FUNCTIONALS OF THE CALCULUS OF VARIATIONS WITH NON STANDARD GROWTH CONDITIONS

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The *Calculus of Variations* is that field of mathematics devoted to the calculation and the analysis of *maxima* and *minima* values of certain applications, said *functionals.*

In every field of the scientific research, there are problems connected to the search of minimizers and maximizers of entities that in some cases can be expressed as functions of real variables, while in other cases are functions defined in function spaces.

The origin of the Calculus of Variations dates back to Zenodoro (two hundred years before Christ) who studied the isoperimetric inequalities; others dates back the origin of the Calculus of Variations to 17^{th} century with the studies of Fermat (1662, determination of the trajectory of a light ray), Newton (1686, determination of the shape of a spherical simmetric body plunged in a liquid, in such a way that it meet less resistance as possible), but the problem that gave great development to the theory was the *brachistochrone* problem, that is the problem of finding the curve descibed by a free falling body, which is also the curve that takes less time to get down (indeed in greek *brakhistos* means just *the shorter*, while *chronos* means *time*). More precisely, the latter consists of determine the course that a material point, subjected only to gravity, has to follow to connect as shortly as possible two assigned points. The problem was suggested and solved by Giovanni Bernoulli in 1696.

The functionals of the Calculus of Variations have a particular expression: they are in fact of integral type. Given an open set $\Omega \subset \mathbb{R}^N$, to each function $u : \Omega \to R^N$, where $n, N \geq 1$, correspond a real number $\mathcal{I}(u)$ defined by

$$\mathcal{I}(u) = \int_\Omega f(x, u(x), Du(x))dx,$$

where $Du(x) = \left(\frac{\partial u^\alpha}{\partial x^i}\right)$, $1 \leq \alpha \leq N$, $1 \leq i \leq n$ is the Jacobian matrix of u and $f : \Omega \times \mathbb{R}^N \times \mathbb{R}^{nN} \to \mathbb{R}$. Of course the function u must be chosen in such a way that the integral $\mathcal{I}(u)$ makes sense.

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The problem is of determinate the lower bound (or the upper one) of $\mathcal{I}$, when u runs in a class of function $\mathfrak{U}$ that we call *admissible*, and possibly the minimum point (maximum point), that is a function $\tilde{u} \in \mathfrak{U}$ such that

$$\mathcal{I}(\tilde{u}) \leq \mathcal{I}(u) \qquad (\mathcal{I}(\tilde{u}) \geq \mathcal{I}(u)) \qquad \forall u \in \mathfrak{U}.$$

There are many applied and phisical problems, for whose passing to a mathematical formalism is a problem of the Calculus of Variations. For instance the problem of the stable equilibrium of a rigid body, the equilibrium position of an elastic membrane, the problem of finding minimal surfaces, the Hamilton principle of minimal action.

Given a minimum problem for a functional of the Calculus of Variation, we can proceed in two different directions.
The first method is based on a generalization of the Fermat theorem:
Let $f : [a, b] \to \mathbb{R}$ a differentiable function. If $x_0 \in [a, b]$ is a relative minimum point for f then $f'(x_0) = 0$.

Let us consider as admissible class of function

$$\mathfrak{U} = \left\{ u \in C^1(\Omega, \mathbb{R}^N) : \; u = u_0 \text{ on } \partial\Omega \right\}.$$

This is what we call a *Dirichlet class* for the functional $\mathcal{I}$. Let $\tilde{u} \in \mathfrak{U}$ be a minimizer of $\mathcal{I}$ in $\mathfrak{U}$. Then for every $\varphi \in C^1_0(\Omega, \mathbb{R}^N)$ it must happens that

$$\mathcal{I}(\tilde{u}) \leq \mathcal{I}(\tilde{u} + t\varphi) \qquad \forall t \in \mathbb{R}.$$

Then in particular, the one variable function $g(t) = \mathcal{I}(\tilde{u} + t\varphi)$ must have a minimum for $t = 0$ and then

$$\delta\mathcal{I}(\varphi) = \left[\frac{d}{dt}\mathcal{I}(\tilde{u} + t\varphi)\right]_{t=0} = 0 \qquad \forall \varphi \in C^1_0(\Omega, \mathbb{R}^N).$$

