DIRECT AND INVERSE SCATTERING FOR THE NONLINEAR SCHRÖDINGER EQUATION WITH A POTENTIAL*

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Resumen

En este artículo se presenta una reseña de resultados recientes en la teoría de dispersión para la ecuación de Schrödinger no lineal en una dimensión y con un potencial. En particular, en la construcción del operador de dispersión para energías pequeñas y en la resolución del problema inverso. Específicamente, damos condiciones en el potencial y en la nolinealidad tales que el operador de dispersión para energías pequeñas determina unívocamente el potencial y la nolinealidad, y damos un método para la reconstrucción de ambos. Estos resultados están basados en la estimación $L^1 - L^\infty$ que demostramos en [10].
Palabras claves: dispersión inversa, ecuación de Schrödinger no lineal.

Abstract

In this paper we review recent results on the scattering theory for the nonlinear Schrödinger equation with a potential on the line. In particular, on the construction of the low-energy scattering operator and on the solution of the inverse scattering problem. Namely, we give conditions on the potential and on the nonlinearity such that the low-energy scattering operator determines uniquely the potential and the nonlinearity, and we give a method for the reconstruction of both. These results are based on the $L^1 - L^\infty$ estimate that we proved in [10].
Key words: inverse scattering, nonlinear Schrödinger equation.
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1 Introduction

In this paper I wish to discuss some recent results in the scattering theory of nonlinear evolution equations of dispersive type. We will consider in particular the nonlinear Schrödinger equation. This is one of the main equations of mathematical physics and it is a typical case that shows many of the main features of other equations. The aim of direct scattering theory is to study the behavior for large times of the solutions, and in particular to prove that -under appropriate conditions- the solutions are asymptotic as $t \to \pm\infty$ to solutions of a simpler, linear equation with constant coefficients. Since the solutions of the later equation are obtained by Fourier transform, direct scattering theory allows us to describe the asymptotic behavior of the solutions to our nonlinear evolution equation in a simple way. Of course, this situation excludes the case of nonlinear bound-states, that are solutions periodic in time. This requires either that the initial data is small or that the interaction is repulsive in a appropriate sense. The operator that to the inital data of the asymptotic solution as $t \to -\infty$ asigns the inital data of the asymptotic solution as $t \to \infty$ is the scattering operator.The purpose of inverse scattering theory is to obtain information on the potential and the nonlinearity, from the scattering operator. In other words, we wish to obtain as much information as possible on the potential and the nonlinearity, from the asymptotic behavior of the solutions; more precisely, from the relation of the asymptotic behaviors as $t \to \pm\infty$. Some of the main problems are uniqueness: does the scattering operator uniquely determines the potential and the nonlinearity?, and reconstruction: to obtain formulae that allow to reconstruct the potential and the nonlinearity from the scattering operator.

We will discuss the following nonlinear Schrödinger equation with a potential,

$$i\frac{\partial}{\partial t}u(t,x) = -\frac{d^2}{dx^2}u(t,x) + V_0(x)u(t,x) + F(x,u), u(0,x) = \phi(x), \tag{1.1}$$

where $t, x \in \mathbf{R}$, the potential, V_0, is a real-valued function and $F(x,u)$ is a complex-valued function. In the case where the potential is zero, $V_0 \equiv 0$, there is a very large literature on the direct scattering problem . See for example [1], [2], [3],[4] [5], [7] and [8]. When $V_0 \equiv 0$ the linearized equation is the free Schrödinger equation,

$$i\frac{\partial}{\partial t}u(t,x) = -\frac{d^2}{dx^2}u(t,x), u(0,x) = \phi(x). \tag{1.2}$$

Let us denote by H_0 the self-adjoint realization of $-\frac{d^2}{dx^2}$ in L^2 with domain the Sobolev space $W_{2,2}$ and let us define e^{-itH_0} by functional calculus. The solution to (1.2) is given by, $e^{-itH_0}\phi$. It follows immediately from the explicit formula for the kernel of e^{-itH_0} that the following $L^1 - L^\infty$ estimate holds,

$$\left\| e^{-itH_0}\phi \right\|_{L^\infty} \leq C \frac{1}{\sqrt{|t|}} \left\| \phi \right\|_{L^1}. \tag{1.3}$$

