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An Explicit Solution for a Two-Phase Stefan Problem with a Similarity Exponential Heat Sources

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Abstract

A two-phase Stefan problem with heat source terms in both liquid and solid phases for a semi-infinite phase-change material is considered. The internal heat source functions are given by $g_j(x,t) = (-1)^{j+1}\dfrac{\rho l}{t}\exp\left(-(\frac{x}{2a_j\sqrt{t}}+d_j)^2\right)$ ($j=1$ solid phase; $j=2$ liquid phase), ρ is the mass density, l is the fusion latent heat by unit of mass, a_j^2 is the diffusion coefficient, x is spatial variable, t is the temporal variable and $d_j \in \mathbb{R}$. A similarity solution is obtained for any data when a temperature boundary condition is imposed at the fixed face $x=0$; when a flux condition of the type $-q_0/\sqrt{t}$ ($q_0>0$) is imposed on $x=0$ then there exists a similarity solution if and only if a restriction on q_0 is satisfied.

Key words : Stefan problem, free boundary problem, Lamé-Clapeyron solution, Neumann solution, phase-change process, fusion, sublimation-dehydration process, heat source, similarity solution.

2000 AMS Subject Classification: 35R35, 80A22, 35C05

I. Introduction.

Sublimation-dehydration, which is commonly known as freeze-dying, is used as a method for removing moisture from biological materials, such as food, pharmaceutical, and biochemical products. Some of the advantages of sublimation-dehydration over evaporative drying are that the structural integrity of the material is maintained and product degradation is minimized [1], [13]. The major disadvantage of the freeze-drying process is that it is generally slow, and consequently, the process is economically unfeasible for certain materials. One means of alleviating this problem is through the use of microwave energy. Several mathematical models have been proposed to describe the freeze-drying process without microwave heating [6], [8]. Only a few studies have also included a microwave heat source in the model [1].

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In [9] the one-phase Lamé-Clapeyron-Stefan problem [7] with a particular type of sources was studied and a generalized Lamé-Clapeyron explicit solution was obtained. Moreover, necessary and sufficient conditions were given in order to characterize the source term which provides a unique solution.

Several applied papers give us the significance in order to consider source terms in the interior of the material which can undergo a change of phase, e.g. [3], [5], [10], [14]. Phase-change problems appear frequently in industrial processes; a large bibliography on the subject was given recently in [16].

In [14] there is a mathematical model for sublimation-dehydration with volumetric heating was presented from which analytical solutions for dimensionless temperature, vapor concentration, and pressure were obtained for two different temperature boundary conditions. It was considered a semi-infinite frozen porous medium with constant thermal properties subject to a sublimation-dehydration process involving a volumetric heat source of the type $g(x,t) = \dfrac{const.}{t} \exp\left(-\left(x+d\right)^2\right)$, and, a sensitivity study was also conducted in which the effects of the material properties inherent in these solutions were analyzed.

In this paper a semi-infinite homogeneous phase-change material initially in solid phase at the uniform temperature $-C < 0$, with a volumetric heat source, is considered. A mathematical description for the temperature within the material is given by

$$\frac{\partial T_2}{\partial t}(x,t) = a_2^2 \frac{\partial^2 T_2}{\partial x^2}(x,t) + \frac{1}{\rho c_2} g_2(x,t), \qquad 0 < x < s(t), \qquad t > 0; \tag{1}$$

$$\frac{\partial T_1}{\partial t}(x,t) = a_1^2 \frac{\partial^2 T_1}{\partial x^2}(x,t) + \frac{1}{\rho c_1} g_1(x,t), \qquad x > s(t), \qquad t > 0; \tag{2}$$

for two given internal source functions ([9], [14]) given by

$$g_j = g_j(x,t) = (-1)^{j+1} \frac{\rho l}{t} \exp\left(-\left(\frac{x}{2a_j\sqrt{t}} + d_j\right)^2\right) \quad j = 1, 2, \tag{3}$$

ρ is the mass density, l is the fusion latent heat per unit of mass, a_j^2 is the diffusion coefficient, c_j is the specified heat per unit of mass and k_j is the thermal conductivity, for $j = 1$ (solid phase), 2 (liquid phase).

