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CONVERGENCE OF THE SOLUTION OF THE ONE-PHASE STEFAN PROBLEM WITH RESPECT TWO PARAMETERS

Adriana C. Briozzo[♭] and Domingo A. Tarzia[†]

[♭]*Depto. Matemática, FCE, Universidad Austral and CONICET, Paraguay 1950, S2000FZF Rosario, Argentina, Abriozzo@austral.edu.ar*

[†]*Depto. Matemática, FCE, Universidad Austral and CONICET, Paraguay 1950, S2000FZF Rosario, Argentina, Dtarzia@austral*

Abstract: A one-phase unidimensional Stefan problem with a convective boundary condition at the fixed face $x = 0$, with a heat transfer coefficient $h > 0$ (proportional to the Biot number) and an initial position of the free boundary $b = s(0) > 0$ is considered. We study the limit of the temperature $\theta = \theta_{b,h}$ and the free boundary $s = s_{b,h}$ when $b \to 0^+$ (for all $h > 0$) and we also obtain an order of convergence. Moreover, we study the limit of the temperature $\theta_{b,h}$ and the free boundary $s_{b,h}$ when $(b, h) \to (0^+, 0^+)$.

Keywords: *Stefan problem, free boundary problem, phase-change process, convective boundary condiction.*
2000 AMS Subject Classification: 35R35, 80A22, 35C55

1 INTRODUCTION

In this paper, we consider the unidimensional free boundary problem (one-phase Stefan problem) with a convective boundary condition at the fixed boundary $\xi = 0$. It consists in determining the temperature $\theta = \theta(\xi, \tau)$ and the free boundary $\xi = s(\tau)$ which satisfy the following conditions

$$\begin{cases} (i)\ \rho c\theta_\tau - k\theta_{\xi\xi} = 0, & 0 < \xi < s(\tau), \tau > 0, \\ (ii)\ k\theta_\xi(0, \tau) = h\left[\theta(0, \tau) - f(\tau)\right], & \tau > 0, \\ (iii)\ \theta(s(\tau), \tau) = 0, & \tau > 0, \\ (iv)\ k\theta_\xi(s(\tau), \tau) = -\rho l \frac{ds}{d\tau}(\tau), & \tau > 0, \\ (v)\ \theta(\xi, 0) = \varphi(\xi), & 0 \leqslant \xi \leqslant b \\ (vi)\ s(0) = b\ (b > 0) & \end{cases} \tag{1}$$

where $b > 0$ is the initial position of the free boundary, $h > 0$ is the thermal transfer coefficient, $\varphi(\xi) \geq 0,\ 0 \leqslant \xi \leqslant b$, is the initial temperature, $f = f(\tau) \geq 0,\ \tau > 0$ is the temperature of the external fluid and the compatibility conditions $k\varphi'(0) = h(\varphi(0) - f(0))$ and $\varphi(b) = 0$ are assumed. The goal of this paper is to study the mathematical behavior of the solution $\theta = \theta_{b,h}(\xi, \tau),\ s = s_{b,h}(\xi, \tau)$ of problem (1) when $b \to 0^+$ (for each $h > 0$) and when $(b, h) \to (0^+, 0^+)$. The Stefan problem was studied in the last decades, see for example, [1], [3], [6], [9], [10], [11] and a large bibliography on the subject was given in [16].

Existence and uniqueness of solution to problem (1) is given in [7]. In [17] the behavior of the solution of the free boundary problem (1) with respect to the heat transfer coefficient h in the one-phase case was studied. A generalization of this result for the two-phase problem was considered in [18]. There, it was proved that the asymptotic behavior when $t \to \infty$ of the one-phase free boundary problem with a convective boundary condition at the fixed face is the same that for the case where the temperature boundary condition, which is depending on time, is given on $x = 0$. Asymptotic behavior for the one-phase problem with temperature boundary condition on the fixed face was given by [4, 5]. For the particular case $f(\tau) = Const > 0$, for the multidimensional case, the study of the asymptotic behavior when $h \to \infty$ is obtained by using the variational inequality [14, 15] and for the one-dimensional case in [13]. In [18] the monotone dependence of the solution with respect to the data and with respect to the thermal transfer coefficient is proved for the two phase Stefan problem. In [12], the classical one-phase Stefan problem is presented in dimensionless form with a time-varying-heat-power boundary condition. The asymptotic behavior of the solution for the generalized form of the Biot number $Bi \to 0$ was studied from a physical point of view. In [2] the mathematical analysis of this asymptotic behavior of the solution with respect to the heat transfer coefficient was considered and an order of convergence was also obtained.