$\delta\mathcal{I}(\varphi)$ is called the *first variation of the functional* $\mathcal{I}$. Since $\mathcal{I}$ has integral form, if it is possible to apply the theorem of derivation under the integral sign, we get a differential equation that in his weak form has the following expression

$$\int_\Omega \left\{ \sum_{i=1}^{n} \sum_{\alpha=1}^{N} f_{z_i^\alpha}(x, \tilde{u}(x), \nabla\tilde{u}(x))\varphi^\alpha_{x_i} + \sum_{\alpha=1}^{N} f_{s^\alpha}(x, \tilde{u}(x), \nabla\tilde{u}(x))\varphi^\alpha \right\} dx = 0$$

for every $\varphi \in C^1_0(\Omega, \mathbb{R}^N)$.
This is called the *Euler-Lagrange equation* for the functional $\mathcal{I}$.
This proceeding has been very useful in the past, since in some cases it gave the explicit expression of the minimizer. For instance, in the simpler case of $n = N = 1$, the Euler-Lagrange equation is an ordinary differential equation and then there are various methods to try to solve it. If $n > 1$ and $N = 1$, we have a partial differential equation, while if $N > 1$ the first variation consists of a system of N partial differential equations, which can be very difficult to solve.
At the beginning of this century, Riemann started using a new method, independent from the theory of partial differential equations, called *Direct Method*, developed sussessfully by Hilbert, Lebesgue, Tonelli and Volterra.

It is well known that the result which signed a new age of the mathematical thought, is the *Gauss algebra theorem.* Gauss proved just the existence of solutions to an algebraic equation, without giving an explicit expression of the solution.
It was the first time that the existence of solutions for a mathematical problem was estabilished and only in a second while the explicit form of solutions and their properties were determined. This new thought obtained very meaningful results in the Calculus of Variations, through the Direct Method.

The Direct Method is based on an extension of the following result due to Weierstrass: *A sequentially lower semicontinuous function in a compact set has a minimum point.* We say that a functional $\mathcal{I} : \mathfrak{U} \to \mathbb{R}$ is sequentially lower semicontinuous with respect to a topology τ, defined on $\mathfrak{U}$, if for every sequence $\{u_h\}_h \subset \mathfrak{U}$ converging to a function u in the topology τ we have

$$\mathcal{I}(u) \leq \liminf_{h\to\infty} \mathcal{I}(u_h).$$

Let us give now an idea of the proceeding concerning with the Direct Methods.

Let us consider a minimizing problem for a functional $\mathcal{I}$ defined in an admissible class of functions $\mathfrak{U}$, equipped with a topology τ. The Direct Method consists substantially of the following steps

(1) prove that the infimum of $\mathcal{I}$ in $\mathfrak{U}$ is finite;
(2) consider minimizing sequences of $\mathcal{I}$ in $\mathfrak{U}$, i.e. $\{u_h\}_h \subset \mathfrak{U}$ such that
$$\lim_{h\to\infty} \mathcal{I}(u_h) = \inf_{u\in\mathfrak{U}} \mathcal{I}(u);$$
(3) determine the compactness of the minimizing sequences, that is the existence of a subsequence of $\{u_h\}_h$, which converges to a function $\tilde{u} \in \mathfrak{U}$ in the topology τ;
(4) ensure the lower semicontinuity of the functional $\mathcal{I}$, that is that for each $\{v_h\}_h \subset \mathfrak{U}$ converging to $v \in \mathfrak{U}$ we have
$$\mathcal{I}(v) \leq \liminf_{h\to\infty} \mathcal{I}(v_h).$$

Once that these conditions are verified, we apply the lower semicontinuity to a minimizing sequence of $\mathcal{I}$ on $\mathfrak{U}$ and we have

$$\mathcal{I}(\tilde{u}) \leq \liminf_{h\to\infty} \mathcal{I}(u_h) = \inf_{u\in\mathfrak{U}} \mathcal{I}(u).$$

Then $\tilde{u}$ is a minimizer for $\mathcal{I}$ in $\mathfrak{U}$.

The most difficult problem is to determine, in the class of admissible functions $\mathfrak{U}$, a topology for which the minimizing sequences are compact (that is a topology with as few open sets as possible) and such that the functional $\mathcal{I}$ turns out to be lower semicontinuous (that is a topology as rich of open sets as possible).

We introduce now the concept of weak derivatives and the Sobolev spaces.

