5.28)_2$ we get

$$\frac{\partial}{\partial r}\Big(r v_\theta\Big) = Cr,$$

so that

$$v_\theta = \frac{D}{r} + \frac{Cr}{2}.$$

Imposing (5.27) we find

$$D = \frac{R_1^2 R_2^2}{R_2^2 - R_1^2}\Big(\omega_1 - \omega_2\Big), \qquad C = \frac{2\Big(\omega_2 R_2^2 - \omega_1 R_1^2\Big)}{R_2^2 - R_1^2}.$$

Therefore

$$v_\theta(r) = \frac{R_1^2 R_2^2}{R_2^2 - R_1^2}\Big(\omega_1 - \omega_2\Big)\frac{1}{r} + \frac{\Big(\omega_2 R_2^2 - \omega_1 R_1^2\Big)}{R_2^2 - R_1^2} r$$

From $(5.28)_1$ we can eventually find the pressure. We notice that, when $\omega_1 = \omega_2 = \omega$

$$v_\theta(r) = \omega r,$$

that is a rigid motion. When $R_2 \to \infty$, imposing $\omega_2 = 0$, we find

$$v_\theta(r) = \frac{\omega_1 R_1^2}{r}.$$

When $R_2 < \infty$ and $\omega_2 = 0$

$$v_\theta(r) = \frac{\omega_1 R_1^2 r}{R_2^2 - R_1^2}\left[\left(\frac{R_2}{r}\right)^2 - 1\right].$$

Exploiting the results of section 5.4, we find that the tangential stress is

$$T_{r\theta} = \mu\left(\frac{\partial v_\theta}{\partial r} - \frac{v_\theta}{r}\right),$$

so that the tangential stress on the inner cylinder (when $\omega_2 = 0$) is

$$T_{r\theta}\Big|_{R_1} = -\frac{2\mu R_2^2 \omega_1}{R_2^2 - R_1^2}. \tag{5.29}$$

To maintain the velocity of the inner cylinder constant we must apply a torque equal to $2\pi R_1^2 |T_{r\theta}| h$, where h is the height of the cylinder. The measurement of this torque allows one to deduce the viscosity by means of (5.29). The device used to this aim is called the *rotating viscometer*.

5.13 Stokes first problem

Let us consider an incompressible Newtonian fluid occupying the domain $x_3 > 0$ as depicted in Fig. 5.4. Let us assume that the boundary $x_3 = 0$ is fixed for $t < 0$ and moves with constant velocity $V\mathbf{e}_1$ for $t \geqslant 0$ (impulsive start). We also suppose that the fluid is at rest for $x_3 = \infty$. We write

$$\begin{cases} \lim_{x_3\to\infty} \mathbf{v} = 0, \\ \mathbf{v}_p = H(t)V\mathbf{e}_1, \end{cases}$$

where $\mathbf{v}_p$ is the velocity of the surface $x_3 = 0$ and where H is the Heaviside function

$$H(t) = \begin{cases} 0 & t < 0, \\ 1 & t \geqslant 0, \end{cases}$$

We suppose that the velocity and the pressure of the fluid have the following form

$$\mathbf{v} = v_1(x_1, x_3, t)\mathbf{e}_1,$$

$$p = p(x_1, x_3, t).$$

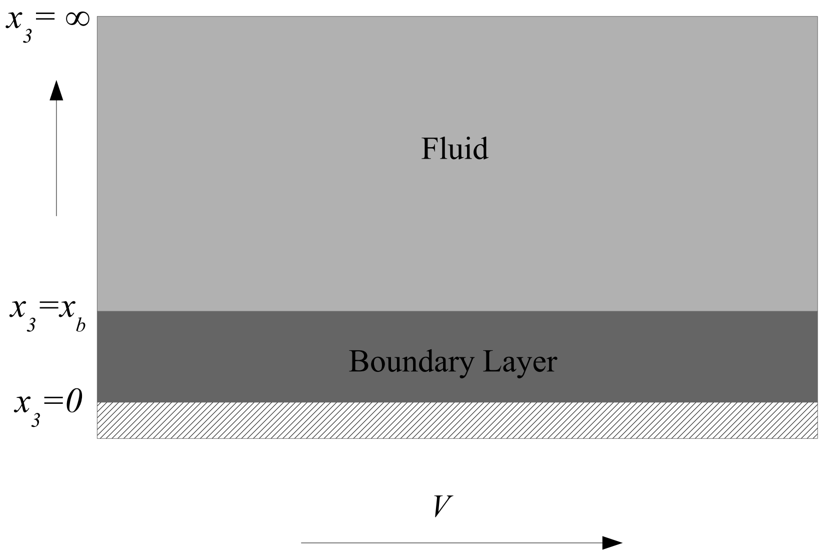


Figure 5.4: Stokes first problem.

Mass balance implies

$$\frac{\partial v_1}{\partial x_1} = 0,$$

so that $v_1 = v_1(x_3, t)$. Moreover the third component of the Navier-Stokes equation yields

$$\frac{\partial p}{\partial x_3} = 0,$$

so that $p = p(x_1, t)$. It is reasonable to require

$$\lim_{x_3 \to \infty} p(x_1, t) = const,$$

so that p is constant everywhere in the fluid. From the first component of the Navier-Stokes equation we get

$$\frac{\partial v_1}{\partial t} = \nu \frac{\partial^2 v_1}{\partial x_3^2}, \tag{5.30}$$

with boundary conditions

$$\begin{cases} \lim_{x_3 \to \infty} v_1(x_3, t) = 0, \\ \\ v_1(0, t) = H(t)V. \end{cases} \tag{5.31}$$

Now we look for a solution of the form

$$v_1(x_3, t) = f(\eta) \qquad\qquad \eta = \frac{x_3}{\sqrt{t}}, .$$

so that

$$\frac{\partial v_1}{\partial t} = -\frac{\eta}{2t} f^{'}(\eta) \qquad\qquad \frac{\partial^2 v_1}{\partial x_3^2} = \frac{1}{t} f^{''}(\eta).$$

Substituting into (5.30) we find

$$2\nu f^{''} + \eta f^{'} = 0,$$

which yields

$$f'(\eta) = c\exp\left\{-\frac{\eta^2}{4\nu}\right\}.$$

To determine the function $f(\eta)$ we introduce the *error function*

$$\text{erf}\,(\eta) = \frac{2}{\sqrt{\pi}}\int_0^\eta \exp\{-\xi^2\}d\xi,$$

with the property

$$\lim_{\eta\to\infty} \text{erf}\,(\eta) = 1.$$

We get

$$f(\eta) = c\sqrt{\nu\pi}\,\text{erf}\,\left(\frac{\eta}{2\sqrt{\nu}}\right) + d.$$

As a consequence

$$v_1(x_3, t) = c\sqrt{\nu\pi}\,\text{erf}\,\left(\frac{x_3}{2\sqrt{\nu t}}\right) + d.$$

Imposing (5.31) we find the solution to our problem

$$v_1(x_3, t) = \left[1 - \text{ erf }\left(\frac{x_3}{2\sqrt{\nu t}}\right)\right] H(t)V.$$

It is easy to check when

$$\frac{x_3}{2\sqrt{\nu t}} \approx 1.8,$$

we get

$$[1 - \text{ erf }(1.8)] \approx \frac{1}{100},$$

meaning that at a distance $x_b \approx 3.6\sqrt{\nu t}$ from $x_3 = 0$, the velocity has reduced by a factor 10^{-2}. Therefore it is reasonable to consider the fluid at rest for $x > x_b$. The strip $[0, x_b]$ is thus a time-dependent *boundary layer*. Outside this boundary layer the effects of the boundary conditions are clearly negligible.

5.14 Stokes flow past a sphere

Let us consider the steady flow of a viscous incompressible fluid around a sphere of radius R, with asymptotic velocity $\mathbf{v}_\infty = v_\infty\mathbf{e}_1$. Suppose $\mathsf{Re} \ll 1$ (*creeping flow*), so that the motion is governed by Stokes equation

$$-\nabla p + \mu\Delta\mathbf{v} = 0. \tag{5.32}$$

Recalling (1.21) we can write the above as

$$\nabla\Big(\frac{p - p_o}{\mu}\Big) = -\text{ curl}\,(\text{ curl}\,\mathbf{v}), \tag{5.33}$$

where p_o is the pressure in static condition. Applying the divergence operator to (5.33) we find

$$\Delta\Big(\frac{p - p_o}{\mu}\Big) = 0. \tag{5.34}$$

Moreover, recalling that $\text{curl}(\nabla p) = 0$, from (5.32)

$$\Delta(\text{curl}\,\mathbf{v}) = 0. \tag{5.35}$$

It is easy to check that a solution of (5.34), (5.35) is given by

$$\frac{p - p_o}{\mu} = -\lambda v_\infty \frac{\mathbf{x} \cdot \mathbf{e}_1}{r^3}, \tag{5.36}$$

$$\text{curl}\,\mathbf{v} = \lambda v_\infty \frac{\mathbf{x} \times \mathbf{e}_1}{r^3}, \tag{5.37}$$

where

$$r = \sqrt{x_1^2 + x_2^2 + x_3^2},$$

and where λ is a constant to be specified. As a consequence, the functions defined in (5.36), (5.37) are harmonic. The velocity field must be determined from (5.37) and from the incompressibility constraint div $\mathbf{v} = 0$, assuming no-slip on the sphere

$$\mathbf{v} = 0, \qquad \text{on } \ r = R.$$

Because of the particular symmetry of the problem (see Fig 5.5) the velocity field can be expressed in the following form

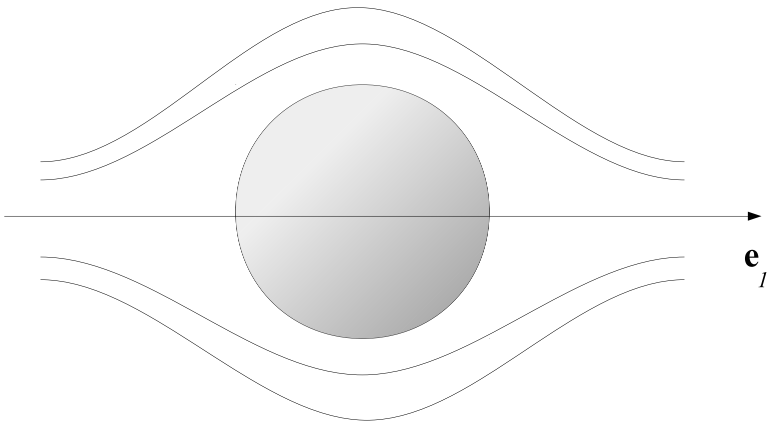


Figure 5.5: Stokes flow pasta a sphere.

$$\frac{\mathbf{v}}{v_\infty} = a(r)\mathbf{e}_1 + x_1 b(r)\mathbf{e}_r, \tag{5.38}$$

where $a(r)$ and $b(r)$ are unknown at this stage and where $\mathbf{e}_r$ is the radial unit vector in spherical coordinate (see Section 1.5). In practice we are assuming that the velocity field is directed along $\mathbf{e}_1$ on $x_1 = 0$. Applying the curl operator to (5.38) we find

$$\text{curl}\,\mathbf{v} = v_\infty\,\text{curl}\,(a\mathbf{e}_1) + v_\infty\,\text{curl}\,(x_1 b\mathbf{e}_r).$$

Now we observe that curl $(\mathbf{e}_1) =$ curl $(\mathbf{e}_r) = 0$. Hence, from (1.19)

$$\text{curl}\,\mathbf{v} = v_\infty[\nabla a \times \mathbf{e}_1] + v_\infty[\nabla(x_1 b) \times \mathbf{e}_r].$$

Recalling (1.33)

$$\nabla a = a^{'}(r)\mathbf{e}_r,$$

$$\nabla(x_1 b) = b(r)\mathbf{e}_1 + x_1 b^{'}(r)\mathbf{e}_r.$$

As a consequence

$$\text{curl}\,\mathbf{v} = v_\infty[a'(r) - b(r)]\mathbf{e}_r \times \mathbf{e}_1. \tag{5.39}$$

Since $\mathbf{x} = r\mathbf{e}_r$, coupling (5.37) with (5.39) we find

$$v_\infty(a' - b)\mathbf{e}_r \times \mathbf{e}_1 = \frac{\lambda v_\infty}{r^2}\mathbf{e}_r \times \mathbf{e}_1,$$

so that

$$a' - b = \frac{\lambda}{r^2}. \tag{5.40}$$

Recalling (1.34) we see that

$$\text{div}\,\mathbf{e}_r = \frac{2}{r},$$

so that, from (5.38) and $\text{div}\,\mathbf{v} = 0$, we get

$$0 = \frac{\text{div}\,\mathbf{v}}{v_\infty} = \nabla a \cdot \mathbf{e}_1 + \nabla(x_1 b) \cdot \mathbf{e}_r + \frac{2x_1 b}{r},$$

which yield

$$0 = a'\mathbf{e}_r \cdot \mathbf{e}_1 + b\mathbf{e}_1 \cdot \mathbf{e}_r + x_1 b'\mathbf{e}_r \cdot \mathbf{e}_r + \frac{2x_1 b}{r}.$$

