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A Brief Survey on Lubrication Problems with Nonlinear Boundary Conditions

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No. 16

A BRIEF SURVEY ON LUBRICATION PROBLEMS WITH NONLINEAR BOUNDARY CONDITIONS

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Abstract. We consider some lubrication problems in a thin domain with thickness of order ε, with mixed boundary conditions and subject to slip phenomenon on a part of the boundary. We study the existence and uniqueness results for the weak solution of each problem, then we establish the asymptotic behavior of its solutions, when the depth of the thin domain tends to zero.

Résumé. Nous considérons quelques problèmes de lubrification dans un domaine mince d'épaisseur ε, avec des conditions aux limites mixtes et soumis au phénomène de glissement de fluide au parois. Nous étudions les résultats d'existence et d'unicité de la solution faible de chaque problème, puis nous établissons le comportement asymptotique des solutions quand l'épaisseur du domaine tend vers zéro.

Resumen. Se consideran algunos problemas de lubricación en un dominio delgado de espesor ε, con condiciones de contorno mixtas y sometido a un fenómeno de deslizamiento sobre una parte de la frontera. Se estudian resultados de existencia y de unicidad de la solución débil de cada problema y luego se establece el comportamiento asintótico de las soluciones cuando el espesor del dominio tiende a cero.

Keywords: Free boundary problems; Lubrication; Non-isothermal fluid; Fluid-solid conditions; slip phenomenon; Roughness phenomenon; Asymptotic approach, Reynolds equation.

Mots clés: Problèmes à frontière libre; Lubrification; Fluide non-isotherme; conditions d'interface fluide-solide; slip Phénomène de glissement de fluide; Phénomène de rugosité; Approche asymptotique; Equation de Reynolds.

Palabras claves: Problemas de frontera libre; Lubricación; Fluidos no-isótermicos; Condiciones fluido-sólido; Fenóneno de deslizamiento; Fenómeno de rugosidad; Comportamiento asintótico; Ecuación de Reynolds.

AMS Subject Classification: 35R35; 76B03; 76D05; 76D07; 76F10; 78M35; 74K35; 76E30.

These Notes are the enlarged content of the conference given by Prof. M. Boukrouche at the Department of Mathematics of FCE-UA, Rosario, for the Congress TEM2005 on 5-7 December 2005. They contain the basic ideas of the existence and uniqueness results of some lubrication problems subject to nonlinear boundary condition and the asymptotic behavior of its solutions when one of the dimension of the fluid domain tends to zero.

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A BRIEF SURVEY ON LUBRICATION PROBLEMS WITH NONLINEAR BOUNDARY CONDITIONS

M. Boukrouche[1]

1. Introduction

This work gives a survey on some results obtained in a series of papers [6, 9, 10, 11, 14, 15, 17] in which we consider a particular cases of the general equations describing the motion of some fluid flows in bounded thin domain, with slip and mixed boundary conditions.
We comment the basic ideas on existence, uniqueness results of the solutions of the associate problems, and also its behavior when the thickness of the thin domain tends to zero. See also [12, 13, 16].
To study lubrication problems or the fluid equations one requires the knowledge of the velocities on the fluid-solid interface. This subject is often a matter of discussion as a lot of physical parameters are involved like micro-roughness of the surface or the rheological properties of the fluid.
No-slip condition, in which the fluid is assumed to have the same velocity as the surrounding solid boundary, is widely used in mathematical studies [54]. Nevertheless, this boundary condition is sometimes overlooked and it is possible to deal with the "slip condition" which allows the fluid to slip on the surface but not to go through it. The normal component of the velocity is equal to zero while the tangential one is proportional to the tangential stresses. Existence and uniqueness theorems for a weak related formulation are easily obtained (see for example) [2]. The intermediate case in which the slip condition only occurs for sufficiently a large ratio between tangential stresses and normal stresses while the no-slip condition is retained for small ratio have also been introduced [23]. This last case is nothing else than a transposition of the well known Coulomb law between two solids [24] to the fluid solid interface and so leads to a free boundary problem model.

An accurate choice of these boundary conditions is of particular interest in the lubrication area which is concerned with thin film flow behavior. In this case, the difference of velocities between the surrounding surfaces is the governing phenomena that allows the pressure in the fluid to build up and prevent the solid surfaces from being in contact which is the main objective of the lubrication. Continuous experimental studies are being conducted [46, 47] but are still difficult due to the thickness of the gap between the solid surfaces which can be as small as 50 nanometers. In such operating conditions, a no slip condition is induced by chemical bonds between the lubricant and the surrounding surfaces. Conversely, tangential stresses are so high that they tend to destroy the chemical bonds and induce a slip phenomena. Such behavior is then close to the Tresca free boundary friction model in solid mechanics [25].
This phenomenon has been related in a lot of mechanical papers for both Newtonian and non Newtonian cases [36, 37, 49, 50, 51, 55]. Although being implicitly used in numerical

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procedures in lubrication problems, a Reynolds thin film equation taking account of such slip phenomena seems to have been posed for the first time in a somewhat mathematical aspect in [52]. This study is restricted to one dimensional problems and the existence of the discretized problem is proved.

The aim of this paper is not only to give existence and uniqueness for this problem but also to obtain rigorously the equation describing such phenomena in a thin film flow by way of an asymptotic analysis in which the small parameter is the width of the gap, following the same ideas as in [4], [20]. The departure point is the Stokes equation with the Tresca boundary conditions [6] and so fall into the scope of the work of [23]. Then we generalize our results to [9, 10, 11, 12, 13, 14, 15, 16, 17]. See also [29, 30, 31, 32] for similar boundary conditions.

This brief survey is organized as follows. In Section 2 we present the derivation of the fluid equations from the three conservation laws of mass, momentum and energy. In Subsection 2.1 we formulate seven problems considered. In Section 3 we give the variational formulation of each considered problems and existence and uniqueness results. In Section 4 we study the asymptotic analysis of the first case to obtain the limit problem, when the thickness of the thin domain becomes very small. In Subsection 4.1 we study the limit problem of the first case. In Subsection 4.2 we study its uniqueness. In Subsection 4.3 we study the second case. In Section 5 we study the asymptotic analysis of the case 3. In Section 6 we study the asymptotic analysis of the case 6.

I would like to thank my friend Professor Domingo Alberto Tarzia, who kindly proposed me to write these notes. I hope that this notes can provide some idea and be useful to who is interested by this subject. Also I would like to thank Guy Bayada, Professor (Insa-Lyon), Grzegorz Łukaszewicz (Warsaw University), Lionel Ciuperca (Lyon University), for fruitful collaborations on this subject.

2. On the fluid equations

We present the derivation of the problems considered from the three *conservation laws of mass, momentum and energy.* Let a bounded domain $\Omega \subset \mathbb{R}^n$ and a times interval $[0,\tau]$. Let $v : [0,\tau] \times \Omega \to \mathbb{R}^n$ such that $(t,x) \mapsto v(t,x)$ be the velocity vector of the continuous medium, $\rho : [0,\tau] \times \Omega \to \mathbb{R}$ such that $(t,x) \mapsto \rho(t,x)$ its density, and $e : [0,\tau] \times \Omega \to \mathbb{R}$ such that $(t,x) \mapsto e(t,x)$ its specific internal energy, which are the unknowns. Let given the external forces $f : [0,\tau] \times \Omega \to \mathbb{R}^n$ and a scalar function R representing the energy contribution by unity of mass and times. It is well known (see for example [41]) that the motion of continuous medium is modeled by the following three conservation laws of mass, momentum and energy respectively

$$\frac{\partial \rho}{\partial t} + v \cdot \nabla \rho + \rho \operatorname{div}(v) = 0, \tag{2.1}$$

$$\rho \left(\frac{\partial v}{\partial t} + v \cdot \nabla v \right) = \operatorname{div}(\sigma) + \rho f, \tag{2.2}$$

$$\rho \left(\frac{\partial e}{\partial t} + v.\nabla e \right) = \sigma : D(v) - \operatorname{div}(q) + R, \tag{2.3}$$

where $\sigma = (\sigma_{ij}$ (for $1 \leq i,j \leq n$) is the stress tensor, $D(v)$ is the strain rate tensor, with components

$$d_{ij}(v) = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right), \quad 1 \leq i,j \leq n \ ,$$

$$\sigma : D(v) = \sum_{i,j=1}^{n} \sigma_{ij} d_{ij}(v), \qquad \operatorname{div}(v) = \sum_{i=1}^{n} \frac{\partial v_i}{\partial x_i}.$$

The first term $\sigma : D(v)$ on the right hand side of (2.3) represents the energy generated by the *deformation of the continuous medium* under the action of the shear forces, the so called *dissipation* term. q is a vector function representing the energy transport, from a macroscopic continuum sense, the heat conduction phenomenon is often described by *Fourier's law*, relating the heat flux q to the temperature T

$$q = -K(T)\nabla T, \tag{2.4}$$

where K is a scalar positive function representing the thermal conductivity, see also a damped version of *Fourier's law* introducing a heat relaxation term, [35, 22]

$$\tau \frac{\partial q}{\partial t} = -(q + K\nabla T),$$

where τ is a relaxation time required to establish a steady state of heat conduction in an element suddenly exposed to heat flux.

The case where the density ρ is not constant in time leads to the compressible Euler equations [41] a physical example is a gas dynamics. Let assume that the continuous medium is an *incompressible fluid* so ρ is constant, then the local mass conservation law (2.1) becomes

$$\operatorname{div}(v) = 0. \tag{2.5}$$

The case where the stress tensor σ is non-symmetric the medium is called *micro-polar fluid* [27, 40]. We assume also in all this study that the stress tensor σ is symmetric [24, 26, 39].

$$\sigma_{ij} = \sigma_{ji} \quad \text{for} \quad 1 \leq i,j \leq n. \tag{2.6}$$

Each stress tensor σ characterizes the kind of the fluid, so for example the power law [26]

$$\sigma = -pI + k(T)\gamma^{r-1}D(v), \qquad \gamma = 2\sqrt{D(v)D(v)}, \tag{2.7}$$

where k is a given positive scalar function, r is the power law index, p is the pressure, I is the $n \times n$ identity matrix, and the product $\mu = \frac{k}{2}\gamma^{r-1}$ is the viscosity of the considered fluid.

Remark that when $r = 1$ the fluid is called *non-isothermal Newtonian.* When $r \neq 1$ the fluid is called *non-isothermal non-Newtonian* and the constitutive equation

$$S = k(T)\gamma^{r-1}D(v)$$

represents *shear thinning* for $r < 1$ and *shear thickening* for $r > 1$ fluids.
With (2.7) the equation (2.2) becomes

$$\frac{\partial v}{\partial t} + v \cdot \nabla v = f + 2^{r-1}\mathrm{div}\Big(k(T)(D(v)D(v))^{\frac{r-1}{2}}D(v)\Big) - \nabla p. \tag{2.8}$$

As $ID(v) = \mathrm{div}\, v$, so from (2.5) $\sigma : D(v) = k(T)\gamma^{r-1}D(v) : D(v)$, then with the Fourier Law (2.4) the energy conservation law (2.3) becomes

$$\frac{\partial e}{\partial t} + e \cdot \nabla e = k(T)\gamma^{r-1}D(v) : D(v) + \mathrm{div}\Big(K(T)\nabla T\Big) + R(T).$$

Assume that the internal energy of the fluid is given by

$$\frac{\partial e}{\partial t} + e \cdot \nabla e = C_v(T)\left(\frac{\partial T}{\partial t} + v \cdot \nabla T\right),$$

where $C_v(T)$ is the specific heat with constant volume, then the energy conservation law becomes

$$C_v(T)\left(\frac{\partial T}{\partial t} + v \cdot \nabla T\right) = 2\mu(T)D(v) : D(v) + \mathrm{div}\Big(K(T)\nabla T\Big) + R(T). \tag{2.9}$$

with the *behavior laws* the equations (2.5), (2.8) (2.9) describe the motion of an incompressible non-isothermal non-Newtonian fluid flow.

2.1. **Formulation of the problems considered.** Let ω be a fixed bounded domain in $\mathbb{R}^2$, for a given function $H : \omega \to \mathbb{R}^+$, we define the surface

$$x_3 = H(x) = H(x_1, x_2).$$

In the lubrication theory it is natural to assume that the fluid film, between the two surfaces ω and $x_3 = H(x)$, is very thin. So we introduce a small positive parameter ε, and a function h such that $H(x) = \varepsilon h(x)$. Then we denote the fluid domain by

$$\Omega^\varepsilon = \{(x, x_3) \in \mathbb{R}^3 \ : \ x \in \omega \ \text{ and } \ 0 < x_3 < \varepsilon h(x)\}, \tag{2.10}$$

with $\partial\Omega^\varepsilon = \bar{\omega} \cup \bar{\Gamma}_1^\varepsilon \cup \bar{\Gamma}_L^\varepsilon$ where $\Gamma_1^\varepsilon = \{(x, x_3) \in \mathbb{R}^3 \ : \ x_3 = \varepsilon h(x)\}$ and Γ_L^ε is the lateral boundary.

In all the following repeated indices means that the summation convention is used.

Case 1. [6] The motion in the fluid is described by the basic stationary Stokes system

$$\frac{\partial \sigma_{ij}^\varepsilon}{\partial x_j} + f_i^\varepsilon = 0, \qquad \mathrm{div}\ (u^\varepsilon) = 0 \quad \text{in} \quad \Omega^\varepsilon, \tag{2.11}$$

with

$$\sigma_{ij}^{\varepsilon} = p^{\varepsilon}\delta_{ij} + 2\nu d_{ij}(u^{\varepsilon}),$$

where ν is a constant viscosity, δ_{ij} is the Kronecker symbol. The velocities on the boundary, except their tangential components, are given in terms of a given function g. The upper surface being assumed to be fixed no slip condition is given so

$$u^{\varepsilon} = g = 0 \quad \text{on } \Gamma_1^{\varepsilon}. \tag{2.12}$$

The velocity is known and parallel to the ω-plane

$$u^{\varepsilon} = g \quad \text{with} \quad g \cdot n = 0 \quad \text{on } \Gamma_L^{\varepsilon}. \tag{2.13}$$

There is no flux across ω so that

$$u^{\varepsilon} \cdot n = g \cdot n = 0 \quad \text{on } \omega. \tag{2.14}$$

The tangential velocity on ω is *unknown* and satisfies the Tresca friction law [25] with the friction coefficient k^{ε}

$$\left.\begin{array}{l} |\sigma_T^{\varepsilon}| = k^{\varepsilon} \Longrightarrow \quad \exists\lambda \geq 0 \quad u_T^{\varepsilon} = s - \lambda\sigma_T^{\varepsilon} \\ |\sigma_T^{\varepsilon}| < k^{\varepsilon} \Longrightarrow \qquad u_T^{\varepsilon} = s \end{array}\right\} \quad \text{on} \quad \omega, \tag{2.15}$$

where $n = (n_1, n_2, n_3)$ is the outward unit normal to Γ^{ε}, $|.|$ denotes the $\mathbb{R}^2$ Euclidean norm, s is the velocity of the lower surface ω; σ_n^{ε} and σ_T^{ε} are, respectively, the normal and the tangential components of the stress tensor

$$\sigma_n^{\varepsilon} = \sigma_{ij}^{\varepsilon} n_i n_j = (\sigma^{\varepsilon}.n).n, \quad \sigma_{T_i}^{\varepsilon} = \sigma_{ij}^{\varepsilon} n_j - \sigma_n^{\varepsilon} n_i,$$

and u_T^{ε}, is the tangential velocity,

$$u_{T_i}^{\varepsilon} = u_i^{\varepsilon} - u_j^{\varepsilon} n_j n_i.$$

The condition (2.15) means that in each point of ω where the Euclidean norm $|\sigma_T^{\varepsilon}|$ reaches the upper limit k^{ε}, there exists an unknown scalar $\lambda \geq 0$, such that the tangential velocity of the fluid u_T^{ε} is braked by $\lambda\sigma_T^{\varepsilon}$ according to the velocity s of the lower surface ω. So in these unknown points of the lower surface ω occur the slip of the fluid according to ω.

Case 2. [9] We consider (2.11)-(2.14) and we change the Tresca boundary conditions (2.15) by the following Coulomb friction law [25] with the friction coefficient k^{ε} :

$$\left.\begin{array}{l} |\sigma_T^{\varepsilon}| = k^{\varepsilon}|\sigma_n^{\varepsilon}| \Longrightarrow \quad \exists\lambda \geq 0 \quad u_T^{\varepsilon} = s - \lambda\sigma_T^{\varepsilon} \\ |\sigma_T^{\varepsilon}| < k^{\varepsilon}|\sigma_n^{\varepsilon}| \Longrightarrow \qquad u_T^{\varepsilon} = s \end{array}\right\} \quad \text{on} \quad \omega. \tag{2.16}$$

Case 3. [10] We consider (2.11), (2.13)-(2.15) and we change the boundary condition (2.12) by the following Fourier's type

$$u^{\varepsilon}.n = 0, \quad \sigma_T^{\varepsilon}(u^{\varepsilon}) + l^{\varepsilon}u^{\varepsilon} = 0 \quad on \quad \Gamma_1^{\varepsilon}, \tag{2.17}$$

where $l^{\varepsilon} > 0$ is a given scalar. This means that on Γ_1^{ε} there are only friction.

Case 4. [11] We consider the Navier-Stokes case with the Reynolds number ε^{γ}

$$\varepsilon^{\gamma} u_j \frac{\partial u_i}{\partial x_j} = f_i + \frac{\partial \sigma_{ij}}{\partial x_j} \quad in \quad \Omega^{\varepsilon}, \quad \gamma \in \mathbb{R} \tag{2.18}$$

where

$$\sigma_{ij}^{\varepsilon} = -p^{\varepsilon}\delta_{ij} + 2\mu d_{ij}(u^{\varepsilon}), \tag{2.19}$$

with the boundary conditions (2.12), (2.14), and we change (2.13), (2.15) by

$$u^{\varepsilon} = \varepsilon^{\beta} g \quad \text{with} \quad g \cdot n = 0 \quad \text{and} \quad \beta \in \mathbb{R} \quad \text{on } \Gamma_L^{\varepsilon}. \tag{2.20}$$

$$\left.\begin{array}{ll}|\sigma_T^\varepsilon| = k^\varepsilon \Longrightarrow & \exists\lambda\geq 0 \quad u_T^\varepsilon = \varepsilon^\beta s - \lambda\sigma_T^\varepsilon \\ |\sigma_T^\varepsilon| < k^\varepsilon \Longrightarrow & u_T^\varepsilon = \varepsilon^\beta s\end{array}\right\} \quad \text{on} \quad \omega. \tag{2.21}$$

Case 5. [14, 16] We consider the Newtonian non-isothermal case (2.11)-(2.15) with

$$\sigma_{ij}^\varepsilon = -p^\varepsilon\delta_{ij} + 2\mu^\varepsilon(T^\varepsilon)d_{ij}(u^\varepsilon),$$

$$\operatorname{div}(K^\varepsilon\nabla T^\varepsilon) + 2\mu(T^\varepsilon)|D(u^\varepsilon)|^2 + R^\varepsilon(T^\varepsilon) = 0, \tag{2.22}$$

$$T^\varepsilon = 0 \quad \text{on} \quad \Gamma_L^\varepsilon \cup \Gamma_1^\varepsilon, \tag{2.23}$$

$$K^\varepsilon\frac{\partial T^\varepsilon}{\partial n} = \theta^\varepsilon(T^\varepsilon) \quad \text{on} \quad \omega, \tag{2.24}$$

where θ^ε is given function on ω.

