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NON-LOCAL DIFFUSION

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1. Introduction.

Let $K : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$ be a nonnegative, smooth function such that $\int_{\mathbb{R}^N} K(x,y)dy = 1$ for all $x \in \mathbb{R}^N$.

Equations of the form

$$u_t(x,t) = \int_{\mathbb{R}^N} K(y,x)u(y,t)dy - u(x,t) \text{ in } \mathbb{R}^N \times [0,\infty) \tag{1.1}$$

have been widely used to model diffusion processes, see [1], [2], [8], [9], [10]. As stated in [8] if $u(x,t)$ is thought of as a density at the point x at time t and $K(y,x)$ is thought of as the probability distribution of jumping from location y to location x, then $\int_{\mathbb{R}^N} K(y,x)u(y,t)dy$ is the rate at which individuals are arriving to position x from all other places. On the other hand $-u(x,t) = -\int_{\mathbb{R}^N} K(x,y)u(x,t)dy$ is the rate at which they are leaving location x to travel to all other sites. This consideration, in the absence of external sources, leads immediately to the fact that the density u satisfies equation (1.1).

In this note we will describe some of the results obtained recently by the authors, in collaboration with J. Coville, S. Martinez, J. Rossi and N. Wolanski, in the topic of non local diffusion. We have decided to describe the results in a rather informal fashion since detailed statements and proofs can be found in the corresponding references.

2. An homogeneous model.

The Cauchy problem in $\mathbb{R}^N$.

A type of kernels that have been widely used in modeling diffusion are kernels of the form

$$K(y,x) = J(x-y)$$

*MAT-Serie A, 15 (2008), 17-24

where $J : \mathbb{R}^N \to \mathbb{R}$ is a smooth non negative function such that $\int_{\mathbb{R}^N} J(x)dx = 1$ and that we will assume symmetric with respect to the origin. Moreover, throughout this note, we will assume that J is supported in the unit ball centered at the origin $B(0,1)$. In this case equation (1.1) takes the form

$$u_t = J * u - u \tag{2.1}$$

where $J*u$ denotes the convolution of J and u. Since J is supported in $B(0,1)$ individuals at location x are not allowed to jump, up to probability zero, off the ball $B(x,1)$. By this reason we refer to this dispersion process as an homogeneous random walk, continuous in time, of step of size one.

Equation (2.1) can be written in integral form

$$u(x,t) = e^{-t}u(x,0) + \int_0^t e^{-(t-s)} \int_{\mathbb{R}^N} J(x-y)u(y,s)dyds. \tag{2.2}$$

It follows that existence and uniqueness of solutions of (2.1) can be obtained by an application of Banach's fixed point theorem to the right hand side operator of (2.2) in a suitable space of functions. It is also a consequence of the proof that if $u(x,0) \geq 0$, then $u(x,t) \geq 0$ for all $t \geq 0$. This plus the fact that the problem is linear imply the following comparison principle for two solutions u_1 and u_2 of (2.1):

$$u_1(x,0) \leq u_2(x,0) \Rightarrow u_1(x,t) \leq u_2(x,t) \text{ for all } t \geq 0.$$

An important aspect of equation (2.1) is its relation, the so called Brownian Motion, with the classic heat equation

$$v_t = \Delta v \text{ in } \mathbb{R}^N \times [0,\infty) \tag{2.3}$$

described in the following theorem which is classical. We give its proof to illustrate the techniques that can be used.

Theorem 2.1 *Let $\varepsilon > 0$ and let u^ε be a solution of the re-scaled problem*

$$u_t^\varepsilon(x,t) = \frac{1}{\varepsilon^2}\left[\int_{\mathbb{R}^{\mathbb{N}}} J(\frac{x-y}{\varepsilon})\frac{u^\varepsilon(y,t)}{\varepsilon^N}dy - u^\varepsilon(x,t)\right].$$

Let v be a solution of

$$v_t = A\Delta v$$

where A is a suitable constant that will be apparent in the proof.

Assume that $u^\varepsilon(x,0) \equiv v(x,0)$ and that $v(\cdot,0)$ is smooth enough. Then

$$\lim_{\varepsilon\to 0} u^\varepsilon = v \textit{ uniformly in } \mathbb{R}^{\mathbb{N}} \times [\nvdash, \mathbb{T}]$$

for any $T > 0$.