The initial temperature and the temperature as $x \to \infty$ are assumed to be constant

$$T_1(x,0) = T_1(+\infty,t) = -C < 0, \qquad x > 0, \qquad t > 0. \tag{4}$$

At $x = 0$, two different temperature boundary conditions are considered, the first is a constant temperature condition

$$T_2(0,t) = B > 0, \qquad t > 0 \tag{5}$$

which is studied in Section II, and the second is an assumed heat flux of the form

$$k_2 \frac{\partial T_2}{\partial x}(0,t) = \frac{-q_0}{\sqrt{t}}, \qquad t > 0 \tag{6}$$

which is studied in Section III. This kind of heat flux condition was also considered in several papers, e.g. [2], [11], [12] and [15].

The phase-change interface condition is determined from an energy balance at the free boundary $x = s(t)$:

$$k_1 \frac{\partial T_1}{\partial x}(s(t),t) - k_2 \frac{\partial T_2}{\partial x}(s(t),t) = \rho l \dot{s}(t), \qquad t > 0, \tag{7}$$

where the temperature conditions at the interface are assumed to be constant:

$$T_1(s(t),t) = T_2(s(t),t) = 0, \qquad t > 0. \tag{8}$$

Moreover, the initial position of the free boundary is

$$s(0) = 0. \tag{9}$$

In section II we obtain an explicit solution for the problem (1)-(5),(7)-(9) for internal heat sources given by (3).

In Section III we solve the same free boundary problem but with the heat flux condition of the type $-\frac{q_0}{\sqrt{t}}$ ($q_0 > 0$) prescribed on the fixed face $x = 0$, and we obtain an explicit solution to this problem if the coefficient q_0 satisfies an appropriate inequality (48) or (49); this restriction on q_0 is new with respect to [14].

II. Solution of the free boundary problem with temperature boundary condition at x=0.

Applying the immobilization domain method (see [4]), we are looking for solutions of the type

$$T_j(x,t) = \theta_j(y) \qquad j = 1, 2, \tag{10}$$

where the new independent spatial variable y is defined by

$$y = \frac{x}{s(t)}. \tag{11}$$

Then, the condition (7) is transformed in

$$k_1\theta_1'(1) - k_2\theta_2'(1) = \rho l s(t) \dot{s}(t), \tag{12}$$

and we must have necessarily that $s(t)\dot{s}(t)$ =const. i.e.,

$$s(t) = 2a_2\lambda\sqrt{t}, \tag{13}$$

where the dimensionless parameter $\lambda > 0$ is unknown.

Next, we define

$$R_j(\eta) = \theta_j\left(\frac{\eta}{\lambda}\right), \qquad j = 1, 2, \qquad \eta = \lambda y, \tag{14}$$

then the problem (1)-(5),(7)-(9) is equivalent to the following one:

$$R_2''(\eta) + 2\eta R_2'(\eta) = \frac{4l}{c_2}\exp\left(-(\eta + d_2)^2\right), \qquad 0 < \eta < \lambda; \tag{15}$$

$$R_1''(\eta) + 2\frac{a_2^2}{a_1^2}\eta R_1'(\eta) = -\frac{4a_2^2 l}{a_1^2 c_2}\exp\left(-\left(\frac{a_2}{a_1}\eta + d_1\right)^2\right), \qquad \eta > \lambda; \tag{16}$$

$$R_1(\lambda) = R_2(\lambda) = 0; \tag{17}$$

$$k_1 R_1'(\lambda) - k_2 R_2'(\lambda) = 2\rho l \lambda a_{2;}^2 \tag{18}$$

$$R_1(+\infty) = -C; \tag{19}$$

$$R_2(0) = B. \tag{20}$$

After some elementary computations, from (15), (17) and (20) we obtain

$$R_2(\eta) = B - (B + \varphi_2(\lambda)) \frac{\operatorname{erf}(\eta)}{\operatorname{erf}(\lambda)} + \varphi_2(\eta), 0 < \eta < \lambda \;, \tag{21}$$

$$\varphi_2(\eta) = \frac{-l\sqrt{\pi}}{c_2 d_2}\left[\operatorname{erf}(\eta + d_2) - \operatorname{erf}(d_2) - \operatorname{erf}(\eta)\exp\left(-d_2^2\right)\right], \; if \; d_2 \neq 0 \tag{22}$$

$$\varphi_2(\eta) = \frac{2l}{c_2}\left[1 - \exp\left(-\eta^2\right)\right], \; if \; d_2 = 0. \tag{23}$$