The goal of this paper is to analize the asymptotic behavior of the solution of the problem (1) when $b \to 0^+$ (for h fixed), and the double asymptotic behavior when $(b,h) \to (0^+, 0^+)$.

We will make the following assumptions on the initial and boundary data:

(i) Let $\varphi = \varphi(\xi)$ be a positive and piecewise continuous function, with $\varphi'(\xi) \leq 0$.

(ii) Let $f = f(\tau)$ a positive bounded piecewise continuous function, with $f'(\tau) \geq 0$

(iii) Compatibility conditions: $f(0) > \varphi(\xi), \forall \xi \in (0,b),\ k\varphi'(0) = h(\varphi(0) - f(0))$and $\varphi(b) = 0$.

If we define the following transformation

$$u(x,t) = \frac{c}{l}\theta(\xi,\tau),\ x = \frac{\xi}{b_0}\ ,\ t = \frac{\alpha}{b_0^2}\tau \tag{2}$$

where $\alpha = \frac{k}{\rho c}$ is the diffusion coefficient and b_0 is a space reference scale, then the free boundary problem (1) becomes

$$\begin{cases} (i)\ u_t - u_{xx} = 0, & 0 < x < S(t),\ t > 0, \\ (ii)\ u_x(0,t) = \frac{hb_0}{k}\left[u(0,t) - F(t)\right], & t > 0, \\ (iii)\ u(S(t),t) = 0, & t > 0, \\ (iv)\ u_x(S(t),t) = -\dot{S}(t), & t > 0, \\ (v)\ u(x,0) = \chi(x) \geq 0, & 0 \leqslant x \leqslant \frac{b}{b_0} \\ (vi)\ S(0) = \frac{b}{b_0} \end{cases} \tag{3}$$

where

$$F(t) = \frac{c}{l}f(\frac{b_0^2 t}{\alpha}) \geq 0,\ H = b_0\frac{h}{k} > 0 \text{ (the Biot number)}, \tag{4}$$

$$\chi(x) = \frac{c}{l}\varphi(b_0 x),\ S(t) = \frac{1}{b_0}s(\frac{b_0^2 t}{\alpha}). \tag{5}$$

In Section 2, we enunciate some preliminary results about of the solution to problem (3). In Section 3 we study the convergence for the solution to problem (3) when $b \to 0^+$ and we give an order of convergence for the corresponding temperature and free boundary. In Section 4, we study the convergence for the solution to problem (3) when $(b,h) \to (0^+, 0^+)$ and we give an order of convergence for the corresponding temperature and the free boundary.

2 Properties of the solution to problem (3)

Under the condition stated in Introduction we have the following results:

Lemma 1 ([*13*], [*17*], [*18*]) *The solution $u = u_{bh}(x,t),\ S = S_{bh}(t)$ to problem (3) has the integral representation given by the following expressions*

$$u_{bh}(x,t) = \int_0^{\frac{b}{b_0}} N(x,t;\xi,0)\chi(\xi)d\xi + \int_0^t N(x,t;S_{bh}(\tau),\tau)V_{bh}(\tau)d\tau \tag{6}$$

$$-\frac{hb_0}{k}\int_0^t N(x,t;0,\tau)v_{bh}(\tau)d\tau + \frac{hb_0}{k}\int_0^t N(x,t;0,\tau)F(\tau)d\tau,$$

$$S_{bh}(t) = \frac{b}{b_0} - \int_0^t V_{bh}(\tau)d\tau \tag{7}$$

where the functions $V_{bh} = V_{bh}(t)$ and $v_{bh} = v_{bh}(t)$, defined as

$$\begin{cases} V_{bh}(t) = u_{bh_x}(S_{bh}(t),t),\ t > 0, \\ v_{bh}(t) = u_{bh}(0,t),\ t > 0, \end{cases} \tag{8}$$