Now, observing that $x_1 = r\mathbf{e}_1 \cdot \mathbf{e}_r$, we get

$$\frac{a' x_1}{r} + \frac{b x_1}{r} + x_1 b' + \frac{2x_1 b}{r} = 0.$$

Exploiting (5.40) we find

$$4b + b' r + \frac{\lambda}{r^2} = 0,$$

whose integration leads to

$$b(r) = \frac{C}{r^4} - \frac{\lambda}{2r^2}.$$

As a consequence

$$a' = \frac{\lambda}{2r^2} + \frac{C}{r^4},$$

and

$$a(r) = -\frac{\lambda}{2r} - \frac{C}{3r^3} + K.$$

Imposing the asymptotic velocity $\mathbf{v}_\infty = v_\infty \mathbf{e}_1$ we have

$$\lim_{r\to\infty} a(r) = 1, \qquad \lim_{r\to\infty} b(r) = 0.$$

The first implies $K = 1$, while the second is satisfied for all $C \in \mathbb{R}$. Finally we impose no-slip on the sphere

$$a(R) = b(R) = 0.$$

The above yields

$$C = \frac{3R^3}{4}, \qquad \lambda = \frac{3R}{2}.$$

Therefore

$$a(r) = 1 - \frac{3}{4}\left(\frac{R}{r}\right) - \frac{1}{4}\left(\frac{R}{r}\right)^3, \tag{5.41}$$

(5.42)
$$b(r) = \frac{3}{4}\left(\frac{R}{r^2}\right)\left[\left(\frac{R}{r}\right)^2 - 1\right].$$

We conclude that velocity field is given by (5.38) with $a(r)$, $b(r)$ given by (5.41), (5.42). From (5.36) we find that pressure is

(5.43)
$$p - p_o = -\frac{3Rv_\infty\mu}{2}\frac{x_1}{r^3}.$$

The velocity field in the spherical coordinate system is

$$v_r = v_\infty \sin\theta\cos\phi(a + rb), \qquad v_\theta = v_\infty a\cos\theta\cos\phi, \qquad v_\phi = -v_\infty a\sin\phi.$$

As a consequence, recalling the form of the stress tensor in spherical coordinates (5.11), the stress on the sphere $r = R$ is given by

$$\mathbf{T} = \begin{bmatrix} -p & \frac{3\mu v_\infty}{2R}\cos\theta\cos\phi & -\frac{3\mu v_\infty}{2R}\sin\phi \\ \frac{3\mu v_\infty}{2R}\cos\theta\cos\phi & -p & 0 \\ -\frac{3\mu v_\infty}{2R}\sin\phi & 0 & -p \end{bmatrix}.$$

The normal stress acting on the sphere is

$$\mathbf{\Phi} = \mathbf{T}\mathbf{e}_r = -p\mathbf{e}_r + \left(\frac{3\mu v_\infty}{2R}\cos\theta\cos\phi\right)\mathbf{e}_\theta - \left(\frac{3\mu v_\infty}{2R}\sin\phi\right)\mathbf{e}_\phi.$$

The drag force due to the motion is obtained replacing p in the above expression with $p - p_o$ obtained in (5.43). We get

$$\mathbf{\Phi} = \left(\frac{3\mu v_\infty}{2R}\right)\left[\sin\theta\cos\phi\mathbf{e}_r + \cos\theta\cos\phi\mathbf{e}_\theta - \sin\phi\mathbf{e}_\phi\right],$$

that is

$$\mathbf{\Phi} = \left(\frac{3\mu}{2R}\right)\mathbf{v}_\infty.$$

This means that the stress exerted by the fluid is constant everywhere on the sphere. In particular the stress is directed along the the asymptotic motion. In conclusion the net drag in the x_1 direction is given by

$$R = \int_{r=R} \mathbf{\Phi}\cdot\mathbf{e}_1 d\sigma = 6\pi\mu R v_\infty,$$

which is the celebrated *Stokes formula.* Notice that the Stokes formula allows to determine the sedimentation rate v_s of a sphere of density ϱ_s falling in a viscous fluid of density ϱ and viscosity μ. Indeed equating the drag and the buoyant force, we find

$$\underbrace{6\pi R\mu v_s}_{\text{Drag}} = \underbrace{\frac{4}{3}\pi R^3 g\Big(\varrho_s - \varrho\Big)}_{\text{Buoyant force}},$$

that is

$$v_s = \frac{2R^2 g}{9\mu}\Big(\varrho_s - \varrho\Big).$$

5.15 Prandtl's boundary layer theory

The failure of potential flow theory to predict drag over an object can be overcome using Prandtl's boundary layer theory which is based on the following assumption: when a viscous fluid flows past an object at high Reynolds number, the flow can be divided in a thin region close to the rigid surface in which viscous effects cannot be neglected and a region away from the object where viscous effects are negligible and potential flow can be used. The former domain is referred to as *Prandtl's boundary layer* (BL), or more simply *boundary layer*.

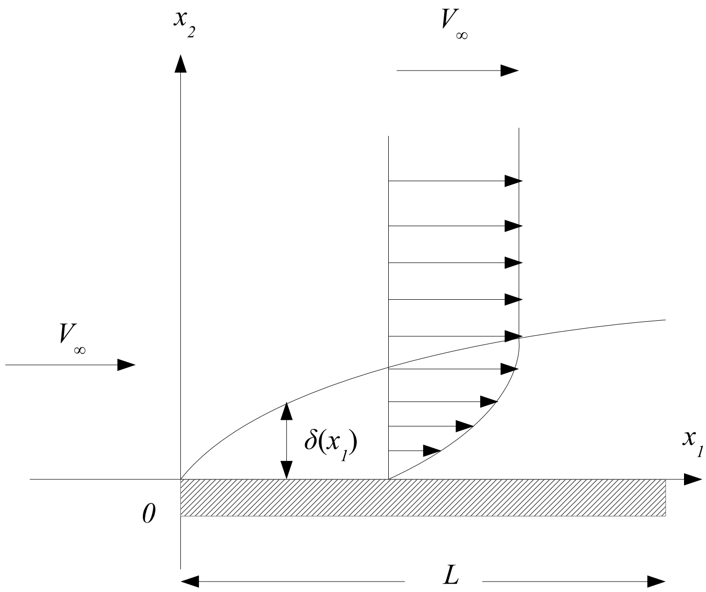


Figure 5.6: Prandtl's boundary layer.

To determine the thickness of the BL we proceed as follows. We consider the planar steady flow of an incompressible Newtonian fluid with velocity given by

$$\mathbf{v} = v_1(x_1, x_2)\mathbf{e}_1 + v_2(x_1, x_2)\mathbf{e}_2.$$

Following Fig. 5.6 we suppose that the boundary layer thickness δ is a function of x_1 and it is small when compared to the characteristic length L of the domain. We write the steady form of the Navier-Stokes equation and the mass balance

$$\begin{cases} \dfrac{\partial v_1}{\partial x_1} + \dfrac{\partial v_2}{\partial x_2} = 0, \\[2ex] v_1 \dfrac{\partial v_1}{\partial x_1} + v_2 \dfrac{\partial v_1}{\partial x_2} = -\dfrac{1}{\varrho}\dfrac{\partial p}{\partial x_1} + \nu\left(\dfrac{\partial^2 v_1}{\partial x_1^2} + \dfrac{\partial^2 v_1}{\partial x_2^2}\right), \\[2ex] v_1 \dfrac{\partial v_2}{\partial x_1} + v_2 \dfrac{\partial v_2}{\partial x_2} = -\dfrac{1}{\varrho}\dfrac{\partial p}{\partial x_2} + \nu\left(\dfrac{\partial^2 v_2}{\partial x_1^2} + \dfrac{\partial^2 v_2}{\partial x_2^2}\right). \end{cases}$$

Away from the boundary layer we may rescale the problem with the following

$$v_1 = V_\infty \tilde{v}_1, \qquad v_2 = V_\infty \tilde{v}_2, \qquad p = \varrho V_\infty^2 \tilde{p}, \qquad x = L\tilde{x}, \qquad y = L\tilde{y}.$$

where V_∞ is the asymptotic velocity. Omitting the tildas, we find

$$\begin{cases} \dfrac{\partial v_1}{\partial x_1} + \dfrac{\partial v_2}{\partial x_2} = 0, \\ v_1 \dfrac{\partial v_1}{\partial x_1} + v_2 \dfrac{\partial v_1}{\partial x_2} = -\dfrac{\partial p}{\partial x_1} + \dfrac{1}{\mathsf{Re}} \left(\dfrac{\partial^2 v_1}{\partial x_1^2} + \dfrac{\partial^2 v_1}{\partial x_2^2} \right), \\ v_1 \dfrac{\partial v_2}{\partial x_1} + v_2 \dfrac{\partial v_2}{\partial x_2} = -\dfrac{\partial p}{\partial x_2} + \dfrac{1}{\mathsf{Re}} \left(\dfrac{\partial^2 v_2}{\partial x_1^2} + \dfrac{\partial^2 v_2}{\partial x_2^2} \right), \end{cases}$$

where

$$\mathsf{Re} = \frac{V_\infty L}{\nu},$$

is the Reynolds number. When $\mathsf{Re} \gg 1$ the system reduces to steady Euler's equation for planar incompressible flows. Within the boundary layer y can no longer be rescaled with L, since the characteristic thickness of the boundary layer is given by some $D \ll L$. Hence we set

$$\varepsilon = \frac{D}{L},$$

and we use the following scaling

$$v_1 = V_\infty \tilde{v}_1, \qquad v_2 = V_\infty \varepsilon \tilde{v}_2, \qquad p = \varrho V_\infty^2 \tilde{p}, \qquad x_1 = L\tilde{x}_1, \qquad x_2 = L\varepsilon\tilde{x}_2.$$

Omitting once again the tildas we find

$$\begin{cases} \dfrac{\partial v_1}{\partial x_1} + \dfrac{\partial v_2}{\partial x_2} = 0, \\ v_1 \dfrac{\partial v_1}{\partial x_1} + v_2 \dfrac{\partial v_1}{\partial x_2} = -\dfrac{\partial p}{\partial x_1} + \dfrac{1}{\varepsilon^2 \mathsf{Re}} \left(\varepsilon^2 \dfrac{\partial^2 v_1}{\partial x_1^2} + \dfrac{\partial^2 v_1}{\partial x_2^2} \right), \\ \varepsilon^2 \left(v_1 \dfrac{\partial v_2}{\partial x_1} + v_2 \dfrac{\partial v_2}{\partial x_2} \right) = -\dfrac{\partial p}{\partial x_2} + \dfrac{1}{\mathsf{Re}} \left(\varepsilon^2 \dfrac{\partial^2 v_2}{\partial x_1^2} + \dfrac{\partial^2 v_2}{\partial x_2^2} \right). \end{cases}$$

Assuming that in the boundary layer the viscous and inertial effects are of the same order we write

$$\varepsilon^2 \mathsf{Re} = O(1).$$

At the leading order (i.e. neglecting the terms containing ε) we get

$$\begin{cases} \dfrac{\partial v_1}{\partial x_1} + \dfrac{\partial v_2}{\partial x_2} = 0, \\ v_1 \dfrac{\partial v_1}{\partial x_1} + v_2 \dfrac{\partial v_1}{\partial x_2} = -\dfrac{\partial p}{\partial x_1} + \alpha \dfrac{\partial^2 v_1}{\partial x_2^2}, \\ -\dfrac{\partial p}{\partial x_2} = 0, \end{cases} \tag{5.44}$$

where

$$\alpha = \frac{1}{\varepsilon^2 \mathsf{Re}} = O(1).$$

The boundary conditions are

$$
(5.45) \qquad \begin{cases} v_1(x_1,0) = v_2(x_1,0) = 0, \quad \text{(No-slip)}, \quad x_1 > 0, \\ \lim_{x_2\to+\infty} v_1(x_1,x_2) = 1, \qquad x_1 > 0 \\ \lim_{x_2\to+\infty} p(x_1,x_2) = p_o, \qquad x_1 > 0 \end{cases}
$$