Proof:

To simplify the notation set $u = u^\varepsilon$ and define

$$L_\varepsilon u = \frac{1}{\varepsilon^2}\left[\int_{\mathbb{R}^{\mathbb{N}}} J(\frac{x-y}{\varepsilon})\frac{u(y,t)}{\varepsilon^N}dy - u(x,t)\right],$$

Set $w = u - v$ and note that$w$ satisfies

$$w_t = L_\varepsilon w + F$$

where

$$F(x,t) = L_\varepsilon v(x,t) - A\Delta v(x,t).$$

Making the change of variables

$$z = \frac{x-y}{\varepsilon}$$

in the integral, noting that $\int_{\mathbb{R}^N} J(z)dz = 1$ and choosing properly the constant A we have

$$F(x,t) = \frac{1}{\varepsilon^2}\left[\int_{\mathbb{R}^N} J(z)\left(v(x+\varepsilon z,t) - v(x,t) - \varepsilon^2|z|^2\Delta v(x,t)\right)dz\right].$$

Using the symmetry of J we obtain

$$F(x,t) =$$

$$\frac{1}{\varepsilon^2}\left[\int_{\mathbb{R}^N} J(z)\left(v(x+\varepsilon z,t) - v(x,t) - \varepsilon\sum_{i=1}^N \frac{\partial v}{\partial z_i}(x,t)z_i - \varepsilon^2\sum_{i,j=1}^N \frac{\partial^2 v}{\partial z_i \partial z_j}(x,t)z_i z_j\right)dz\right].$$

Finally using Taylor's expansion for the regular function v we get

$$F(x,t) = \varepsilon B(x,t)$$

where B is a function which is bounded independently of ε.

So w satisfies

$$w_t \le L_\varepsilon w + \varepsilon M$$

and

$$w(x,0) = 0.$$

Since the function $h(t) = t\varepsilon M$ satisfies

$$h_t = L_\varepsilon h + \varepsilon M$$

and

$$h(x,0) = 0$$

by comparison we have

$$|u^\varepsilon(x,t) - v(x,t)| \le t\varepsilon M$$

and the proof is finished. □

Remark 2.1 *The kernel in the re-scaled problem has been modified in such a way that the size of the step now is ε. The shortening of the step size is compensated by the multiplication by $\frac{1}{\varepsilon^2}$ of the right hand side. This factor represents an speed up of the walk necessary to compensate reduction of the size of the step.*

The Neumann problem.

The following model for the Neumann problem has been proposed in [6] and [7]. Let Ω be a domain in $\mathbb{R}^{\mathbb{N}}$ and $g : \overline{\mathbb{R}^{\mathbb{N}} \setminus \Omega} \to \mathbb{R}$ smooth. Consider the problem

$$u_t(x,t) = \int_\Omega J(x-y)(u(y,t)-u(x,t))dy + \int_{\mathbb{R}^{\mathbb{N}}\setminus\Omega} J(x-y)g(y,t). \tag{2.4}$$

In this model we have that the first integral takes into account the diffusion inside Ω. In fact, the integral $\int_\Omega J(x-y)(u(y,t)-u(x,t))dy$ takes into account the individuals arriving or leaving position x from or to other places. Since we are integrating in Ω, we are imposing that diffusion takes place only inside Ω. The last term takes into account the prescribed flux (given by the data $g(x,t)$) of individuals from outside (that is individuals that enter or leave the domain according to the sign of g). This is what is called Neumann boundary conditions.

Existence, uniqueness and some qualitative properties, such as the asymptotic behavior, of the solutions of problem (2.4) with suitable initial conditions have been studied in [6].

With respect to its relation with the classical Neumann problem the following result has been obtained in [7]: Let u^ε be a solution of the re-scaled problem

$$u_t^\varepsilon(x,t) = \frac{1}{\varepsilon^{N+2}} \int_\Omega J(\frac{x-y}{\varepsilon})(u^\varepsilon(y,t)-u^\varepsilon(x,t))dy$$
$$+\frac{1}{\varepsilon^{N+1}} \int_{\mathbb{R}^{\mathbb{N}}\setminus\Omega} J(\frac{x-y}{\varepsilon})g(y,t).$$

and let v be the solution of

$$v_t = A\Delta v \text{ in } \Omega\times[0,\infty),$$
$$v = g \text{ on } \partial\Omega\times[0,\infty).$$

Assume that for all ε one has

$$u^\varepsilon(x,0) = v(x,0) \text{ on } \Omega.$$

Then u^ε converges to v as $\varepsilon \to 0$. The convergence is uniform on compact subsets in the case that $g \equiv 0$ and takes place weakly in $C([0,T], L^1(\Omega))$ in the general case.

The Dirichlet problem.

A non local Dirichlet problem has been proposed in [5] as follows: Let $\Omega \subset \mathbb{R}^N$ and $h; \mathbb{R}^N \to \mathbb{R}$. Consider the problem

$$\begin{aligned} u_t(x,t) &= \int_{\mathbb{R}^N} J(x-y)u(y,t)dy - u(x,t) \text{ for } x\in\Omega, \\ u(x,t) &= h(x) \text{ for } x\in\mathbb{R}^N\setminus\Omega \ \text{ and } t\geq 0 \end{aligned} \tag{2.5}$$

with initial condition $u(x,0)$.