are the solutions of the following system of integral equations:

$$V_{bh}(t) = 2\int_0^{\frac{b}{b_0}} \chi'(\xi)\, G(S_{bh}(t),t,\xi,0)d\xi - 2\int_0^t \frac{hb_0}{k}[v_{bh}(\tau) - F(\tau)]N_x(S_{bh}(t),t,0,\tau)d\tau$$

$$+2\int_0^t V_{bh}(\tau)N_x(S_{bh}(t),t,S_{bh}(\tau),\tau)d\tau, \tag{9}$$

$$v_{bh}(t) = \int_0^{\frac{b}{b_0}} \chi(\xi)\, N(0,t,\xi,0)d\xi - \int_0^t \frac{b_0 h}{k}[v_{bh}(\tau) - F(\tau)]N(0,t,0,\tau)d\tau \tag{10}$$

$$+\int_0^t V_{bh}(\tau)N(0,t,S_{bh}(\tau),\tau)d\tau,$$

for $0 < x < S_{bh}(t),\ 0 < t < T$, *where G and N are the Green and Neumann functions defined by:*

$$G(x,t,\xi,\tau) = K(x,t,\xi,\tau) - K(-x,t,\xi,\tau) \tag{11}$$

$$N(x,t,\xi,\tau) = K(x,t,\xi,\tau) + K(-x,t,\xi,\tau) \tag{12}$$

with

$$K(x,t,\xi,\tau) = \begin{cases} \frac{1}{2\sqrt{\pi(t-\tau)}} \exp\left(-\frac{(x-\xi)^2}{4(t-\tau)}\right) & t > \tau \\ 0 & t \le \tau. \end{cases} \tag{13}$$

For simplicity of notation, in what follows we denote $u = u_b$ and $S = S_b$ when we analize the dependence of the solution to problem (3) with respect to b. We denote $u = u_h$ and $S = S_h$ when we analize the dependence of the solution to problem (3) with respect to h.

Lemma 2 ([*13*], [*17*], [*18*])*The solution* $u = u_{bh}(x,t),\ S = S_{bh}(t)$ *to problem (3) satisfies the following inequalities:*

(a) $0 \le u_{bh}(x,t) \le F(t)$;

(b) $h_1 < h_2 \Rightarrow u_{h_1}(x,t) \le u_{h_2}(x,t)$;

(c) $h_1 < h_2 \Rightarrow S_{h_1}(t) \le S_{h_2}(t)$;

(d) $u_{bh}(x,t) \ge 0\ ,\ u_{bh_x} \le 0,\ u_{bh_t} = u_{bh_{xx}} \ge 0$

(e) $0 \le \dot{S}_{bh}(t) \le H.F(t)$

(f) for $0 < x < S_{bh}(t),\ t > 0$ *we have the following integral relations:*

$$S_{bh}(t) = \frac{b}{b_0} + \int_0^{\frac{b}{b_0}} \chi(x)dx - \frac{hb_0}{k}\int_0^t [u_{bh}(0,\tau) - F(\tau)]\, d\tau - \int_0^{S_{bh}(t)} u_{bh}(x,t)dx, \tag{14}$$

$$S_{bh}^2(t) = \frac{b^2}{b_0^2} - 2\int_0^{S_{bh}(t)} x\, u_{bh}(x,t)dx + 2\int_0^{\frac{b}{b_0}} x\, \chi(x)dx + 2\int_0^t u_{bh}(0,\tau)d\tau, \tag{15}$$

$$\int_0^{S_{bh}(t)} u_{bh}^2(x,t)dx - \int_0^{\frac{b}{b_0}} \chi^2(x)dx + 2\int_0^t\int_0^{S_{bh}(\tau)} u_{bh_x}^2(x,\tau)dxd\tau \le \frac{hb_0}{k}\int_0^t F^2(\tau)d\tau. \tag{16}$$

Lemma 3 *If* $b_1 < b_2$ *we have* $S_{b_1}(t) < S_{b_2}(t)$ *and* $u_{b_1}(x,t) < u_{b_2}(x,t)$, *for all* $t > 0, 0 < x < S_{b_1}(t)$.