The estimate (1.3) expresses in a quantitative way a important property of the solutions to (1.2). Namely, that as $t \to \pm\infty$ the solutions not only propagate to spacial infinity, but they also *spread* uniformly in space. This is known in the physics literature as *wave packet spreading.* This spreading is essential in the study of the nonlinear Schrödinger equation. This can be seen as follows. For simplicity, suppose that $\phi \in L^1 \cap W_{1,2}$. Then, we have that,

$$\left\| e^{-itH_0}\phi \right\|_{L^\infty} \leq C \frac{1}{\sqrt{1+|t|}} \left(\left\| \phi \right\|_{L^1} + \left\| \phi \right\|_{W_{1,2}} \right).$$

Suppose moreover, that $F(x,u) = \lambda |u|^{(p-1)} u$, for some constant λ and some large positive p. Then, if ϕ is small enough; initially F will be small and the solution to (1.1) with $V_0 \equiv 0$ will propagate for small times essentially like a solution to the free Schrödinger equation (1.2). Moreover, because of the *spreading* the solution will be even smaller as t increases, and this will give the nonlinear term no chance to become very big and to make the solution blow-up in a finite time. If the initial data is not small, but the nonlinearity is repulsive -in a proper sense- a similar phenomenon takes place. In both cases there is a balance within the linear and the nonlinear terms that makes possible the existence of solutions global in time, and that permits the analysis of the large time asymptotics of the solutions in terms of the solutions to the free Schrödinger equation (1.2), i.e., scattering takes place. But what happens when the the potential V_0 is not identically zero? In this case the linearized equation is the following Schrödinger equation with a potential,

$$i\frac{\partial}{\partial t}u(t,x) = -\frac{d^2}{dx^2}u(t,x) + V_0(x)u(t,x), u(0,x) = \phi(x), \tag{1.4}$$

and we would need that an estimate as (1.3) holds with H_0 replaced with the perturbed Hamiltonian, $H := H_0 + V_0$. The problem is that such an estimate is at worst not true and at best quite hard to come by. Suppose for example that H has an eigenvalue, E, with eigenvector ϕ, i.e., that $H\phi = E\phi$. Then, $e^{-itH}\phi = e^{-itE}\phi$. This solution is periodic in time and it does not spreads at all; $\left\| e^{-itH}\phi \right\|_{L^\infty} = \|\phi\|_{L^\infty}$. However, we can hope that an estimate as (1.3) holds for initial data in $\mathcal{H}_c$, where $\mathcal{H}_c$ is the subspace of continuity for H, that is to say, the orthogonal complement of the subspace spanned by all eigenvectors of

H. In one dimension this is rather delicate estimate-due to the singularity at low energy-that has been proved only recently in [10]. We give this result below, but first we state some standard notations and definitions. For any $\gamma \in \mathbf{R}$, let us denote by L^1_γ the Banach space of all complex-valued measurable functions, ϕ, defined on $\mathbf{R}$ and such that

$$\|\phi\|_{L^1_\gamma} := \int |\phi(x)|\,(1+|x|)^\gamma\,dx < \infty.$$

We say that $V_0 \in L^1_1$ is *generic* if the Jost solutions to the stationary Schrödinger equation at zero energy are linearly independent, and we say that V_0 is *exceptional* if they are linearly dependent. See [10] for details. We denote by P_c the orthogonal projector in L^2 onto $\mathcal{H}_c$.

THEOREM 1.1. *(The $L^1 - L^\infty$ estimate [10]). Suppose that $V \in L^1_\gamma$ where in the* generic case $\gamma > 3/2$ *and in the* exceptional case $\gamma > 5/2$. *Then for some constant C,*

$$\left\| e^{-itH} P_c \right\|_{\mathcal{B}(L^1, L^\infty)} \leq \frac{C}{\sqrt{|t|}}. \tag{1.5}$$

COROLLARY 1.2. *(The $L^p - L^{\acute{p}}$ estimate [10]). Suppose that the conditions of Theorem 1.1 are satisfied. Then for $1 \leq p \leq 2$ and $\frac{1}{p} + \frac{1}{\acute{p}} = 1$,*

$$\left\| e^{-itH} P_c \right\|_{\mathcal{B}\left(L^p, L^{\acute{p}}\right)} \leq \frac{C}{|t|^{\left(\frac{1}{p}-\frac{1}{2}\right)}}. \tag{1.6}$$