Case 6. [15] We consider the non-Newtonian non-isothermal case (2.11)-(2.15) with

$$\sigma_{ij}^\varepsilon = -p^\varepsilon\delta_{ij} + 2\mu(T^\varepsilon)|D(u^\varepsilon)|^{r-2}d_{ij}(u^\varepsilon),$$

$$\operatorname{div}(K^\varepsilon\nabla T^\varepsilon) + 2\mu(T^\varepsilon)|D(u^\varepsilon)|^r + R^\varepsilon T^\varepsilon = 0, \tag{2.25}$$

$$T^\varepsilon = 0 \quad \text{on} \quad \Gamma_L^\varepsilon \cup \Gamma_1^\varepsilon, \tag{2.26}$$

$$K^\varepsilon\frac{\partial T^\varepsilon}{\partial n} = b^\varepsilon \quad \text{on} \quad \omega, \tag{2.27}$$

where $r \in \mathbb{R}$ such that $1 < r < \infty$ and b^ε is given function on ω.

Case 7. We consider also in [17] the equations (2.11)-(2.15) taking the roughness phenomenon so the small parameter ε is now related to the roughness wavelength and also to the thickness of the gap between the surfaces $z_3 = 0$ and $z_3 = \lambda\varepsilon h(z, \frac{z}{\varepsilon})$, such that the domain occupied by the fluid is

$$\Omega^\varepsilon = \{(z, z_3) \in \mathbb{R}^3 : \quad z \in \omega \qquad 0 < z_3 < \lambda\varepsilon h^\varepsilon(z)\}$$

where

$$h^\varepsilon(z) = h(z, \frac{z}{\varepsilon}) \qquad z \in \omega, \tag{2.28}$$

and $\lambda > 0$ is a fixed constant.

3. Existences uniqueness results

We assume that the function $g \in (H^{\frac{1}{2}}(\Gamma^\varepsilon))^3$ and such that

$$\int_{\Gamma^\varepsilon} g\cdot n d\sigma = 0, \quad g_3 = 0 \quad on \quad \Gamma_L^\varepsilon, \quad g = 0 \quad on \quad \Gamma_1^\varepsilon, \quad g\cdot n = 0 \quad on \quad \omega. \tag{3.1}$$

So following [33] (lemma 2.2 p.24), there exists a function G^ε such that

$$G^\varepsilon \in (H^1(\Omega^\varepsilon))^3, \quad div(G^\varepsilon) = 0 \quad \text{in}\, \Omega^\varepsilon, \quad G^\varepsilon = g \quad \text{on} \quad \Gamma^\varepsilon. \tag{3.2}$$

Let define now the following notations

$$V^\varepsilon = \Big\{v \in (H^1(\Omega^\varepsilon))^3 \quad : \quad v = G^\varepsilon \text{ on } \Gamma_L^\varepsilon \cup \Gamma_1^\varepsilon \quad , \quad v.n = 0 \text{ on } \omega\Big\},$$

$$V_0^\varepsilon = \Big\{v \in (H^1(\Omega^\varepsilon))^3 \quad : \quad v = 0 \text{ on } \Gamma_L^\varepsilon \cup \Gamma_1^\varepsilon \quad , \quad v.n = 0 \text{ on } \omega\Big\},$$

$$V_{div}^\varepsilon = \Big\{v \in V^\varepsilon : \operatorname{div}(v) = 0 \quad \text{in} \quad \Omega^\varepsilon\Big\},$$

$$L_0^2(\Omega^\varepsilon) = \{q \in L^2\left(\Omega^\varepsilon\right) \ : \ \int_{\Omega^\varepsilon} q \, dx dx_3 = 0\},$$

(u, v) denotes the scalar product in $L^2(\Omega^\varepsilon)$

$$a(u,\varphi) = \sum_{i,j=1}^{3} 2\nu \int_{\Omega^\varepsilon} D_{ij}(u) D_{ij}(\varphi) dx d_3, \qquad j(\varphi) = \int_{\omega} k^\varepsilon |\varphi - s| dx dx_3$$

So **Case 1**, leads [24] to the following variational formulation: *For* G^ε *as in* (3.2)*, find* $u^\varepsilon \in V^\varepsilon_{div}$ *and* $p^\varepsilon \in L^2_0(\Omega^\varepsilon)$*, such that*

$$a(u^\varepsilon, \varphi - u^\varepsilon) - (p^\varepsilon, div(\varphi)) + j(\varphi) - j(u^\varepsilon) \geq (f^\varepsilon, \varphi - u^\varepsilon) \quad \forall \varphi \in V^\varepsilon. \tag{3.3}$$

Theorem 1. *Assuming that* f^ε *in* $(L^2(\Omega^\varepsilon))^3$*, and the friction coefficient* k^ε *is a non negative function in* $L^\infty(\omega)$*, then there exists a unique* u^ε *and there exists a unique (up to an additive constant)* p^ε *such that* $(u^\varepsilon, p^\varepsilon)$ *in* $V^\varepsilon_{div} \times L^2_0(\Omega^\varepsilon)$ *is a solution to the variational inequality* (3.3)*.*

Proof. [6] The existence and uniqueness of u^ε in V^ε_{div} satisfying the variational inequality of the second kind (3.3) is well known and follows (for example) from [18]. To get p^ε, we apply the duality results of convex optimisation ([28] theorem 4.1 and remark 4.2). recalling [6] we can rewrite (3.3) so that it is defined on the whole of $V(\Omega^\varepsilon)$ by introducing the indicator functions:

$$\psi_{V^\varepsilon_{div}} : (L^2(\Omega^\varepsilon))^3 \to \overline{\mathbb{R}} \quad \text{such that} \quad \varphi \mapsto \psi_{V^\varepsilon_{div}}(\varphi) = \begin{cases} 0 & \text{if } \varphi \in V^\varepsilon_{div} \\ +\infty & \text{if } \varphi \notin V^\varepsilon_{div}, \end{cases}$$

and

$$H : L^2(\Omega^\varepsilon) \to \overline{\mathbb{R}} \quad \text{such that} \quad q \mapsto H(q) = \begin{cases} 0 & \text{if } q = 0 \\ +\infty & \text{if } q \neq 0, \end{cases}$$

so (3.3) is equivalent to

$$\begin{aligned} a(u, \varphi - u) + j(\varphi) - j(u) + \psi_{V^\varepsilon_{div}}(\varphi) - \psi_{V^\varepsilon_{div}}(u) \quad &\geq \\ \geq (f^\varepsilon, \varphi - u) \qquad \forall \varphi \in V^\varepsilon_0, \quad div(\varphi) = 0&, \end{aligned} \tag{3.4}$$

and the unique solution of (3.3) minimizes the functional:

$$\inf_{\varphi \in V^\varepsilon_0} \left\{ \frac{1}{2} a(\varphi, \varphi) - (f^\varepsilon, \varphi) + j(\varphi) + H(div(\varphi)) + \psi_{V^\varepsilon_{div}}(\varphi) \right\} \tag{3.5}$$

which can be write in the following form

$$\inf_{\varphi \in V^\varepsilon_0} \{F(\varphi) + G(\Lambda(\varphi))\}, \quad \text{where}$$

$$F : V^\varepsilon_0 \to \mathbb{R} \quad \textit{such that} \quad \varphi \mapsto F(\varphi) = \frac{1}{2} a(\varphi, \varphi) - (f, \varphi),$$

$$\begin{aligned} \Lambda : V^\varepsilon_0 \to Y \quad &= \quad L^2(\omega) \times L^2(\Omega^\varepsilon) \times V^\varepsilon_0, \\ \varphi \mapsto \Lambda(\varphi) \quad &= \quad (\Lambda_1 \varphi, \Lambda_2 \varphi, \varphi) = (\varphi_{|\omega} \ , \ div(\varphi) \ , \ \varphi), \end{aligned}$$

$$G : Y \to \overline{\mathbb{R}} \quad \textit{such that} \quad q \mapsto G(q) = j(q_1) + H(q_2) + \psi_{V^\varepsilon_{div}}(q_3).$$

Then, the dual problem (to (3.5)) is given by: *Find* $p^\star$ *in* $Y^\star = L^2(\omega) \times L^2(\Omega^\varepsilon) \times V^{\varepsilon\star}_0$ *solution of the problem*

$$\sup_{q^\star \in Y^\star} \left\{ -F^\star(\Lambda^\star q^\star) - G^\star(-q^\star) \right\}, \tag{3.6}$$

where

$$F^\star(\Lambda^\star q^\star) = \sup_{\varphi\in V_0^\varepsilon}\{<\Lambda_1^\star q_1^\star,\varphi> + <\Lambda_2^\star q_2^\star,\varphi> + <\Lambda_3^\star q_3^\star,\varphi> - F(\varphi)\},$$

$$\begin{aligned} G^\star(-q^\star) :&= \sup_{q\in Y}\{<-q^\star,q> - G(q)\} = \sup_{q_1\in L^2(\omega)}\{<-q_1^\star,q_1> - j(q_1)\} + \\ &+ \sup_{q_2\in L^2(\Omega^\varepsilon)}\{<-q_2^\star,q_2> - H(q_2)\} + \sup_{q_3\in V_0^\varepsilon}\{<-q_3^\star,q_3> - \psi_{V^\varepsilon_{div}}(q_3)\}, \end{aligned}$$

and from the definition of H, we have for any $q = (q_1, q_2, q_3)$ in $Y = L^2(\omega)\times L^2(\Omega^\varepsilon)\times V_0^\varepsilon$

$$G^\star(-q^\star) \geq \{<-q_1^\star,q_1> - j(q_1)\} + \{<-q_3^\star,q_3> - \psi_{V^\varepsilon_{div}}(q_3)\}.$$

As the function $G^\star$ from $Y^\star \to \mathbb{R}$, is continuous, then the hypothesis of [28] (see chap.III, Theorem 4.1), are satisfied for the dual problem (3.6), and imply the existence of $p^\star$ in $Y^\star$ satisfying

$$\{F(u^\varepsilon) + G(\Lambda(u^\varepsilon)\} + \{F^\star(\Lambda^\star p^\star) + G^\star(-p^\star)\} = 0,$$

which can be written

$$\begin{aligned} &\left\{F(u^\varepsilon) + j(\Lambda_1 u^\varepsilon) + H(\Lambda_2 u^\varepsilon) + \psi_{V^\varepsilon_{div}}(\Lambda_3 u^\varepsilon)\right\} \\ &+ \left\{F^\star(\Lambda^\star p^\star) + j^\star(-p_1^\star) + (\psi_{V^\varepsilon_{div}})^\star(-p_3^\star)\right\} = 0. \end{aligned}$$

Let us remark from the definition of H and by choosing $q = \Lambda\varphi$ for any φ in V^ε that

$$\begin{aligned} F(u^\varepsilon) - F(\varphi) + j(\Lambda_1 u^\varepsilon) - j(\Lambda_1\varphi) + \psi_{V^\varepsilon_{div}}(\Lambda_3 u^\varepsilon) - \psi_{V^\varepsilon_{div}}(\Lambda_3\varphi) + <p_2^\star,\Lambda_2\varphi> \\ - <p_2^\star,\Lambda_2 u^\varepsilon> \quad \leq \quad \{H(\Lambda_2 u^\varepsilon) - <p_2^\star, div(u^\varepsilon)>\} \leq 0, \end{aligned}$$

which is exactly

$$\begin{aligned} &a(u^\varepsilon,\varphi-u^\varepsilon) + j(\varphi) - j(u^\varepsilon) + \psi_{V^\varepsilon_{div}}(\Lambda_3\varphi) - \psi_{V^\varepsilon_{div}}(\Lambda_3 u^\varepsilon) \\ &- <p_2^\star\,,\,div(\varphi-u^\varepsilon)> \quad \geq (f^\varepsilon,\,\varphi-u^\varepsilon) \quad \forall\varphi\in V_0^\varepsilon. \end{aligned} \tag{3.7}$$

So taking in (3.7) φ and u^ε in V^ε_{div}, we get exactly (3.3).
Using Green's formula with $\varphi = u^\varepsilon \pm \phi$ for any ϕ in $(H_0^1(\Omega^\varepsilon))^3$, (3.7) induces

$$\nabla p_2^\star = \nu\Delta u^\varepsilon + f^\varepsilon \quad a.e. \quad in \quad \Omega^\varepsilon,$$

then as u^ε is unique in V^ε_{div}, we deduce the uniqueness (up to an additive constant) of $p_2^\star$ in $L^2(\Omega^\varepsilon)$. □

Case 2, leads to the similar variational inequality (3.3) where the functional j is now

$$j(\varphi) = \int_\omega k^\varepsilon|\sigma_n^\varepsilon|\,|\varphi - s|dx.$$

Since $j(u^\varepsilon)$ has no meaning for $u^\varepsilon \in V^\varepsilon_{div}$, we consider (cf. [24]) a regularization operator S from $H^{-\frac{1}{2}}(\omega)$ into $L^2_+(\omega)$ defined, for all $\tau \in H^{-\frac{1}{2}}(\omega)$ and $S(\tau)\in L^2_+(\omega)$, by

$$S(\tau)(x) = |<\tau,\psi(x-t)>_{H^{-\frac{1}{2}}(\omega),H^{\frac{1}{2}}_{00}(\omega)}|, \quad \forall x\in\omega, \tag{3.8}$$

where $\psi \in D(\omega)$ is a given positive function. Here $H^{-\frac{1}{2}}(\omega)$ is the topological dual space of $H^{\frac{1}{2}}_{00} = \{\psi_{|\omega} :\ \psi\in H^1(\Omega)\,,\psi = 0\,\text{on}\ \Gamma_1\cup\Gamma_L\}$, $L^2_+(\omega)$ is the subspace of $L^2(\omega)$ of non-negative functions. So we put $S(\sigma_n^\varepsilon)$ instead of $|\sigma_n^\varepsilon|$ in the value of functional j, which give a correct meaning of $j(\varphi)$.

We have the same Theorem 1. For the proof, we apply first Tichonov's fixed point theorem to deduce the existence of u^ε, and then the existence of p^ε is obtained using the same duality results of convex optimisation. As for the solid-solid Coulomb interface law, cf., for example, [5], [43], the uniqueness is obtained for small k^ε.

In this case we state see [9] the main results concerning the existence of a weak limit $(u^\star, p^\star)$ of $(u^\varepsilon, p^\varepsilon)$, the strong convergence of u^ε to $u^\star$ in a convenient space, a specific Reynolds equation in a weak form, the limit form of the Coulomb boundary conditions, and the uniqueness of $(u^\star, p^\star)$.

Case 3 [10], leads to : *For G^ε as in (3.2), find $u^\varepsilon \in V^\varepsilon_{div}$ and $p^\varepsilon \in L^2_0(\Omega^\varepsilon)$, such that*

$$a_1(u^\varepsilon, \varphi - u) - (p^\varepsilon\,,\, div\,\varphi) + j(\varphi) - j(u^\varepsilon) \geq (f^\varepsilon, \varphi - u^\varepsilon) \quad \forall \varphi \in V^\varepsilon \tag{3.9}$$

where

$$a_1(u, \varphi) = a(u, \varphi) + \int_{\Gamma_1^\varepsilon} l^\varepsilon u\, \varphi ds$$

the integral on Γ_1^ε comes from the Fourier condition (2.17). The bilinear form a_1 is continuous symmetric and coercive indeed following [53] suppose that a_1 is not coercive so there exists a subsequence $(w_n) \in V^\varepsilon_{div}$ such that $a_1(w_n, w_n) < \dfrac{1}{n}\|w_n\|_{H^1(\Omega^\varepsilon)}$ $\forall n \in \mathbb{N}^\star$. Let $u_n = \dfrac{w_n}{\|w_n\|_{H^1(\Omega^\varepsilon)}}$, then $\|u_n\|_{H^1(\Omega^\varepsilon)} = 1$ and $a_1(u_n, u_n) < \dfrac{1}{n}$ for all $n \in \mathbb{N}^\star$. As

$$\|u_n\|^2_{H^1(\Omega^\varepsilon)} = a(u_n, u_n) + \|u_n\|^2_{L^2(\Omega^\varepsilon)} \leq a_1(u_n, u_n) + \|u_n\|^2_{L^2(\Omega^\varepsilon)} \leq \frac{1}{n} + 1 \leq 2$$

so there exists $u \in H^1(\Omega^\varepsilon)$ such that $u_n \rightharpoonup u$ in $H^1(\Omega^\varepsilon)$ weak and then in $L^2(\Omega^\varepsilon)$ strong thus $\|u\|_{L^2(\Omega^\varepsilon)} = 1$. And by

$$0 \leq \int_{\Gamma_1^\varepsilon} l^\varepsilon u_n^2 ds \leq \liminf_{n \to +\infty} \int_{\Gamma_1^\varepsilon} l^\varepsilon u_n^2 ds \leq \liminf_{n \to +\infty} \frac{1}{n} = 0$$

$$0 \leq a(u, u) \leq \liminf_{n \to +\infty} a(u_n, u_n) \leq \liminf_{n \to +\infty} \frac{1}{n} = 0$$

we obtain that $u = 0$ in Ω^ε which is impossible with $\|u\|_{L^2(\Omega^\varepsilon)} = 1$. Thus a_1 is coercive so here also the theorem 1 remains valid for this problem 2.