Proof. If $b_1 < b_2$ we have $S_{b_1}(0) < S_{b_2}(0)$. We suppose that the assertion of Lemma 3 is false, that is there exists $t_0 > 0$ such that

$$S_{b_1}(t) < S_{b_2}(t),\ \forall\ 0 < t < t_0; \quad S_{b_1}(t_0) = S_{b_2}(t_0). \tag{17}$$

If we define

$$v(x,t) = u_{b_1}(x,t) - u_{b_2}(x,t),\ 0 < x < S_{b_1}(t),\ 0 < t < t_0$$

we have the following properties:

$$\begin{cases} (i)\ v_t - v_{xx} = 0,\ 0 < x < S_{b_1}(t), 0 < t < t_0 \\ (ii)\ v_x(0,t) = \frac{hb_0}{k}\left[u_{b_1}(0,t) - u_{b_2}(0,t)\right],\ 0 < t < t_0 \\ (iii)\ v(S_{b_1}(t),t) = -u_{b_2}(S_{b_1}(t),t) < 0, 0 < t < t_0 \\ (iv)\ v_x(S_{b_1}(t),t) = u_{b_2 x}(S_{b_1}(t),t) - \dot{S}_{b_1}(t), 0 < t < t_0 \\ (v)\ S_{b_1}(0) = 0 \end{cases} \tag{18}$$

and

$$v(S_{b_1}(t_0), t_0) = 0.$$

If $v(0,t) = u_{b_1}(0,t) - u_{b_2}(0,t) > 0, 0 < t < t_0$ by the condition (iii) we deduce that $v_x(0,t) < 0$ which is a contradiction by using the condition (ii), then $v(0,t) \leq 0, 0 < t < t_0$.
Therefore we have a maximum value $v(S_{b_1}(t_0), t_0) = 0$ and then we get $v_x(S_{b_1}(t_0), t_0) \geq 0$. But in other hand we have $v_x(S_{b_1}(t_0), t_0) < 0$ which is a contradiction.
Hence $S_{b_1}(t) < S_{b_2}(t)$ and by maximum principle we have $u_{b_1}(x,t) < u_{b_2}(x,t)$, for all $t > 0, 0 < x < S_{b_1}(t)$.

□

3 ASYMPTOTIC BEHAVIOR OF THE SOLUTION (u_b, s_b) WHEN $b \to 0^+$.

In this section we will study the behavior of the solution $u = u_{bh}(x,t),\ S = S_{bh}(t)$ to problem (3) when $b \to 0^+$. We will prove that the solution to problem (3) converges to the solution of the following parabolic free boundary problem (19) :

$$\begin{cases} (i)\ u_{0h_t} - u_{0h_{xx}} = 0, & 0 < x < S_{0h}(t),\ t > 0, \\ (ii)\ u_{0h_x}(0,t) = \frac{hb_0}{k}\left[u_{0h}(0,t) - F(t)\right], & t > 0, \\ (iii)\ u_{0h}(S_{0h}(t),t) = 0, & t > 0, \\ (iv)\ u_{0h_x}(S_{0h}(t),t) = -\dot{S}_{0h}(t), & t > 0, \\ (v)\ S_{0h}(0) = 0 \end{cases} \tag{19}$$

when $b \to 0^+$. The problem (19) has the following integral representation

$$u_{0h}(x,t) = \int_0^t N(x,t; S_{0h}(\tau), \tau) u_{0h_x}(S_{0h}(\tau), \tau) d\tau - \frac{b_0 h}{k} \int_0^t [u_{0h}(0,\tau) - F(\tau)] N(x,t,0,\tau) d\tau. \tag{20}$$

We will use some integral relations satisfied by the solutions $u = u_{bh}(x,t),\ S = S_{bh}(t)$, and $u = u_{0h}(x,t),\ S = S_{0h}(t)$ to problems (3) and (19) respectively.