The $L^1 - L^\infty$ estimates have many applications. See for example [1]. In fact, in page 27 of [1] the problem of proving estimate (1.5) was posed as an interesting open problem of independent importance, but actually [10] already existed in preprint form when [1] was published. Corollary 1.2 follows from Theorem 1.1 using the fact that e^{-itH} is unitary in L^2 and by interpolation. The proof of (1.5) given in [10] is based in a careful analysis of the low energy behavior of the Jost solutions that uses the fact that the Jost solutions are obtained as solutions of integral equations of Volterra type and techniques of ordinary differential equations. The $L^1 - L^\infty$ estimate (1.5) is the key to the solution of the low-energy direct scatering problem and of the inverse scattering problem in [10] and [13]. We state the results below after we introduce some more standard notations. As usual, we say that $F(x,u)$ is a C^k function of u in the real sense if for each $x \in \mathbf{R}$, $\Re F$ and $\Im F$ are C^k functions with respect to the real and imaginary parts of u.We assume that F is C^2 in the real sense and that $\left(\frac{\partial}{\partial x} F\right)(x,u)$ is C^1 in the real sense. If $F = F_1 + iF_2$ with F_1, F_2 real-valued, and $u = r + is, r, s \in \mathbf{R}$ we denote,

$$F^{(2)}(x,u) := \sum_{j=1}^{2}\left[\left|\frac{\partial^2}{\partial r^2}F_j(x,u)\right| + \left|\frac{\partial^2}{\partial r\partial s}F_j(x,u)\right| + \left|\frac{\partial^2}{\partial s^2}F_j(x,u)\right|\right],$$

$$\left(\frac{\partial}{\partial x}F\right)^{(1)}(x,u) := \sum_{j=1}^{2}\left[\left|\frac{\partial}{\partial r}\left(\frac{\partial}{\partial x}F_j\right)(x,u)\right| + \left|\frac{\partial}{\partial s}\left(\frac{\partial}{\partial x}F_j\right)(x,u)\right|\right]$$

Let us define,

$$M := \left\{u \in C(\mathbf{R}, W_{1,p+1}) : \sup_{t\in\mathbf{R}}(1+|t|)^d \|u\|_{W_{1,p+1}} < \infty\right\},$$

with norm

$$\|u\|_M := \sup_{t\in\mathbf{R}}(1+|t|)^d \|u\|_{W_{1,p+1}},$$

where $p \geq 1$, and $d := \frac{1}{2}\frac{p-1}{p+1}$. For functions $u(t,x)$ defined in $\mathbf{R}^2$ we denote $u(t)$, for $u(t,\cdot)$.

THEOREM 1.3. *(The low-energy scattering operator [13]) Suppose that $V_0 \in L^1_\gamma$, where in the* generic case *$\gamma > 3/2$ and in the* exceptional case *$\gamma > 5/2$, that H has no eigenvalues, and that*

$$N(V_0) := \sup_{x\in\mathbf{R}} \int_x^{x+1} |V_0(y)|^2\, dy < \infty.$$

Furthermore, assume that F is C^2 in the real sense, that $F(x,0) = 0$, and that for each fixed $x \in \mathbf{R}$ all the first order derivatives, in the real sense, of F vanish at zero. Moreover, suppose that $\frac{\partial}{\partial x}F$ is $C^{(1)}$ in the real sense. We assume that the following estimates hold:

$$F^{(2)}(x,u) = O\left(|u|^{p-2}\right),\ \left(\frac{\partial}{\partial x}F\right)^{(1)}(x,u) = O\left(|u|^{p-1}\right),\ u \to 0,\ \textit{uniformly for } x \in \mathbf{R},$$

for some $\rho < p < \infty$, and where ρ is the positive root of $\frac{1}{2}\frac{\rho-1}{\rho+1} = \frac{1}{\rho}$. Then, there is a $\delta > 0$ such that for all $\phi_- \in W_{2,2} \cap W_{1,1+\frac{1}{p}}$ with $\|\phi_-\|_{W_{2,2}} + \|\phi_-\|_{W_{1,1+\frac{1}{p}}} \leq \delta$, there is a unique solution, u, to (1.1) such that $u \in C(\mathbf{R}, W_{1,2}) \cap M$ and,

$$\lim_{t\to-\infty} \|u(t) - e^{-itH}\phi_-\|_{W_{1,2}} = 0.$$

Moreover, there is a unique $\phi_+ \in W_{1,2}$ such that

$$\lim_{t\to\infty} \|u(t) - e^{-itH}\phi_+\|_{W_{1,2}} = 0.$$

Furthermore, $e^{-itH}\phi_\pm \in M$ and

$$\left\|u - e^{-itH}\phi_\pm\right\|_M \leq C\left\|e^{-itH}\phi_\pm\right\|_M^p,$$

$$\|\phi_+ - \phi_-\|_{W_{1,2}} \leq C \left[\|\phi_-\|_{W_{2,2}} + \|\phi_-\|_{W_{1,1+\frac{1}{p}}} \right]^p .$$

The scattering operator, $S_{V_0} : \phi_- \hookrightarrow \phi_+$ *is injective on* $W_{1,1+\frac{1}{p}} \cap W_{2,2}$.