From $(5.44)_3$ and $(5.45)_3$ we find $p = p_o$ (constant), so that $\partial p/\partial x_1 = 0$. We look for a solution in the form

$$
v_1(x_1,x_2) = f'(s), \qquad s = \frac{x_2}{\sqrt{\alpha x_1}},
$$

where we write $f'(s)$ instead of $f(s)$ because the corresponding differential equation is simpler with this selection. The boundary conditions are

$$
(5.46) \qquad f'(0) = 0, \qquad \lim_{s\to+\infty} f'(s) = 1.
$$

We have

$$
(5.47) \qquad \frac{\partial v_1}{\partial x_1} = -\frac{s}{2x_1} f''(s), \qquad \frac{\partial v_1}{\partial x_2} = \frac{1}{\sqrt{\alpha x_1}} f''(s),
$$

$$
\frac{\partial^2 v_1}{\partial x_2^2} = \frac{1}{\alpha x_1} f'''(s), \qquad \frac{\partial v_2}{\partial x_2} = \frac{s}{2x_1} f''(s).
$$

Moreover

$$
\frac{\partial v_2}{\partial s} = \frac{\partial v_2}{\partial x_2}\frac{\partial x_2}{\partial s} = \frac{f''(s) s\alpha}{2\sqrt{\alpha x_1}},
$$

so that

$$
(5.48) \qquad v_2 = \frac{\alpha}{2\sqrt{\alpha x_1}} \int f''(s) s ds = \frac{\alpha}{2\sqrt{\alpha x_1}} \Big[s f'(s) - f(s) \Big].
$$

Recalling $(5.45)_1$, (5.46) we observe that $f(0) = 0$. Substitution of (5.47)-(5.48) into $(5.44)_2$ leads to

$$
(5.49) \qquad \begin{cases} f'''(s) + \frac{1}{2} f(s) f'(s) = 0, \\ f(0) = f'(0) = 0, \\ \lim_{s\to\infty} f'(s) = 1. \end{cases}
$$

Equation $(5.49)_1$ is the so-called *Blasius equation.* It is a third order autonomous nonlinear ODE, which can be only solved numerically. In particular it can be shown that $f'(5) = 0.99$, meaning that for $s > 5$ the longitudinal dimensional velocity is essentially equal to V_∞ (see Fig. 5.7). The thickness of the boundary layer $\delta(x_1)$ is conventionally obtained setting $s = 5$. Hence, going back to dimensional variables

$$
5 = \frac{\left(\dfrac{\delta}{D}\right)}{\sqrt{\dfrac{x_1}{\varepsilon^2 \mathsf{Re}} L}},
$$

so that

$$
\delta(x) = 5\sqrt{\frac{\nu x}{V_\infty}}.
$$

For $y > \delta(x)$ the influence of the boundary conditions can be safely neglected.

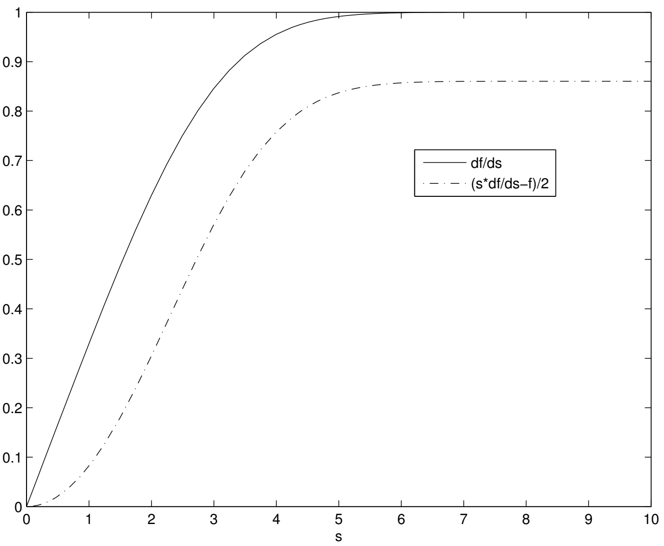


Figure 5.7: Solution of the Blasius equation.

Chapter 6

Non-Newtonian fluids

In the previous chapter we have defined Newtonian fluids as continua where the viscous stress is linearly proportional to the local strain rate, the constitutive equation being (5.3).

Newtonian fluid is the simplest mathematical model in which viscosity is taken into account. Though no real fluid fits the definition perfectly, a vast number of real fluids (or gases) can be adequately described by Newtonian constitutive equation (5.3). A *non-Newtonian fluid* is any

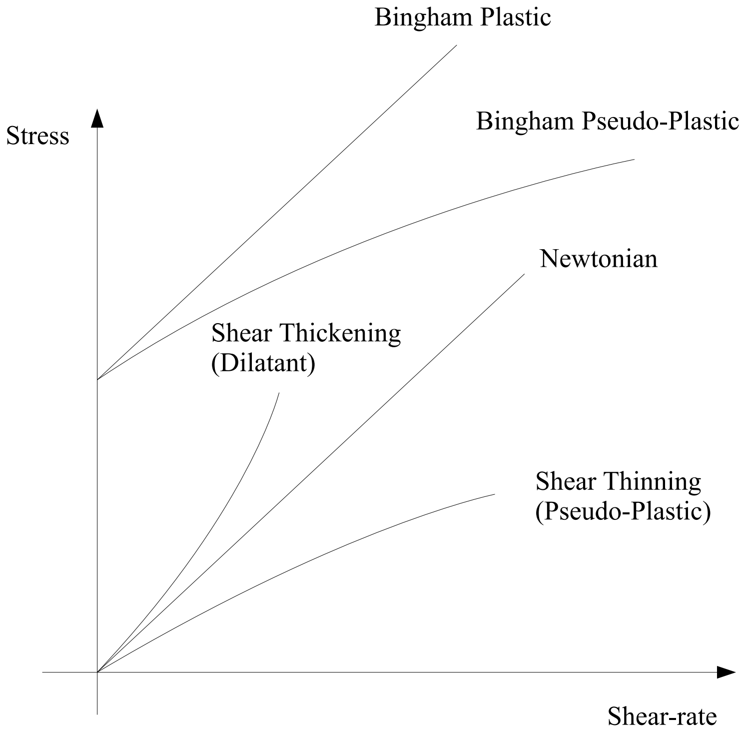


Figure 6.1: Shear-strain relation of some non-Newtonian fluids.

fluid that departs from the linear behavior between stress and strain rate. Not all non-Newtonian fluids behave in the same way when the stress is applied. Some, for instance, become more rigid when the stress increases, others become more fluid. Some react depending on the length of time in which the stress is applied. Molten polymers, toothpaste, paints, blood, foams, lubricants, inks, drilling muds, oils, colloidal suspensions, etc. exhibit non-Newtonian behavior. A sketch of some stress-strain relation of common non-Newtonian fluids is displayed in Fig. 6.1.

In shear-thickening fluids, for instance, viscosity increases with the applied stress, while in shear thinning fluids viscosity decreases with the applied stress. In Bingham plastics and Bingham pseudo-plastics a finite yield stress must be overcome before the fluid begins to flow. There are also fluids in which the apparent viscosity decreases with the duration of the stress (thixotropic fluids).

In this chapter we present some models for non-Newtonian fluids, analyzing their behavior in several important flow fields such as steady flow in straight channel and pipes, steady cone and plate flow, steady flow between concentric cylinders. Reference books on this topic are [2], [4], [25], [26].

6.1 Reiner-Rivlin fluids

We begin by considering fluids whose constitutive relation is

$$\mathbf{T} = \mathbf{T}\Big(\varrho, \mathbf{L}\Big),$$

where the current state of the stress depends on the velocity gradient and not on any previous deformation the fluid might have undergone. From the requirement of stress invariance and isotropy, the most general form of $\mathbf{T}$ is the one of stokesian fluids

$$\mathbf{T} = \alpha\mathbf{I} + \beta\mathbf{D} + \gamma\mathbf{D}^2, \tag{6.1}$$

where α, β, γ depend on ϱ and on the principal invariants of $\mathbf{D}$. When the fluid is incompressible equation (6.1) reduces to the following

$$\mathbf{T} = -p\mathbf{I} + \phi_1\mathbf{D} + \phi_2\mathbf{D}^2, \tag{6.2}$$

where $-p(\mathbf{x}, t)$ is the Lagrange multiplier due to the incompressibility constraint and where now ϕ_1 and ϕ_2 depend only on the second and third invariant of $\mathbf{D}$, since $\mathrm{tr}\mathbf{D} = 0$. Fluids of type (6.2) are commonly known as *Reiner-Rivlin* (see [26]) fluids. Incompressible Newtonian fluids are a special subclass with $\phi_2 = 0$ and ϕ_1 constant. Recalling (4.59) and recalling the incompressibility constraint we get

$$(\mathbf{T} + p\mathbf{I}) \cdot \mathbf{D} = \mathbf{T} \cdot \mathbf{D} = \Big(\phi_1\mathbf{D} + \phi_2\mathbf{D}^2\Big) \cdot \mathbf{D} = \mathrm{tr}\,\Big(\phi_1\mathbf{D}^2 + \phi_2\mathbf{D}^3\Big) \geqslant 0.$$

Exploiting Theorem 1, we get

$$\mathbf{D}^3 = -i_2(\mathbf{D})\mathbf{D} + i_3(\mathbf{D})\mathbf{I},$$

so that

$$\mathbf{T} \cdot \mathbf{D} = \mathrm{tr}\,\Big(\phi_1\mathbf{D}^2 + \phi_2 i_3(\mathbf{D})\mathbf{I}\Big) = \phi_1 \mathrm{tr}\,\mathbf{D}^2 + 3\phi_2 \det\mathbf{D} \geqslant 0. \tag{6.3}$$

It is then clear that some restrictions must be imposed on the coefficients ϕ_1 and ϕ_2 in order to satisfy (6.3). To determine the dependence of ϕ_1 and ϕ_2 on the stress invariants $i_2(\mathbf{D})$, $i_3(\mathbf{D})$ we consider the simple shear motion considered in Section 2.7. For this flow the velocity field is given by

$$\mathbf{v} = \dot{\gamma}x_2\mathbf{e}_1,$$

where $\dot{\gamma} = V/h$ and V is the velocity of the upper plate that drives the motion. We get

$$\mathbf{D} = \begin{bmatrix} 0 & \frac{\dot{\gamma}}{2} & 0 \\ \frac{\dot{\gamma}}{2} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad \mathbf{D}^2 = \begin{bmatrix} \frac{\dot{\gamma}^2}{4} & 0 & 0 \\ 0 & \frac{\dot{\gamma}^2}{4} & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

so that

$$\begin{cases} i_1(\mathbf{D}) = \operatorname{tr}\mathbf{D} = 0, \\ i_2(\mathbf{D}) = \dfrac{1}{2}\left[(\operatorname{tr}\mathbf{D})^2 - \operatorname{tr}(\mathbf{D}^2)\right] = -\dfrac{\dot{\gamma}^2}{4}, \\ i_3(\mathbf{D}) = \det\mathbf{D} = 0. \end{cases}$$

We get

$$\mathbf{T} = \begin{bmatrix} -p + \phi_2\frac{\dot{\gamma}^2}{4} & \phi_1\frac{\dot{\gamma}}{2} & 0 \\ \phi_1\frac{\dot{\gamma}}{2} & -p + \phi_2\frac{\dot{\gamma}^2}{4} & 0 \\ 0 & 0 & -p \end{bmatrix},$$

where ϕ_1 and ϕ_2 are even functions of $\dot{\gamma}$. The components

$$\begin{cases} T_{12} = \phi_1\dfrac{\dot{\gamma}}{2}, \\ T_{11} - T_{22} = 0, \\ T_{22} - T_{33} = \phi_2\dfrac{\dot{\gamma}^2}{4}, \end{cases}$$

are called the *shear stress*, the *first normal stress* and the *second normal stress*. From experimental results on real fluids (see [2]) there is no evidence that fluids exhibiting a zero value for the first normal stress has a non zero second normal stress. Hence we confine ourselves to fluids in which $\phi_2 = 0$. Moreover, since for the vast majority of viscometric flows (i.e. the class of flows we are interested in) the quantity $\det\mathbf{D}$ is identically zero, we get

$$\mathbf{T} = -p\mathbf{I} + \phi_1\left(-\frac{1}{2}\operatorname{tr}\mathbf{D}^2\right)\mathbf{D}. \tag{6.4}$$

From thermodynamics condition (6.3), we get $\phi_1 \geqslant 0$. Fluids of type (6.4) are called *generalized Newtonian fluids*.