Let $\Omega \subset \mathbb{R}^n$ be an open set and $u \in L^1_{loc}(\Omega)$. If $1 \leq i \leq n$, we say that the function $v_i \in L^1_{loc}(\Omega)$ is the weak derivative of u with respect to x_i if

$$\int_\Omega v_i \varphi dx = -\int_\Omega u \frac{\partial\varphi}{\partial x_i} dx \qquad \forall \varphi \in C_0^\infty(\Omega).$$

If a function u is differentiable in Ω, the classical derivative coincides with the weak one. We call *weak gradient* of u the vector $\nabla u = \left(\frac{\partial u}{\partial x_i}\right)$ in the weak sense.

We introduce the Sobolev space $W^{1,p}(\Omega)$, as the space of the functions $u \in L^p(\Omega)$, which have weak derivatives that are L^p-functions. It is a Banach space endowed with the norm

$$\|u\|_{W^{1,p}(\Omega)} = \left(\|u\|^p_{L^p(\Omega)} + \|\nabla u\|^p_{L^p(\Omega)}\right)^{\frac{1}{p}}\ [1];$$

nevertheless, the topology inducted by the $W^{1,p}-$norm is not good for our goals. We need to introduce in $W^{1,p}$ a so called *weak topology*, that is defined as the weaker topology (that is the one with less open sets) that makes all the linear functional on $W^{1,p}$ continuous.
Given a sequence of functions $\{u_h\}_h \subset W^{1,p}$, we say that $\{u_h\}_h$ weakly converges to a function u in $W^{1,p}$ (we will write $u_h \rightharpoonup u$ to mean such a convergence) if and only if

$$\lim_{h\to\infty}\int_\Omega u_h v dx = \int_\Omega uv dx, \qquad \lim_{h\to\infty}\int_\Omega \frac{\partial u_h}{\partial x_i} dx = \int_\Omega \frac{\partial u}{\partial x_i} v dx$$

for every $i = 1, \dots, n$ and for every $v \in L^q(\Omega)$, where q is such that $\frac{1}{p} + \frac{1}{q} = 1$.

We will use an infinite dimensional spaces extension of Bolzano-Weierstrass theorem: *Let $p > 1$ and $\|u_h\|_{W^{1,p}} \leq M \quad \forall h$. Then there exists a subsequence that weakly converge in $W^{1,p}$.*
Clearly all the definitions given above for Sobolev spaces, can be extended to the case of vector valued functions, i.e. when $N > 1$.

Let us consider the integral functional $\mathcal{I}$ and assume, for sake of simplicity that the integrand function is of the type $f = f(x, z)$. Furthermore we assume that f is a *Carathéodory function*, that is

(1) $f(\cdot, z)$ is measurable in Ω, for every $z \in \mathbb{R}^{nN}$;
(2) $f(x, \cdot)$ is continuous in $\mathbb{R}^{nN}$, for a.e. $x \in \Omega$.

These conditions ensure that for every measurable function w in Ω the integral

$$\int_\Omega f(x, w(x))dx$$

is well defined.

Now let us consider the following functional

$$\mathcal{I}(u) = \int_\Omega f(x, Du)dx.$$

For such functionals it is opportune to introduce the so called *natural growth conditions*

$$|z|^p - c_1 \leq f(x, z) \leq c_2(1 + |z|^p), \tag{1}$$

where $c_1, c_2 > 0$ and $p > 1$.
Condition (1) allows to say that $\mathcal{I}$ is well defined. Indeed since f is a Carathéodory function, the upper bound in (1) implies that $f(x, Du(x))$ is integrable in Ω and $\mathcal{I}$ is finite in $W^{1,p}$.

[1] for simplicity of notation we write $\|\nabla u\|_{L^p}$ instead of $\|\ |\nabla u|\ \|_{L^p}$.

Starting from some results by Tonelli in the 20's and from furhter researchs of the last 50 years, when $N = 1$, the functional $\mathcal{I}$ is lower semicontinuous in the weak topology of $W^{1,1}$ (and then in $W^{1,p}$ $p > 1$ also) if and only if f is convex in the variables z.
In the case $N > 1$ we have to replace the convexity assumption with the weaker condition of *quasiconvexity* due to Morrey [35].
Let $z_0 \in \mathbb{R}^{nN}$, $f : \Omega \times \mathbb{R}^{nN} \to \mathbb{R}$ is quasiconvex if for a.e. $x \in \Omega$ and for every $\varphi \in C_0^\infty(\Omega, \mathbb{R}^N)$ we have

$$f(x_0, z_0) \leq \frac{1}{|\Omega|} \int_\Omega f(x_0, z_0 + D\varphi) dx$$

Condition (1) implies that $\mathcal{I}$ is coercive in $W^{1,p}$, i.e.