Case 4, leads to : Find $u^\varepsilon \in V^\varepsilon_{div}$, $p^\varepsilon \in L^2_0(\Omega^\varepsilon)$, such that

$$a(u^\varepsilon, \phi - u^\varepsilon) + \varepsilon^\gamma b(u^\varepsilon, u^\varepsilon, \phi - u^\varepsilon) - (p^\varepsilon, div(\phi)) + j(\phi) - j(u^\varepsilon) \geq (f^\varepsilon, \phi - u^\varepsilon) \;\; \forall \phi \in V^\varepsilon,$$

where

$$b:\ V^\varepsilon \times V^\varepsilon \times V^\varepsilon \to \mathbb{R} \quad : \quad (u, v, w) \to b(u, v, w) = \int_{\Omega^\varepsilon} u_i v_{j,i} w_j dx dx_3,$$

$$j(v) = \int_\omega k^\varepsilon \mid v - \varepsilon^\beta s \mid ds.$$

Theorem 2. *There exists μ_0 such that for $\mu > \mu_0$, this problem has at least one solution $(u^\varepsilon, p^\varepsilon)$, under the condition $\beta \geq \frac{1}{2} - \gamma$. There exists $\varepsilon^1 > 0$ such that for $\varepsilon \leq \varepsilon^1$, then u^ε, such that $(u^\varepsilon, p^\varepsilon)$ is solution of this problem, is unique.*

Proof. The condition $\beta \geq \frac{1}{2} - \gamma$ allow us to obtain the existence of a constant $C > 0$ such that the following application will be well defined

$$\Lambda \quad : B_C \rightarrow \; B_C : \qquad \text{such that} \qquad \xi \rightarrow u^\varepsilon$$

where B_C is the $(H^1(\Omega^\varepsilon))^3$ closed ball of radius C, u^ε is the unique solution of the following variational inequality:

$$a(u^\varepsilon, \phi - u^\varepsilon) + \varepsilon^\gamma b(\xi, u^\varepsilon, \phi - u^\varepsilon) + j(\phi) - j(u^\varepsilon) \geq (f^\varepsilon, \phi - u^\varepsilon) \;\; \forall \phi \in V^\varepsilon_{div}. \qquad (3.10)$$

then by Schauder fixed point theorem, there exists at least one solution u^ε for the following variational inequality:

$$a(u^\varepsilon, \phi - u^\varepsilon) + \varepsilon^\gamma b(u^\varepsilon, u^\varepsilon, \phi - u^\varepsilon) + j(\phi) - j(u^\varepsilon) \geq (f^\varepsilon, \phi - u^\varepsilon) \;\; \forall \phi \in V^\varepsilon_{div}. \qquad (3.11)$$

the existence of the pressure p^ε comes as in Theorem 1 (see also [6]). The uniqueness follows from some estimates on the gradient of the velocity and it is valid under the condition that the fluid domain must be thin enough. For all the proof see [11]. □

Case 5, leads to the non-isothermal coupled problem : Find

$$u^\varepsilon \in V^\varepsilon_{div} \cap (H^2(\Omega^\varepsilon))^3, \quad p^\varepsilon \in L^2_0(\Omega^\varepsilon) \cap H^1(\Omega^\varepsilon), \quad T^\varepsilon \in H^1_{\Gamma^\varepsilon_L \cup \Gamma^\varepsilon_1}(\Omega^\varepsilon) \cap C^{0,1}(\overline{\Omega^\varepsilon}),$$

such that

$$a(T^\varepsilon; u^\varepsilon, \varphi - u^\varepsilon) - (p^\varepsilon, \operatorname{div}(\varphi)) + j(\varphi) - j(u^\varepsilon) \geq (f^\varepsilon, \varphi - u^\varepsilon), \quad \forall \varphi \in V^\varepsilon, \qquad (3.12)$$

$$\int_{\Omega^\varepsilon} K^\varepsilon \nabla T^\varepsilon \nabla \psi + R^\varepsilon(T^\varepsilon)\psi \, dx, = \int_{\Omega^\varepsilon} 2\mu^\varepsilon(T^\varepsilon)|D(u)|^2 \psi \, dx + \int_\omega \theta^\varepsilon(T^\varepsilon)\psi \, ds, \quad \forall \psi \in H^1_{\Gamma^\varepsilon_L \cup \Gamma^\varepsilon_1}(\Omega^\varepsilon), \qquad (3.13)$$

where

$$H^1_{\Gamma^\varepsilon_L \cup \Gamma^\varepsilon_1}(\Omega^\varepsilon) = \Big\{ \chi \in H^1(\Omega^\varepsilon) \quad : \quad \chi = 0 \text{ on } \Gamma^\varepsilon_L \cup \Gamma^\varepsilon_1 \Big\}.$$

$$a(T; u, v) = \int_{\Omega^\varepsilon} 2\mu^\varepsilon(T) D(u) : D(v) \, dx' dx_3,$$

Note that the first term of $c(u; T, \psi)$ is well defined for $u \in V^\varepsilon \cap (H^2(\Omega^\varepsilon))^3$.

We study first the two following intermediate problems:

Given $T \in H^1_{\Gamma^\varepsilon_L \cup \Gamma^\varepsilon_1}(\Omega^\varepsilon) \cap C^{0,1}(\overline{\Omega^\varepsilon})$, find $v^\varepsilon \in V^\varepsilon_{div} \cap (H^2(\Omega^\varepsilon))^3$ such that

$$a(T; v^\varepsilon, \varphi - v^\varepsilon) - (p^\varepsilon, \operatorname{div}(\varphi)) + j(\varphi) - j(v^\varepsilon) \geq (f^\varepsilon, \varphi - v^\varepsilon), \quad \forall \varphi \in V^\varepsilon.$$

Given $u \in V^\varepsilon_{div} \cap (H^2(\Omega^\varepsilon))^3$, find T^ε in $H^1_{\Gamma^\varepsilon_L \cup \Gamma^\varepsilon_1}(\Omega^\varepsilon) \cap C^{0,1}(\overline{\Omega^\varepsilon})$ such that (3.13) hold.

The main result in this case is to establish the needed regularity results. Note that the boundary $\partial\Omega^\varepsilon$ is decomposed of three connected compact components ω, Γ^ε_1, and Γ^ε_L, the angles at the corner at the intersections $\Gamma^\varepsilon_1 \cap \Gamma^\varepsilon_L$ and $\omega \cap \Gamma^\varepsilon_L$ are less or equal to 90 degree. So we use the local regularity theory in a neighborhood of the boundary [44] and the partition of unity.

We obtain the regularity results in interior and near Γ^ε_1, Γ^ε_L and ω, following [21, 38, 48, 8, 34, 42], the difference here is that the coefficient of our bilinear form $a(.,.)$ depends on T^ε. So we obtain the following estimate

$$\|v^\varepsilon\|_{2,\Omega^\varepsilon} + \|p^\varepsilon\|_{1,\Omega^\varepsilon} \leq C \left(\|f^\varepsilon|_{0,\Omega^\varepsilon} + \|k^\varepsilon\|_{1/2,\omega} + \|G^\varepsilon\|_{2,\Omega^\varepsilon} \right) \qquad (3.14)$$

but the constant C depend on some data μ_*, μ^*, C_{μ^ε}, C_K, Ω^ε and also on $\|T^\varepsilon\|_{C^{0,1}(\overline{\Omega^\varepsilon})}$. Thus we establish the needed regularity result of the temperature T^ε, then we deduce with (3.14) the needed regularity of the velocity v^ε. Then with the Banach fixed point theorem we establish the existence and uniqueness results of the weak solution to the above coupled problem [14].

Case 6, we first assume that the function $g \in (W^{1-\frac{1}{r},r}(\Gamma^\varepsilon))^3$. So with (3.1) it is well known [3] (Lemma 3.3) that there exists a function G^ε such that

$$G^\varepsilon \in (W^{1,r}(\Omega^\varepsilon))^3 \ \text{ with } div(G^\varepsilon) = 0 \ \text{ in } \ \Omega^\varepsilon, \ and \ \ G^\varepsilon = g \ \text{ on } \ \Gamma^\varepsilon. \tag{3.15}$$

So we consider the following functional framework on Ω^ε:

$$W^{1,r}_{\Gamma_1^\varepsilon\cap\Gamma_L^\varepsilon}(\Omega^\varepsilon) = \{\psi \in W^{1,r}(\Omega^\varepsilon) \quad : \quad \psi = 0 \ \text{ on } \Gamma_1^\varepsilon \cap \Gamma_L^\varepsilon\}$$

$$V^\varepsilon = \{v \in (W^{1,r}(\Omega^\varepsilon))^3 \quad : \quad v = G^\varepsilon \quad \text{ on } \Gamma_1, \quad v\cdot n = 0 \quad \text{ on } \omega\}$$

$$L^{r'}_0(\Omega^\varepsilon) = \{\varphi \in L^{r'}(\Omega^\varepsilon) : \quad \int_{\Omega^\varepsilon} \varphi(x)dx = 0\}$$

then we denote by r', q' the conjugates of r and q respectively. This non-Newtonian non-isothermal leads to the following variational formulation : Find $u^\varepsilon \in V^\varepsilon_{div}$, $p^\varepsilon \in L^{r'}_0(\Omega^\varepsilon)$, and $T^\varepsilon \in W^{1,q}_{\Gamma_L^\varepsilon\cup\Gamma_1^\varepsilon}(\Omega^\varepsilon)$ such that

$$a(T^\varepsilon; u^\varepsilon, \varphi - u^\varepsilon) + (p^\varepsilon, \mathrm{div}(\varphi)) + j(\varphi) - j(u^\varepsilon) \geq (f^\varepsilon, \varphi - u^\varepsilon), \quad \forall \varphi \in V^\varepsilon, \tag{3.16}$$

$$\int_{\Omega^\varepsilon} K^\varepsilon \nabla T^\varepsilon \nabla\psi + R^\varepsilon T^\varepsilon dx = 2\int_{\Omega^\varepsilon} \mu^\varepsilon(T^\varepsilon)|D(u^\varepsilon)|^r \psi dx + \int_\omega b^\varepsilon \psi ds, \quad \forall \psi \in W^{1,q'}_{\Gamma_L^\varepsilon\cup\Gamma_1^\varepsilon}(\Omega^\varepsilon), \tag{3.17}$$

where

$$a(T^\varepsilon; u^\varepsilon, \varphi) = 2\int_{\Omega^\varepsilon} \mu^\varepsilon(T^\varepsilon)|D(u^\varepsilon)|^{r-2} D(u^\varepsilon) : D(\psi)dx$$

This non-Newtonian case is a direct generalized of the Newtonian **Case 5**. Here by the Sobolev inequalities [1], for $q' = q/(q-1) > 3$ the injection of $W^{1,q'}_{\Gamma_L^\varepsilon\cup\Gamma_1^\varepsilon}(\Omega^\varepsilon)$ in $L^\infty(\Omega^\varepsilon)$ make sense to the first term of the right-hand side of (3.13). This idea is not possible for the Newtonian **Case 5**.

We assume that there exist $\mu_\star$, $\mu^\star$, $K^\varepsilon_\star$, $K^\star_\varepsilon$, $R^\varepsilon_\star$, $R^\star_\varepsilon$, C^ε_b in $\mathbb{R}$ such that

$$\left.\begin{array}{l} \mu^\varepsilon \in \mathcal{C}^1(\mathbb{R}), \qquad 0 < \mu_\star \leq \mu^\varepsilon \leq \mu^\star \ ; \\ \\ f^\varepsilon \in (W^{1,r'}(\Omega^\varepsilon))^3 \ ; \ 0 < K^\varepsilon_\star \leq K^\varepsilon \leq K^\star_\varepsilon \ ; \\ \\ 0 < R^\varepsilon_\star \leq R^\varepsilon \leq r^\star_\varepsilon \ ; \ |b^\varepsilon| \leq C^\varepsilon_b. \end{array}\right\} (H)$$

we have the following result

Lemma 1. [15] *Let $\theta \in W^{1,q}_{\Gamma_1^\varepsilon\cup\Gamma_L^\varepsilon}$. We denote by $u^\varepsilon_\theta \in V^\varepsilon_{div}$ the solution of the following inequality*

$$a(\theta; u^\varepsilon_\theta, \phi - u^\varepsilon_\theta) + j^\varepsilon(\phi) - j^\varepsilon(u^\varepsilon_\theta) \geq (f^\varepsilon, \phi - u^\varepsilon_\theta) \ \forall \phi \in V^\varepsilon_{div}.$$

Then there exists C^ε constante independent of θ such that

$$\| \nabla u^\varepsilon_\theta \|_{L^r(\Omega^\varepsilon)} \leq C^\varepsilon. \tag{3.18}$$

And the application :

$$\theta \longrightarrow u^\varepsilon_\theta \in V^\varepsilon_{div}$$

is strongly continuous.

Theorem 3. *Assume (H) hold. For all $r > 1$ the problem* (3.16)-(3.17) *has at last one solution $(u^\varepsilon, p^\varepsilon, T^\varepsilon)$ in $V^\varepsilon_{div} \times L^{r'}_0(\Omega^\varepsilon) \times W^{1,q}_{\Gamma_1 \cup \Gamma_L}(\Omega^\varepsilon)$.*

Proof. [15] Let $\delta > 0$, we consider the function

$$(\theta, u^\varepsilon_\theta) \mapsto m_\delta(\theta) = \frac{2\mu^\varepsilon(\theta) \mid D(u^\varepsilon_\theta) \mid^r}{1 + 2\delta\mu^\varepsilon(\theta) \mid D(u^\varepsilon_\theta) \mid^r} \tag{3.19}$$

where u^ε_θ satisfies the variational inequality

$$a(\theta; u^\varepsilon_\theta, \phi - u^\varepsilon_\theta) + j^\varepsilon(\phi) - j^\varepsilon(u^\varepsilon_\theta) \geq (f^\varepsilon, \phi - u^\varepsilon_\theta) \ \forall \phi \in V^\varepsilon_{div}.$$

Using (3.18), $\exists C^\varepsilon_1$ a constant independent of δ, θ, u^ε_θ such that

$$\parallel m_\delta \parallel_{L^1(\Omega^\varepsilon)} \leq C^\varepsilon_1. \tag{3.20}$$

Now, we consider the following problem : Find T^ε_δ such that

$$\int_{\Omega^\varepsilon} K^\varepsilon \nabla T^\varepsilon_\delta \nabla \psi + R^\varepsilon T^\varepsilon_\delta \psi = \int_{\Omega^\varepsilon} m_\delta \psi, \ + \int_\omega b^\varepsilon \psi \qquad \forall \psi \in H_{\Gamma^\varepsilon_1 \cup \Gamma^\varepsilon_L}(\Omega^\varepsilon). \tag{3.21}$$

This problem has a unique solution by Lax-Milgram Lemma. Let define the application

$$\gamma : B(0, \tilde{C}) \cap W^{1,q}_{\Gamma^\varepsilon_1 \cup \Gamma^\varepsilon_L}(\Omega^\varepsilon) \rightarrow B(0, \tilde{C}) \cap W^{1,q}_{\Gamma^\varepsilon_1 \cup \Gamma^\varepsilon_L}(\Omega^\varepsilon)$$

$$\theta \mapsto T^\varepsilon_\delta$$

where $B(0, \tilde{C})$ is a closed ball in $W^{1,q}_{\Gamma^\varepsilon_1 \cup \Gamma^\varepsilon_L}(\Omega^\varepsilon)$. We must look for $\tilde{C} > 0$ such that γ be well defined. So we choose $\psi = \varphi(T^\varepsilon_\delta)$ in (3.21), where φ is defined by

$$\varphi(t) = \xi \ sign(t) \int_0^{|t|} \frac{d\tau}{(1+\tau)^{\xi+1}} = sign(t) \big[1 - \frac{1}{(1+ \mid t \mid)^\xi} \big], \tag{3.22}$$

with $\xi > 0$. we deduce from (3.20)-(3.21) after some calculations that

$$\int_{\Omega^\varepsilon} \mid \nabla T^\varepsilon_\delta \mid^q \leq 2^{\frac{(q^\star - 1)(2-q)}{2}} \Big(\frac{C^\varepsilon_1}{\xi K^\varepsilon_\star} \Big)^{\frac{q}{2}} \times \Big(\mid \Omega^\varepsilon \mid^{\frac{(2-q)}{2}} + A \Big). \tag{3.23}$$

A is independent of δ. So we can choose

$$\tilde{C} = \Big[2^{(q^\star - 1)(2-q)/2} \Big(\frac{C^\varepsilon_1}{\xi K^\varepsilon_\star} \Big)^{q/2} \Big(\mid \Omega^\varepsilon \mid^{(2-q)/2} + A \Big) \Big]^{\frac{1}{q}}$$

Using now lemma 1 and Schauder's fixed point theorem with the application γ, so there exists

$$(u^\varepsilon_\delta, p^\varepsilon_\delta, T^\varepsilon_\delta) \in V^\varepsilon_{div} \times L^{r'}_0(\Omega^\varepsilon) \times H^1(\Omega^\varepsilon) \cap B(0, \tilde{C})$$

where $(u^\varepsilon_\delta, p^\varepsilon_\delta)$ solves (3.12), with

$$u^\varepsilon_\delta = u^\varepsilon_{T^\varepsilon_\delta}, \quad p^\varepsilon_\delta = p^\varepsilon_{T^\varepsilon_\delta}, \qquad T^\varepsilon_\delta \text{ solves (3.21)}$$

and $m_\delta = m_\delta(u^\varepsilon_\delta, T^\varepsilon_\delta)$ defined by (3.19). We obtain also the following estimation for the pressure p^ε_δ (as in [13])

$$\parallel \frac{\partial p^\varepsilon_\delta}{\partial x_i} \parallel_{W^{-1,r'}(\Omega^\varepsilon)} \leq C \ \ i = 1, 2, 3, \tag{3.24}$$

using (3.18), (3.23), (3.24), and taking a subsequence $\delta \longrightarrow 0$, we obtain

$$u_\delta^\varepsilon \rightharpoonup u^\varepsilon \quad \text{weakly in} \quad W^{1,r}_{\Gamma_1^\varepsilon \cup \Gamma_L^\varepsilon},$$
$$p_\delta^\varepsilon \rightharpoonup p^\varepsilon \quad \text{weakly in} \quad L^{r'}(\Omega^\varepsilon),$$
$$T_\delta^\varepsilon \rightharpoonup T^\varepsilon \quad \text{weakly in} \quad W^{1,q}(\Omega^\varepsilon) \quad \text{and} \quad \text{strongly in } L^q(\Omega^\varepsilon).$$

From Lemma 1, $u_\delta^\varepsilon \to u^\varepsilon$ strongly in V_{div}^ε consequently as $\mu^\varepsilon \in C^1(\mathbb{R})$ we get

$$m_\delta = m_\delta(T_\delta^\varepsilon, u_\delta^\varepsilon) \to 2\mu^\varepsilon(T^\varepsilon) \mid D(u^\varepsilon) \mid^r \quad \text{in} \quad L^1(\Omega^\varepsilon),$$

we conclude that the limit, $(u^\varepsilon, p^\varepsilon, T^\varepsilon)$ solves Problem (3.16)-(3.17). □

Case 7, leads to the same variational inequality (3.3) so the existence and uniqueness of the weak solution comes from Theorem 1. The main difficulties is to study the behavior of the weak solution and especially how to pass to the two-scale limit in the variational inequality, due to the term coming from the Tresca fluid-solid boundary conditions. This difficulty induce us to prove (see [17]) a needed result of lower-semicontinuity for the two-scale convergence, using some results on subdifferential and regularization of convex functions.