Existence, uniqueness and some properties of the solutions of problem (2.5) have been proved in [5]. Moreover it is proved there that if u^ε is the solution of the corresponding, to this case, re-scaled problem with $u^\varepsilon(x,0) = v(x,0)$ where v is the solution of

$$v_t = A\Delta v \text{ in } \Omega \times [0,\infty),$$

$$v = h \text{ on } \partial\Omega \times [0,\infty).$$

Then u^ε converges to v uniformly on compact subsets of $\Omega \times [0,\infty)$.

3. A non homogeneous model.

In [3] a non homogeneous dispersal model in the real line was studied. Kernels of the form

$$K(x,y) = J\left(\frac{x-y}{g(y)}\right)\frac{1}{g(y)}$$

were considered where $g : \mathbb{R} \to \mathbb{R}$ is a continuous bounded function that satisfies: The set $\{x \in \mathbb{R} \mid g(x) = 0\} \cap [-K, K]$ is finite for any $K > 0$. If $g(\bar{x}) = 0$, then there exist $r > 0$, $C > 0$ and $0 < \alpha < 1$ such that $g(x) \geq C|x - \bar{x}|^\alpha$ for all $x \in [\bar{x} - r, \bar{x} + r]$.

The evolution equation considered in this case is

$$u_t(x,t) = \int_{\mathbb{R}} J\left(\frac{x-y}{g(y)}\right)\frac{u(y,t)}{g(y)}dy - u(x,t). \tag{3.1}$$

Under the above mentioned hypotheses it is proved that (3.1) has a globally defined mass preserving solution for any given $u(\cdot,0) \in L^1(\mathbb{R})$. Moreover even though g can vanish at some points, these solutions have an infinite speed of propagation in the sense that if $u(x,0) \geq 0$ and $u(x,0) \neq 0$, then $u(x,0) > 0$ for all x and all $t > 0$.

In order to study the asymptotic behavior of solutions of (3.1) we are lead to the analysis stationary solutions, namely solutions of the equation

$$p(x) = \int_{\mathbb{R}} J\left(\frac{x-y}{g(y)}\right)\frac{p(y)}{g(y)}dy. \tag{3.2}$$

It is proved the existence of bounded positive solutions of (3.2) and that solutions of (3.1) converge to solutions of (3.2) as $t \to \infty$.

An important role in the study of problem (3.2) is played by the following lemma that can be of independent interest.

Lemma 3.1 *A continuous function p is a solution of* (3.2) *if and only if there exist constants P and Q such that*

$$\int_0^1 \int_{x-w}^{x+w} p(s) \int_{\frac{w}{g(s)}}^1 J(z)dzdsdw \equiv Px + Q.$$

Proof: Differentiate twice the left hand side to obtain 0 if and only if p is a solution of (3.2). $\square$

The hypothesis that if $g(\bar{x}) = 0$, then there exist $r > 0$, $C > 0$ and $0 < \alpha < 1$ such that $g(x) \geq C|x - \bar{x}|^\alpha$ for all $x \in [\bar{x} - r, \bar{x} + r]$ is important in our arguments. Nothing is known, to the best of our knowledge, if it does not hold.

4. A non linear model.

In this section we will deal with a non linear model that was introduced in [4]. We propose to use a kernel where the size of the step depends on the density at the point. The simplest model, with $N = 1$, is

$$K(y, x) = J\left(\frac{x-y}{u(y,t)}\right)\frac{1}{u(y,t)}.$$

The equation that governs the dispersal becomes in this case

$$u_t(x,t) = \int_{\mathbb{R}} J\left(\frac{x-y}{u(y,t)}\right)\frac{1}{u(y,t)}dy - u(x,t). \tag{4.1}$$

One of the main features of solutions of (4.1) is that if the support of the initial condition $u(\cdot, 0)$ is compact, then the support of $u(\cdot, t)$ remains compact for all $t > 0$. This gives rise to a free boundary like in the case of the porous medium equation $u_t = (u^m)_{xx}$ with $m > 1$.

With the additional hypothesis that J is decreasing in the interval $[0, 1]$ one can prove existence, uniqueness and a comparison principle for solutions of (4.1). Moreover it is trivial to check that the constant functions are solutions of (4.1) and hence if $u(x, 0) \geq 0$, then $u(x,t) \geq 0$ for all $t > 0$.

We state now as a theorem the fact that the compactness of the support is preserved and provide a proof.

Theorem 4.2 *If $u(\cdot, 0) \geq 0$ is compactly supported and bounded then $u(\cdot, t)$ is also compactly supported for all $t \geq 0$.*

Proof: Due to the scaling invariance of the equation, namely if $u(x, t)$ is a solution then for any $\lambda > 0$ the function $v_\lambda(x,t) = \lambda u(\frac{x}{\lambda}, t)$ is also a solution, we can restrict ourselves to initial data supported in $[-1, 1]$ and such that $\sup_{x\in\mathbb{R}} u(x, 0) \leq 1$.