Note that, $\rho \approx 3.56$.

The wave operators for the linear scattering problem corresponding to equation (1.1) with $F \equiv 0$ and equation (1.2) are given by:

$$W_\pm := s - \lim_{t \to \pm\infty} e^{itH}\, e^{-itH_0}.$$

It is proven in [6] that the limits above exit in the strong topology in L^2 and that Range$W_\pm = \mathcal{H}_c$. The corresponding scattering operator is given by,

$$S_L := W_+^* \, W_- .$$

The scattering operator below relates asymptotic states that are solutions to the linear Schrödinger equation and it is the appropriate one for the reconstruction of V_0.

$$S := W_+^* \, S_{V_0} \, W_- . \tag{1.7}$$

In the following theorem we reconstruct S_L from S.

THEOREM 1.4. *([13]) Suppose that the assumptions of Theorem 1.3 are satisfied. Then for every* $\phi_- \in W_{2,2} \cap W_{1,1+\frac{1}{p}}$

$$\frac{d}{d\epsilon} S(\epsilon\phi)\bigg|_{\epsilon=0} = S_L \phi, \tag{1.8}$$

where the derivative in the left-hand side of (1.8) exists in the strong convergence in $W_{1,2}$.

COROLLARY 1.5. *([13]) Under the conditions of Theorem 1.3 the scattering operator,* S*, determines uniquely the potential* V_0.

■

In the case where $F(x,u) = \sum_{j=1}^{\infty} V_j(x)|u|^{2(j_0+j)}u$ we can also reconstruct the $V_j, j = 1, 2, \cdots$.

LEMMA 1.6. *([13]) Suppose that the conditions of Theorem 1.3 are satisfied, and moreover, that* $F(x,u) = \sum_{j=1}^{\infty} V_j(x)|u|^{2(j_0+j)}u$, *where* j_0 *is an integer such that,* $j_0 \geq (p-3)/2$, *for* $|u| \leq \eta$, *for some* $\eta > 0$, *and where* $V_j \in W_{1,\infty}$ *with* $\|V_j\|_{W_{1,\infty}} \leq C^j, j = 1, 2, \cdots$, *for some constant* C. *Then, for any* $\phi \in W_{2,2} \cap W_{1,1+\frac{1}{p}}$ *there is an* $\epsilon_0 > 0$ *such that for all* $0 < \epsilon < \epsilon_0$*:*

$$i\left((S_V - I)(\epsilon\phi),\, \phi\right)_{L^2} = \sum_{j=1}^{\infty} \epsilon^{2(j_0+j)+1}\left[\int\int dt\, dx\, V_j(x)\left|e^{-itH}\phi\right|^{2(j_0+j+1)} + Q_j\right], \qquad (1.9)$$

where $Q_1 = 0$ *and* $Q_j, j > 1$, *depends only on* ϕ *and on* V_k *with* $k < j$. *Moreover, for any* $\acute{x} \in \mathbf{R}$, *and any* $\lambda > 0$, *we denote,* $\phi_\lambda(x) := \phi(\lambda(x - \acute{x}))$. *Then, if* $\phi \neq 0$*:*

$$V_j(\acute{x}) = \frac{\lim_{\lambda\to\infty} \lambda^3 \int\int\, dt\, dx\, V_j(x)\left|e^{-itH}\phi_\lambda\right|^{2(j_0+j+1)}}{\int\int\, dt\, dx\, |e^{-itH_0}\phi|^{2(j_0+j+1)}}. \qquad (1.10)$$

COROLLARY 1.7. *([13]) Under the conditions of Lemma 1.6 the scattering operator, S, determines uniquely the potentials* $V_j, j = 0, 1, \cdots$.

The method to reconstruct the potentials $V_j, j = 0, 1, \cdots$, is as follows. First S_L is obtained from S using (1.8). By any standard method for inverse scatering for the linear Schrödinger equation on the line (recall that H has no eigenvalues) we reconstruct V_0. We then reconstruct S_{V_0} from S using (1.7). Finally (1.9) and (1.10) give us, recursively, $V_j, j = 1, 2, \cdots$.

For the proof of these results, the extension to the multidimensional case and to the nonlinear Klein-Gordon equation see [9], [10], [11], [12], [13], [14] and [15]. In these papers also a discussion of the literature is given.

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