$$\lim_{\|u\| \to \infty} \mathcal{I}(u) = +\infty.$$

In fact,

$$\int_\Omega |Du|^p dx \leq \mathcal{I}(u) + c_1,$$

so if $\{u_h\}_h$ is a minimizing sequence, we have that $\|Du_h\|_{L^p} \leq c$. Then if we have that also $\|u_h\|_{W^{1,p}}$ is bounded, by results about weak convergence, it follows that $\{u_h\}_h$ has a subsequence weakly converging to a function u.
For instance, let $N = 1$ and $\mathcal{I}$ be defined in the Dirichlet class $u_0 + W_0^{1,p}(\Omega)$ with f convex in z and satisfying (1). Then there exists a minimizer.

There are many functionals that can be considered in the Calculus of Variations, whose integrand functions don't satisfy growth condition (1).
For instance, let us consider the following convex integrand functions:

$$f_1(z) = |z|^p \log^\alpha(1 + |z|) \qquad p \geq 1,\ \alpha > 0$$

$$f_2(z) = (1 + |z|) \log(1 + |z|) - |z|$$

$$f_3(z) = |z|^{a + b \sin \log \log(|z| + e)} \qquad b > 0,\ a > 1 + b\sqrt{2}$$

$$f_4(z) = (1 + |z|^2)^{\frac{p}{2}} + \sum_{i=1}^n |z_i|^{p_i} \qquad p_i \geq p\ \forall i = 1, \dots, n$$

$$f_5(z) = |z|^{\alpha(x)}$$

$$f_6(z) = [h(|z|)]^{\alpha(x)}$$

(in the definition of f_5 and f_6, $\alpha : \Omega \to \mathbb{R}$ is a bounded measurable function. Moreover $h : \overline{\mathbb{R}^+} \to \mathbb{R}$ is a convex function)

$$f_7(z) = e^{|z|^\alpha} \qquad \alpha > 0$$

As it can be easily checked, the functions $f_1 - f_6$ satisfy growth conditions that are known as $(p,q)-$*growth conditions*, that is for a.e. $x \in \Omega$

$$|z|^p - c_1 \leq f(x, z) \leq c_2(1 + |z|^q) \tag{2}$$

where $p < q$ and $c_1, c_2 > 0$.

Functionals with $(p, q)-$growth of type (2) came out naturally in problems concerning non linear elasticity. A typical integrand function for these problems is, if $n = N$,

$$f(x, z) = a(x)|z|^p + |\det(z)|, \qquad z \in \mathbb{R}^{N^2}, \ p < N$$

that satisfy (2) with $q = N$.
Clearly there are also integrand functions of different type, that cannot be described using $(p, q)-$growth conditions. In particular there are functions for which there exists two positive convex functions g_1, g_2 (eventually $g_1 = g_2$) such that for a.e. $x \in \Omega$

$$g_1(|z|) - c_1 \leq f(x, z) \leq c_2(1 + g_2(|z|)), \tag{3}$$

where $c_1, c_2 > 0$. We say that f has general growth conditions if (3) is satisfied.

In the last ten years also the study of functionals under non natural growth conditions has been developed, in particular to prove regularity of minimizers and lower semicontinuity in the vectorial case.

The aim of this short note is an overview of our recent contribution to problem of the Calculus of Variations with non standard growth conditions.
We focus mainly on the regularity properties of minima of integral functionals in the scalar $(N = 1)$ as well as in the vectorial $(N > 1)$ case. We also try to sketch the main ideas and techniques underlying the most relevant proofs. Moreover we present the basic of the theory of Orlicz and Orlicz-Sobolev spaces, which turned out to be one of the most effective tools of analysis of some classes of problems.
Of course, due to the complexity of such problems and the huge amount of available results, we gave up any ambition to be exhaustive. Nevertheless, we think that the list of references is rather complete:
items [1] to [19] refers mainly to Orlicz spaces and their applications to Partial Differential Equations;
items [20] to [39] to the case of "natural" growth conditions;
lastly, items [40] to [78] to the case of "non standard' ones.

Studies about the lower semicontinuity will not be treated at all; however items [79] to [92] in the cited references cover those we feel to be the most important works of these twenty years on the subject.

REGULARITY PROBLEM: The scalar case.