We note first that

$$u_t(x,t) \leq \int_{\mathbb{R}} J\left(\frac{x-y}{u(y,t)}\right) dy. \tag{4.2}$$

Therefore, since $0 \leq u \leq 1$, we get by (4.2) that

$$u(x,t) \leq \frac{1}{2} \text{ for all } t \leq \frac{1}{2} \text{ and all } x \text{ such that } |x| \geq 1.$$

Now if $|x| \geq 2$ and $t \leq \frac{1}{2}$ we have that $|x - y| \leq u(y, t)$ implies that $|y| \geq 1$ and hence $u(y,t) \leq \frac{1}{2}$. Therefore, again by (4.2), we have

$$u(x,t) \leq \frac{1}{4} \text{ for all } t \leq \frac{1}{2} \text{ and all } x \text{ such that } |x| \geq 2.$$

We look now at the case $|x| \geq 2 + \frac{1}{2}$ and $t \leq \frac{1}{2}$. In this case $|x - y| \leq u(y,t)$ implies that $|y| \geq 2$ and hence $u(y,t) \leq \frac{1}{4}$. Again by (4.2), we have

$$u(x,t) \leq \frac{1}{8} \text{ for all } t \leq \frac{1}{2} \text{ and all } x \text{ such that } |x| \geq 2 + \frac{1}{2}.$$

Repeating this procedure we obtain by induction that for any integer $n \geq 1$ one has

$$u(x,t) \leq \frac{1}{2^{n+2}} \text{ for all } t \leq \frac{1}{2} \text{ and all } x \text{ such that } |x| \geq 2 + \sum_{k=1}^{n} \frac{1}{2^k}.$$

It follows that the support of $u(\cdot,t)$ is contained in the interval $[-3,3]$ for all $t \leq \frac{1}{2}$ as we wanted to prove. □

We will give now a formal argument that suggest what relation is expected to exist between these non linear random walks and the porous medium equation.

Consider the re-scaled problem

$$u_t(x,t) = \frac{1}{\varepsilon^2}\left[\int_{\mathbb{R}} J\left(\frac{|x-y|}{\varepsilon u(y,t)}\right)\frac{dy}{\varepsilon} - u(x,t)\right]$$

and assume that their solutions u^ε converge to a function v as $\varepsilon \to 0$. In order to do not overload the notation we set $u^\varepsilon = u$.

We take the Fourier transform

$$\hat{u}_t(\xi,t) = \frac{1}{\varepsilon^2}\left[\int_{\mathbb{R}}\int_{\mathbb{R}} J\left(\frac{x-y}{\varepsilon u(y,t)}\right)e^{-ix\xi}\frac{dydx}{\varepsilon} - \hat{u}(\xi,t)\right].$$

Setting

$$z = \frac{x-y}{\varepsilon u(y,t)}$$

we have

$$\hat{u}_t(x,t) = \frac{1}{\varepsilon^2}\left[\int_{\mathbb{R}}\int_{\mathbb{R}} J(z)\, e^{-i\xi\varepsilon u(y,t)z}e^{-iy\xi}u(y,t)dzdy - \hat{u}(\xi,t)\right].$$

Or

$$\hat{u}_t(\xi,t) = \frac{1}{\varepsilon^2}\int_{\mathbb{R}}\left[\hat{J}\left(\xi\varepsilon u(y,t)\right) - 1\right]e^{-iy\xi}u(y,t)dy.$$

Taking the Taylor expansion of $\hat{J}$ about zero we get

$$\hat{u}_t(\xi,t) = \frac{1}{\varepsilon^2}\int_{\mathbb{R}}\left[\hat{J}''(0)\xi^2\varepsilon^2u^2(y,t)\right]e^{-iy\xi}u(y,t)dy + \frac{O(\varepsilon^3)}{\varepsilon^2}.$$

Or

$$\hat{u}_t(\xi,t) = C\int_{\mathbb{R}}(-\xi^2)u^3(y,t)e^{-iy\xi}dy + O(\varepsilon).$$

As $\varepsilon \to 0$

$$\hat{v}_t(x,t) = C\int_{\mathbb{R}}(-\xi^2)v^3(y,t)e^{-iy\xi}dy$$

which means

$$\hat{v}_t(x,t) = C\widehat{(v^3)_{xx}}.$$

Hence the solutions of the re-scaled problems should converge, as $\varepsilon \to 0$, to a solution of the porous medium equation

$$v_t = (v^3)_{xx}.$$

There are several questions that can be raised and for which we do not have an answer. For example: Do the free boundaries converge to the free boundary?

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