and, from (16), (17) and (19), we have

$$R_1(\eta) = -\frac{(C + \varphi_1(+\infty))}{\operatorname{erf}c\left(\frac{a_2}{a_1}\lambda\right)} \frac{2}{\sqrt{\pi}} \int_{\frac{a_2}{a_1}\lambda}^{\frac{a_2}{a_1}\eta} \exp(-u^2)du + \varphi_1(\eta), \quad \eta > \lambda, \tag{24}$$

where

$$\begin{aligned} \varphi_1(\eta) \;=\; & \frac{l\sqrt{\pi}}{c_1 d_1}\exp\left(-d_1^2\right)[\exp(-2\frac{a_2}{a_1}\lambda d_1)\left(\operatorname{erf}(\frac{a_2}{a_1}\lambda) - \operatorname{erf}(\frac{a_2}{a_1}\eta)\right) + \\ & + \exp\left(d_1^2\right)\left(\operatorname{erf}(\frac{a_2}{a_1}\eta + d_1) - \operatorname{erf}(\frac{a_2}{a_1}\lambda + d_1)\right)], \text{ if } d_1 \neq 0 \end{aligned} \tag{25}$$

$$\varphi_1(+\infty) = \tfrac{l\sqrt{\pi}}{c_1 d_1}\exp\left(-d_1^2\right)[\exp\left(d_1^2\right)\operatorname{erf}c(\tfrac{a_2}{a_1}\lambda + d_1) - \exp(-2\tfrac{a_2}{a_1}\lambda d_1)\operatorname{erf}c(\tfrac{a_2}{a_1}\lambda)], \text{ if } d_1 \neq 0 \tag{26}$$

or

$$\begin{aligned} \varphi_1(\eta) = & \frac{2l\sqrt{\pi}}{c_1}[\frac{a_2}{a_1}\lambda\left(\operatorname{erf}(\frac{a_2}{a_1}\eta) - \operatorname{erf}(\frac{a_2}{a_1}\lambda)\right) + \\ & + \frac{1}{\sqrt{\pi}}\left(\exp\left(-\left(\frac{a_2}{a_1}\eta\right)^2\right) - \exp\left(-\left(\frac{a_2}{a_1}\lambda\right)^2\right)\right)], \text{ if } d_1 = 0 \end{aligned} \tag{27}$$

$$\varphi_1(+\infty) = \frac{2l\sqrt{\pi}}{c_1}[\frac{a_2}{a_1}\lambda \operatorname{erf}c(\frac{a_2}{a_1}\lambda) + -\frac{1}{\sqrt{\pi}}\exp\left(-\left(\frac{a_2}{a_1}\lambda\right)^2\right)], \text{ if } d_1 = 0 \tag{28}$$

where λ is the unknown coefficient which must verify the condition (18). Furthermore, the Eq.18 for λ is equivalent to the following equation

$$f_1(x) = f_2(x) \;, \quad x > 0. \tag{29}$$

where

$$f_1(x) = F_0(x)h_1(x) \;\;, \;\; f_2(x) = Q\left(\frac{a_2}{a_1}x\right) \; h_2(x) \tag{30}$$

with

$$Q(x) = \sqrt{\pi} x \exp(x^2) \operatorname{erf} c(x) \ , \quad F_0(x) = x \operatorname{erf}(x) \exp(x^2), \tag{31}$$

$$h_1(x) = Ste_1 - \frac{\sqrt{\pi}\exp(-d_1^2)}{d_1} \left[\exp(-\tfrac{2d_1 a_2}{a_1} x) \operatorname{erf} c(\tfrac{a_2}{a_1} x) - \exp(d_1^2) \operatorname{erf} c(\tfrac{a_2}{a_1} x + d_1) \right] \tag{32}$$

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x \exp(-u^2) du, \ \ \operatorname{erf} c(x) = 1 - \operatorname{erf}(x) \ , \ h_2(x) = \frac{Ste_2}{\sqrt{\pi}} - F(x), \tag{33}$$

with

$$F(x) = F_0(x) + \frac{\exp(-d_2^2)}{d_2} \left[\exp(d_2^2) \left(\operatorname{erf}(x + d_2) - \operatorname{erf}(d_2) \right) - \exp(-2xd_2) \operatorname{erf}(x) \right], \tag{34}$$

and

$$Ste_1 = \frac{Cc_1}{l} \ , \ Ste_2 = \frac{Bc_2}{l} \tag{35}$$

are the Stefan number for phase $j = 1$ and $j = 2$ respectively.