Let us consider a scalar integral functional $(N = 1)$

$$\mathcal{I}(u) = \int_\Omega f(x, Du(x))dx, \tag{4}$$

where $f : \Omega \times \mathbb{R}^n \to \mathbb{R}$ is convex in z and satisfy natural growth conditions (1). As we have seen above, using the proceeding of the Direct Methods, we have that there exists a minimizer for $\mathcal{I}$ in the Sobolev space $W^{1,p}(\Omega)$, i.e. a function u that has weak derivatives in L^p_{loc}. From here one could ask if this minimizer has more regularity.
The regularity problem has been open for a long time and has come to a solution, almost in the scalar case, by a celebrated result due to De Giorgi (1957 [22]), in which it is

proved the Hölder-continuity of solutions of an elliptic differential equation in divergence form. In many cases these type of equations can be seen as the Euler-Lagrange equation of a functional of the Calculus of Variations.
The De Giorgi theorem has been generalized in many directions and it applies to wider classes of functionals that allow the first variation. A fundamental contribution to this aim has been given by Giaquinta-Giusti (1980-1984 [25], [26], [27]). They proved that the minimizers of functionals of type (4), under natural growth conditions (1), satisfy some inequalities of the kind of the ones introduced by De Giorgi, without passing to the first variation of the functional, but using directly the minimizing properties.
More precisely, let Q_R a cube strictly contained in Ω and let us define the families of sets

$$A_{k,R} = \{x \in Q_R : \; u(x) < k\},$$
$$B_{k,R} = \{x \in Q_R : \; u(x) > k\}.$$

Then a minimum $u \in W^{1,p}_{loc}(\Omega)$ of functional (4), under natural growth conditions (1) satisfies

$$\int_{A_{k,\rho}} |Du|^p dx \le \frac{c_1}{(R-\rho)^p} \int_{A_{k,R}} (u-k)^p dx + c_2 |A_{k,R}|^\alpha \tag{5}$$

and

$$\int_{B_{k,\rho}} |Du|^p dx \le \frac{c_3}{(R-\rho)^p} \int_{B_{k,R}} (k-u)^p dx + c_4 |B_{k,R}|^\alpha \tag{6}$$

where $0 < \rho < R$, $c_1, \dots, c_4, \alpha$ are positive constants and $|E|$ denotes the Lebesgue measure of a subset $E \subset \mathbb{R}^n$.
The main properties of inequalities (5) and (6) is that they contain the whole information relative to the continuity of the minimizer.

A different proof of the Hölder continuity of solutions to elliptic equations has been obtained by Moser (1961, [36]), through an inequality, due to Harnack, that extends a classical result for armonic functions. More precisely it proves that each positive solution u satisfies

$$\inf_{Q_R} u > c \sup_{Q_R} u.$$

Di Benedetto and Trudinger (1984, [23]) has proved that the positive minimizers of a functional $\mathcal{I}$ under growth conditions (1) and without differentiability hypothesis, satisfy the Harnack inequality.
In the regularity results we have recalled above and in each of their many extensions, the natural growth conditions (1) are essential.

The study of regularity of minimizers under non standard growth conditions, has begun about ten years ago. A considerable contribution is due to P.Marcellini that in a group of papers starting from 1989, [65]-[69], studied the regularity of minimizers under $(p,q)-$growth conditions and under general growth conditions and in the case of functionals with exponential growth. More precisely, he gave some interesting Lipschitz regularity results for the gradient of the minimizers, i.e. $Du \in W^{1,\infty}_{loc}(\Omega)$.
Notice that using a standard method, we obtain higher regularity. Indeed, if $u \in W^{1,\infty}_{loc}$, then by applying De Giorgi theorem to some suitable differential equations satisfied by the derivatives of u, we find that $u \in C^{1,\alpha}_{loc}$. Then recursively we obtain that $u \in C^{k,\alpha}_{loc}$ for every k and finally $u \in C^\infty$ according to regularity of data.

The regularity of minimizers of functionals under non standard growth conditions has been intensely studied in the last years (see references [40]-[78]).