Lemma 4 *For problem (*19*), we have the following integral relations:*

$$S_{0h}(t) = -\int_0^{S_{0h}(t)} u_0(x,t)dx - \frac{hb_0}{k} \int_0^t \left[u_0(0,\tau) - F(\tau)\right] d\tau,\ 0 < x < S_0(t),\ t > 0, \tag{21}$$

$$S_{0h}^2(t) = -2\int_0^{S_{0h}(t)} x\, u_{0h}(x,t)dx + 2\int_0^t u_{0h}(0,\tau)d\tau, \tag{22}$$
$$0 < x < S_{0h}(t),\ t > 0,$$

$$\int_0^{S_{0h}(t)} u_{0h}^2(x,t)dx + 2\int_0^t \int_0^{S_{0h}(\tau)} u_{0h_x}^2(x,\tau)dx d\tau \leq \frac{hb_0}{k} \int_0^t F^2(\tau)d\tau; \tag{23}$$
$$0 < x < S_{0h}(t),\ t > 0.$$

Proof. See [3], [8], [18] . □

Lemma 5 *We have* $S_{0h}(t) < S_{bh}(t)$ *and* $u_{0h}(0,t) < u_{bh}(0,t)$, *for all* $t > 0,\ b > 0.$

Proof. Following the same way in Lemma 3, we suppose that the assertion of the Lemma 5 is false, that is there exists $t_1 > 0$ such that

$$S_{0h}(t) < S_{bh}(t),\ \forall\ 0 < t < t_1, \quad S_{0h}(t_1) = S_{bh}(t_1). \tag{24}$$

If we define

$$w(x,t) = u_{bh}(x,t) - u_{0h}(x,t),\ 0 \leq x \leq S_{0h}(t),\ 0 < t < t_1$$

we have the following properties:

$$\begin{cases} (i)\ w_t - w_{xx} = 0,\ 0 < x < S_{0h}(t), 0 < t < t_1 \\ (ii)\ w_x(0,t) = u_{bh_x}(0,t) - u_{0h_x}(0,t) = \frac{hb_0}{k}\left[u_{bh}(0,t) - u_{00}(0,t)\right],\ 0 < t < t_1 \\ (iii)\ w(S_{0h}(t),t) = u_{bh}(S_{0h}(t),t) \geq 0, 0 < t < t_1 \\ (iv)\ w_x(S_{0h}(t),t) = u_{bh_x}(S_{0h}(t),t) + \dot{S}_{0h}(t), 0 < t < t_1 \\ (v)\ S_{0h}(0) = 0 \end{cases} \tag{25}$$

and

$$w(S_{0h}(t_1),t_1) = 0.$$

If $w(0,t) = u_{bh}(0,t) - u_{0h}(0,t) < 0, 0 < t < t_1$ we deduce that the minimum is attained on $x = 0$ and $w_x(0,t) \geq 0$ which is a contradiction by using the condition (25)(ii). Then $w(0,t) \geq 0, 0 < t < t_1$. Therefore we have a minimum value $w(S_0(t_1),t_1) = 0$ and then we get $w_x(S_{0h}(t_1),t_1) < 0$. In other hand we have

$$w_x(S_{0h}(t_1),t_1) = u_{bh_x}(S_{0h}(t_1),t_1) - u_{0h_x}(S_{0h}(t_1),t_1) = -\dot{S}_{bh}(t_1) + \dot{S}_{0h}(t_1) \geq 0$$

which is a contradiction. □

Lemma 6 *We have*

$$u_{bh}(x,t) \geq u_{0h}(x,t),\ \textit{for all}\ 0 < x < S_{0h}(t),\ t > 0. \tag{26}$$

Proof. If we consider

$$w(x,t) = u_{bh}(x,t) - u_{0h}(x,t),\ \ 0 \leq x \leq S_{0h}(t),\ t > 0$$

then w satisfies the following properties:

$$\begin{cases} (i)\ w_t - w_{xx} = 0, 0 < x < S_{0h}(t),\ t > 0 \\ (ii)\ w_x(0,t) = u_{bh_x}(0,t) - u_{0h_x}(0,t) = \frac{hb_0}{k}\left[u_{bh}(0,t) - u_{0h}(0,t)\right] > 0,\ t > 0 \\ (iii)\ w(S_{0h}(t),t) = u_{bh}(S_{0h}(t),t) \geq 0,\ t > 0 \\ (iv)\ w_x(S_{0h}(t),t) = -u_x(S_{0h}(t),t) + \dot{S}_{0h}(t),\ t > 0 \\ (vi)\ S_{0h}(0) = 0. \end{cases} \tag{27}$$

By the maximum principle we obtain that $w(x,t) \geq 0$ and attains its minimum on the parabolic boundary. Then we get (26).