Theorem 1 *The Eq.29 has a unique solution $\lambda > 0$. Moreover, the free boundary problem with heat source terms (1)-(5),(7)-(9) has an explicit solution given by*

$$\begin{aligned} &T_1(x,t) = \frac{-\left(C + \varphi_1(+\infty)\right)}{\operatorname{erf} c\left(\frac{a_2}{a_1}\lambda\right)} \left[\operatorname{erf}\left(\tfrac{x}{2a_1\sqrt{t}}\right) - \operatorname{erf}\left(\tfrac{a_2}{a_1}\lambda\right) \right] + \varphi_1\left(\frac{x}{2a_2\sqrt{t}}\right), \\ &\textit{for} \ \ x > s(t), \qquad t > 0; \\ &T_2(x,t) = B - (B + \varphi_2(\lambda)) \frac{\operatorname{erf}(\frac{x}{2a_2\sqrt{t}})}{\operatorname{erf}(\lambda)} + \varphi_2(\tfrac{x}{2a_2\sqrt{t}}) \ , \\ &\textit{for} \ \ 0 < x < s(t), \qquad t > 0; \end{aligned} \tag{36}$$

*where $\varphi_1(\eta)$ and $\varphi_2(\eta)$ are defined in $(25)-(28)$ and $(22)-(23)$ respectively. The free boundary $s(t)$ is given by (13) where the coefficient λ is the unique solution of Eq.*29.

Proof. Taking into account the Lemma 2 (see below) we can prove that Eq.29 has a unique solution $\lambda > 0$. We invert the relations (14), (10) and (11) in order to obtain an explicit solution of problem (1)-(5),(7)-(9) with the source terms g_j defined by (3).■

Lemma 2 *A) Functions $Q(x), F_0(x)$ and $F(x)$ satisfy the following properties:*

(i) $Q(0) = 0, \ Q(+\infty) = 1, \ Q'(x) > 0, \ \forall x > 0, Q'(0) = \sqrt{\pi}.$

(ii) $F_0(0) = 0 \ , \ \ F_0(+\infty) = +\infty \ , \ \ F_0'(x) > 0, \ \forall x > 0.$

(iii) $F(0) = 0 \ , \ \ F(+\infty) = +\infty \ , \ \ \frac{\partial F}{\partial x}(x) > 0, \ \forall x > 0.$

B) (a) Function $f_1(x)$, satisfies the following properties:

(i) $f_1(0^+) = 0 \ , \ \ $ *(ii)* $f_1(+\infty) = +\infty,$

(iii) if condition (47) *is verified then* $f_1(x) > 0,\ \forall x > 0$,
$\dfrac{\partial f_1}{\partial x}(x) > 0$ *and* $\dfrac{\partial f_1}{\partial x}(0^+) = 0^+$,
(iv) if conditions (47) *is not verified then* $f_1(\xi_1) = 0$ *and* $f_1(x)$ *is negative in* $(0, \xi_1)$, *and is positive in* $(\xi_1, +\infty)$; *then there exists* $x_1 \in (0, \xi_1)$ *such that* $\dfrac{\partial f_1}{\partial x}(x_1) = 0$. *Moreover we have* $\dfrac{\partial f_1}{\partial x}(x) > 0$ $\forall\, x > \xi_1$.

(b) Function $f_2(x)$ *satisfies the following properties:*

$(i)\ f_2(0^+) = 0\ ,\ (ii)\ f_2(+\infty) = -\infty\ ,\ (iii)\ f_2(\xi_2) = 0$,

$(iii)\ \dfrac{\partial f_2}{\partial x}(x) = \dfrac{a_2}{a_1} Q'\left(\tfrac{a_2}{a_1}x\right)\ h_2(x) + Q\left(\tfrac{a_2}{a_1}x\right)\ \dfrac{\partial h_2}{\partial x}(x)$,

$(iv)\ \dfrac{\partial f_2}{\partial x}(0^+) = \dfrac{a_2}{a_1} Ste_2 > 0$,

(v) there exists $x_2 \in (0, \xi_2)$ *such that* $\dfrac{\partial f_2}{\partial x}(x_2) = 0$,

$(vi)\ \dfrac{\partial f_2}{\partial x}(x) < 0\ ,\ \forall\, x > \xi_2$.