4. Asymptotic analysis

To be able to compare the solutions for various ε and provide the asymptotic analysis, we use the change of variables $y = x_3/\varepsilon$ to define the fixed domain

$$\Omega = \{(x, y) \quad \text{such that} \quad x \in \omega, \quad \text{and} \quad 0 < y < h(x)\},$$

and we denote by $\Gamma = \bar{\omega} \cup \bar{\Gamma}_L \cup \bar{\Gamma}_1$ its boundary. For the three cases 1-3, we define the following functions in Ω

$$\hat{u}_i^\varepsilon(x, y) = u_i^\varepsilon(x, x_3)\, 1 \le i \le 2, \quad \hat{u}_3^\varepsilon(x, y) = \frac{1}{\varepsilon} u_3^\varepsilon(x, x_3),$$
$$\hat{p}^\varepsilon(x, y) = \varepsilon^2 p^\varepsilon(x, x_3).$$

Let us define first the $\varepsilon-$independent vector

$$\hat{f}(x, y) = (\hat{f}_1(x, y), \hat{f}_2(x, y), \hat{f}_3(x, y)),$$

then assume the following dependence (with respect to ε) of the data

$$\hat{f}(x, y) = \varepsilon^2 f^\varepsilon(x, x_3), \quad \hat{g}(x, y) = g(x, x_3). \tag{4.1}$$

$$\hat{k} = \varepsilon k^\varepsilon \text{ for the Tresca cases 1, 3, but } \hat{k} = \varepsilon^{-1} k^\varepsilon \text{ for the Coulomb case 2.} \tag{4.2}$$

The first assumption in (4.1) means that the body forces cannot be too big. In (4.2) the first one means that k^ε, the upper limit for the tangential stress has the same order of magnitude as the actual stress inside the fluid, which is the ratio of the tangential velocity and of the gap: $\frac{s}{\varepsilon h}$, while the second one means that, roughly speaking, the friction coefficient $k^\varepsilon \sim \varepsilon$ is the ratio of the tangential stress inside the fluid σ_T^ε and of the normal stress σ_n^ε, while $\sigma_T^\varepsilon \sim \frac{1}{\varepsilon}$ (ratio between the tangential velocity and the thickness of the gap) and $\sigma_n^\varepsilon \sim \frac{1}{\varepsilon^2}$ (order of magnitude of the actual pressure).

Let us define the $\varepsilon-$independent vector $\hat{G}(x, y) = (\hat{G}_1(x, y), \hat{G}_2(x, y), \hat{G}_3(x, y))$ such that

$$\frac{\partial \hat{G}_1}{\partial x_1} + \frac{\partial \hat{G}_2}{\partial x_2} + \frac{\partial \hat{G}_3}{\partial y} = 0 \quad in\, \Omega, \quad \hat{G} = \hat{g} \quad \text{on} \quad \Gamma,$$

and recalling that $g_3 = 0$ on Γ_L, then we can choose as G^ε the lift defined by

$$G_i^\varepsilon(x, x_3) = \hat{G}_i(x, y) \quad for \quad i = 1, 2 \quad and \quad G_3^\varepsilon(x, x_3) = \varepsilon\hat{G}_3(x, y).$$

Now we define function spaces and sets on Ω we need in our considerations.

$$V = \{\varphi \in (H^1(\Omega)^3 \ : \varphi.n = 0 \quad \text{on} \quad \omega, \quad v = \hat{G} \quad \text{on}\, \Gamma_L \cup \Gamma_1\},$$

$$V_{div} = \{\varphi \in V : \quad div\, \varphi = 0 \text{ in } \Omega\}$$

$$V_0(\Omega) = \{v \in (H^1(\Omega)^3 \ : \ v.n = 0 \quad \text{on} \quad \omega,\, v = 0\, \text{on}\, \Gamma_L \cup \Gamma_1\},$$

$$L_0^2(\Omega) = \{q \in L^2\,(\Omega) \ : \ \int_\Omega q dx dy = 0\},$$

Then assuming (4.1) and the first of (4.2), there exists a unique $\hat{u}^\varepsilon$ in V_{div} and $\hat{p}^\varepsilon$ in $L_0^2(\Omega)$, such that the variational inequality (3.3) leads to the following form:

$$\begin{aligned}
&\sum_{i,j=1}^{2} \int_\Omega \left(\varepsilon^2\nu(\frac{\partial \hat{u}_i^\varepsilon}{\partial x_j} + \frac{\partial \hat{u}_j^\varepsilon}{\partial x_i}) - \hat{p}^\varepsilon \delta_{i,j}\right) \frac{\partial}{\partial x_j}(\varphi_i - \hat{u}_i^\varepsilon) dx dy + \\
&\sum_{i=1}^{2} \int_\Omega \nu(\frac{\partial \hat{u}_i^\varepsilon}{\partial y} + \varepsilon^2 \frac{\partial \hat{u}_3^\varepsilon}{\partial x_i}) \frac{\partial}{\partial y}(\varphi_i - \hat{u}_i^\varepsilon) dx dy + \int_\Omega (2\nu\varepsilon^2 \frac{\partial \hat{u}_3^\varepsilon}{\partial y} - \hat{p}^\varepsilon) \frac{\partial}{\partial y}(\varepsilon^{-1}\varphi_3 - \hat{u}_3^\varepsilon) dx dy \\
&+ \sum_{j=1}^{2} \int_\Omega \varepsilon^2 \nu(\varepsilon^2 \frac{\partial \hat{u}_3^\varepsilon}{\partial x_j} + \frac{\partial \hat{u}_j^\varepsilon}{\partial y}) \frac{\partial}{\partial x_j}(\varepsilon^{-1}\varphi_3 - \hat{u}_3^\varepsilon) dx dy \geq \sum_{i=1}^{2} \int_\Omega \hat{f}_i(\varphi_i - \hat{u}_i^\varepsilon) dx dy \\
&+ \int_\Omega \varepsilon \hat{f}_3(\varepsilon^{-1}\varphi_3 - \hat{u}_3^\varepsilon) dx dy + \int_\omega \hat{k}(|\varphi - s|) - |\hat{u}^\varepsilon - s|) dx \quad \forall \varphi \in K. \qquad (4.3)
\end{aligned}$$

Theorem 4. *Assuming* (4.1) *and the first of* (4.2) *we have the following estimate on* $\hat{u}^\varepsilon$

$$\begin{aligned}
&\frac{\nu\varepsilon^2}{2}\|\frac{\partial \hat{u}_1^\varepsilon}{\partial x_1}\|^2 + \frac{\nu\varepsilon^2}{2}\|\frac{\partial \hat{u}_1^\varepsilon}{\partial x_2}\|^2 + \frac{\nu\varepsilon^2}{2}\|\frac{\partial \hat{u}_2^\varepsilon}{\partial x_1}\|^2 + (\frac{\nu}{2} - \frac{\delta^2}{4})\|\frac{\partial \hat{u}_1^\varepsilon}{\partial y}\|^2 \\
&+(\frac{\nu}{2} - \frac{\delta^2}{4})\|\frac{\partial \hat{u}_2^\varepsilon}{\partial y}\|^2 + \frac{\nu\varepsilon^2}{2}\|\frac{\partial \hat{u}_2^\varepsilon}{\partial x_2}\|^2 + \varepsilon^2(\frac{\nu}{2} - \frac{\delta^2}{4})\|\frac{\partial \hat{u}_3^\varepsilon}{\partial y}\|^2 \\
&+\frac{\nu\varepsilon^4}{2}\|\frac{\partial \hat{u}_3^\varepsilon}{\partial x_1}\|^2 + \frac{\nu\varepsilon^4}{2}\|\frac{\partial \hat{u}_3^\varepsilon}{\partial x_2}\|^2 \leq C_0. \qquad (4.4)
\end{aligned}$$

where $\|.\|$ *denotes here the* L^2*-norm in* Ω*,* δ *is the diameter of* Ω*, and* C_0 *is an independent constant of* ε*.*

Proof. [6] Putting $\varphi_i = \hat{G}_i$ for $i = 1, 2$ and $\varphi_3 = \varepsilon \hat{G}_3$, in (5.8), leads to

$$\sum_{i,j=1}^{2} \int_\Omega \left(\varepsilon^2 \nu (\frac{\partial \hat{u}_i^\varepsilon}{\partial x_j} + \frac{\partial \hat{u}_j^\varepsilon}{\partial x_i}) - \hat{p}^\varepsilon \delta_{i,j} \right) \frac{\partial \hat{u}_i^\varepsilon}{\partial x_j} dxdy + \int_\Omega (2\nu \varepsilon^2 \frac{\partial \hat{u}_3^\varepsilon}{\partial y} - \hat{p}^\varepsilon) \frac{\partial \hat{u}_3^\varepsilon}{\partial y} dxdy$$

$$+ \sum_{i=1}^{2} \int_\Omega \nu (\frac{\partial \hat{u}_i^\varepsilon}{\partial y} + \varepsilon^2 \frac{\partial \hat{u}_3^\varepsilon}{\partial x_i}) \frac{\partial \hat{u}_i^\varepsilon}{\partial y} dxdy + \sum_{j=1}^{2} \int_\Omega \varepsilon^2 \nu (\varepsilon^2 \frac{\partial \hat{u}_3^\varepsilon}{\partial x_j} + \frac{\partial \hat{u}_j^\varepsilon}{\partial y}) \frac{\partial \hat{u}_3^\varepsilon}{\partial x_j} dxdy$$

$$\leq \sum_{i,j=1}^{2} \int_\Omega \left(\varepsilon^2 \nu (\frac{\partial \hat{u}_i^\varepsilon}{\partial x_j} + \frac{\partial \hat{u}_j^\varepsilon}{\partial x_i}) - \hat{p}^\varepsilon \delta_{i,j} \right) \frac{\partial \hat{G}_i}{\partial x_j} dxdy + \int_\Omega (2\nu \varepsilon^2 \frac{\partial \hat{u}_3^\varepsilon}{\partial y} - \hat{p}^\varepsilon) \frac{\partial \hat{G}_3}{\partial y} dxdy$$

$$+ \sum_{i=1}^{2} \int_\Omega \nu (\frac{\partial \hat{u}_i^\varepsilon}{\partial y} + \varepsilon^2 \frac{\partial \hat{u}_3^\varepsilon}{\partial x_i}) \frac{\partial \hat{G}_i}{\partial y} dxdy + \sum_{j=1}^{2} \int_\Omega \varepsilon^2 \nu (\varepsilon^2 \frac{\partial \hat{u}_3^\varepsilon}{\partial x_j} + \frac{\partial \hat{u}_j^\varepsilon}{\partial y}) \frac{\partial \hat{G}_3}{\partial x_j} dxdy +$$

$$+ \int_\omega \hat{k} |\hat{G} - s| dx - \sum_{i=1}^{2} \int_\Omega \hat{f}_i (\hat{G}_i - \hat{u}_i^\varepsilon) dxdy - \int_\Omega \varepsilon \hat{f}_3 (\hat{G}_3 - \hat{u}_3^\varepsilon) dxdy, \tag{4.5}$$

as $\hat{k}$ is positive.
Using (3.3), the Poincaré inequality, $\varepsilon \leq 1$, and $2ab \leq a^2 + b^2$ we deduce

$$\frac{\nu \varepsilon^2}{2} \| \frac{\partial \hat{u}_1^\varepsilon}{\partial x_1} \|^2 + \frac{\nu \varepsilon^2}{2} \| \frac{\partial \hat{u}_1^\varepsilon}{\partial x_2} \|^2 + \frac{\nu \varepsilon^2}{2} \| \frac{\partial \hat{u}_2^\varepsilon}{\partial x_1} \|^2 + (\frac{\nu}{2} - \frac{\delta^2}{4}) \| \frac{\partial \hat{u}_1^\varepsilon}{\partial y} \|^2 + (\frac{\nu}{2} - \frac{\delta^2}{4}) \| \frac{\partial \hat{u}_2^\varepsilon}{\partial y} \|^2$$

$$+ \frac{\nu \varepsilon^2}{2} \| \frac{\partial \hat{u}_2^\varepsilon}{\partial x_2} \|^2 + \varepsilon^2 (\frac{\nu}{2} - \frac{\delta^2}{4}) \| \frac{\partial \hat{u}_3^\varepsilon}{\partial y} \|^2 + \frac{\nu \varepsilon^4}{2} \| \frac{\partial \hat{u}_3^\varepsilon}{\partial x_1} \|^2 + \frac{\nu \varepsilon^4}{2} \| \frac{\partial \hat{u}_3^\varepsilon}{\partial x_2} \|^2 \leq$$

$$\leq \nu \| \frac{\partial \hat{G}_1}{\partial x_1} \|^2 + \nu \| \frac{\partial \hat{G}_1}{\partial x_2} \|^2 + \nu \| \frac{\partial \hat{G}_2}{\partial x_1} \|^2 + \nu \| \frac{\partial \hat{G}_1}{\partial y} \|^2 + \nu \| \frac{\partial \hat{G}_2}{\partial y} \|^2$$

$$+ \nu \| \frac{\partial \hat{G}_2}{\partial x_2} \|^2 + \nu \| \frac{\partial \hat{G}_3}{\partial y} \|^2 + \nu \| \frac{\partial \hat{G}_3}{\partial x_1} \|^2 + \nu \| \frac{\partial \hat{G}_3}{\partial x_2} \|^2$$

$$+ \|\hat{f}_1\| \|\hat{G}_1\| + \|\hat{f}_2\| \|\hat{G}_2\| + \|\hat{f}_3\| \|\hat{G}_3\| + (\|\hat{f}_1\|^2 + \|\hat{f}_2\|^2 + \|\hat{f}_3\|^2) + const. \|\hat{k}\|_{L^\infty(\omega)} = C_0,$$

thus (4.4) follows. □

Theorem 5. *Assuming* (4.1), *and* $\nu > \frac{\delta^2}{2}$ *or* $f = 0$, *the following estimates on* p^ε *are satisfied.*

$$\| \frac{\partial \hat{p}^\varepsilon}{\partial x_i} \|_{H^{-1}(\Omega)} \leq C_1 \qquad (i = 1, 2) \tag{4.6}$$

$$\| \frac{\partial \hat{p}^\varepsilon}{\partial y} \|_{H^{-1}(\Omega)} \leq \varepsilon . C_2, \tag{4.7}$$

where C_1 *and* C_2 *denote independent constants of* ε.

Proof. [6] Let ψ in $H_0^1(\Omega)$, putting in (5.8) $\varphi_i = \hat{u}_i^\varepsilon$ (for $i = 1, 2$), and $\varphi_3 = \varepsilon\hat{u}_3^\varepsilon \pm \psi$, we deduce

$$-\int_\Omega \hat{p}^\varepsilon \frac{\partial \psi}{\partial y} dxdy = -\int_\Omega 2\nu\varepsilon^2 \frac{\partial \hat{u}_3^\varepsilon}{\partial y}\frac{\partial \psi}{\partial y} dxdy -$$

$$-\sum_{j=1}^{2}\int_{\Omega^\varepsilon} \varepsilon^2\nu(\varepsilon^2\frac{\partial \hat{u}_3^\varepsilon}{\partial x_j} + \frac{\partial \hat{u}_j^\varepsilon}{\partial y})\frac{\partial \psi}{\partial x_j} dxdy + \int_\Omega \varepsilon \hat{f}_3\psi dxdy \tag{4.8}$$

Taking in (5.8) $\varphi_1 = \hat{u}_1^\varepsilon \pm \psi$, ψ in $H_0^1(\Omega)$, $\varphi_2 = \hat{u}_2^\varepsilon$, $\varphi_3 = \varepsilon\hat{u}_3^\varepsilon$, we get

$$-\int_\Omega \hat{p}^\varepsilon \frac{\partial \psi}{\partial x_1} dxdy = -\int_\Omega 2\varepsilon^2\nu \frac{\partial \hat{u}_1^\varepsilon}{\partial x_1}\frac{\partial \psi}{\partial x_1} dxdy - \int_\Omega \varepsilon^2\nu\left(\frac{\partial \hat{u}_1^\varepsilon}{\partial x_2} + \frac{\partial \hat{u}_2^\varepsilon}{\partial x_1}\right)\frac{\partial \psi}{\partial x_1} dxdy$$

$$-\int_\Omega \nu\left(\frac{\partial \hat{u}_1^\varepsilon}{\partial y} + \varepsilon^2\frac{\partial \hat{u}_3^\varepsilon}{\partial x_1}\right)\frac{\partial \psi}{\partial y} dxdy + \int_\Omega \hat{f}_1\psi dxdy \quad \forall \psi \in H_0^1(\Omega) \tag{4.9}$$

In the same way, the choice $\varphi_1 = \hat{u}_1^\varepsilon$, $\varphi_2 = \hat{u}_2^\varepsilon \pm \psi$, ψ in $H_0^1(\Omega)$, $\varphi_3 = \varepsilon\hat{u}_3^\varepsilon$, leads to

$$-\int_\Omega \hat{p}^\varepsilon \frac{\partial \psi}{\partial x_2} dxdy = -\int_\Omega 2\varepsilon^2\nu \frac{\partial \hat{u}_2^\varepsilon}{\partial x_2}\frac{\partial \psi}{\partial x_2} dxdy - \int_\Omega \varepsilon^2\nu\left(\frac{\partial \hat{u}_1^\varepsilon}{\partial x_2} + \frac{\partial \hat{u}_2^\varepsilon}{\partial x_1}\right)\frac{\partial \psi}{\partial x_1} dxdy$$

$$-\int_\Omega \nu\left(\frac{\partial \hat{u}_2^\varepsilon}{\partial y} + \varepsilon^2\frac{\partial \hat{u}_3^\varepsilon}{\partial x_2}\right)\frac{\partial \psi}{\partial y} dxdy + \int_\Omega \hat{f}_2\psi dxdy \quad \forall \psi \in H_0^1(\Omega) \tag{4.10}$$

then from (4.8) using (4.4) we get (4), and from (4.9)-(4.10) using (4.4) we get (5.2). □

We define now the Banach space

$$V_y = \{v \in (L^2(\Omega))^2 : \quad \frac{\partial v}{\partial y} \in (L^2(\Omega))^2, \quad v = 0 \quad on \quad \Gamma_1\}$$

with its norm

$$\|v\|_{V_y} = \sum_{i=1}^{2}\left(\|v_i\|^2_{L^2(\Omega)} + \|\frac{\partial v_i}{\partial y}\|^2_{L^2(\Omega)}\right).$$