In the case of $(p,q)-$growth conditions, we have that the functional could be not well defined in the same space in which it is coercive, moreover in this case it is interesting to investigate the regularity properties of the minimizers.
Actually one finds that the exponents p and q cannot be too far from (more precisely $q < p^* = \frac{np}{n-p}$). To understand this fact the following counterexample to regularity in the scalar case is fundamental

Example (P.Marcellini [64] - M.Giaquinta [57]). Let $B = B_1(0)$ and let us consider the functional

$$\int_B \left[\sum_{i=1}^{n-1} \left| \frac{\partial u}{\partial x_i} \right|^2 + \frac{1}{2} \left| \frac{\partial u}{\partial x_n} \right|^4 \right] dx.$$

Here we assume that $n \geq 2$ and $u : B \to \mathbb{R}$.
It can be checked that the function

$$u(x) = \sqrt{\frac{n-4}{24}} \frac{x_n^2}{\sqrt{\sum_{i=1}^{n-1} x_i^2}}$$

is a minimizer of functional (2.3), almost if $n \geq 6$. Furthermore this function is clearly not continuous on the half-lines $(0, \dots, 0, x_n)$, with $x_n \neq 0$.

More generally, the problem that arises in the above counterexample can be generalized if one considers a functional of the form

$$\int_\Omega \left[|Du|^p + \sum_{i=1}^{n} |D_i u|^{p_i} \right] dx. \tag{7}$$

In this case in fact, it has been proved (see [74]) that if $1 \leq p < p_i$, for $i = 1, \dots, n$, to get some regularity result for the minimizers of such a functional, it is needed that the non homogeneous exponents p_i satisfy an opportune upper bound. More precisely, if we introduce the harmonic mean

$$\frac{1}{\bar{p}} = \frac{1}{n} \sum_{i=1}^{n} \frac{1}{p_i}$$

and its Sobolev conjugate:

$$\bar{p}^* = \begin{cases} \dfrac{n\bar{p}}{n-\bar{p}} & \text{if } \bar{p} < n \\ & \\ \text{any } \ q > p & \text{otherwise.} \end{cases}$$

then one finds that a minimizer of functional (7) is a bounded function (that is $u \in L^\infty_{loc}(\Omega)$) if and only if

$$\max_{1 \leq i \leq n} p_i \leq \bar{p}^*.$$

It is not known if this bound is also sufficient to get higher regularity, but it remains a possible comparison term, when one tries to find some regularity properties of the minimizers.

When we deal with a functional wich has general growth conditions of type (3), one can ask what are the conditions to be satisfied by g_1 and g_2.
These conditions pass through the concepts of $N-$function and the Orlicz-Sobolev spaces.

A convex function $g : [0, +\infty) \to \mathbb{R}$ is said to be a Young function if it satisfies

$$g(0) = 0 \qquad \lim_{t\to+\infty} g(t) = +\infty.$$

If φ is the right derivative of g, then φ is non decreasing and left continuous. Moreover

$$g(t) = \int_0^t \varphi(s)ds \qquad \forall t \in [0, +\infty).$$

By the convexity of g we have

$$g(t) \leq t\varphi(t) \qquad \forall t \in [0, +\infty). \tag{8}$$

In the following we will denote as positive Young functions those that vanish only for $t = 0$.
We will say that a Young function g is a $N-$function if it is a positive Young function and satisfies

$$\lim_{t\to 0^+} \frac{g(t)}{t} = 0, \qquad \lim_{t\to+\infty} \frac{g(t)}{t} = +\infty.$$

If g is a $N-$function, then (8) holds with φ a non decreasing function, left continuous, $\varphi(0) = 0$, $\varphi(t) > 0$ for every $t > 0$ and $\varphi(t) \to +\infty$ as $t \to +\infty$.

We will say that two positive Young functions g, h are equivalent near to $+\infty$ if there exists $t_0, k_1, k_2 > 0$ such that

$$h(k_1 t) \leq g(t) \leq h(k_2 t) \qquad \forall t \geq t_0.$$

Let $m > 1$. A positive Young function g is of class Δ_2^m if one of three equivalent conditions is satisfied:

$$g(\lambda t) \leq \lambda^m g(t) \qquad \forall \lambda > 1, \quad t \geq 0, \tag{i}$$

$$\frac{g(t)}{t^m} \quad \text{is decreasing,} \tag{ii}$$

$$\varphi(t)t \leq mg(t) \qquad \forall t \geq 0. \tag{iii}$$

It is easy to check that $g(t) = t^m$, with $m > 1$ is a $N-$function of class Δ_2^m.
The function $g(t) = t^m \log^\alpha(1 + t)$, where $m \geq 1$ and $\alpha > 0$ is equivalent at $+\infty$ to a $N-$function of class $\Delta_2^{m+\varepsilon}$ $(\varepsilon > 0)$.
The function $g(t) = t^{a+b\sin\log\log(e+t)}$ is equivalent near to $+\infty$, to a $N-$function of class Δ_2^{a+b}, if $b > 0$ and $a > 1 + \sqrt{2}b$.
If $g \in \Delta_2^m$, it is easy to see that $g(t) \leq ct^m$ for $t \geq 1$.
Observe that the class Δ_2^1 contains just the linear functions.