C) *Function* $W(x)$ *satisfies the following properties:*

$$(i)\ W(0) = \frac{a_1}{a_2\sqrt{\pi}}[Ste_1 - 2\sqrt{\pi}P(d_1)] \text{ if } d_1 \neq 0,\ \textit{where } P \textit{ is defined by}$$
$$P(x) = \frac{\exp(-x^2) - \operatorname{erf} c(x)}{2x},$$

$$(ii)\ W(0) = \frac{a_1}{a_2\sqrt{\pi}}[Ste_1 - 2] \text{ if } d_1 = 0, \tag{37}$$

$$(iii)\ W(+\infty) = +\infty,$$

$$(iv)\ \textit{if condition } (47) \textit{ is verified then } W(0) \geq 0 \textit{ and } \frac{\partial W}{\partial x}(x) > 0,\ \forall x > 0.$$

D) *Function* $V(x)$ *satisfies the following properties:*

$$(i)\ V(0) = \frac{q_0}{\rho l a_2}\ ,\ (ii)\ V(+\infty) = -\infty\ ,\ (iii)\ \frac{\partial V}{\partial x}(x) < 0,\ \forall x > 0.$$

III. Solution of the free boundary problem with a heat flux condition on the fixed face x=0.

In this section we consider the problem (1)-(5),(7)-(9), but condition (5) will be replaced by condition (6) (see [12], [15]). We can define the same transformations (10),(11) and (14) as were done for the previous problem, and we obtain (15)-(19) and

$$R_2'(0) = \frac{-2q_0}{\rho c_2 a_2} \tag{38}$$

It is easy to see that the free boundary must be of the type $s(t) = 2a_2\mu\sqrt{t}$ where μ is a dimensionless constant to be determined. The solution to the problem (15)-(19) and (38) is given by

$$R_1(\eta) = -\frac{(C + \varphi_3(+\infty))}{\operatorname{erf} c\left(\frac{a_2}{a_1}\mu\right)}\left[\operatorname{erf}\left(\frac{a_2}{a_1}\eta\right) - \operatorname{erf}\left(\frac{a_2}{a_1}\mu\right)\right] + \varphi_3(\eta), \quad \eta > \mu, \tag{39}$$

where

$$\begin{aligned}\varphi_3(\eta) &= \frac{l\sqrt{\pi}}{c_1 d_1}\exp\left(-d_1^2\right)[\exp(-2\frac{a_2}{a_1}\mu d_1)\left(\operatorname{erf}(\frac{a_2}{a_1}\mu) - \operatorname{erf}(\frac{a_2}{a_1}\eta)\right) + \\ &\quad + \exp\left(d_1^2\right)\left(\operatorname{erf}(\frac{a_2}{a_1}\eta + d_1) - \operatorname{erf}(\frac{a_2}{a_1}\mu + d_1)\right)], \text{ if } d_1 \neq 0\end{aligned} \tag{40}$$

$$\varphi_3(+\infty) = \tfrac{l\sqrt{\pi}}{c_1 d_1}\exp\left(-d_1^2\right)[\exp\left(d_1^2\right)\operatorname{erf} c(\tfrac{a_2}{a_1}\mu + d_1) - \exp(-2\tfrac{a_2}{a_1}\mu d_1)\operatorname{erf} c(\tfrac{a_2}{a_1}\mu)], \text{ if } d_1 \neq 0 \tag{41}$$

or

$$\begin{aligned}\varphi_3(\eta) &= \frac{2l\sqrt{\pi}}{c_1}[\frac{a_2}{a_1}\mu\left(\operatorname{erf}(\frac{a_2}{a_1}\eta) - \operatorname{erf}(\frac{a_2}{a_1}\mu)\right) + \\ &\quad + \frac{1}{\sqrt{\pi}}\left(\exp\left(-\left(\frac{a_2}{a_1}\eta\right)^2\right) - \exp\left(-\left(\frac{a_2}{a_1}\mu\right)^2\right)\right)], \text{ if } d_1 = 0\end{aligned} \tag{42}$$

$$\varphi_3(+\infty) = \frac{2l\sqrt{\pi}}{c_1}[\frac{a_2}{a_1}\mu \operatorname{erf} c(\frac{a_2}{a_1}\mu) - \frac{1}{\sqrt{\pi}}\exp\left(-\left(\frac{a_2}{a_1}\mu\right)^2\right)], \text{ if } d_1 = 0 \tag{43}$$