Corollary 1. *Let the assumptions of* Theorem 1 *and* Theorem 5 *hold, then there exists* $u_i^\star$ *in* V_y *(*$i = 1, 2$*), and* $p^\star$ *in* $L_0^2(\Omega)$ *such that*

$$\hat{u}_i^\varepsilon \rightharpoonup u_i^\star \quad (1 \le i \le 2) \quad \text{weakly in} \quad V_y \tag{4.11}$$

where $V_y = \{\psi \in L^2(\Omega)$ such that $\frac{\partial \psi}{\partial y} \in L^2(\Omega)\}$.

$$\varepsilon\frac{\partial \hat{u}_i^\varepsilon}{\partial x_j} \rightharpoonup 0 \quad (1 \le i, j \le 2) \quad \text{weakly in} \quad L^2(\Omega) \tag{4.12}$$

$$\varepsilon\frac{\partial \hat{u}_3^\varepsilon}{\partial y} \rightharpoonup 0 \qquad \text{weakly in} \quad L^2(\Omega) \tag{4.13}$$

$$\varepsilon^2\frac{\partial \hat{u}_3^\varepsilon}{\partial x_i} \rightharpoonup 0 \quad (1 \le i \le 2) \quad \text{weakly in} \quad L^2(\Omega) \tag{4.14}$$

$$\varepsilon\hat{u}_3^\varepsilon \rightharpoonup 0 \qquad \text{weakly in} \quad L^2(\Omega) \tag{4.15}$$

$$\hat{p}^\varepsilon \rightharpoonup p^\star \qquad \text{weakly in} \quad L_0^2(\Omega) \tag{4.16}$$

Proof. [6] From (4.4) there exists a fixed constant C which does not depend on ε such that

$$\|\frac{\partial \hat{u}_i^\varepsilon}{\partial y}\| \leq C \quad (1 \leq i \leq 2)$$

using the above estimate and the Poincare inequality in the domain Ω we deduce (4.11). Also (4.12)-(4.14) follows from (4.4), and (4.16) follows from (5.2), (4) and [54]. To prove (4.15), as in [7] we choose q such that $q(x,y) = y\theta(x) - \gamma$ where θ in $C_0^\infty(\omega)$ and

$$\gamma = \left(\int_\Omega y\theta dxdy\right) / \left(\int_\Omega dxdy\right).$$

Using the Green formula, the boundary conditions on Γ imply

$$-\sum_{i=1}^{2}\int_\Omega y\hat{u}_i^\varepsilon \frac{\partial \theta}{\partial x_i} dxdy - \int_\Omega \theta \hat{u}_3^\varepsilon dxdy = 0,$$

As $\hat{u}_i^\varepsilon \rightharpoonup u_i^\star$ in V_y $(i = 1,2)$, then (4.15) holds. □

4.1. **Study of the limit problem of the 1rst Case.** We give both the equations satisfied by $p^\star$ and $u^\star$ in Ω and the inequalities for the trace of the velocity $u^\star(x,0)$ and the stress $\dfrac{\partial u^\star}{\partial y}(x,0)$ on $\partial\omega$.

Theorem 6. *With the same assumptions as in* Theorem 5, $(u^\star, p^\star)$ *satisfy*

$$p^\star \in H^1(\omega), \tag{4.17}$$

$$-\nu \frac{\partial^2 u_i^\star}{\partial y^2} + \frac{\partial p^\star}{\partial x_i} = \hat{f}_i \quad (i = 1,2) \quad \text{in} \quad L^2(\Omega). \tag{4.18}$$

Proof. [6] We choose in (5.8) $\varphi_3 = \hat{u}_3^\varepsilon \pm \psi$ with ψ in $H_0^1(\Omega)$ we deduce

$$\sum_{j=1}^{2}\int_{\Omega^\varepsilon} \varepsilon^2\nu(\varepsilon^2 \frac{\partial \hat{u}_3^\varepsilon}{\partial x_j} + \frac{\partial \hat{u}_j^\varepsilon}{\partial y})\frac{\partial \psi}{\partial x_j} dxdy + \int_\Omega (2\nu\varepsilon^2 \frac{\partial \hat{u}_3^\varepsilon}{\partial y} - \hat{p}^\varepsilon)\frac{\partial \psi}{\partial y} dxdy$$

$$= \int_\Omega \varepsilon f_3 \psi dxdy$$

Using (4.14) (4.11) (4.13) and the hypothesis of this theorem we obtain

$$\int_\Omega p^\star \frac{\partial \psi_3}{\partial y} dxdy = 0 \quad \forall \psi \in H_0^1(\Omega),$$

then

$$\frac{\partial p^\star}{\partial y} = 0 \quad in \quad H^{-1}(\Omega). \tag{4.19}$$

Choosing now $\varphi_i = \hat{u}_i^\varepsilon \pm \psi_i$, for $(i = 1,2)$ with ψ_i in $H_0^1(\Omega)$ and $\varphi_3 = \varepsilon \hat{u}_3^\varepsilon$, in (5.8), leads to

$$\sum_{i,j=1}^{2}\int_\Omega \left(\varepsilon^2\nu(\frac{\partial \hat{u}_i^\varepsilon}{\partial x_j} + \frac{\partial \hat{u}_j^\varepsilon}{\partial x_i}) - \hat{p}^\varepsilon \delta_{i,j}\right)\frac{\partial \psi_i}{\partial x_j} dxdy +$$

$$+\sum_{i=1}^{2}\int_{\Omega}\nu(\frac{\partial \hat{u}_i^{\varepsilon}}{\partial y}+\varepsilon^2\frac{\partial \hat{u}_3^{\varepsilon}}{\partial x_i})\frac{\partial \psi_i}{\partial y}dxdy=\sum_{i=1}^{2}\int_{\Omega}\hat{f}_i\psi_i dxdy \tag{4.20}$$

Using (4.12) (4.16) (4.11) (4.14) and the hypothesis of this theorem, we deduce first with $\psi_1 = 0$ and ψ_2 in $H_0^1(\Omega)$, then with $\psi_2 = 0$ and ψ_1 in $H_0^1(\Omega)$, the following equality

$$-\sum_{i=1}^{2}\int_{\Omega}p^{\star}\frac{\partial \psi_i}{\partial x_i}dxdy+\sum_{i=1}^{2}\int_{\Omega}\nu\frac{\partial u_i^{\star}}{\partial y}\frac{\partial \psi_i}{\partial y}dxdy=\sum_{i=1}^{2}\int_{\Omega}\hat{f}_i\psi_i dxdy \tag{4.21}$$

then using the Green formula, we obtain

$$-\nu\frac{\partial^2 u_i^{\star}}{\partial y^2}+\frac{\partial p^{\star}}{\partial x_i}=\hat{f}_i \quad (i=1,2) \quad \text{in} \quad H^{-1}(\Omega). \tag{4.22}$$

To prove that $p^{\star}$ is in $H^1(\omega)$, let us recall first that $p^{\star}$ is a function of (x_1, x_2) only from (5), then following [7] we choose ψ_i in (4.21) such that $\psi_i(x, y) = y(y - h(x))\theta(x)$ with θ in $H_0^1(\omega)$, and using the Green formula we deduce

$$\frac{1}{6}\int_{\omega}p^{\star}\frac{\partial(h^3\theta)}{\partial x_i}dx-2\nu\int_{\omega}h\tilde{u_i^{\star}}\theta dx=\int_{\omega}\tilde{f}_i\theta dx$$

where

$$\tilde{u_i^{\star}}(x)=\frac{1}{h(x)}\int_0^{h(x)}u_i^{\star}(x,y)dy, \quad \textit{and} \quad \tilde{f}_i(x)=\int_0^{h(x)}y(y-h(x))\hat{f}_i(x,y)dy.$$

Whence

$$2\nu h\tilde{u_i^{\star}}-\frac{1}{6}h^3\frac{\partial p^{\star}}{\partial x_i}=\tilde{f}_i \qquad (i=1,2) \quad \text{in} \quad H^{-1}(\omega). \tag{4.23}$$

As f_i is in $L^2(\Omega)$, $u_i^{\star}$ in V_y then in $L^2(\Omega)$, therefore $\tilde{f}_i$ and $\tilde{u}_i^{\star}$ are in $L^2(\omega)$, then from (4.23) we get $p^{\star}$ in $H^1(\omega)$, then (5) follows. So as f_i belongs to $L^2(\Omega)$, then from (5.10) we have $\frac{\partial^2 u_i^{\star}}{\partial y^2}$ in $L^2(\Omega)$. Whence (5.2) holds, and we also have $\frac{\partial u_i^{\star}}{\partial y}$ in V_y. □

For convenience, we will denote by $s^{\star}(x) = u^{\star}(x, 0)$ and $\tau^{\star}(x) = \frac{\partial u^{\star}}{\partial y}(x, 0)$, as $\frac{\partial u^{\star}}{\partial y}$ in V_y then $\tau^{\star}$ belongs to $L^2(\omega)$, and we have :

Theorem 7. *Under the same hypothesis of* Theorem 5, *$(s^{\star}, \tau^{\star})$ satisfy the following inequalities*

$$\int_{\omega}\hat{k}(|\psi+s^{\star}-s|)-|s^{\star}-s|)dx-\int_{\omega}\nu\tau^{\star}\psi dx\geq 0 \qquad \forall\psi\in(L^2(\omega))^2 \tag{4.24}$$

$$\left.\begin{array}{l}\nu|\tau^{\star}|=\hat{k}\Longrightarrow \quad \exists\lambda\geq 0 \quad s^{\star}=s+\lambda\tau^{\star}\\ \nu|\tau^{\star}|<\hat{k}\Longrightarrow \qquad s^{\star}=s\end{array}\right\} \quad a.e. \quad in \quad \omega \tag{4.25}$$

where $|.|$ denotes the $\mathbb{R}^2$ Euclidean norm.

Proof. [6] Choosing $\varphi = (\varphi_1, \varphi_2, \varepsilon\hat{u}_3^{\varepsilon})$ with $\varphi_i = \hat{u}_i^{\varepsilon} + \psi_i$, for $(i = 1, 2)$ and ψ_i in $H^1_{\Gamma_1\cap\Gamma_L}(\omega)$ where $H^1_{\Gamma_1\cap\Gamma_L}(\omega) = \{v \in H^1(\Omega) : v = 0 \,\text{on}\, \Gamma_1 \cap \Gamma_L\}$, in (5.8), leads to

$$\sum_{i,j=1}^{2}\int_{\Omega}\left(\varepsilon^2\nu(\frac{\partial \hat{u}_i^{\varepsilon}}{\partial x_j}+\frac{\partial \hat{u}_j^{\varepsilon}}{\partial x_i})-\hat{p}^{\varepsilon}\delta_{i,j}\right)\frac{\partial \psi_i}{\partial x_j}dxdy+\sum_{i=1}^{2}\int_{\Omega}\nu\left(\frac{\partial \hat{u}_i^{\varepsilon}}{\partial y}+\varepsilon^2\frac{\partial \hat{u}_3^{\varepsilon}}{\partial x_i}\right)\frac{\partial \psi_i}{\partial y}dxdy$$

$$+\int_\omega \hat{k}(|\psi+\hat{u}^\varepsilon-s|-|\hat{u}^\varepsilon-s|)dx \geq \sum_{i=1}^{2}\int_\Omega \hat{f}_i\psi_i dxdy. \tag{4.26}$$

Using Corollary 1, we can pass to the limit in (4.26), to obtain

$$\sum_{i=1}^{2}\int_\Omega -p^\star \frac{\partial \psi_i}{\partial x_i}dxdy + \sum_{i=1}^{2}\int_\Omega \nu \frac{\partial u_i^\star}{\partial y}\frac{\partial \psi_i}{\partial y}dxdy +$$

$$+\int_\omega \hat{k}(|\psi+s^\star-s|-|s^\star-s|)dx \geq \sum_{i=1}^{2}\int_\Omega \hat{f}_i\psi_i dxdy$$

Using now the Green formula, the equality (5.2) and the fact that $\psi_i=0$ on $\Gamma_1\cap\Gamma_L$ and $cos(n,x_i)=0$ on ω, we deduce

$$\int_\omega \hat{k}(|\psi+s^\star-s|-|s^\star-s|)dx - \int_\omega \nu\tau^\star\psi dx \geq 0 \ \forall\psi\in (H^1_{\Gamma_1\cup\Gamma_L}(\Omega))^2. \tag{4.27}$$

This inequality remains valid for any ψ in $(D(\omega))^2$ (using the same notations for the trace) and by density of $D(\omega)$ in $L^2(\omega)$ for any ψ in $(L^2(\omega))^2$. Then (5.1) follows.
To prove (4.25), we take $\psi_i=\pm(s_i^\star-s_i)$, in (5.1), we obtain

$$\int_\omega \Big(\hat{k}|s^\star-s|-\nu\tau^\star(s^\star-s)\Big)\,dx=0, \tag{4.28}$$

taking $\psi=\phi-(s^\star-s)$ with ϕ in $(L^2(\omega))^2$, in (5.1), we obtain

$$\int_\omega \Big(\hat{k}|\phi|-\nu\tau^\star\phi\Big)\,dx \geq \int_\omega \Big(\hat{k}|s^\star-s|-\nu\tau^\star(s^\star-s)\Big)\,dx.$$

And from (4.28) we deduce

$$\int_\omega \Big(\hat{k}|\phi|-\nu\tau^\star\phi\Big)\,dx \geq 0 \quad \forall\phi\in(L^2(\omega))^2, \tag{4.29}$$

taking first $\phi=(\varphi_1,\varphi_2)$ such that $\varphi_i\geq 0$ $i=1,2$, in (4.29), we obtain :

$$\int_\omega \Big(\hat{k}|\phi|-\nu|\tau^\star|.|\phi|cos(\tau^\star,\phi)\Big)\,dx = \int_\omega \Big(\hat{k}-\nu|\tau^\star|cos(\tau^\star,\phi)\Big)\,|\phi|dx \geq 0,$$

then:

$$\nu|\tau^\star|cos(\tau^\star,\phi)\leq \hat{k} \quad a.e. \quad \text{on} \quad \omega, \tag{4.30}$$

taking now $-\phi$, with $\phi=(\varphi_1,\varphi_2)$ such that $\varphi_i\geq 0$ $i=1,2$, in (4.29), we obtain :

$$\int_\omega \Big(\hat{k}|\phi|+\nu|\tau^\star|.|\phi|cos(\tau^\star,\phi)\Big)\,dx = \int_\omega \Big(\hat{k}+\nu|\tau^\star|cos(\tau^\star,\phi)\Big)\,|\phi|dx \geq 0,$$

whence

$$\nu|\tau^\star|cos(\tau^\star,\phi)\geq -\hat{k} \quad a.e. \quad \text{on} \quad \omega, \tag{4.31}$$

from (4.30) and (4.31) we get:

$$\nu|\tau^\star|\leq \hat{k} \quad a.e. \quad \text{on} \quad \omega, \tag{4.32}$$

then

$$\hat{k}|s^\star-s|\geq \nu|\tau^\star|.|s^\star-s|\geq \nu\tau^\star.(s^\star-s) \quad a.e. \quad \text{on} \quad \omega$$

so

$$\hat{k}|s^\star - s| - \nu\tau^\star.(s^\star - s) \geq 0 \quad a.e. \quad \text{on} \quad \omega$$

and from (4.28) we deduce that

$$\hat{k}|s^\star - s| - \nu\tau^\star.(s^\star - s) = 0 \quad a.e. \quad \text{on} \quad \omega. \tag{4.33}$$

If $\nu|\tau^\star| = \hat{k}$, then from (4.33) we have

$$\nu|\tau^\star|.|s^\star - s| = \nu\tau^\star.(s^\star - s) \quad a.e. \quad \text{on}\,\omega,$$

then $cos(s^\star - s, \nu\tau^\star) = 1$, which implies the existence of $\lambda \geq 0$ such that $s^\star - s = \lambda\nu\tau^\star$. And if $\nu|\tau^\star| < \hat{k}$, then from (4.33) we have

$$\hat{k}|s^\star - s| - \nu\tau^\star(s^\star - s) = 0 \geq (\hat{k} - |\nu\tau^\star|)|s^\star - s| \quad a.e. \quad \text{on} \quad \omega,$$

whence $s^\star - s = 0 \quad a.e. \quad \text{on} \quad \omega$. then (4.25) follows. □

Theorem 8. *Under the same hypothesis of in Theorem 5, and assuming that $\hat{f}$ is a function of x only, we have*

$$\frac{h^2}{2}\nabla p^\star(x) + \nu s^\star(x) + \nu h\tau^\star(x) - \frac{h^2}{2}\hat{f}(x) = 0 \quad a.e. \quad \text{in} \quad \omega. \tag{4.34}$$

$$\int_\omega \left(h^2\tau^\star(x) + 4hs^\star(x)\right)\nabla\varphi(x)dx = 6\int_{\partial\omega}\varphi(x)\tilde{g}(x).n \ \ \forall\varphi \in H^1(\omega). \tag{4.35}$$

Proof. [6] Integrate twice (5.2) between 0 and y we obtain

$$\nu u_i^\star(x,y) = \frac{y^2}{2}\frac{\partial p^\star(x)}{\partial x_i} + \nu u_i^\star(x,0) + \nu y\frac{\partial u_i^\star(x,0)}{\partial y} - \frac{y^2}{2}\hat{f}_i(x),$$

and as $u_i^\star(x,h) = 0$, then (4.34) follows. On the other hand, taking the average of the preceding expression we have

$$h\nu\tilde{u}_i^\star(x) = \int_0^{h(x)}\nu u_i^\star(x,y)dy = \frac{h^3}{6}\frac{\partial p^\star(x)}{\partial x_i} + \nu h u_i^\star(x,0) + \nu\frac{h^2}{2}\frac{\partial u_i^\star(x,0)}{\partial y} - \frac{h^3}{6}\hat{f}_i(x) \tag{4.36}$$