Let $r \geq 1$. A positive Young function g is said of class ∇_2^r if one of the following three conditions is satisfied

$$g(\lambda t) \geq \lambda^r g(t) \qquad \forall \lambda > 1, \quad t \geq 0, \tag{i}$$

$$\frac{g(t)}{t^r} \quad \text{is increasing,} \tag{ii}$$

$$\varphi(t)t \geq rg(t) \qquad \forall t \geq 0. \tag{iii}$$

Every positive Young function is of class ∇_2^1. We have that:
$g(t) = t^r$ $(r > 1)$ is a function of class ∇_2^r.
$g(t) = t^r \log^\alpha(1+t)$, where $r > 1$ and $\alpha > 0$, is equivalent at $+\infty$ to a $N-$function of class ∇_2^r.
$g(t) = t^{a+b\sin\log\log(e+t)}$ is equivalent near to $+\infty$, to a $N-$function of class ∇_2^{a-b}, if $b > 0$ and $a > 1 + \sqrt{2}b$.
$g(t) = \exp(t^\alpha)$ is of class ∇_2^r for all $r \geq 1$, but $g \notin \Delta_2^m$.

Let g be a $N-$function. We introduce the *Fenchel's conjugate* of g, that is a $N-$function denoted by $\tilde{g}$, defined by

$$\tilde{g}(s) = \max_{s>0}\{st - g(s)\}.$$

The following *Young inequality* holds

$$st \leq g(s) + \tilde{g}(t) \qquad \forall s, t \geq 0.$$

In the following, when no confusion arise we will not specify the exponent m, writing simply $g \in \Delta_2$ to say that g is a function of the class Δ_2^m.

We have that

$$g \in \Delta_2 \Longleftrightarrow \tilde{g} \in \nabla_2$$

and then of course

$$g \in \Delta_2 \cap \nabla_2 \Longleftrightarrow g, \tilde{g} \in \Delta_2.$$

Given an open set with finite measure $\Omega \subset \mathbb{R}^n$, we introduce the Orlicz class of functions $K_g(\Omega, \mathbb{R}^N)$ associated to a $N-$function g, as the set of all the measurable functions in Ω such that

$$\int_\Omega g(|u(x)|)dx < +\infty.$$

As usual we identify functions wich differ on a set of zero measure. Since g is a convex function, it is easy to prove that K_g is a convex set, but in general $K_g(\Omega, \mathbb{R}^N)$ is not a linear space.
Indeed the following result holds

K_g is a linear space if and only if $g \in \Delta_2$

We define the Orlicz functions space $L_g(\Omega, \mathbb{R}^N)$ as the smallest linear space containing K_g. We have $L_g(\Omega, \mathbb{R}^N) = \{\alpha u : \ \alpha \in \mathbb{R}, \ u \in K_g(\Omega, \mathbb{R}^N)\}$. Moreover $K_g \equiv L_g$ if and only if $g \in \Delta_2$.
If $g(t) = t^p$ we have that $L_g(\Omega, \mathbb{R}^N) \equiv L^p(\Omega, \mathbb{R}^N)$.

The Orlicz space $L_g(\Omega, \mathbb{R}^N)$ is a Banach space when endowed with the norm

$$\|u\|_{L_g(\Omega,\mathbb{R}^N)} = \inf\left\{k > 0 : \ \int_\Omega g\left(\frac{|u(x)|}{k}\right)dx \leq 1\right\}. \tag{10}$$

Also a Hölder-type inequality holds: let $u \in L_g$ and $v \in L_{\tilde{g}}$, then $uv \in L^1$ and

$$\|uv\|_{L^1} \leq 2\|u\|_{L_g}\|v\|_{L_{\tilde{g}}}.$$

The Orlicz spaces arises as a generalization of L^p spaces and appear for the first time in a monography of Zygmud in 1935. The first almost complete text about these spaces was by Krasnosel'skii-Rutickii [14], while more recently we refer to Rao-Ren [17] (see also [1]).