6.2 Some example of generalized Newtonian fluids

Generalized Newtonian fluids are characterized by constitutive equation (6.4) in which the shear stress is a nonlinear function of the shear rate. We may rewrite the constitutive equation in the following form

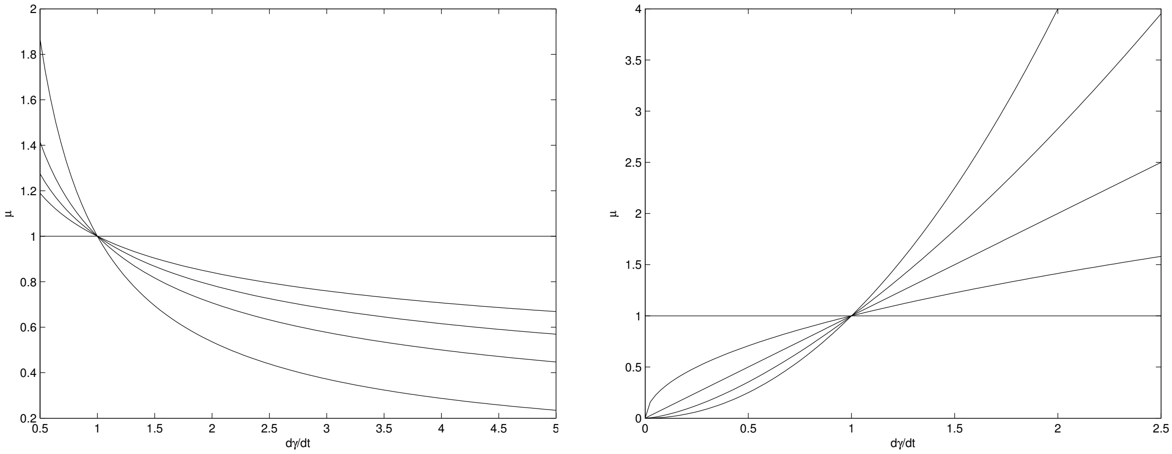
$$\mathbf{T} = -p\mathbf{I} + 2\mu(\dot{\gamma})\mathbf{D}, \tag{6.5}$$

where the function $\mu(\dot{\gamma})$ is called the *apparent viscosity* and

$$\dot{\gamma} = \sqrt{\frac{1}{2}\mathbf{D}\cdot\mathbf{D}}.$$

Power-law. These fluids are characterized by an apparent viscosity of the form

$$\mu(\dot{\gamma}) = k\dot{\gamma}^{n-1},$$



(a) Power law - Shear thinning (b) Power law - Shear thickening

Figure 6.2

and a constitutive equation of the type

$$\mathbf{T} = -p\mathbf{I} + 2k\dot{\gamma}^{n-1}\mathbf{D},$$

where the dimensionless parameter n is called the *flow behaviour index* and k is the so-called *consistency*. When $n = 1$ the classical incompressible Newtonian model is recovered. When $n < 1$ the constitutive equation is shear thinning, whereas in the case $n > 1$ the constitutive equation is shear thickening. The apparent viscosity is such that

$$n < 1 \quad \text{(Shear thinning)} \quad \lim_{\dot{\gamma}\to 0} \mu(\dot{\gamma}) = \infty \qquad \lim_{\dot{\gamma}\to \infty} \mu(\dot{\gamma}) = 0$$

$$n > 1 \quad \text{(Shear thickening)} \quad \lim_{\dot{\gamma}\to 0} \mu(\dot{\gamma}) = 0 \qquad \lim_{\dot{\gamma}\to \infty} \mu(\dot{\gamma}) = \infty$$

The unboundness of the apparent viscosity and the absence of non zero viscosity for vanishing shear rate actually limits the applicability of power law fluids.

Prandtl-Erying. These fluids are such that

$$\mu(\dot{\gamma}) = \mu_c \frac{\sinh^{-1}(2k\dot{\gamma})}{2k\dot{\gamma}},$$

where

$$\sinh^{-1}(z) = \ln(z + \sqrt{1+z^2}),$$

and

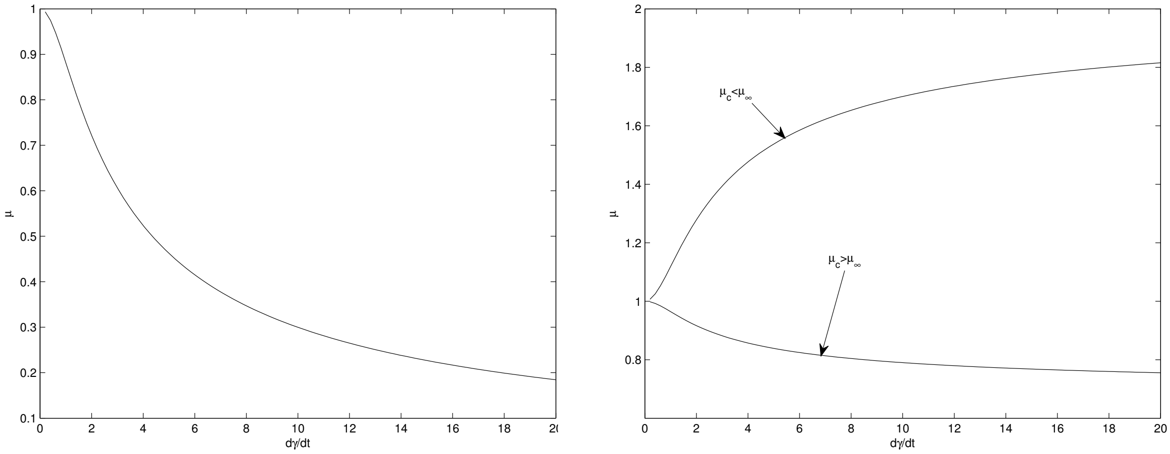
$$\mathbf{T} = -p\mathbf{I} + \mu_c \frac{\sinh^{-1}(2k\dot{\gamma})}{k\dot{\gamma}}\mathbf{D},$$

and where μ_c and k are material constants. The apparent viscosity (see Fig. 6.3a) tends to zero as $\dot{\gamma}$ tends to infinity

$$\lim_{\dot{\gamma}\to 0} \mu(\dot{\gamma}) = \mu_c \qquad \lim_{\dot{\gamma}\to \infty} \mu(\dot{\gamma}) = 0.$$

Powell-Eyring. These fluids are characterized by a three constant model. The apparent viscosity is bounded by a non zero value at both the upper and the lower limits

$$\mu(\dot{\gamma}) = \mu_\infty + (\mu_c - \mu_\infty) \frac{\sinh^{-1}(2k\dot{\gamma})}{2k\dot{\gamma}},$$



(a) Prandtl-Eyring fluid

(b) Powell-Eyring fluid

Figure 6.3

and

$$\mathbf{T} = -p\mathbf{I} + \left[2\mu_\infty + (\mu_c - \mu_\infty)\frac{\sinh^{-1}(2k\dot{\gamma})}{k\dot{\gamma}}\right]\mathbf{D},$$

where μ_c, μ_∞ and k are material constants. The apparent viscosity (see Fig. 6.3b) is such that

$$\lim_{\dot{\gamma}\to 0}\mu(\dot{\gamma}) = \mu_c \qquad \lim_{\dot{\gamma}\to\infty}\mu(\dot{\gamma}) = \mu_\infty.$$

Cross. This is a four constant model where the apparent viscosity is

$$\mu(\dot{\gamma}) = \mu_\infty + (\mu_c - \mu_\infty)\left[\frac{1}{1 + (2k\dot{\gamma})^{n-1}}\right],$$

and

$$\mathbf{T} = -p\mathbf{I} + 2\left[\mu_\infty + (\mu_c - \mu_\infty)\frac{1}{1 + (2k\dot{\gamma})^{n-1}}\right]\mathbf{D},$$

and where μ_c, μ_∞ and k are material constants. The apparent viscosity (see Fig. 6.4a) is such that

$$n > 1 \qquad \lim_{\dot{\gamma}\to 0}\mu(\dot{\gamma}) = \mu_c \qquad \lim_{\dot{\gamma}\to\infty}\mu(\dot{\gamma}) = \mu_\infty$$

$$n < 1 \qquad \lim_{\dot{\gamma}\to 0}\mu(\dot{\gamma}) = \mu_\infty \qquad \lim_{\dot{\gamma}\to\infty}\mu(\dot{\gamma}) = \mu_c$$

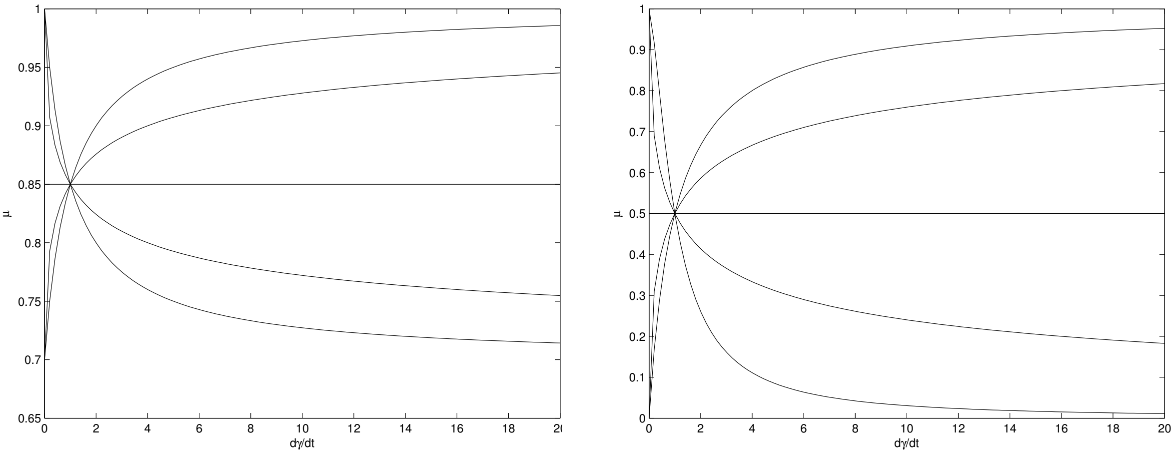
Ellis. Since often the flow regime of interest is such that a non zero μ_∞ is not required, we may set $\mu_\infty = 0$ in the Cross model, obtaining the so called Ellis model

$$\mu(\dot{\gamma}) = \left[\frac{\mu_c}{1 + (2k\dot{\gamma})^{n-1}}\right],$$

with

$$\mathbf{T} = -p\mathbf{I} + 2\left[\frac{\mu_c}{1 + (2k\dot{\gamma})^{n-1}}\right]\mathbf{D},$$

and where μ_c and k are material constants. The main advantage of Ellis model is that it is possible to construct analytical solution for some simple flows (see [4]).



(a) Cross fluid (b) Ellis fluid

Figure 6.4

6.3 The Bingham model

An interesting generalized Newtonian fluid is the so-called *Bingham plastic.* This type of fluid is characterized by the presence of stress threshold below which the fluid is incapable of flowing. This threshold is termed *yield stress* and must be overcome in order to observe a non zero strain rate. The apparent viscosity of a Bingham fluid is a multivalued function for zero strain rate.

In a Bingham fluid the stress $\mathbf{T} = -p\mathbf{I} + \mathbf{S}$ is defined as

$$
\begin{cases} \mathbf{S} = \left(2\mu_y + \dfrac{\tau_y}{\dot{\gamma}}\right)\mathbf{D}, & \tau \geqslant \tau_y, \\ \mathbf{D} = 0, & \tau \leqslant \tau_y, \end{cases} \tag{6.6}
$$

where μ_y is the viscosity, τ_y is the yield stress and

$$\tau = \sqrt{\frac{1}{2}\mathbf{S}\cdot\mathbf{S}}.$$

Therefore we can write

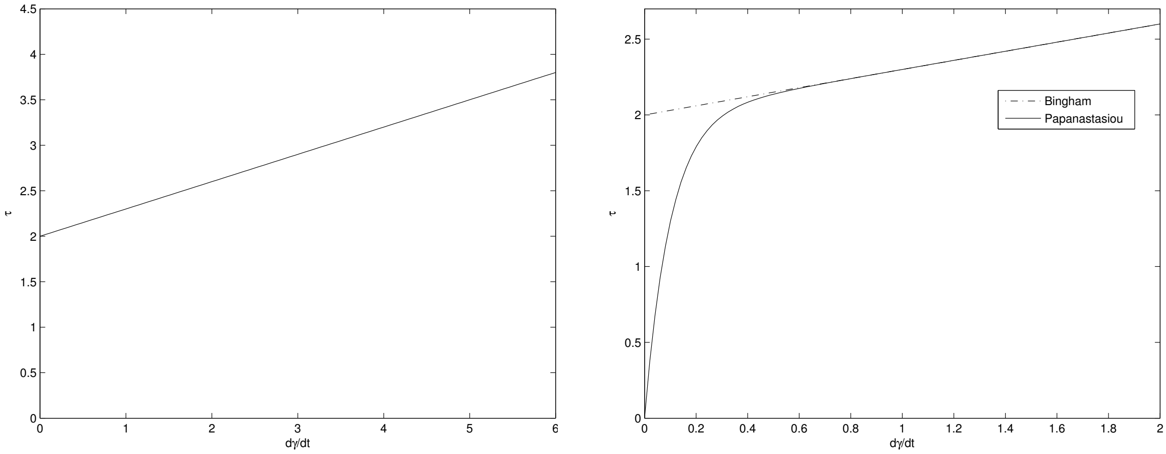
$$(\tau - \tau_y)_+ = 2\mu_y\dot{\gamma}.$$

In Fig. 6.5a the stress-strain relation of the Bingham model is shown. As one can notice, when $\dot{\gamma} = 0$ the stress τ is indeterminate. In complex flows where the shear stress is not constant throughout the flow, there may be regions where the yield criterion is reached (and the fluid is flowing) and other regions where $\tau < \tau_y$ and the fluid behaves like a rigid body, i.e. with $\mathbf{D} = 0$. Note that $\mathbf{D} = 0$ does not necessarily imply $\mathbf{v} = 0$. In some peculiar cases analytical solutions of the Bingham model can be found.