Otherwise, for all φ in $H^1(\omega)$, and as $div(\hat{u^\varepsilon}) = 0$ in Ω we have:

$$\int_\Omega \varphi div(\hat{u^\varepsilon})dxdy = 0 = \int_\omega \varphi(x)\int_0^h\left(\sum_{i=1}^2\frac{\partial\hat{u}_i^\varepsilon}{\partial x_i} + \frac{\partial\hat{u}_3^\varepsilon}{\partial y}dx\right)dy =$$

$$= \int_\omega \varphi(x)\sum_{i=1}^2\left(\frac{\partial(h\,\tilde{\hat{u}}_i^\varepsilon)}{\partial x_i} + \hat{u}_3^\varepsilon(x,h) - \hat{u}_3^\varepsilon(x,0)\right)dx$$

then as $\hat{u}_3^\varepsilon = 0$ on $\partial\Omega = \bar{\omega}\cup\bar{\Gamma}_1\cup\bar{\Gamma}_L$, we have

$$\int_\omega \varphi(x)\sum_{i=1}^2\frac{\partial(h\,\tilde{\hat{u}}_i^\varepsilon)}{\partial x_i}dx = 0, \quad where \quad \tilde{\hat{u}}_i^\varepsilon(x) = \frac{1}{h(x)}\int_0^{h(x)}\hat{u}_i^\varepsilon(x,y)dy, \quad \forall x \in \omega,$$

$$and \quad \tilde{g}_i(x) = \int_0^{h(x)}\hat{g}_i(x,y)dy = h(x)\,\tilde{\hat{u}}_i^\varepsilon(x), \quad \forall x \in \partial\omega.$$

Using Green's formula we have

$$\sum_{i=1}^{2}\int_{\omega} h\tilde{\hat{u}}_i^{\varepsilon}\frac{\partial\varphi}{\partial x_i}dx=\sum_{i=1}^{2}\int_{\partial\omega} h\tilde{\hat{u}}_i^{\varepsilon}\varphi.cos(n,x_i)=\sum_{i=1}^{2}\int_{\partial\omega}\tilde{g}_i(x)\varphi.cos(n,x_i)$$

as $\hat{u}_i^{\varepsilon}\rightharpoonup u_i^{\star}$ in V_y then in $L^2(\omega)$, therefore $\tilde{\hat{u}}_i^{\varepsilon}\rightharpoonup\tilde{u}_i^{\star}$ in $L^2(\omega)$, and as $\partial\omega\subset\partial\Omega$, we deduce :

$$\sum_{i=1}^{2}\int_{\omega} h\tilde{u}_i^{\star}\frac{\partial\varphi}{\partial x_i}dx=\sum_{i=1}^{2}\int_{\partial\omega}\varphi(x)\tilde{g}_i(x)cos(n,x_i)\qquad\forall\varphi\in H^1(\omega).\tag{4.37}$$

From (4.36) we have

$$\int_{\omega}\left(\frac{h^3}{6\nu}\nabla p^{\star}+hs^{\star}+\frac{h^2}{2}\tau^{\star}-\frac{h^3}{6\nu}\hat{f}\right)\nabla\varphi dx=\int_{\partial\omega}\varphi\tilde{g}.n.\tag{4.38}$$

Then using (4.34) and (4.38), we obtain the weak formulation of the Reynolds equation:

$$\int_{\omega}\left(\frac{h^3}{12\nu}\nabla p^{\star}-\frac{h}{2}s^{\star}-\frac{h^3}{12\nu}\hat{f}\right)\nabla\varphi dx=\int_{\partial\omega}\varphi\tilde{g}.n.\tag{4.39}$$

Using once again (4.34) and (4.39) we get (4.35). □

4.2. **Study of the uniqueness.** In this subsection, we will give another formulation of the limit inequalities for $s^{\star}$ and $\tau^{\star}$ on ω which enables us to express $s^{\star}$ as a solution of a variational inequality of the second kind with a convenient decomposition. The basic idea is that we have three unknowns $s^{\star}$, $\tau^{\star}$ and $\nabla p^{\star}$ and three relations (4.34) (4.35) and (5.1). A test function in (4.35) appears only to be a gradient function. So it is only possible to control the "gradient" part of $s^{\star}$ and $\tau^{\star}$ by this equation which is obtained by a slightly modified version of the well known decomposition of $L^2(\omega)^2$, due to the non constant $h(x)$ coefficients.

Lemma 2. *Let h in $L^{\infty}(\omega)\cap H^1(\omega)$ such that $h\geq\alpha>0$. Every function ψ in $(L^2(\omega))^2$ has the following orthogonal decomposition:*

$$\psi=h^2\nabla\varphi+h^{-1}\mathbf{curl}(\theta)\tag{4.40}$$

where φ in $H^1(\omega)/\mathbb{R}$ is the only solution of the problem

$$\int_{\omega}h^3\nabla\varphi\nabla\mu\,dx=\int_{\omega}h\psi\nabla\mu\,dx\quad\forall\mu\in H^1(\omega),\tag{4.41}$$

and θ in $H_0^1(\omega)$ is the only solution of the problem

$$\int_{\omega}\mathbf{curl}(\theta)\mathbf{curl}(\xi)dx=\int_{\omega}(h\psi-h^3\nabla\varphi)\mathbf{curl}(\xi)\,dx\quad\forall\xi\in H_0^1(\omega).\tag{4.42}$$

Proof. [6] As h in $L^{\infty}(\omega)$, for all ψ in $(L^2(\omega))^2$, we have $h\psi$ in $(L^2(\omega))^2$, following [33](theorem 3.2), the Neumann's problem (4.41) has a unique solution φ in $H^1(\omega)/\mathbb{R}$. This solution φ satisfies $\nabla(h\psi-h^3\nabla\varphi)=0$ in $H^{-1}(\omega)$. Hence $h\psi-h^3\nabla\varphi$ is a divergence-free vector of $H(div,\omega)$. Moreover, Green's formula applied to (4.41) yields:

$$0=\int_{\omega}(h\psi-h^3\nabla\varphi)\nabla\mu dx=\int_{\partial\omega}(h\psi-h^3\nabla\varphi).n\mu\quad\forall\mu\in H^1(\omega),$$

implying that $(h\psi - h^3\nabla\varphi).n = 0$ in $H^{-1/2}(\partial\omega)$. Whence $h\psi - h^3\nabla\varphi$ lies in the space $H = \{v \in (L^2(\omega))^2 : \quad div(v) = 0, \quad v.n = 0\}$. Moreover, as ω is connected, we deduce, from [33] (Theorem 3.1 and its corollary), that the space H is characterized by $H = \{\mathbf{curl}(\mu) \quad : \mu \in H_0^1(\omega)\}$, and the mapping **curl** is an isomorphism from $H_0^1(\omega)$ onto H. So there exists a unique stream function θ in $H_0^1(\omega)$ of $h\psi - h^3\nabla\varphi$ satisfying (4.40) and (4.42). □

Theorem 9. *Let h in $L^\infty(\omega) \cap H^1(\omega)$. Under the same hypothesis of* Theorem 5*, $s^\star$ is uniquely given by $s^\star = h^2\nabla C + h^{-1}\mathbf{curl}(D)$, where $U = (C, D)$ is the unique solution of the following variational problem: Find U in $H^1(\omega) \times H_0^1(\omega)$ such that*

$$a(U, \phi - U) + J(\phi) - J(U) \geq L(\phi - U) \quad \forall \phi = (\varphi, \theta) \in H^1(\omega) \times H_0^1(\omega), \tag{4.43}$$

$$\text{where} \quad a(U, \phi) = \int_\omega 4\nu h^3 \nabla C \nabla \varphi dx + \int_\omega \nu h^{-3}\mathbf{curl}(D)\mathbf{curl}(\theta) dx,$$

$$J(\phi) = \int_\omega \hat{k}(|h^2\nabla\varphi + h^{-1}\mathbf{curl}(\theta) - s|)dx,$$

$$L\phi = \frac{1}{2}\int_\omega \hat{f}\mathbf{curl}(\theta)dx + \int_{\partial\omega} 6\nu\tilde{g}.n\varphi.$$

Proof. [6] From (5.1) and the orthogonal decomposition of ψ, we have

$$\int_\omega \hat{k}\left(|h^2\nabla\varphi + h^{-1}\mathbf{curl}(\theta) + s^\star - s| - |s^\star - s|\right) dx \geq$$

$$\geq \int_\omega \nu\tau^\star h^2 \nabla\varphi + \int_\omega \nu\tau^\star h^{-1}\mathbf{curl}(\theta)dx \qquad \forall(\varphi,\theta) \in H^1(\omega) \times H_0^1(\omega) \tag{4.44}$$

and from (4.35), we have

$$\int_\omega \nu h^2 \tau^\star \nabla\varphi = -\int_\omega 4\nu h s^\star \nabla\varphi + \int_{\partial\omega} 6\nu\tilde{g}.n\varphi \quad \forall\varphi \in H^1(\omega) \tag{4.45}$$

then from (4.44) and (4.45), we have for all (φ, θ) in $H^1(\omega) \times H_0^1(\omega)$

$$\int_\omega \hat{k}\left(|h^2\nabla\varphi + h^{-1}\mathbf{curl}(\theta) + s^\star - s| - |s^\star - s|\right) dx \geq$$

$$\geq -\int_\omega 4\nu h s^\star \nabla\varphi + \int_{\partial\omega} 6\nu\tilde{g}.n\varphi + \int_\omega \nu\tau^\star h^{-1}\mathbf{curl}(\theta)dx. \tag{4.46}$$

Now as $s^\star$ in $(L^2(\omega))^2$, we can use its orthogonal decomposition as $s^\star = h^2\nabla C + h^{-1}\mathbf{curl}(D)$, then we deduce for all (φ, θ) in $H^1(\omega) \times H_0^1(\omega)$

$$\int_\omega \hat{k}|h^2\nabla\varphi + h^{-1}\mathbf{curl}(\theta) + h^2\nabla C + h^{-1}\mathbf{curl}(D) - s|dx$$

$$-\int_\omega \hat{k}|h^2\nabla C + h^{-1}\mathbf{curl}(D) - s|dx \geq -\int_\omega 4\nu h^3 \nabla C \nabla\varphi$$

$$-4\nu \int_\omega \mathbf{curl}(D)\nabla\varphi + \int_{\partial\omega} 6\nu\tilde{g}.n\varphi + \int_\omega \nu\tau^\star h^{-1}\mathbf{curl}(\theta)dx. \tag{4.47}$$

Using (4.34) we have

$$\int_\omega \nu\tau^\star h^{-1}\mathbf{curl}(\tau^\star)dx = \int_\omega (-\frac{1}{2}\nabla p^\star - \frac{\nu}{h^2}s^\star + \frac{1}{2}\hat{f})\mathbf{curl}(\theta)dx,$$

then

$$\int_\omega \nu\tau^\star h^{-1}\mathbf{curl}(\theta)dx = -\int_\omega \frac{1}{2}\mathbf{curl}(\theta)\nabla p^\star dx - \int_\omega \nu\mathbf{curl}(\theta)\nabla C dx$$

$$-\int_\omega \nu h^{-3}\mathbf{curl}(D)\mathbf{curl}(\theta)dx + \frac{1}{2}\int_\omega \hat{f}\mathbf{curl}(\theta)dx.$$

Using Green's formula and that θ in $H_0^1(\omega)$, we have

$$\int_\omega \mathbf{curl}(\theta)\nabla p^\star dx = -< p^\star, div(\mathbf{curl}(\theta)) > + \int_{\partial\omega} \mathbf{curl}(\theta).np^\star dx = 0$$

by the same argument we also have

$$\int_\omega \nu\mathbf{curl}(\theta)\nabla C = \int_\omega \mathbf{curl}(D)\nabla\varphi = 0.$$

Then from (4.47) $U = (C, D)$ satisfies for all $\phi = (\varphi, \theta)$ in $H^1(\omega) \times H_0^1(\omega)$

$$\int_\omega \left\{4\nu h^3\nabla C\nabla\varphi + \nu h^{-3}\mathbf{curl}(D)\mathbf{curl}(\theta)\right\} dx$$

$$+\int_\omega \hat{k}|h^2\nabla(\varphi + C) + h^{-1}\mathbf{curl}(\theta + D) - s|dx$$

$$-\int_\omega \hat{k}|h^2\nabla C + h^{-1}\mathbf{curl}(D) - s|dx \geq \int_\omega \frac{1}{2}\hat{f}\mathbf{curl}(\theta)dx + \int_{\partial\omega} 6\nu\tilde{g}.n\varphi,$$

taking $\tilde{\varphi} = \varphi + C$ and $\tilde{\theta} = \theta + D$ we deduce the variational inequality (5.11).
As the bilinear form $a(,)$ is continuous and coercive, the functional J is convex, proper and continuous, and the linear form L is continuous, the existence and uniqueness of (C, D) in $H^1(\omega) \times H_0^1(\omega)$ follows, and implies the existence and uniqueness of $s^\star$ in $(L^2(\omega))^2$. □

Theorem 10. *Under the same hypothesis of* Theorem 9*, there exists a unique solution* $p^\star$ *in* $H^1(\omega)$ *satisfying the weak formulation of the Reynolds equation* (4.39)*. Then* $\tau^\star$ *is then unique.*

Proof. [6] From Theorem 9 $s^\star$ is unique in $(L^2(\omega))^2$, then the uniqueness of $p^\star$ follows from (4.39). Finally $\tau^\star$ is unique from the uniqueness of $p^\star$ and $s^\star$ using (4.34). □

4.3. **Case 2.** We introduce in this case see [9] the definition that $v = (v_1, v_2) \in (L^2(\Omega))^2$ satisfies the condition (D') if

$$\int_\Omega \left(v_1 \frac{\partial \theta}{\partial x_1} + v_2 \frac{\partial \theta}{\partial x_2} \right) dxdy = 0, \quad \forall \theta \in C_0^\infty(\omega).$$

And consider

$$\Pi(V_{div}) = \{\bar{\varphi} \in (H^1(\Omega))^2 \ : \ \bar{\varphi} = (\varphi_1, \varphi_2), \quad \varphi_i = \hat{G}_i \quad \text{on} \quad \Gamma_1 \cup \Gamma_L, i = 1, 2\}$$

$$\tilde{\Pi}(V_{div}) = \{\bar{\varphi} \in \Pi(V_{div}) : \bar{\varphi} \quad \text{satisfies} \quad \text{condition} \quad (D')\},$$

and also the Banach space

$$V_y = \{v = (v_1, v_2) \in (L^2(\Omega))^2 : \quad \frac{\partial v_i}{\partial y} \in L^2(\Omega), i = 1, 2, \quad v = 0 \quad \text{on} \quad \Gamma_1\}$$

with its norm $\| \cdot \|_{V_y}$,

$$\|v\|_{V_y}^2 = \sum_{i=1}^{2} \left(\|v_i\|_{L^2(\Omega)}^2 + \|\frac{\partial v_i}{\partial y}\|_{L^2(\Omega)}^2 \right),$$

and define its linear subspace (endowed with the same topology)

$$\tilde{V}_y = \{v \in V_y : v \quad \text{satisfies} \quad \text{condition} \quad (D')\}.$$

We obtain also the following similar main results

Theorem 11. [9] *There exist $u^\star = (u_1^\star, u_2^\star)$ in $\tilde{V}_y$, $p^\star$ in $L_0^2(\Omega)$, and a subsequence $\varepsilon \to 0$ such that* (4.11)-(4.16) *hold. We have also, $\hat{u}_i^\varepsilon \to u_i^\star$ strongly in V_y for $i = 1, 2$. The limit functions $u^\star, p^\star$ satisfy* (5)-(5.2)*. Moreover $u^\star, p^\star$ satisfy the inequality*

$$\int_\Omega \nu \frac{\partial u^\star}{\partial y} \frac{\partial (\varphi - u^\star)}{\partial y} dxdy - \int_\Omega p^\star \left(\frac{\partial \varphi_1}{\partial x_1} + \frac{\partial \varphi_2}{\partial x_2} \right) dxdy$$

$$+ \int_\omega \hat{k} S(-p^\star)(|\varphi - s| - |u^\star - s|) dx \geq \sum_{i=1}^{2} \int_\Omega \hat{f}_i (\varphi_i - u_i^\star) dxdy \qquad \forall \varphi \in \Pi(K). \tag{4.48}$$

Theorem 12. [9] *The pair $(u^\star, p^\star)$ satisfies the same weak form of the Reynolds equation* (4.39) *where $u^\star = u^\star(.\,, 0)$. Moreover, the traces $\tau^\star = \frac{\partial u^\star}{\partial y}(x, 0)$ and $u^\star(x, 0)$ satisfy the following limit form of the Coulomb boundary conditions* (2.16)

$$\left.\begin{array}{l} \nu|\tau^\star| = \hat{k} S(-p^\star) \Longrightarrow \ \exists \lambda \geq 0 \quad u^\star = s + \lambda \tau^\star \\ \nu|\tau^\star| < \hat{k} S(-p^\star) \Longrightarrow \qquad u^\star = s \end{array}\right\} \quad a.e. \quad in \quad \omega.$$

Theorem 13. [9] *There exists a positive constant $k^\star$ such that for $\|\hat{k}\|_{L^\infty(\omega)} \leq k^\star$ the solution $(u^\star, p^\star)$ in $\tilde{V}_y \times (L_0^2(\omega) \cap H^1(\omega))$ of inequality ((5.1) is unique.*

5. Study of the **case 3**

For this case (see [10]) two technical difficulties to study the asymptotic anlysis of this problem. The first, we cannot use the usual Korn inequality as we do not assume that the velocity vanishes at one of the boundaries (on the top or the bottom) as is usually assumed in lubrication problems. We thus derive an analogue of the Korn inequality suitable for our boundary conditions and such that the constants can be controlled appropriately as the gap between the surfaces approach zero. This leads us to the main uniform estimate of the velocity fields and to the limit variational inequality, in consequence. The second, to be able to make use of the latter, we have to characterize precisely the limit solution space and the set of admissible test functions. As the limit variational inequality is written in terms of the first two components of the velocity field, we have to characterize - in this very limit case - projections of the convexes appearing in the weak form of the Stokes flow. This allows us, in particular, to obtain a stronger convergence of the velocity fields as in usually expected. We give here only the main results

Lemma 3. [10] *"Poincaré inequality"*

$$\int_{\Omega^\varepsilon} |u|^2 \leq 2\varepsilon h_M \int_{\Gamma_1^\varepsilon} |u|^2 + 2(\varepsilon h_M)^2 \int_{\Omega^\varepsilon} |\frac{\partial u}{\partial x_3}|^2. \tag{5.1}$$

Proof.

$$u(x,t) = u(x, h^\varepsilon(x)) - \int_t^{h^\varepsilon(x)} \frac{\partial u}{\partial z}(x,z)dz$$

We integrate over $t \in [0, h^\varepsilon(x)]$ to get

$$\int_0^{h^\varepsilon(x)} |u(x,t)|^2 dt \leq 2h^\varepsilon(x)|u(x,h^\varepsilon(x))|^2 + 2(h^\varepsilon(x))^2 \int_0^{h^\varepsilon(x)} |\frac{\partial u}{\partial z}(x,z)|^2 dz,$$

and, after integration over ω we get (5.1). □

Lemma 4. [10] *"Korn's Inegality" assumming that $h \in C^2(\bar{\omega})$, we have*

$$\int_{\Omega^\varepsilon} |\nabla(u - G^\varepsilon)|^2 \leq a(u - G^\varepsilon, u - G^\varepsilon) + C(\Gamma_1^\varepsilon) \int_{\Gamma_1^\varepsilon} |u - G^\varepsilon|^2,$$

where

$$C(\Gamma_1^\varepsilon) = 2\varepsilon ||D_2 h||_{C(\bar{\omega})}(1 + \varepsilon^2 ||D_1 h||^2_{C(\bar{\omega})}).$$

Observe that $C(\Gamma_1^\varepsilon))$ est d'ordre ε.