□

Lemma 7 *We have*

$$\lim_{b \to 0^+} S_{bh}(t) = S_{0h}(t) \tag{28}$$

for each t belongs to a compact set in $\mathbb{R}^+$, *with the following order of convergence given by:*

$$0 \leq S_{bh}(t) - S_{0h}(t) \leq \frac{b}{b_0} + \int_0^{\frac{b}{b_0}} \chi(x)dx,\ \ h > 0, t > 0 \tag{29}$$

Proof. According to Lemma 2 it follows

$$S_{bh}(t)-S_{0h}(t)=\frac{b}{b_0}+\int_0^{\frac{b}{b_0}}\chi(x)dx+\int_0^{S_{0h}(t)}u_{0h}(x,t)dx-\int_0^{S_{bh}(t)}u_{bh}(x,t)dx$$

$$-\frac{b_0h}{k}\int_0^t\left[u_{bh}(0,\tau)-u_{0h}(0,\tau)\right]d\tau$$

$$=\frac{b}{b_0}+\int_0^{\frac{b}{b_0}}\chi(x)dx+\int_0^{S_{bh}(t)}\left[\widetilde{u}_{0h}(x,t)-u_{bh}(x,t)\right]dx+\frac{hb_0}{k}\int_0^t\left[u_{0h}(0,\tau)-u_{bh}(0,\tau)\right]d\tau$$

where $\widetilde{u}_{0h}$ is defined as an extension of u_{0h} by 0 as follows:

$$\widetilde{u}_{0h}(x,t)=\begin{cases}u_{0h}(x,t),\ 0\le x\le S_{0h}(t),\ t>0\\ 0,\ \ S_{0h}(t)<x\le S_{bh}(t),\ t>0.\end{cases}$$

Since $\widetilde{u}_{0h}(x,t)-u_{bh}(x,t)\le 0$ and $u_{0h}(0,t)-u_{bh}(0,t)\le 0$ the thesis holds. □

Theorem 1 *We have*

$$\lim_{b\to 0^+}u_{bh}(x,t)=u_{0h}(x,t),$$

for all compact set in the domain $0<x<S_{0h}(t),\ t>0$ *and the following estimation holds*

$$0\le\int_0^{S_{0h}(t)}\left[u_{bh}(x,t)-u_{0h}(x,t)\right]\ dx\le\frac{b}{b_0}+\frac{b}{b_0}\left\|\chi\right\|.$$

Proof. Taking into account Lemma 2 we have

$$\int_0^{S_{0h}(t)}\left[u_{bh}(x,t)-u_{0h}(x,t)\right]\ dx=\frac{b}{b_0}+\int_0^{\frac{b}{b_0}}\chi(x)dx-\int_{S_{0h}(t)}^{S_{bh}(t)}u_{bh}(x,t)dx$$

$$-\frac{hb_0}{k}\int_0^t\left[u_{bh}(0,\tau)-u_0(0,\tau)\right]d\tau-\left[S_{bh}(t)-S_{0h}(t)\right].$$

By using, Lemmas 1, 3 and 4, we have

$$0\le\int_0^{S_{0h}(t)}\left[u_{bh}(x,t)-u_{0h}(x,t)\right]\ dx\le\frac{b}{b_0}+\int_0^{\frac{b}{b_0}}\chi(x)\ dx$$

$$\le\frac{b}{b_0}+\frac{b}{b_0}\left\|\chi\right\| \tag{30}$$

and the thesis holds. □

4 Asymptotic behavior of the solution $(u_{bh},\ S_{bh}\ )$ when $(b,h)\to(0^+,0^+)$.

In this section we will prove the convergence of u_{bh} and S_{bh} when $(b,h)\to$ to $(0^+,0^+)$.