In some particular geometries the Bingham model may lead to paradoxes (e.g. lubrication paradox, see [19]) consisting in the unyielded phase that does not behave as a rigid body. Recently Fusi et al. [8], [9], [10], [11] have explained and overcome such paradoxes.

6.4 Papanastasiou model

Even though the Bingham model appears to well model some real fluids (such as foams, pastes, mayonnaise, suspensions etc), the constitutive equation is in general complex to model numeri-



(a) Bingham model

(b) Papanastasiou model

Figure 6.5

cally because of the difficulty in tracking the yield surface. This is essentially due to the singularity of the apparent viscosity when $\tau \leqslant \tau_y$. The Papanastasiou model overcomes this difficulty by smoothing the singularity. Indeed in the Papanastasiou model

$$\mathbf{S} = \left[2\mu_y + \frac{\tau_y \left(1 - e^{-n\dot{\gamma}}\right)}{\dot{\gamma}} \right] \mathbf{D},$$

and

$$\tau = 2\mu_y\dot{\gamma} + \tau_y(1 - e^{-n\dot{\gamma}}),$$

as shown in Fig. 6.5b. In the limit $n \to \infty$ we recover the classical Bingham model.

6.5 Flow of a generalized Newtonian fluid in a pipe

Here we consider the steady flow in a pipe of circular cross section, as the one considered in Section 5.11 for Newtonian fluids. We assume

$$\mathbf{v} = v_z(r)\mathbf{e}_z. \tag{6.7}$$

The volumetric flow rate is given by

$$Q = 2\pi \int_0^R v_z r dr. \tag{6.8}$$

When a closed form solution can be calculated for a given apparent viscosity $\mu(\dot{\gamma})$, then one can calculate the volumetric flow rate from (6.8). Unfortunately most generalized Newtonian flows does not admit a closed form solution, so that Q can be evaluated only numerically. In some cases it is possible to extrapolate a relation between the pressure gradient, the flow rate and $\mu(\dot{\gamma})$. A velocity field of the form (6.7) identically satisfies the incompressibility condition.

Moreover

$$\mathbf{D} = \begin{bmatrix} 0 & 0 & \frac{1}{2}\left(\frac{\partial v_z}{\partial r}\right) \\ 0 & 0 & 0 \\ \frac{1}{2}\left(\frac{\partial v_z}{\partial r}\right) & 0 & 0 \end{bmatrix}, \qquad \mathbf{D}^2 = \begin{bmatrix} \frac{1}{4}\left(\frac{\partial v_z}{\partial r}\right)^2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \frac{1}{4}\left(\frac{\partial v_z}{\partial r}\right)^2 \end{bmatrix},$$

so that

$$\dot\gamma = \sqrt{\frac{1}{2}\mathbf{D}\cdot\mathbf{D}} = -\frac{1}{2}\frac{\partial v_z}{\partial r}, \tag{6.9}$$

where we have taken the minus sign since we expect velocity to be a decreasing function of r. The principal invariants are

$$i_1(\mathbf{D}) = 0, \qquad i_2(\mathbf{D}) = -\dot\gamma^2 \qquad i_3(\mathbf{D}) = 0.$$

Recalling (6.5) the stress tensor is

$$\mathbf{T} = \begin{bmatrix} -p & 0 & -2\mu(\dot\gamma)\dot\gamma \\ 0 & -p & 0 \\ -2\mu(\dot\gamma)\dot\gamma & 0 & -p \end{bmatrix},$$

From the balance of linear momentum in cylindrical coordinates derived in Section 3.9 we find

$$\begin{cases} 0 = \dfrac{\partial p}{\partial r}, \\ 0 = \dfrac{1}{r}\dfrac{\partial p}{\partial \theta}, \\ 0 = -\dfrac{\partial p}{\partial z} + \dfrac{1}{r}\dfrac{\partial}{\partial r}(rT_{rz}), \end{cases} \tag{6.10}$$

Therefore $p = p(z,t)$ and $\partial p/\partial z$ must not depend on z, namely

$$p = -\Theta z + f(t).$$

Integration of $(6.10)_3$ yields

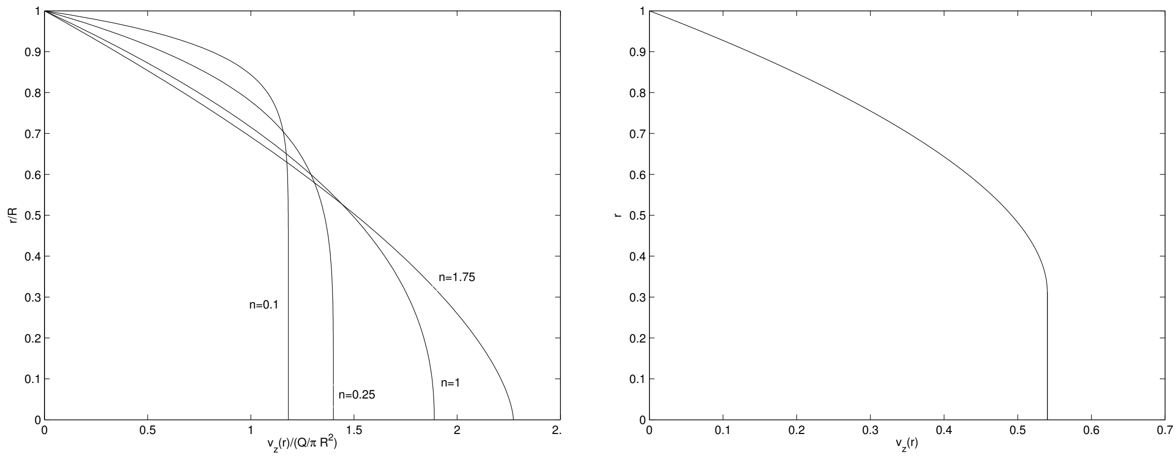
$$T_{rz} = -\frac{\Theta r}{2}, \tag{6.11}$$

where we have exploited the symmetry condition $T_{rz} = 0$. As a consequence

$$\mu(\dot\gamma)\dot\gamma = \frac{\Theta r}{4}. \tag{6.12}$$

To proceed further we must specify the viscosity function. If we consider, for instance, power law fluids where

$$\mu(\dot\gamma) = k\dot\gamma^{n-1},$$



(a) Power law non dimensional velocity profile

(b) Bingham velocity profile

Figure 6.6

then

$$\dot{\gamma} = -\frac{1}{2}\frac{\partial v_z}{\partial r} = \left(\frac{\Theta r}{4k}\right)^{1/n}.$$

Integrating with the no-slip condition $v_z(R) = 0$ we get

$$v_z(r) = \left(\frac{\Theta R}{4k}\right)^{1/n} \frac{2R}{1/n+1}\left[1 - \left(\frac{r}{R}\right)^{1/n+1}\right].$$

Exploiting (6.8) we also find

$$Q = \left(\frac{\Theta R}{4k}\right)^{1/n} \frac{2\pi R^3}{3+1/n}.$$

In conclusion we can express the velocity field as a function of Q

$$v_z(r) = \frac{Q}{\pi R^2}\left[\frac{1/n+3}{1/n+1}\right]\left[1 - \left(\frac{r}{R}\right)^{1/n+1}\right]. \tag{6.13}$$

In Fig. 6.6a we have plotted the nondimensional velocity profiles $v_z(r)/(Q/\pi R^2)$ as a function of the nondimensional radius r/R, for different values of n. We notice that, for the same flow rate, the velocity field of the shear thinning power-law model ($n < 1$) is flatter than the Newtonian model $n = 1$. When considering a Bingham fluid

$$\tau = |T_{rz}| = \frac{\Theta r}{2},$$

which we will be the largest at $r = R$. Therefore, when $\Theta R 2^{-1} < \tau_y$ the yield criterion is never met and $\mathbf{D} = 0$ throughout the channel. In this case, from the no-slip condition, $\mathbf{v} = 0$ everywhere. If, on the other hand $\Theta R 2^{-1} > \tau_y$ then there will be a region adjacent to the wall where the criterion is met and the fluid flows as a viscous Newtonian fluid and an inner core where $\mathbf{D} = 0$. The interface separating the two regions is given by

$$|T_{rz}| = \tau_y \qquad \Longleftrightarrow \qquad r_y = \frac{2\tau_y}{\Theta}.$$

Hence recalling $(6.10)_3$ and (6.6) we have

$$\frac{\partial p}{\partial z} = \frac{1}{r}\frac{\partial}{\partial r}\left[\frac{r}{2}\frac{\partial v_z}{\partial r}\left(2\mu_y + \frac{\tau_y}{\dot\gamma}\right)\right],$$

or equivalently

$$\frac{\partial p}{\partial z} = \frac{1}{r}\frac{\partial}{\partial r}\left[r\left(\mu_y\frac{\partial v_z}{\partial r} + \tau_y \,\mathrm{sgn}\,\frac{\partial v_z}{\partial r}\right)\right].$$

Since we are looking for solutions where $\partial v_z/\partial r < 0$ we get

$$-\Theta r = \frac{\partial}{\partial r}\left[r\left(\mu_y\frac{\partial v_z}{\partial r} - \tau_y\right)\right].$$

Integrating between r_y and $r > r_y$ and recalling that $\partial v_z/\partial r = 0$ on $r = r_y$ we find

$$\mu_y\frac{\partial v_z}{\partial r} = \frac{\Theta}{2}\left(r_y - r\right). \tag{6.14}$$

Integrating once more between r and R with the no-slip condition $v_z(R) = 0$ we may write the velocity field for the whole domain as

$$\begin{cases} v_z(r) = \dfrac{\Theta R^2}{4\mu_y}\left[1 - \left(\dfrac{r}{R}\right)^2\right] - \dfrac{\tau_y R}{\mu}\left[1 - \dfrac{r}{R}\right] & r \in [r_y, R], \\ v_z(r) = \dfrac{\Theta R^2}{4\mu_y}\left[1 - \dfrac{r_y}{R}\right]^2 & r \in [0, r_y]. \end{cases}$$

The volumetric flow rate can be easily calculated from (6.8). In Fig. 6.6b the velocity profile of the Bigham fluid flowing in a cylindrical duct is shown.

6.6 Evaluation of Q in a cylindrical duct

For the large majority of generalized Newtonian fluids it is impossible to determine a closed form solution of the velocity field. Here we show how to evaluate the volumetric flow rate without obtaining an explicit expression of velocity. Recalling definition (6.8) and integrating by parts we get

$$Q = 2\pi\left[\underbrace{v_z(r)\frac{r^2}{2}\Big|_0^R}_{=0} - \int_0^R \frac{r^2}{2}\left(\frac{\partial v_z}{\partial r}\right)dr\right] = -\pi\int_0^R r^2\left(\frac{\partial v_z}{\partial r}\right)dr. \tag{6.15}$$

Integrating once more by parts we find

$$Q = \frac{\pi}{3}\left[-R^3\,\frac{\partial v_z}{\partial r}\Big|_R + \int_0^R r^3\left(\frac{\partial^2 v_z}{\partial r^2}\right)dr\right]. \tag{6.16}$$

Now recall (6.9) and set

$$\dot\gamma_w = \dot\gamma(R) = -\frac{1}{2}\,\frac{\partial v_z}{\partial r}\Big|_R.$$

Relation (6.16) can be rewritten as

$$Q = \frac{\pi}{3}\left[2R^3\dot{\gamma}_w + \int_0^R r^3\left(\frac{\partial^2 v_z}{\partial r^2}\right)dr\right]. \tag{6.17}$$

Exploiting the substitution

$$\dot{\gamma}(r) = -\frac{1}{2}\left.\frac{\partial v_z}{\partial r}\right|_r \qquad \frac{d\dot{\gamma}}{dr} = -\frac{1}{2}\frac{\partial^2 v_z}{\partial r^2},$$

we can rewrite (6.17) as

$$Q = \frac{2\pi}{3}\left[R^3\dot{\gamma}_w - \int_0^{\dot{\gamma}_w} r^3(\dot{\gamma})d\dot{\gamma}\right].$$

Equation (6.12) provides

$$r = r(\dot{\gamma}) = \frac{4\mu(\dot{\gamma})\dot{\gamma}}{\Theta},$$

while $\dot{\gamma}_w$ is found solving

$$\mu(\dot{\gamma}_w)\dot{\gamma}_w = \frac{\Theta R}{4}. \tag{6.18}$$

In conclusion

$$Q = \frac{2\pi}{3}\left[R^3\dot{\gamma}_w - \frac{64}{\Theta^3}\int_0^{\dot{\gamma}_w} \mu^3\dot{\gamma}^3 d\dot{\gamma}\right]. \tag{6.19}$$

Once $\dot{\gamma}_w$ is obtained from (6.18), we can integrate (6.19) and get Q as a function of the pressure gradient Θ. Notice that, when μ is constant (Newtonian fluid) we get

$$Q = \frac{\pi\Theta R^4}{8\mu}, \tag{6.20}$$

which is exactly the volumetric flow rate obtained in Section 5.11.