Proof. We have

$$\begin{aligned}
a(v,v) &= \frac{1}{2}\int_{\Omega^\varepsilon} (\frac{\partial v_i}{\partial x_k} + \frac{\partial v_k}{\partial x_i})^2 = \int_{\Omega^\varepsilon} (\frac{\partial v_i}{\partial x_k}\frac{\partial v_i}{\partial x_k} + \frac{\partial v_i}{\partial x_k}\frac{\partial v_k}{\partial x_i}) = \int_{\Omega^\varepsilon} |\nabla v|^2 + \int_{\Omega^\varepsilon} \frac{\partial v_i}{\partial x_k}\frac{\partial v_k}{\partial x_i} \\
&= \int_{\Omega^\varepsilon} |\nabla v|^2 - \int_{\Omega^\varepsilon} \frac{\partial^2 v_i}{\partial x_k \partial x_i} v_k + \int_{\partial\Omega^\varepsilon} \frac{\partial v_i}{\partial x_k} v_k n_i \\
&= \int_{\Omega^\varepsilon} |\nabla v|^2 + \int_{\Omega^\varepsilon} \frac{\partial v_i}{\partial x_i}\frac{\partial v_k}{\partial x_k} - \int_{\partial\Omega^\varepsilon} \frac{\partial v_i}{\partial x_i} v_k n_k + \int_{\partial\Omega^\varepsilon} \frac{\partial v_i}{\partial x_k} v_k n_i.
\end{aligned}$$

As $div\, v = 0$ then we have

$$\int_{\Omega^\varepsilon} |\nabla v|^2 = a(v,v) + \int_{\partial\Omega^\varepsilon} \frac{\partial v_i}{\partial x_i} v_k n_k - \int_{\partial\Omega^\varepsilon} \frac{\partial v_i}{\partial x_k} v_k n_i.$$

For $v = u - G^\varepsilon$, we have $(u - G^\varepsilon)n|_{\partial\Omega^\varepsilon} = 0$, so

$$\int_{\partial\Omega^\varepsilon} \frac{\partial v_i}{\partial x_i} v_k n_k = 0,$$

in other hand, $u - G^\varepsilon = 0$ sur Γ_L^ε. So

$$\int_{\partial\Omega^\varepsilon} \frac{\partial v_i}{\partial x_k} v_k n_i = \int_{\Gamma_1^\varepsilon \cup \omega} \frac{\partial v_i}{\partial x_k} v_k n_i = \int_{\Gamma_1^\varepsilon \cup \omega} (u_k - G_k^\varepsilon) \frac{\partial (u_i - G_i^\varepsilon)}{\partial x_k} n_i.$$

as

$$(u - G^\varepsilon) \cdot n_{|\Gamma_1^\varepsilon \cup \omega} = 0, \quad alors \quad \frac{\partial}{\partial x_k}\{(u - G^\varepsilon) \cdot n\} = 0,$$

that is

$$\frac{\partial (u_i - G_i^\varepsilon)}{\partial x_k} n_i = -(u_i - G_i^\varepsilon) \frac{\partial n_i}{\partial x_k}.$$

but $\frac{\partial n_i}{\partial x_k}_{|\omega} = 0$, then from (5.2), (5) we have

$$|\int_{\partial\Omega^\varepsilon} \frac{\partial v_i}{\partial x_k} v_k n_i| = |\int_{\Gamma_1^\varepsilon} (u_k - G_k^\varepsilon) \frac{\partial (u_i - G_i^\varepsilon)}{\partial x_k} n_i| \leq 2 \max_{q \in \Gamma_1^\varepsilon} |\frac{\partial n_i}{\partial x_k}(q)| \int_{\Gamma_1^\varepsilon} |u - G^\varepsilon|^2,$$

Γ_1^ε is given by $x_3 = h^\varepsilon(x_1, x_2)$, then the unit normal vector exterior to Γ_1^ε can be written

$$n(q) = \frac{\left(-\frac{\partial h^\varepsilon}{\partial x_1}(x_1,x_2), -\frac{\partial h^\varepsilon}{\partial x_2}(x_1,x_2), 1\right)}{\sqrt{1 + |\nabla h^\varepsilon(x)|^2}} = n(x_1, x_2)$$

For $i = 1, 2$, we have

$$\frac{\partial n_3}{\partial x_i}(x_1,x_2) = -(1 + |\nabla h^\varepsilon|^2)^{-\frac{3}{2}} \frac{\frac{\partial h^\varepsilon}{\partial x_1}\frac{\partial^2 h^\varepsilon}{\partial x_i \partial x_1} + \frac{\partial h^\varepsilon}{\partial x_2}\frac{\partial^2 h^\varepsilon}{\partial x_i \partial x_2}}{(1 + |\nabla h^\varepsilon|^2)^{\frac{3}{2}}}$$

hence

$$|\frac{\partial n_3}{\partial x_i}| \leq 2|D_1 h^\varepsilon|.|D_2 h^\varepsilon| \leq |D_2 h^\varepsilon|(1 + |D_1 h^\varepsilon|^2)$$

similarly

$$|\frac{\partial n_j}{\partial x_i}| \leq |D_2 h^\varepsilon|(1 + |D_1 h^\varepsilon|^2).$$

□

Lemma 5.

$$\int_{\Omega^\varepsilon} |\nabla u|^2 \leq \frac{4}{\nu}\, a(u,u) + 10 \int_{\Omega^\varepsilon} |\nabla G^\varepsilon|^2 + 4C(\Gamma_1^\varepsilon) \left\{ \int_{\Gamma_1^\varepsilon} |u|^2 + \int_{\Gamma_1^\varepsilon} |G^\varepsilon|^2 \right\}. \tag{5.2}$$

Proof. [10] We have

$$\int_{\Omega^\varepsilon} |\nabla u|^2 \leq 2 \int_{\Omega^\varepsilon} |\nabla (u - G^\varepsilon)|^2 + 2 \int_{\Omega^\varepsilon} |\nabla G^\varepsilon|^2.$$

and from 4

$$\int_{\Omega^\varepsilon} |\nabla u|^2 \leq \frac{2}{\nu} a(u - G^\varepsilon \,,\, u - G^\varepsilon) + 2C(\Gamma_1^\varepsilon) \int_{\Gamma_1^\varepsilon} |u - G^\varepsilon|^2 + 2 \int_{\Gamma_1^\varepsilon} |\nabla G^\varepsilon|^2$$

or

$$a(u - G^\varepsilon \,,\, u - G^\varepsilon) \leq 2a(u \,,\, u) + 2a(G^\varepsilon \,,\, G^\varepsilon) \leq 2a(u \,,\, u) + 4\nu \int_{\Omega^\varepsilon} |\nabla G^\varepsilon|^2$$

so (5.2) follows. □

5.1. **Scaling and uniform estimates.** Let $\Omega = \Omega^1$, $y = \frac{x_3}{\varepsilon}$ and assume that

$$\hat{l} = \varepsilon l^\varepsilon, \quad \hat{k} = \varepsilon k^\varepsilon, \qquad \hat{f}(x, y) = \varepsilon^2 f^\varepsilon(x, x_3).$$

Let consider also $\hat{G}(x, y) = (\hat{G}_1(x, y), \hat{G}_2(x, y), \hat{G}_3(x, y)) \in (H^1(\Omega))^3$ such that

$$\hat{G}.n = 0 \quad sur \quad \Gamma_1 \cup \omega \quad (\Gamma_1 = \Gamma_1^1)$$

$$\frac{\partial \hat{G}_1}{\partial x_1} + \frac{\partial \hat{G}_2}{\partial x_2} + \frac{\partial \hat{G}_3}{\partial y} = 0 \quad (div\, \hat{G} = 0)$$

we define $G^\varepsilon \in (H^1(\Omega^\varepsilon)^3$, by

$$G_i^\varepsilon(x, x_3) = \hat{G}_i(x, y), \quad i = 1, 2, \quad G_3^\varepsilon(x, x_3) = \varepsilon \hat{G}_3(x, y),$$

hence

$$\int_{\Omega^\varepsilon} |\nabla G^\varepsilon|^2 dx dx_3 \leq \frac{1}{\varepsilon} \int_{\Omega} |\nabla \hat{G}|^2 dx dy. \tag{5.3}$$

$$\int_{\Gamma_1^\varepsilon} |G^\varepsilon|^2 \leq C_0(\Omega) \int_{\Omega} \left(|\hat{G}|^2 + |\nabla \hat{G}|^2 \right). \tag{5.4}$$

with $C_0(\Omega)$ independant of ε.

$$\|f^\varepsilon\|^2_{0,\Omega^\varepsilon} = \frac{1}{\varepsilon^3} \|\hat{f}\|^2_{0,\Omega} \tag{5.5}$$

Lemma 6. *Assume that there exists $\hat{l} \in \mathbb{R}^+$ such that*

$$\varepsilon l^\varepsilon = \hat{l} \quad and \quad \frac{C(\Gamma_1^\varepsilon)}{l^\varepsilon} < \frac{3}{\nu}, \tag{5.6}$$

then there exists a constante $C > 0$ independent of ε such that

$$\varepsilon \int_{\Omega^\varepsilon} |\nabla u|^2 \leq C. \tag{5.7}$$

Proof. [10] Choosing $\varphi = G^\varepsilon$ in (3.9), and using Young's inegality we obtain

$$\frac{1}{2}a(u,u) + \frac{l^\varepsilon}{2}\int_{\Gamma_1^\varepsilon} u^2 + \int_\omega k^\varepsilon|u-s| \le \frac{5l^\varepsilon}{4}\int_{\Gamma_1^\varepsilon}|G^\varepsilon|^2 + \frac{33\nu}{32}\int_{\Omega^\varepsilon}|\nabla G^\varepsilon|^2$$
$$+\left(\frac{32(h_M^\varepsilon)^2}{\nu} + \frac{4h_M^\varepsilon}{l^\varepsilon}\right)\int_{\Omega^\varepsilon}|f^\varepsilon|^2 + \frac{\nu}{32}\int_{\Omega^\varepsilon}|\nabla u|^2. \tag{5.8}$$

From Lemma 5 we have

$$\int_{\Omega^\varepsilon}|\nabla u|^2 \le \frac{4}{\nu}a(u\,,\,u) + 10\int_{\Omega^\varepsilon}|\nabla G^\varepsilon|^2 + 4C(\Gamma_1^\varepsilon)\left\{\int_{\Gamma_1^\varepsilon}|u|^2 + \int_{\Gamma_1^\varepsilon}|G^\varepsilon|^2\right\}. \tag{5.9}$$

From (5.8) and (5.9) we get

$$\int_{\Omega^\varepsilon}|\nabla u|^2 \le \left(\frac{1}{4} + \frac{\nu C(\Gamma_1^\varepsilon)}{4l^\varepsilon}\right)\int_{\Omega^\varepsilon}|\nabla u|^2 + 10\left(\frac{1}{\nu} + C(\Gamma_1^\varepsilon)\right)\int_{\Omega^\varepsilon}|G^\varepsilon|^2$$
$$+\left(\frac{73}{4} + \frac{33}{4}\frac{\nu C(\Gamma_1^\varepsilon)}{l^\varepsilon}\right)\int_{\Omega^\varepsilon}|\nabla G^\varepsilon|^2 + 8\left(\frac{1}{\nu} + \frac{C(\Gamma_1^\varepsilon)}{l^\varepsilon}\right)\left(\frac{32(h_M^\varepsilon)^2}{\nu} + \frac{4h_M^\varepsilon}{l^\varepsilon}\right)\int_{\Omega^\varepsilon}|f^\varepsilon|^2$$

from (5.3), (5.4), (5.5) and (5.6) we deduce the result. □

Assuming (4.1) and the first of (4.2) then from (5.7) we get the same estimates (5.2) and (5.2). Other estimates follows from the "Poincaré inequality" (Lemma 3) with the new variables,

$$\int_\Omega |\hat{u}_i|^2 \le 2h_M\int_{\Gamma_1}|\hat{u}_i|^2 + 2h_M^2\int_\Omega\left|\frac{\partial\hat{u}_i}{\partial y}\right|^2,$$

for $i = 1,2,3$, hence,

$$||\hat{u}_i||^2_{L^2(\Omega)} \le C_3, \qquad \varepsilon^2\|\hat{u}_3\|^2_{L^2(\Omega)} \le C_4. \tag{5.10}$$

So we obtain the same convergence as in Corollary 1. So we obtain the limit variational inequality,

$$\sum_{i=1}^2\int_\Omega \frac{1}{2}\frac{\partial u_i^\star}{\partial y}\frac{\partial(\hat{\varphi}_i - u_i^\star)}{\partial y} - p^\star(x)\frac{\partial\hat{\varphi}_i}{\partial x_i}dxdy - \sum_{i=1}^2\int_\omega p^\star(x)\hat{\varphi}_i(x,h)\frac{\partial h}{\partial x_i}(x)dx +$$
$$\sum_{i=1}^2\hat{l}\int_\omega u_i^\star(x,h)\left(\hat{\varphi}_i(x,h) - u_i^\star(x,h)\right)dx + \int_\omega\hat{k}\left(|\hat{\varphi}-s| - |u^\star - s|\right)dx$$
$$\ge \sum_{i=1}^2\int_\omega \hat{f}_i(\hat{\varphi} - u_i^\star)dxdy.\forall\bar{\varphi} = (\hat{\varphi}_1,\hat{\varphi}_2) \in \mathbf{\Pi}(\hat{V}_{div}), \tag{5.11}$$

$$\begin{aligned}\mathbf{\Pi}(\hat{V_{div}}) &= \{\bar{\varphi} = (\hat{\varphi}_1,\hat{\varphi}_2) \in (H^1(\Omega))^2 : \exists\hat{\varphi}_3 \in H^1(\Omega),\, \hat{\varphi} = (\hat{\varphi}_1,\hat{\varphi}_2,\hat{\varphi}_3) \in \hat{V}_{div}\}\\ &= \{\bar{\varphi} = (\hat{\varphi}_1,\hat{\varphi}_2) \in (H^1(\Omega))^2 :\, (\bar{\varphi} - \bar{G})_{|\Gamma_L} = 0\}.\end{aligned}$$

Lemma 7. [10]

$$F_1 = \{u^\star = (u_1^\star, u_2^\star) \in (L^2(\Omega))^2 : \frac{\partial u_i^\star}{\partial y} \in L^2(\Omega), \quad i = 1, 2, \; u^\star \text{ satisfait } (D')\}$$

is contained in the closure of $\Pi(\hat{V}_{div})$ *in the topology of* $V_y \times V_y$.

Lemma 8. *The convergence* $\hat{u}_i \rightharpoonup u_i^\star$ *when* $\varepsilon \to 0$, *for* $i = 1, 2$ *is strong in* V_y.

Proof. [10] From the "Korn inequality" (5.2) we have pour $\hat{\varphi} \in \hat{V}_{div}$

$$\sum_{i=1}^{2} \|\frac{\partial(\hat{u}_i - \hat{\varphi}_i)}{\partial y}\|^2_{L^2(\Omega)} \le \hat{a}(\hat{u} - \hat{\varphi}, \hat{u} - \hat{\varphi}) + C(\Gamma_1, h) \int_{\Gamma_1} |\hat{u} - \hat{\varphi}|^2$$

and

$$\hat{a}(\hat{u} - \hat{\varphi}, \hat{u} - \hat{\varphi}) = \hat{a}(\hat{\varphi}, \hat{\varphi} - \hat{u}) + \hat{a}(\hat{u}, \hat{u} - \hat{\varphi}) \le \hat{a}(\hat{\varphi}, \hat{\varphi} - \hat{u}) + \hat{l} \int_{\Gamma_1} \hat{u}(\hat{\varphi} - \hat{u})$$
$$+ \hat{k} \int_{\omega} (|\hat{\varphi} - s| - |\hat{u} - s|) dx + \int_{\Omega} \hat{f}(\hat{\varphi} - \hat{u})$$

so

$$\sum_{i=1}^{2} \|\frac{\partial(\hat{u}_i - \hat{\varphi}_i)}{\partial y}\|^2_{L^2(\Omega)} + \hat{k} \int_{\omega} (|\bar{u} - s| - |\bar{\varphi} - s|) dx \le \int_{\Omega} \frac{\partial \bar{\varphi}}{\partial y} \frac{\partial(\bar{\varphi} - u^\star)}{\partial y} dx dy$$
$$+ \hat{l} \int_{\Gamma_1} u^\star(\bar{\varphi} - u^\star) + C(\Gamma_1, h) \int_{\Gamma_1} |u^\star - \bar{\varphi}|^2 + \sum_{i=1}^{2} \int_{\Omega} f_i(\bar{\varphi}_i - u_i^\star).$$

using Lemma 7 we can pass to the limit $\bar{\varphi}_i \to u_i^\star$ in the right part to obtain the strong convergence of $\hat{u}_i \to u_i^\star$ in the left part. □

Theorem 14. *The limit functions* $u^\star, p^\star$ *satisfy*

$$p^\star(x_1, x_2, y) = p^\star(x_1, x_2) \quad a.e. \quad in \quad \Omega, \quad p^\star \in H^1(\omega),$$

$$-\frac{\partial^2 u_i^\star}{\partial y^2} + \frac{\partial p^\star}{\partial x_i} = f_i \quad (i = 1, 2) \quad \text{ds} \quad L^2(\Omega).$$

satisfies the following weak form of the Reynolds equation

$$\int_{\omega} \left(\frac{h^3}{12} \nabla p^\star - \frac{h}{2} s^\star - \frac{h}{2} s_h^\star + \tilde{\bar{f}} \right) \nabla \varphi dx + \int_{\omega} s_h^\star \nabla h \varphi dx + \int_{\partial\omega} \varphi \tilde{g}.n. = 0 \quad \forall \varphi \in H^1(\omega).$$

where

$$s^\star(x) := u^\star(x, 0), \; s_h^\star(x) := u^\star(x, h(x))$$

$$\tilde{g}(x) := \int_0^{h(x)} \hat{g}(x, y) dy \quad \forall x \in \partial\omega.$$