Lemma 8 *If* $\int_0^{+\infty}F(\tau)d\tau<\infty$ *then we have the following limit:*

$$\lim_{(b,h)\to(0^+,0^+)}S_{bh}(t)=0 \tag{31}$$

for each t *in a compact set in* $\mathbb{R}^+$, *with the following order of convergence given by:*

$$0\le S_{bh}(t)\le\frac{b}{b_0}(1+\left\|\chi\right\|)+\frac{hb_0}{k}\int_0^tF(\tau)d\tau. \tag{32}$$

Proof. Taking into account (14) and Lemma 2, we have:

$$0 \leq S_{bh}(t) = \frac{b}{b_0} + \int_0^{\frac{b}{b_0}} \chi(x)dx - \frac{hb_0}{k}\int_0^t \left[u_{bh}(0,\tau) - F(\tau)\right] d\tau - \int_0^{S_{bh}(t)} u_{bh}(x,t)dx$$

$$\leq \frac{b}{b_0} + \int_0^{\frac{b}{b_0}} \chi(x)dx + \frac{hb_0}{k}\int_0^t F(\tau)d\tau.$$

If $\int_0^{+\infty} F(\tau)d\tau < \infty$ then the thesis holds. □

Theorem 2 *If $\int_0^{+\infty} F(\tau)d\tau < \infty$ then we have*

$$\lim_{(b,h)\to(0^+,0^+)} u_{bh}(x,t) = 0,$$

and the following order of convergence

$$0 \leq u_{bh}(x,t) \leq \frac{b}{b_0\sqrt{\pi t}}\|\chi\| + \frac{2hb_0}{k\sqrt{\pi}}\|F\|\sqrt{t},\ \ t > 0.$$

Proof. We consider the integral representation of $u_{b,h}$ given by (6). Taking into account Lemma 2 we have:

$$0 \leq u_{bh}(x,t) \leq \int_0^{\frac{b}{b_0}} N(x,t;\xi,0)\chi(\xi)d\xi + \frac{hb_0}{k}\int_0^t N(x,t;0,\tau)F(\tau)d\tau, 0 < x < S_{bh}(t), t > 0$$

Moreover

$$N(x,t;\xi,0) = K\left(x,t,\xi,0\right) + K\left(-x,t,\xi,0\right)$$
$$= \frac{1}{2\sqrt{\pi t}}\exp\left(-\frac{(x-\xi)^2}{4t}\right) + \frac{1}{2\sqrt{\pi t}}\exp\left(-\frac{(x+\xi)^2}{4t}\right) \leq \frac{1}{\sqrt{\pi t}}$$

and

$$N(x,t;0,\tau) = K\left(x,t,0,\tau\right) + K\left(-x,t,0,\tau\right)$$
$$= \frac{1}{\sqrt{\pi(t-\tau)}}\exp\left(-\frac{x^2}{4(t-\tau)}\right).$$

Then

$$\int_0^{\frac{b}{b_0}} N(x,t;\xi,0)\chi(\xi)d\xi \leq \frac{b}{b_0\sqrt{\pi t}}\|\chi\|$$

and

$$\frac{hb_0}{k}\int_0^t N(x,t;0,\tau)F(\tau)d\tau \leq \frac{2hb_0}{k\sqrt{\pi}}\|F\|\sqrt{t}.$$

Therefore

$$0 \leq u_{bh}(x,t) \leq \frac{b}{b_0\sqrt{\pi t}}\|\chi\| + \frac{2hb_0}{k\sqrt{\pi}}\|F\|\sqrt{t}$$

and

$$u_{bh}(x,t) \to 0,\ when\,(b,h) \to (0^+,0^+).$$

□

5 Conclusions

The asymptotic behavior of the solution to the Stefan problem with a convective boundary condition at the fixed face when the heat transfer coefficient (proportional to the Biot number) and the initial position of the free boundary go to zero has been obtained with an order of convergence.

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