6.7 Obtaining $\mu(\dot{\gamma})$ from pressure drop and flow rate

Capillary viscometer are tools designed to reproduce the fully developed flow of type (6.7) in cylindrical ducts. Their use is based on the idea that measures of pressure drops and flow rates can provide, via balance of linear momentum, the expression of $\mu(\dot{\gamma})$. We shall see that the problem can be reduced to obtaining μ as a function of a characteristic shear rate $\dot{\gamma}_c$ and of the shear stress at the wall τ_w, where

$$\dot{\gamma}_c = \frac{2Q}{\pi R^3}, \tag{6.21}$$

is the shear rate at the wall for a Newtonian fluid. Recalling (6.9) and (6.15) we find

$$\dot{\gamma}_c = \frac{4}{R^3}\int_0^R r^2\dot{\gamma}(r)dr. \tag{6.22}$$

From (6.11) we find

$$\tau_w = T_{rz}\Big|_R = -\frac{\Theta R}{2}, \tag{6.23}$$

so that τ_w can be easily evaluated from pressure drop measurements. We get

$$r = -\frac{2T_{rz}}{\Theta} = \frac{RT_{rz}}{\tau_w}. \tag{6.24}$$

From (6.12)

$$\mu(\dot\gamma)\dot\gamma = \frac{\Theta R}{4}\frac{T_{rz}}{\tau_w}.$$

When the above is invertible

$$\dot\gamma = \dot\gamma(T_{rz}).$$

Hence, from (6.22)

$$\dot\gamma_c = \frac{4}{R^3}\int_0^R \dot\gamma(T_{rz}) r^2(T_{rz}) dr. \tag{6.25}$$

Exploiting the substitution (6.24)

$$T_{rz} = \frac{r\tau_w}{R} \qquad \frac{dT_{rz}}{dr} = \frac{\tau_w}{R},$$

we find

$$\dot\gamma_c = \frac{4}{\tau_w^3}\int_0^{\tau_w} \dot\gamma(T_{rz}) T_{rz}^2 dT_{rz}.$$

If we now differentiate the above w.r.t. τ_w we get

$$\frac{d\dot\gamma_c}{d\tau_w} = -3\frac{\dot\gamma_c}{\tau_w} + 4\frac{\dot\gamma(\tau_w)}{\tau_w},$$

that, after some algebra can be rewritten as

$$4\left(\frac{\dot\gamma_w}{\dot\gamma_c}\right) = 3 + \frac{d\ \ln(\dot\gamma_c)}{d\tau_w}\cdot\frac{d\tau_w}{d\ \ln(\tau_w)}$$

where we have defined $\dot\gamma_w = \dot\gamma(\tau_w)$. Setting

$$\frac{1}{m} = \frac{d\ \ln(\dot\gamma_c)}{d\ \ln(\tau_w)},$$

we get

$$\dot\gamma_w = \dot\gamma_c\left(\frac{3m+1}{4m}\right). \tag{6.26}$$

Equation (6.26) is the *Mooney-Rabinowitsch equation*. Experiments can be run with different pressure drops to obtain a curve of $\dot\gamma_c$ as a function of τ_w, recall (6.21) and (6.23). Using a log scale the value m can be obtained from the slope of the curve. From the definition of the apparent viscosity evaluated at the wall (6.18) we find

$$\mu(\dot\gamma_w) = -\frac{\tau_w}{2\dot\gamma_w}, \tag{6.27}$$

which provides the apparent viscosity for the specific flow considered.

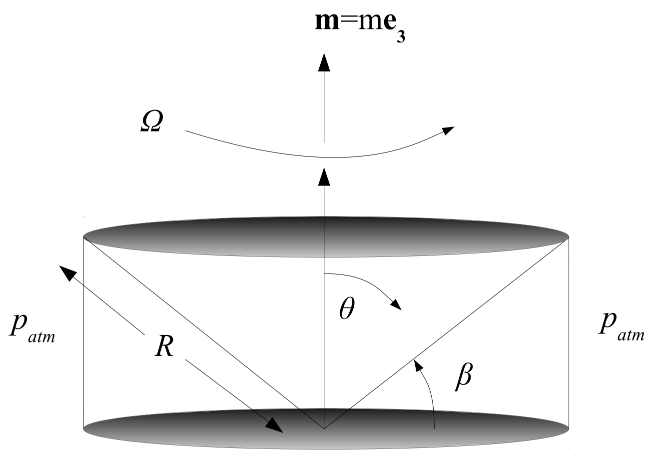


Figure 6.7: Schematic representation of the cone and plate rheometer. The angle β is commonly less that 10 degrees.

6.8 Cone and plate flow for a generalized Newtonian fluid

The cone and plate rheometer is a device used to measure viscosity and normal stress coefficients. A schematic representation of this device is shown in Fig. 6.7. The flow is driven by the rotation of the cone about the $\mathbf{e}_3$ axis. The fundamental relationships for this kind of rheometer can be developed using the spherical polar coordinates introduced in Section 1.5. In particular we look for a velocity field of the form

$$\mathbf{v} = v_\phi(r, \theta)\mathbf{e}_\phi,$$

where θ represent the polar angle. The boundary conditions are

$$\left\{ \begin{array}{l} \mathbf{v}\left(r, \dfrac{\pi}{2}\right) = 0, \\ \\ \mathbf{v}\left(r, \dfrac{\pi}{2} - \beta\right) = \Omega r \cos\beta\, \mathbf{e}_\phi, \\ \\ \mathbf{Tn} = -p_{atm}\mathbf{n} \end{array} \right. \tag{6.28}$$

where Ω is the angular velocity of the rotating cone, β is the angle formed between the cone and the plate, p_{atm} is the atmospheric pressure acting on the free lateral surface of the fluid whose normal is $\mathbf{n}$. From the boundary condition $(6.28)_2$ it is clear that the dependence of v_ϕ on r must be linear, that is

$$\mathbf{v} = r\omega(\theta)\mathbf{e}_\phi. \tag{6.29}$$

Recalling the definition of the gradient of a vector in spherical coordinates given in Section 1.5, we can easily prove that

$$\mathbf{D} = \frac{1}{2}\begin{bmatrix} 0 & 0 & 0 \\ \\ 0 & 0 & \omega^{'}(\theta) - \cot\theta\omega(\theta) \\ \\ 0 & \omega^{'}(\theta) - \cot\theta\omega(\theta) & 0 \end{bmatrix},$$

$$\mathbf{D}^2 = \frac{1}{4}\begin{bmatrix} 0 & 0 & 0 \\ 0 & [\omega'(\theta) - \cot\theta\omega(\theta)]^2 & 0 \\ 0 & 0 & [\omega'(\theta) - \cot\theta\omega(\theta)]^2 \end{bmatrix}.$$

Moreover

$$\dot{\gamma} = \sqrt{\frac{1}{2}\mathbf{D}\cdot\mathbf{D}} = -\sin\theta\frac{d}{d\theta}\left(\frac{\omega(\theta)}{\sin\theta}\right) = \cot\theta\omega(\theta) - \omega'(\theta), \tag{6.30}$$

where we have taken the minus sign because we want

$$\frac{d}{d\theta}\left(\frac{\omega(\theta)}{\sin\theta}\right) < 0.$$

The only non-zero stress components of the extra stress are

$$T_{\theta\phi} = T_{\phi\theta} = -\mu(\dot{\gamma})\dot{\gamma}.$$

Next we observe that $\dot{\gamma}$ is a function of θ only and so is $T_{\theta\phi}$. The equations of linear momentum (see Section 3.9) reduce to

$$\begin{cases} -\varrho r\omega^2 = -\dfrac{\partial p}{\partial r}, \\ -\varrho r^2\omega^2\cot\theta = -\dfrac{\partial p}{\partial \theta}, \\ 0 = -\dfrac{1}{\sin\theta}\dfrac{\partial p}{\partial \phi} + \dfrac{1}{\sin^2\theta}\dfrac{\partial}{\partial\theta}(T_{\theta\phi}\sin^2\theta). \end{cases} \tag{6.31}$$

Since $T_{\theta\phi}$ depends only on on θ, we conclude that p must be linear in ϕ

$$p = A(\theta)\phi + B(r,\theta).$$

Recalling boundary condition $(6.28)_3$ we see that $A \equiv 0$ and $p = p(r,\theta)$. Differentiating $(6.31)_1$ w.r.t θ and $(6.31)_2$ w.r.t r and subtracting we get

$$2\varrho r\omega[\omega' - \omega\ \cot\theta],$$

implying either $\omega = 0$ or

$$\frac{\omega'}{\omega} = \cot\theta \qquad \Longleftrightarrow \qquad \omega = A\sin\theta,$$

with A constant of integration. Unfortunately neither of these solutions satisfy the boundary conditions. This seems to indicate that our specific form of the velocity field is meaningful only when inertial effects are negligible, that is when the left hand sides of equations (6.31) are set to zero. In this case p is independent of θ and r and therefore constant and equal to p_{atm} throughout the fluid. From the third component of (6.31) we get

$$T_{\theta\phi} = -\frac{C}{\sin^2\theta}, \tag{6.32}$$

with C constant of integration. Let us see how to relate this constant to the applied torque $\mathbf{m} = m\mathbf{e}_3$. We recall that in our case the torque is given by

$$\mathbf{m} = m\mathbf{e}_3 = \int_S \Big(\mathbf{x}\times\mathbf{T}\mathbf{n}\Big)d\sigma,$$

where S is the lateral surface of the cone whose height is (see Fig. 6.7)

$$H = R\sin\beta.$$

The level arm $\mathbf{x}$ evaluated at the lateral surface of the cone S is

$$\mathbf{x} = (r\cos\beta\cos\phi)\,\mathbf{e}_1 + (r\cos\beta\sin\phi)\,\mathbf{e}_2,$$

or

$$\mathbf{x} = (r\cos^2\beta)\,\mathbf{e}_r + (r\cos\beta\sin\beta)\,\mathbf{e}_\theta,$$

when expressed in spherical coordinates. The stress the fluid is acting on S is

$$\mathbf{Tn}\Big|_S = p_a\,\mathbf{e}_\theta - T_{\theta\phi}\Big|_S\mathbf{e}_\phi,$$

so that

$$\Big(\mathbf{x}\times\mathbf{Tn}\Big)\Big|_S = p_a r\cos^2\beta\mathbf{e}_\phi + \Big(T_{\theta\phi} r\cos^2\beta\mathbf{e}_\theta - T_{\theta\phi} r\cos\beta\sin\beta\mathbf{e}_r\Big)\Big|_S.$$

Recalling (1.32) it is easy to show that

$$\Big(T_{\theta\phi} r\cos^2\beta\mathbf{e}_\theta - T_{\theta\phi} r\cos\beta\sin\beta\mathbf{e}_r\Big)\Big|_S = -T_{\theta\phi}\Big|_S r\cos\beta\mathbf{e}_3.$$

so that

$$\Big(\mathbf{x}\times\mathbf{Tn}\Big)\Big|_S = p_a r\cos^2\beta\Big(-\sin\phi\mathbf{e}_1 + \cos\phi\mathbf{e}_2\Big) - T_{\theta\phi}\Big|_S r\cos\beta\mathbf{e}_3.$$

As a consequence

$$\mathbf{m} = \left[\int_0^{2\pi} d\phi\int_0^R dr\left[p_a r^2\cos^3\beta\Big(-\sin\phi\mathbf{e}_1 + \cos\phi\mathbf{e}_2\Big) - T_{\theta\phi}\Big|_S r^2\cos^2\beta\mathbf{e}_3\right]\right]\mathbf{e}_3,$$

which reduces to

$$\mathbf{m} = 2\pi\left[\int_0^R\left[-T_{\theta\phi}\Big|_S r^2\cos^2\beta\right]dr\right]\mathbf{e}_3.$$

From (6.32)

$$T_{\theta\phi}\Big|_S = -\frac{C}{\cos^2\beta},$$

hence

$$\mathbf{m} = \frac{2\pi C R^3}{3}\mathbf{e}_3.$$

The torque $\mathbf{m} = m\mathbf{e}_3$ is such that

$$C = \frac{3m}{2\pi R^3},$$

and

$$T_{\theta\phi} = -\frac{3m}{2\pi R^3\sin^2\theta}. \tag{6.33}$$

We have obtained an expression relating the shear stress with the applied torque (the relation depends also on the geometry). To determine $\mu(\dot\gamma)$ we need to use (6.33) in conjunction with the expression of $T_{\theta\phi}$ for a specific constitutive equation.