Moreover, the traces

$$\tau^\star := \frac{\partial u^\star}{\partial y}(., 0), \quad \tau_h^\star := \frac{\partial u^\star}{\partial y}(., h(x)),$$
$$s_h^\star := u^\star(., h(x)), \quad s^\star := u^\star(., 0)$$

satisfy the following limit form of the Tresca and Fourier boundary conditions

$$\left.\begin{array}{l} |\tau^\star| = \hat{k} \Longrightarrow \quad \exists\lambda \geq 0 \quad u^\star = s + \lambda\tau^\star \\ |\tau^\star| < \hat{k} \Longrightarrow \qquad u^\star = s \end{array}\right\} \; p.p. \; sur\, \omega.$$

$$\tau_h^\star \nabla h.n + \hat{l} s_h^\star = 0 \; p.p. \; sur\, \Gamma_1.$$

6. Study of the case 6

To study the asymptotic behavior of the solutions to this Case, we use the same scaling so we introduce the change of the variable $z = \dfrac{x_3}{\varepsilon}$ and obtain a fixed domain which is independent of ε,

$$\Omega = \{(x', z) \in \mathbb{R}^3 \; : \; (x', 0) \in \omega, \;\; 0 < z < h(x')\},$$

$$\partial\Omega = \Gamma = \bar{\omega} \cup \bar{\Gamma_1} \cup \bar{\Gamma}$$

then we define the following functions in Ω

$$\hat{u_i^\varepsilon}(x', z) = u_i^\varepsilon(x', x_3) \text{ for } i = 1, 2, \qquad \hat{u_3^\varepsilon}(x', z) = \varepsilon^{-1} u_3^\varepsilon(x', x_3),$$

$$\hat{p^\varepsilon}(x', z) = \varepsilon^r p^\varepsilon(x', x_3) \; ; \; \hat{T^\varepsilon}(x', z) = T^\varepsilon(x', x_3).$$

And assume the dependence of the data on ε

$$\hat{K}(x', z) = \varepsilon^{-2+r+\alpha} K^\varepsilon(x', x_3), \qquad \hat{R}(x', x_3) = \varepsilon^{r+\alpha} R^\varepsilon(x', z), \qquad \hat{\mu} = \mu^\varepsilon$$

with

$$\alpha = \frac{3(2-q)}{3-q}.$$

We suppose that

$$K_\star \leq \hat{K} \leq K^\star.$$

Let $\hat{G} = (\hat{G}_1, \hat{G}_2, \hat{G}_3)$ be independent on ε :

$$div_z(\hat{G}) = \frac{\partial \hat{G}_1}{\partial x_1} + \frac{\partial \hat{G}_2}{\partial x_2} + \frac{\partial \hat{G}_3}{\partial z} = 0, \;\; \text{and } \hat{G} = \hat{g} \text{ on } \Gamma.$$

Thus the extension G^ε of g is defined by

$$G_i^\varepsilon(x', x_3) = \hat{G}_i(x', z) \; i = 1, 2 \; ; \; G_3^\varepsilon(x', x_3) = \varepsilon \hat{G}_3^\varepsilon.$$

Injecting the new data and unknown in (3.12)-(3.13), we deduce that $(\hat{u^\varepsilon}, \hat{p^\varepsilon}, \hat{T^\varepsilon})$ satisfies the following problem

$$\hat{a}(\hat{T^\varepsilon}; \hat{u^\varepsilon}, \hat{\phi} - \hat{u^\varepsilon}) + (\hat{p^\varepsilon}, div_z(\hat{\phi} - \hat{u^\varepsilon})) + \hat{j}(\hat{\phi}) - \hat{j}(\hat{u^\varepsilon}) \geq \sum_{i=1}^{2} (\hat{f}_i, \hat{\phi}_i - \hat{u_i^\varepsilon}) + \varepsilon(\hat{f}_3, \hat{\phi}_3 - \hat{u_3^\varepsilon})$$

$$\forall \hat{\phi} \in V, \quad (6.1)$$

$$\int_\Omega \varepsilon^2 \hat{K} \;\; \nabla_\varepsilon \hat{T^\varepsilon} \nabla_\varepsilon \hat{\psi} dx' dz + \int_\Omega \hat{R} \hat{T^\varepsilon} \hat{\psi} = 2 \int_\Omega \varepsilon^\alpha \hat{\mu}(\hat{T^\varepsilon}) \mid \hat{D}(\hat{u}^\varepsilon) \mid^r \hat{\psi} dx' dz$$

$$\forall \hat{\psi} \in W^{1,q'}(\Omega), \qquad (6.2)$$

where

$$V = \Big\{ v \in (W^{1,r}(\Omega))^3 \; : \; v = \hat{G} \;\text{ on }\; \Gamma_L \cup \Gamma_1, \quad v.n_{|\omega} = 0 \Big\}$$

$$\hat{a}(\hat{T^\varepsilon};\hat{u^\varepsilon},\hat{\phi}) = +\varepsilon^2 \sum_{1\le i,j\le 2}\int_\Omega \hat{\mu}(\hat{T^\varepsilon}) \mid \hat{D}(\hat{u^\varepsilon}) \mid^{r-2} \left(\frac{\partial \hat{u_i^\varepsilon}}{\partial x_j}+\frac{\partial \hat{u_j^\varepsilon}}{\partial x_i}\right)\frac{\partial \hat{\phi}_i}{\partial x_j} +$$

$$+\sum_{i=1}^{2}\int_\Omega \hat{\mu}(\hat{T^\varepsilon}) \mid \hat{D}(\hat{u^\varepsilon}) \mid^{r-2} \left(\frac{\partial \hat{u_i^\varepsilon}}{\partial z}+\varepsilon^2\frac{\partial \hat{u_3^\varepsilon}}{\partial x_i}\right)\left(\frac{\partial \hat{\phi}_i}{\partial z}+\varepsilon^2\frac{\partial \hat{\phi}_3}{\partial x_i}\right) +$$

$$+\varepsilon^2\int_\Omega \hat{\mu}(\hat{T^\varepsilon}) \mid \hat{D}(\hat{u^\varepsilon}) \mid^{r-2} \frac{\partial \hat{u_3^\varepsilon}}{\partial z}\frac{\partial}{\partial z}(\hat{\phi}_3-\hat{u_3^\varepsilon})dx'dz,$$

$$\mid \hat{D}(\hat{u^\varepsilon}) \mid^2 = \varepsilon^2\left(\frac{1}{4}\sum_{1\le i,j\le 2}(\frac{\partial \hat{u_i^\varepsilon}}{\partial x_j}+\frac{\partial \hat{u_j^\varepsilon}}{\partial x_i})^2+(\frac{\partial \hat{u_3^\varepsilon}}{\partial z})^2\right)+\frac{1}{2}\sum_{i=1}^{2}(\frac{\partial \hat{u_i^\varepsilon}}{\partial z}+\varepsilon^2\frac{\partial \hat{u_3^\varepsilon}}{\partial x_i})^2=\varepsilon^2 \mid D(u^\varepsilon) \mid^2$$

$$\nabla_\varepsilon v = (\frac{\partial v_1}{\partial x_1},\frac{\partial v_2}{\partial x_2},\frac{1}{\varepsilon}\frac{\partial v_3}{\partial z})^t, \qquad \hat{j}(\hat{\phi}) = \int_\omega \hat{k} \mid \hat{\phi}-s \mid dx'.$$

Let's now introduce the linear subspace

$$V_z^q = \{v \in W^{1,q}(\Omega) \ : \ \frac{\partial v}{\partial z} \in L^q(\Omega), \quad v_{\Gamma_1}=0\}.$$

Theorem 15. [15] *Assume (H) hold, and* $\alpha = \dfrac{3(2-q)}{3-q}$, *there exists a constant C, independent of* ε, *such that*

$$\sum_{i,j=1}^{2}\parallel \varepsilon\frac{\partial \hat{u_i^\varepsilon}}{\partial x_j}\parallel^r_{L^r(\Omega)}+\parallel \varepsilon\frac{\partial \hat{u_3^\varepsilon}}{\partial z}\parallel^r_{L^r(\Omega)}+\sum_{i=1}^{2}\big(\parallel \frac{\partial \hat{u_i^\varepsilon}}{\partial z}\parallel^r_{L^r(\Omega)}+\parallel \varepsilon^2\frac{\partial \hat{u_3^\varepsilon}}{\partial x_i}\parallel^r_{L^r(\Omega)}\big)\le C \quad (6.3)$$

$$\parallel \frac{\partial \hat{p^\varepsilon}}{\partial x_i}\parallel_{W^{-1,r'}(\Omega)}\le C, \ \ i=1,2, \qquad \parallel \frac{\partial \hat{p^\varepsilon}}{\partial z}\parallel_{W^{-1,r'}(\Omega)}\le C.\varepsilon, \quad (6.4)$$

$$\parallel \frac{\partial \hat{T^\varepsilon}}{\partial z}\parallel_{W^{1,q}(\Omega)}\le C, \qquad \sum_{i=1}^{2}\parallel \varepsilon\frac{\partial \hat{T^\varepsilon}}{\partial x_i}\parallel_{W^{1,q}(\Omega)}\le C. \quad (6.5)$$

Proof. From (6.1) and (6.2), we obtain (6.3)-(6.4), the main difficulty here is to obtain (6.5) which need the technical condition $\alpha = \dfrac{3(2-q)}{3-q}$ see [15] for all the proof. □

So the following weak convergences hold

Theorem 16. [15] *Assume (H), there exist*

$$u_i^\star \in V_z^r \quad i=1,2, \quad p^\star \in L_0^{r'}(\Omega),$$
$$and \quad T^\star \in W^{1,q}(\Omega) \quad with \quad T^\star_{|\Gamma_1}=0,$$

such that

$$\hat{u_i^\varepsilon} \rightharpoonup u_i^\star \quad (1\le i\le 2) \quad weakly \quad in \ V_z^r, \quad (6.6)$$

$$\varepsilon\frac{\partial \hat{u_i^\varepsilon}}{\partial x_j} \rightharpoonup 0 \ \ (1\le i,j\le 2) \ \ weakly \ \ in \ L^r(\Omega), \quad (6.7)$$

$$\varepsilon\frac{\partial \hat{u_3^\varepsilon}}{\partial z} \rightharpoonup 0 \ \ (1\le i,j\le 2) \quad in \ \ L^r(\Omega), \quad (6.8)$$

$$\varepsilon^2 \frac{\partial \hat{u}_3^\varepsilon}{\partial x_i} \rightharpoonup 0 \quad (1 \le i \le 2) \quad in\ L^r(\Omega), \tag{6.9}$$

$$\varepsilon \hat{u}_3^\varepsilon \rightharpoonup 0 \quad (1 \le i, j \le 2) \quad in \quad L^r(\Omega), \tag{6.10}$$

$$\hat{p^\varepsilon} \rightharpoonup p^\star \quad in\ L_0^{r'}(\Omega), \tag{6.11}$$

$$\hat{T^\varepsilon} \rightharpoonup T^\star \quad in\ V_z^q, \tag{6.12}$$

$$\varepsilon \frac{\partial \hat{T}^\varepsilon}{\partial x_i} \rightharpoonup 0 \quad in\ L^q(\Omega), \quad i = 1, 2. \tag{6.13}$$

Proof. Readily, from to (6.3)-(6.4) we obtain (6.6)-(6.11), while (6.12)-(6.13) follow from (6.5). □

Then we can pass to the limit in (6.1) for $\varepsilon \to 0$ using Minty's Lemma, and in (6.2), to obtain

Theorem 17. [15] *Assume (H), and also that $\hat{K} \in \mathcal{C}^1(\mathbb{R}) : (\hat{K})' \in L^\infty(\mathbb{R})$, then $u^\star$, $p^\star$, $T^\star$ satisfy*

$$\sum_{i=1}^{2} \int_\Omega \hat{\mu}(T^\star)\Big(\frac{1}{2}\sum_{i=1}^{2}\big(\frac{\partial u_i^\star}{\partial z}\big)^2\Big)^{\frac{r-2}{2}} \frac{\partial u_i^\star}{\partial z}\frac{\partial}{\partial z}(\hat{\phi}_i - u_i^\star)dx'dz - \sum_{i=1}^{2}(p^\star, \frac{\partial}{\partial x_i}(\hat{\phi}_i - u_i^\star))$$

$$+\hat{j}(\hat{\phi}) - \hat{j}(u^\star) \ge \sum_{i=1}^{2}(\hat{f}_i, \hat{\phi}_i - u_i^\star) \quad \forall \hat{\phi} \in \Pi(V), \tag{6.14}$$

$$-\frac{\partial}{\partial z}\Big(\hat{\mu}(T^\star)\big(\frac{1}{2}\sum_{i=1}^{2}(\frac{\partial u_i^\star}{\partial z})^2\big)^{\frac{r-2}{2}}\frac{\partial u_i^\star}{\partial z}\Big) + \frac{\partial p^\star}{\partial x_i} = \hat{f}_i, \quad i = 1, 2 \quad in \quad L^{r'}(\Omega), \tag{6.15}$$

$$p^\star(x_1, x_2, z) = p^\star(x_1, x_2) \ \ a.e\ in\ \Omega, \qquad p^\star \in W^{1,r'}(\omega), \tag{6.16}$$

$$-\frac{\partial}{\partial z}\big(\hat{K}\frac{\partial T^\star}{\partial z}\big) + \hat{R}T^\star = 0 \ \ in\ \ L^q(\Omega). \tag{6.17}$$

$$T^\star = 0 \quad on \quad \Gamma_1, \quad and \quad -\hat{K}\frac{\partial T^\star}{\partial z} = \hat{b} \quad on \quad \omega. \tag{6.18}$$

Then we obtain the limit problem

Theorem 18. [15] *Under the same hypothesis as in theorem 4, the traces*

$$s^\star = u^\star(x', 0), \quad \zeta^\star = T^\star(x', 0)$$

$$\tau^\star = \left(\frac{1}{2}\sum_{i=1}^{2}(\frac{\partial u^\star}{\partial z}(x', 0))^2\right)^{\frac{r-2}{2}} \frac{\partial u_i^\star}{\partial z}(x', 0),$$

satisfy the following inequality

$$\int_\omega \hat{k}(\mid \psi + s^\star - s \mid - \mid s^\star - s \mid)dx' - \int_\omega \hat{\mu}(\zeta^\star)\tau^\star\psi dx' \ge 0 \qquad \forall \psi \in (L^r(\omega))^2, \tag{6.19}$$

the limit of Tresca's boundary condition on ω gives:

$$\left.\begin{array}{l} \hat{\mu}(\zeta^\star) \mid \tau^\star \mid < \hat{k} \ \Rightarrow \ s^\star = s \\ \hat{\mu}(\zeta^\star) \mid \tau^\star \mid = \hat{k} \ \Rightarrow \ \exists\ \lambda \ge 0 \ : \ s^\star = s + \lambda\tau^\star \end{array}\right\} \text{ a.e on } \omega.$$

Also $u^\star, p^\star$ and $T^\star$ satisfy the specific weak Reynolds equation

$$\int_\omega \left[\int_0^h \int_0^y \hat{\mu}(T^\star(x',\xi)) A^\star(x',\xi) \frac{\partial u^\star}{\partial \xi}(x',\xi) d\xi dy \right] \cdot \nabla\phi(x') dx'$$

$$- \int_\omega \frac{h}{2} \left[\int_0^h \hat{\mu}(T^\star(x',\xi)) A^\star(x',\xi) \frac{\partial u^\star}{\partial \xi}(x',\xi) d\xi \right] \cdot \nabla\phi(x') dx'$$

$$+ \int_\omega \left[\frac{h^3}{12} \nabla p^\star(x') + \tilde{F}(x') \right] \cdot \nabla\phi(x') dx' = 0 \quad \forall \phi \in W^{1,r}(\omega) \tag{6.20}$$

and the following equation

$$\int_\Omega \hat{K} \frac{\partial T^\star}{\partial z} \frac{\partial \hat{\psi}}{\partial z} dx'dz + \int_\Omega \hat{R} T^\star \hat{\psi} dx'dz = 0 \;\; \forall \hat{\psi} \in W^{1,q'}_{\Gamma_1 \cup \Gamma_L}(\Omega), \tag{6.21}$$

where

$$A^\star(x',\xi) = \left(\frac{1}{2} \sum_{i=1}^{2} (\frac{\partial u^\star}{\partial \xi}(x',\xi))^2 \right)^{\frac{r-2}{2}}$$

$$\tilde{F}(x') = \int_0^h \int_0^y \int_0^\xi \hat{f}(x',y) dy d\xi dz - \frac{h}{2} \int_0^h \int_0^\xi \hat{f}(x',y) dy d\xi. \tag{6.22}$$

Theorem 19. *The solution $(u^\star, T^\star, p^\star)$ of our limit problem is unique.*

Proof. [15] Let (U^1, T^1, p^1), (U^2, T^2, p^2) be two solutions of the limit problem. Then

$$T = T^1 - T^2$$

satisfies the problem

$$-\frac{\partial}{\partial z} \left(\hat{K} \frac{\partial T}{\partial z} \right) + \hat{R} T = 0, \quad T_{|\Gamma_1} = 0, \quad \hat{K} \frac{\partial T}{\partial z} = 0 \quad on \quad \omega$$

so $T = 0$, thus $T^1 = T^2 = T^\star$.
Taking $\phi = U^2$ and $\phi = U^1$ respectively, as test functions in (6.14) we get

$$\sum_{i=1}^{2} \int_\Omega \hat{\mu}(T^\star) \left[(\frac{1}{2} \sum_{i=1}^{2} (\frac{\partial U_i^1}{\partial z})^2)^{\frac{r-2}{2}} \frac{\partial U_i^1}{\partial z} - (\frac{1}{2} \sum_{i=1}^{2} (\frac{\partial U_i^2}{\partial z})^2)^{\frac{r-2}{2}} \frac{\partial U_i^2}{\partial z} \right] \times$$

$$\times \frac{\partial}{\partial z}(U_i^1 - U_i^2) dx'dz \leq 0, \tag{6.23}$$

using some inequality [45]) we obtain for $r \geq 1$

$$\| \frac{\partial}{\partial z}(U^1 - U^2) \|_{(L^r(\Omega))^2} = 0,$$

using the Poincare inequality we deduce that

$$\| U^1 - U^2 \|_{V_z} = 0,$$

so $u^\star$ is unique. The uniqueness of $p^\star$ in $L_0^{r'}(\omega) \cap W^{1,r'}(\omega)$ follows then from the *specific weak Reynolds equation* (6.20), indeed we obtain first

$$\int_\omega \frac{h^3}{12} \nabla(p^1 - p^2) \nabla\phi \, dx' = 0,$$

taking

$$\phi = p^1 - p^2$$

and by Poincare's inequality we get

$$\| p^1 - p^2 \|_{L^{r'}(\omega)} = 0.$$

This ends the proof of the uniqueness. □

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