and

$$R_2(\eta) = \frac{q_0\sqrt{\pi}}{\rho c_2 a_2}(\operatorname{erf}(\mu) - \operatorname{erf}(\eta)) + \varphi_2(\eta) - \varphi_2(\mu), 0 < \eta < \mu \tag{44}$$

where φ_2 was defined in (22)-(23) and the unknown μ must satisfy the following equation

$$W(x) = V(x) \quad , \quad x > 0 \tag{45}$$

where

$$W(x) = \tfrac{x\exp(x^2)}{Q\left(\frac{a_2}{a_1}x\right)}[Ste_1 - \tfrac{\sqrt{\pi}\exp(-d_1^2)}{d_1}\left(\exp\left(-\tfrac{2a_2}{a_1}xd_1\right)\operatorname{erf} c\left(\tfrac{a_2}{a_1}x\right) - \exp(d_1^2)\operatorname{erf} c\left(\tfrac{a_2}{a_1}x + d_1\right)\right)]$$

if $d_1 \neq 0$,

$$W(x) = \frac{x\exp(x^2)\exp\left(-\left(\frac{a_2}{a_1}x\right)^2\right)}{Q\left(\frac{a_2}{a_1}x\right)}[Ste_1\exp\left(\tfrac{a_2}{a_1}x\right)^2 + 2Q\left(\tfrac{a_2}{a_1}x\right) - 2], \text{ if } d_1 = 0,$$

and

$$V(x) = \frac{q_0}{\rho l a_2} - x\exp(x^2) + \frac{\exp(-d_2^2)}{d_2}\left(\exp(-2d_2x) - 1\right), \text{ if } d_2 \neq 0, \tag{46}$$

$$V(x) = \frac{q_0}{\rho l a_2} - x\exp(x^2) - 2x, \text{ if } d_2 = 0.$$

Theorem 3 *(a) If*

$$Ste_1 \geqslant 2\ ,\ \ \textit{if}\ d_1 \geqslant 0 \quad \textit{or} \quad Ste_1 \geqslant 2\sqrt{\pi}P(d_1),\ \textit{if}\ d_1 < 0 \tag{47}$$

then Eq.45 has a unique solution $\mu > 0$ if and only if q_0 satisfies the following inequality

$$q_0 \ \geq\ 2a_1\rho l\left[\frac{Ste_1}{2\sqrt{\pi}} - P(d_1)\right]\ \textit{if}\ d_1 \neq 0, \tag{48}$$

$$q_0 \ \geq\ \frac{a_1\rho l}{\sqrt{\pi}}\left[Ste_1 - 2\right]\ \ \textit{if}\ d_1 = 0, \tag{49}$$

where P was defined in $(37i)$.

(b) The free boundary problem with sources term (1)-(4), (6)-(9) has an explicit solution given by

$$T_1(x,t) = \frac{-\left(C + \varphi_3(+\infty)\right)}{\operatorname{erf} c\left(\frac{a_2}{a_1}\mu\right)}\left[\operatorname{erf}\left(\frac{x}{2a_1\sqrt{t}}\right) - \operatorname{erf}\left(\frac{a_2}{a_1}\mu\right)\right] + \varphi_3\left(\frac{x}{2a_2\sqrt{t}}\right) \tag{50}$$
for $x > s(t), \qquad t > 0$

$$T_2(x,t) = \frac{q_0\sqrt{\pi}}{\rho c_2 a_2}\left[\operatorname{erf}(\mu) - \operatorname{erf}\left(\frac{x}{2a_2\sqrt{t}}\right)\right] + \varphi_2\left(\frac{x}{2a_2\sqrt{t}}\right) - \varphi_2\left(\mu\right) \tag{51}$$
for $0 < x < s(t), \qquad t > 0;$

where φ_3 and φ_2 are defined in (40)-(43) and (22)-(23) respectively, the free boundary is given by

$$s(t) = 2a_2\mu\sqrt{t},$$

and μ is the unique solution given in (a).
Proof. To prove (a) we use the definitions of the functions W and V, and Lemma 2. We invert the relations (14), (10) and (11) in order to obtain (50)-(51).■

A more general case for internal heat sources of the non-exponential type will be given in a forthcoming paper.

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