Small angle approximation Let us consider now the special case in which the angle β is much less than one, i.e. $\beta \ll 1$. It is convenient to introduce the angle

$$\alpha = \frac{\pi}{2} - \theta.$$

In the fluid domain $\alpha < \beta \ll 1$ and

$$\cos\theta = \sin\alpha = \alpha + O(\alpha^2) = \alpha + O(\beta^2),$$

$$\sin\theta = \cos\alpha = 1 + O(\alpha^2) = 1 + O(\beta^2),$$

We have

$$T_{\theta\phi} = -\frac{C}{\sin^2\theta} = -\frac{C}{\cos^2\alpha} = -C + O(\beta^2).$$

From (6.30)

$$\dot{\gamma} = \Big(1 + O(\alpha^2)\Big)\frac{\partial}{\partial\alpha}\Big[w\Big(1 + O(\alpha^2)\Big)\Big]$$

Recalling that $\alpha \ll 1$ we expand ω as

$$\omega(\alpha) = K + J\alpha + O(\alpha^2),$$

where the constant of integration can be obtained imposing the boundary conditions (6.28)

$$\begin{cases} \omega\Big|_{\theta=\pi/2} = \omega\Big|_{\alpha=0} = K = 0, \\ \\ \omega\Big|_{\theta=\pi/2-\beta} = \omega\Big|_{\alpha=\beta} = J\beta = \Omega. \end{cases}$$

Therefore

$$\omega(\alpha) = \frac{\Omega\alpha}{\beta} + O(\beta^2).$$

The approximate value of $\dot{\gamma}$ is

$$\dot{\gamma} = \frac{\Omega}{\beta} + O(\beta^2).$$

As a consequence

$$\mu\left(\frac{\Omega}{\beta}\right) = \frac{3m}{2\pi R^3}\left(\frac{\beta}{\Omega}\right). \tag{6.34}$$

Hence, for a given geometry (R, β) we may determine the form of the functional μ by changing m and Ω. Note that in the small angle approximation (where inertial effects are ignored) both $T_{\theta\phi}$ and $\dot{\gamma}$ can be considered constant throughout the fluid domain.

Finite angle approximation: power law fluids Relation (6.34) relates μ to the applied torque, Ω, β and R in the case of small angle β and negligible inertial effects. What happens when these assumptions are relaxed? Whereas things get really more complicated when inertial effects are considered, we may still get some closed form solutions even in the case in which the angle β is not small. To obtain something similar to (6.34) we must consider particular constitutive equations. Let us focus, for instance, on the power-law models introduced earlier.

A power-law fluid undergoing a flow motion of type (6.29) is such that

$$T_{\theta\phi} = -\mu(\dot{\gamma})\dot{\gamma} = -k\dot{\gamma}^n. \tag{6.35}$$

Recalling (6.29) and (6.32) we find

$$\dot{\gamma} = -\sin\theta \frac{d}{d\theta}\left(\frac{\omega}{\sin\theta}\right) = \left(-\frac{T_{\theta\phi}}{k}\right)^{1/n} = \left(\frac{C}{k}\right)^{1/n}\left[\sin\theta\right]^{-2/n}, \tag{6.36}$$

so that

$$\frac{d}{d\theta}\left(\frac{\omega}{\sin\theta}\right) = -\left(\frac{C}{k}\right)^{1/n}\left[\sin\theta\right]^{-1-2/n}.$$

Integrating with boundary condition $\omega(\pi/2) = 0$, we find

$$\omega(\theta) = \left(\frac{C}{k}\right)^{1/n}\sin\theta \int_0^{\pi/2}\left[\sin\xi\right]^{-1-2/n} d\xi. \tag{6.37}$$

We can obtain Ω evaluating ω at the cone surface and exploiting $(6.28)_2$. The integral in (6.37) can be easily computed when $1 + 2/n$ is an integer. Then, defining

$$\mathcal{F}(\beta) = \int_{\pi/2-\beta}^{\pi/2}\left[\sin\xi\right]^{-1-2/n} d\xi,$$

we get, recalling the boundary condition $(6.28)_2$,

$$\Omega = \left(\frac{C}{k}\right)^{1/n}\mathcal{F}(\beta).$$

From (6.36) we find

$$\dot{\gamma} = \frac{\Omega}{\mathcal{F}(\beta)\left[\sin\theta\right]^{2/n}} \qquad \dot{\gamma}_w = \frac{\Omega}{\mathcal{F}(\beta)\left[\cos\beta\right]^{2/n}}.$$

From (6.35) we easily find that

$$\mu(\dot{\gamma}) = k\left(\frac{\Omega}{\mathcal{F}(\beta)}\right)^{n-1}\left[\cos\beta\right]^{2/n-2}$$

When $n = 2$ we find

$$\mathcal{F}(\beta) = \int_{\pi/2-\beta}^{\pi/2}\left[\sin\xi\right]^{-2} d\xi = \tan\beta,$$

and

$$\mu(\dot{\gamma}) = k\left(\frac{\Omega}{\sin\beta}\right).$$

Recalling (6.32) we get

$$T_{\theta\phi}\Big|_w = -\mu(\dot{\gamma}_w)\dot{\gamma}_w = -\frac{3m}{2\pi R^3\cos^2\beta},$$

so that

$$\mu(\dot{\gamma}_w) = \frac{3m}{2\pi R^3\cos^2\beta}\frac{\sin\beta}{\Omega}. \tag{6.38}$$

Index

Bibliography

[1] Apostol T.M., Calculus, Vol. II, Wiley 1969.

[2] Astarita G., Marrucci G., Principles of Non-Newtonian fluid mechanics, Mac-Graw Hill, 1974.

[3] Batchelor G.K., An introduction to fluid dynamics, Cambridge University Press, 2000.

[4] Bird R.B., Useful Non-Newtonian models, Ann. Rev. Fluid Mech., 8, 13-34, 1976.

[5] Bowen R.M., Wang C.C., Introduction to vectors and tensors, Vol. 1, Plenum, 1976.

[6] Brinkmann H.W., Klotz E.A., Linear algebra and analytic geometry, Addison-Wesley, 1971.

[7] Chorin A.J., Marsden J. E., A mathematical introduction to fluid mechanics, Springer, 1980.

[8] Fusi L., Farina A., Rosso F., On the mathematical paradoxes for the flow of a viscoplastic film down an inclined surface, Int. J. Nonlin. Mech., 58, 139-150, (2014).

[9] Fusi L., Farina A., Rosso F., Retrieving the Bingham model from a bi-viscous model: some explanatory remarks, Applied Mathematical Letters, 27, 11–14, 2014.

[10] Fusi L., Farina A., Rosso F., Roscani S., Pressure driven lubrication flow of a Bingham Fluid in a channel: A Novel Approach, Journal of Non-Newtonian Fluid Mechanics, 221, 66-75, 2015.

[11] Fusi L., Farina A., Rosso F., Planar squeeze flow of a Bingham fluid, Journal of Non-Newtonian Fluid Mechanics, 225, 1-9, 2015.

[12] Gurtin M.E., An introduction to continuum mechanics, Academic Press, 1981.

[13] Kellogg O.D., Foundations of potential theory, Dover, 1953.

[14] Korn G. A., Korn, T. M., Mathematical Handbook for Scientists and Engineers, McGraw-Hill, 1968.

[15] Lamb H., Hydrodynamics, Dover 1945.

[16] Landau L.D., Lifschitz E.M., Fluid Mechanics, Pergamon, 1987.

[17] Lang S. , Complex analysis, Springer 1975.

[18] Lighthill M.J., An Informal Introduction to Theoretical Fluid Mechanics, Clarendo, 1986.

[19] Lipscomb G., Denn M., Flow of Bingham fluids in complex geometries, Journal of Non-Newtonian Fluid Mechanics, 142, 183-198, (1984).

[20] Marsden J. E., Tromba A., Vector calculus, W.H. Freeman, 2003.

[21] Meyer R.E., Introduction to mathematical fluid dynamics, Wiley, 1971.

[22] Prandtl L., Essentials of Fluid Dynamics, Hafner, 1952.

[23] Serrin J., Mathematical principles of classical fluid mechanics, Springer, 1959.

[24] Smith D.R., An introduction to continuum mechanics, Springer 1993.

[25] Truesdell C., Noll W., The non-linear field theories of mechanics, Springer, 1965.

[26] Truesdell C., Rajagopal K.R.., An introduction to fluid mechanics, Birkhäuser, 2000.

MAT - INDICE GENERAL

Ver: http://web.austral.edu.ar/cienciasEmpresariales-investigacion-mat-A.asp

Serie A: CONFERENCIAS, SEMINARIOS Y TRABAJOS DE MATEMÁTICA (ISSN 1515-4904, 2468-9734)

1 (2000): Elvira Mascolo – Francesco Siepe, "Functionals of the Calculus of Variations with non Standard Growth Conditions", 1-22.
2 (2000): Domingo A. Tarzia, "A Bibliography on Moving-Free Boundary Problems for the Heat-Diffusion Equation. The Stefan and Related Problems", 1-297.
3 (2001): Domingo A. Tarzia (Ed.), "VI Seminario sobre Problemas de Frontera Libre y sus Aplicaciones", Primera Parte:

- Ma. Cristina Sanziel, "Conditions to obtain a waiting time for a discrete two-phase Stefan problem", 1-6.
- Ariel L. Lombardi – Domingo A. Tarzia, "On similarity solutions for thawing processes", 7-12.
- Ricardo Weder, "Direct and inverse scattering for the nonlinear Schrödinger equation with a potential", 13-20.
- Domingo A. Tarzia, "Stefan problem for a non-classical heat equation", 21-26.
- Pedro Morin – Rubén D. Spies, "A quasilinearization approach for parameter identification in nonlinear abstract Cauchy problems", 27-41.

4 (2001): Domingo A. Tarzia (Ed.), "VI Seminario sobre Problemas de Frontera Libre y sus Aplicaciones", Segunda Parte:

- Omar Gil, "El problema de Hele-Shaw como un problema límite para la ecuación de los medios porosos", 1-10.
- Juan C. Reginato – Domingo A. Tarzia, "Estimations of nutrient uptakes by roots of crops through a moving boundary model", 11-16.
- Oscar D. Quiroga – Luis T. Villa – Fernando Suarez, "Problemas de frontera libre en procesos de transferencia de materia y energía con reacción química", 17-22.
- Edgardo A. Spiazzi – Rodolfo H. Mascheroni, "Modelo de deshidratación osmótica de alimentos vegetales", 23-32.
- Eduardo A. Santillan Marcus – Domingo A. Tarzia, "Exact solutions for phase change processes in humid porous half spaces", 33-38.

5 (2001): Domingo A. Tarzia (Ed.), "VI Seminario sobre Problemas de Frontera Libre y sus Aplicaciones", Tercera Parte:

- Adriana C. Briozzo – Domingo A. Tarzia, "On a two-phase Stefan problem with nonlinear thermal coefficients", 1-10.
- Germán Torres – Cristina V. Turner, "Métodos de diferencias finitas para un problema de Bingham unidimensional", 11-26.
- Analía Gastón – Gustavo Sánchez Sarmiento – Horacio Reggiardo, "Un problemas de frontera libre: Fusión de una vaina de acero dentro de una cuchara de acería", 27-32.
- Ma. Fernanda Natale – Domingo A. Tarzia, "An exact solution for a one-phase Stefan problem with nonlinear thermal coefficient", 33-36.
- Claudia Lederman – Juan L. Vazquez – Noemí Wolanski, "Uniqueness of solution to a free boundary problem from combustion with transport", 37-41.

6 (2002): Federico Talamucci, "Some Problems Concerning with Mass and Heat Transfer in a Multi-Component System", 1-28.
7 (2004): Domingo A. Tarzia (Ed.), "Primeras Jornadas sobre Ecuaciones Diferenciales, Optimización y Análisis Numérico", Primera Parte:

- Adriana B. Verdiell – María C. Maciel – Susana L. Orofino – Tatiana I. Gibelli, "A survey of the spectral gradient method", 1-14.
- María F. Natale – Domingo A. Tarzia, "An integral equation in order to solve a one-phase Stefan problem with nonlinear thermal conductivity", 15-24.
- María C. Sanziel – Domingo A. Tarzia, "Optimization on the heat flux in a mixed elliptic problem with temperature constraints", 25-30.
- Claudia M. Gariboldi – Domingo A. Tarzia, "A new proof of the convergence of distributed optimal controls on the internal energy in mixed elliptic problems", 31-42.

8 (2004): Domingo A. Tarzia (Ed.), "Primeras Jornadas sobre Ecuaciones Diferenciales, Optimización y Análisis Numérico", Segunda Parte:

- Rubén D. Spies, "Differentiability of the solutions of a semilinear abstract Cauchy problem with respect to parameters", 1-10.
- Adriana C. Briozzo – María F. Natale – Domingo A. Tarzia, "An explicit solution for a two-phase Stefan problem with a similarity exponencial heat sources", 11-19.
- Domingo A. Tarzia, "An explicit solution for a two-phase unidimensional Stefan problem with a convective boundary condition at the fixed face", 21-27.

9 (2005): Micol Amar – Roberto Gianni, "A Brief Survey on Homogenization with a Physical Application", 1-20.
10 (2005): Domingo A. Tarzia – Cristina V. Turner (Eds.), "Segundas Jornadas sobre Ecuaciones Diferenciales, Optimización y Análisis Numérico":

- Marcos Gaudiano – Cristina Turner, "Difusión de un solvente en un polímero vidrioso con una condición de contorno del tipo creciente en el tiempo", 1-9.
- Adriana C. Briozzo – María F. Natale – Domingo A. Tarzia, "A one-phase Lamé-Clapeyron-Stefan problem with nonlinear thermal coefficients", 11-16.
- Eduardo A. Santillan Marcus - Domingo A. Tarzia, "Un caso de determinación de coeficientes térmicos desconocidos de un material semiinfinito poroso a través de un problema de desublimación con acoplamiento de temperatura y humedad", 17-22.

11 (2005): Antonio Fasano, "Mathematical Models of Some Diffusive Processes with Free Boundaries", 1-128.

12 (2006): Arturo De Pablo, “An Introduction to the Problem of Blow-up for Semilinear and Quasilinear Parabolic Equations”, 1-28.
13 (2006): Graciela G. Garguichevich – C112laudia M. Gariboldi – Pedro R. Marangunic – Diego Pallara, “Direct methods in the calculus of variations”, 1-41.
14 (2007): María C. Maciel – Domingo A. Tarzia (Eds.), “Terceras Jornadas sobre Ecuaciones Diferenciales, Optimización y Análisis Numérico”:

- Tatiana I. Gibelli – María C. Maciel, “Large-scale algorithms for minimizing a linear function with a strictly convex quadratic constraint”, 1-12.
- María C. Maciel – Elvio A. Pilotta – Graciela N. Sottosanto, “Thickness optimization of an elastic beam”, 13-23.
- María F. Natale – Eduardo A. Santillan Marcus – Domingo A. Tarzia, “Determinación de dos coeficientes térmicos a través de un problema de desublimación con acoplamiento de temperatura y humedad”, 25-30.
- Rubén D. Spies – Karina G. Temperini, “Sobre la no convergencia del método de mínimos cuadrados en dimension infinita”, 31-34.
- Juan C. Reginato – Domingo A. Tarzia, “An alternative method to compute Michaelis-Menten parameters from nutrient uptake data”, 35-40.

15 (2008): D.A. Tarzia – R.H. Mascheroni (Eds.), “Workshop on Mathematical Modelling of Energy and Mass Transfer Processes, and Applications”:

- María F. Natale – Domingo A. Tarzia, “The classical one-phase Stefan problem with temperature-dependent termal conductivity and a convective term”, 1-16.
- Carmen Cortazar – Manuel Elgueta, “Non-local diffusion”, 17-24.
- Luis T. Villa, “Consideraciones sobre el buen planteo de un modelo de frontera libre-móvil descriptivo de un proceso de freído por inmersión”, 25-36.
- Ricardo Simpson – Isabel Figueroa – Arthur Teixeira, “Simple, practical, and efficient on-line correction of process deviations in batch retort through simulation”, 37-53.

16 (2009): Mahdi Boukrouche, “A brief survey on lubrication problems with nonlinear boundary conditions”, 1-38.
17 (2010): Carlos Vázquez, “An introduction to Black-Scholes modeling and numerical methods in derivatives pricing”, 1-41.
18 (2011): Pablo M. Jacovkis, “A glimpse into mathematical modelling”, 1-30.
19 (2014): D.A. Tarzia (Ed.), “VII Italian – Latin American Conference on Industrial and Applied Mathematics”, Part I:

- Graciela M. Croceri – Graciela N. Sottosanto, “Comparación de dos algoritmos de lagrangiano aumentado para el problema de optimización con restricciones de igualdad”, 1-7.
- Martha Hilda Timoteo Sánchez – Yolanda Santiago Ayala, “Sobre la existencia y unicidad de la solución de un modelo de propagación del sonido en un fluido compresible”, 9-16.
- Carlos Andrés Trujillo-Salazar – Hernán Darío Toro-Zapata, “Análisis teórico de la transmisión y el control del VIH/SIDA en un centro de reclusión”, 7-26.
- Santiago C. Rojas Romero, “Sobre el espectro de Fucik para un sistema acoplado”, 27-34.
- Manuel Maurette, “A topological approach to the repulsive central motion problem”, 35-42.

20 (2015): D.A. Tarzia (Ed.), “VII Italian – Latin American Conference on Industrial and Applied Mathematics”, Part II:

- Yolanda Santiago Ayala, “Sobre la estabilidad de un sistema elástico dinámico en cristales cúbicos”, 1-8.
- Matías Barber – Martín Maas – Francisco Grings – Haydee Karzembaum, “Un enfoque Bayesiano para la estimación de humedad del suelo a partir de datos SAR polarimétricos en banda L”, 9-16.
- Luis T. Villa – Angélica C. Bouciguez – Ricardo F. Lozano, “Estrategia de cálculo para la resolución de la primera etapa del proceso de freído por inmersión”, 17-21.
- Rodrigo Castro – Pablo Jacovkis, “Global models and applied mathematics”, 23-30.
- Adriana C. Briozzo – Domingo A. Tarzia, “Convergence of the solution of the one-phase Stefan problem with respect two parameters”, 31-38.

21 (2016): Lorenzo Fusi, “An introduction to fluid mechanics”, 1-110.

Serie B: CURSOS Y SEMINARIOS PARA EDUCACIÓN MATEMÁTICA (ISSN 1515-4912)

1 (2000): Domingo A. Tarzia, “Cómo pensar, entender, razonar, demostrar y crear en Matemática”, 1-76.
2 (2003): Domingo A. Tarzia, “Matemática: Operaciones numéricas y geometría del plano”, 1-80.

INFORMACION PARA LOS AUTORES

Los trabajos han de estar escritos en español o inglés. Excepcionalmente el Director y el Comité Editorial podrán admitir trabajos escritos en otros idiomas ampliamente utilizados. Deberá presentarse el texto mecanografiado o elaborado mediante un procesador de textos, con caracteres de 12 puntos, en un rectángulo de 16cm×24cm y en una sola cara del papel. Trabajos escritos en LATEX o en MS-WORD serán bienvenidos y en ese caso el autor deberá adjuntar un diskette con los archivos correspondientes, o bien enviarlos por correo electrónico.

En cada trabajo deberá constar, en la primera página, a partir de la quinta línea, el título en letras mayúsculas y sin punto final, el nombre del o de los autores, su identificación institucional y su correspondiente dirección postal y electrónica. Se acompañará un resumen que no exceda las 200 palabras en español y otro en inglés, añadiendo en ambos las palabras claves. También se solicita la inclusión de la corresponiente AMS-Mathematics Subject Classification. Al final de la última página deberán dejarse al menos dos líneas en blanco para incluir los datos de recepción. Las tablas y gráficos deberán insertarse en el texto y estar numeradas en forma correlativa.

Las referencias bibliográficas se compondrán sólo de los trabajos mencionados en el texto y se incluirán al final, por orden alfabético de autores y en orden cronológico, si existieran varios trabajos del mismo autor; cada una precedida por el correspondiente número de orden, entre corchetes. Las citas en el texto se efectuarán según los siguientes modelos: [1]; Caffarelli & Vazquez [1]; Caffarelli & Vazquez (1995, [1]). Y en la referencia final:

[1] L. A. CAFFARELLI & J.L. VAZQUEZ, *A free-boundary problem for the heat equation arising inflame propagation*, Trans. Amer. Math. Soc., 347 (1995), pp. 411-441.

[2] A. FASANO & M. PRIMICERIO, *Blow-up and regularization for the Hele-Shaw problem,* in *Variational and free boundary problems*, Friedman A. & Spruck J. (Eds.), IMA Math. Appl. Vol. 53, Springer Verlag, New York (1993), pp. 73-85.

[3] J.F. RODRIGUES, *Obstacle problems in mathematical physics*, North-Holland, Amsterdam (1987).

INTERCAMBIOS

Departamento de Matemática – Biblioteca, Servicio de Canje
Facultad de Ciencias Empresariales -Universidad Austral
Paraguay 1950, S2000FZF ROSARIO, ARGENTINA

NUMEROS APARECIDOS en Serie A:

- #1 (2000): E. Mascolo – F. Siepe, "Functionals of the Calculus of Variations with non Standard Growth Conditions".
- #2 (2000): D.A. Tarzia, "A Bibliography on Moving-Free Boundary Problems for the Heat-Diffusion Equation. The Stefan and Related Problems".
- # 3 (2001): D.A. Tarzia (Ed.), "VI Seminario sobre Problemas de Frontera Libre y sus Aplicaciones", Primera Parte.
- #4 (2001): D.A. Tarzia (Ed.), "VI Seminario sobre Problemas de Frontera Libre y sus Aplicaciones", Segunda Parte.
- #5 (2001): D.A. Tarzia (Ed.), "VI Seminario sobre Problemas de Frontera Libre y sus Aplicaciones", Tercera Parte.
- #6 (2002): F. Talamucci,"Some Problems Concerning with Mass and Heat Transfer in a Multi-Component System".
- #7 (2004): D.A. Tarzia (Ed.), "Primeras Jornadas sobre Ecuaciones Diferenciales, Optimización y Análisis Numérico", Primera Parte.
- #8 (2004): D.A. Tarzia (Ed.), "Primeras Jornadas sobre Ecuaciones Diferenciales, Optimización y Análisis Numérico", Segunda Parte.
- #9 (2005): M. Amar – R. Gianni, "A Brief Survey on Homogenization with a Physical Application".
- #10 (2005): D.A. Tarzia – C.V. Turner (Eds.), "Segundas Jornadas sobre Ecuaciones Diferenciales, Optimización y Análisis Numérico".
- #11 (2005): A. Fasano, "Mathematical Models of Some Diffusive Processes with Free Boundaries".
- #12 (2006): A. De Pablo, "An Introduction to the Problem of Blow-up for Semilinear and Quasilinear Parabolic Equations".
- #13 (2006): G.G. Garguichevich – C.M. Gariboldi – P.R. Marangunic – D. Pallara, "Direct methods in the calculus of variations".
- #14 (2007): M.C. Maciel – D.A. Tarzia (Eds.), "Terceras Jornadas sobre Ecuaciones Diferenciales, Optimización y Análisis Numérico".
- #15 (2008): D.A. Tarzia – R.H. Mascheroni (Eds.), "Workshop on Mathematical Modelling of Energy and Mass Transfer Processes, and Applications".
- #16 (2009): M. Boukrouche, "A brief survey on lubrication problems with nonlinear boundary conditions".
- #17 (2010): C. Vázquez, "An introduction to Black-Scholes modeling and numerical methods in derivatives pricing".
- #18 (2011): J.M. Jacovkis, "A glimpse into mathematical modelling".
- #19 (2014): D.A. Tarzia (Ed.), "VII Italian-Latin American Conference on Industrial and Applied Mathematics", Part 1.
- #20 (2015): D.A. Tarzia (Ed.), "VII Italian-Latin American Conference on Industrial and Applied Mathematics", Part 2.
- #21 (2016): L. Fusi, "An introduction to fluid mechanics".

NUMEROS APARECIDOS en Serie B:

- #1(2000): D.A. Tarzia, "Cómo pensar, entender, razonar, demostrar y crear en Matemática".
- #2(2003): D.A. Tarzia, "Matemática: Operaciones numéricas y geometría del plano".

DEPARTAMENTO DE MATEMÁTICA
FACULTAD DE CIENCIAS EMPRESARIALES
UNIVERSIDAD AUSTRAL
Paraguay 1950 - S2000FZF ROSARIO - ARGENTINA
TEL: +54 - 341- 522-3000 ; FAX: +54 - 341-522-3001
E-mail: revistamat@austral.edu.ar