The closure of the space $C_0^\infty(\Omega, \mathbb{R}^N)$ with respect to the norm (10) is denoted by $E_g(\Omega, \mathbb{R}^N)$. It represents the largest linear space contained in the Orlicz class K_g. Then we can argue that

$$E_g \subseteq K_g \subseteq L_g.$$

The equality holds if and only if $g \in \Delta_2$.

The Orlicz-Sobolev functions space $W^1L_g(\Omega, \mathbb{R}^N)$, consists of all the functions $u \in L_g(\Omega, \mathbb{R}^N)$ whose derivatives in the sense of distributions are in L_g. W^1L_g is still a Banach space, with the norm

$$\|u\|_{W^1L_g(\Omega,\mathbb{R}^N)} = \|u\|_{L_g(\Omega,\mathbb{R}^N)} + \|Du\|_{L_g(\Omega,\mathbb{R}^N)}. \tag{11}$$

As usual, $W_0^1L_g(\Omega, \mathbb{R}^N)$ denotes the completion of $C_c^\infty(\Omega, \mathbb{R}^N)$ with respect to the topology inducted by the norm (11).

Let us recall some embedding results in Orlicz-Sobolev spaces.
If g is a $N-$function such that

$$\int_0^1 \frac{g^{-1}(s)}{s^{1+\frac{1}{n}}} ds < +\infty \tag{12}$$

$$\int_1^{+\infty} \frac{g^{-1}(s)}{s^{1+\frac{1}{n}}} ds < +\infty. \tag{13}$$

We define for every $t \in [0, +\infty)$ the function

$$h(t) = \int_0^t \frac{g^{-1}(s)}{s^{1+\frac{1}{n}}} ds.$$

We will say that the inverse function of h is the *Sobolev-conjugate* of g, that is $g_* = h^{-1}$. It can be proved that:
if $g(t) = t^p$, where $1 < p < n$, then $g_*(t) = ct^{p^*}$, where $p^* = \frac{np}{n-p}$ and c is such that $\frac{1}{c} = p^{*p^*}$.

It can be easily checked that g_* is still a $N-$function, such that

$$\lim_{t \to +\infty} \frac{g(kt)}{g_*(t)} = 0 \qquad \forall k > 0$$

and then $L_g \subseteq L_{g_*}$, $1 < m < n$ and $g \in \Delta_2^m$, then $g_* \in \Delta_2^{m^*}$. Analougusly $g^* \in \nabla_2^{r^*}$ if $g \in \nabla_2^r$ with $1 \leq r < n$.
The following embedding result holds

If $\Omega \subset \mathbb{R}^n$ is an open set with Lipschitz boundary and if g is a $N-$function satisfying (12) and (13), then

$$W^1L_g \hookrightarrow L_{g_*}.$$

The proof of this theorem and of other connected results can be found in Adams [1] (see also Donaldson-Trudinger [8], Trudinger [19]; recently embedding results have been obtained by Cianchi [4],[5]).

Let g be a $N-$function; if $g(|Du|) \in L^1(\Omega, \mathbb{R}^N)$ we have the following Poincaré-type inequality (see Bhattacharya-Leonetti [42])

$$\int_\Omega g\left(\frac{|u(x)-u_\Omega|}{d}\right)dx \le \left(\frac{\omega_n d^n}{|\Omega|}\right)^{1-\frac{1}{n}} \int_\Omega g(|Du|)dx \tag{14}$$

where $d = diam(\Omega)$, ω_n is the volume of the unit ball in $\mathbb{R}^n$ and u_Ω is the average of u in Ω i.e.

$$u_\Omega = \frac{1}{|\Omega|}\int_\Omega u(x)dx.$$

Let us introduce the weak* topology in $L_g(\Omega, \mathbb{R}^N)$. Since the Orlicz space L_g is the dual space of $E_{\tilde{g}}(\Omega, \mathbb{R}^N)$, then we say that a sequence $\{u_h\}_h \subset L_g(\Omega, \mathbb{R}^N)$ weakly converge to a function u in the weak* topology of L_g if

$$\lim_{h\to\infty}\int_\Omega u_h v dx = \int_\Omega uvdx \qquad \forall v \in E_{\tilde{g}}(\Omega, \mathbb{R}^N).$$

Therefore, since we have that $E_{\tilde{g}} \equiv L_{\tilde{g}}$ if and only if $\tilde{g} \in \Delta_2$, we have that L_g is a reflexive space if and only if, assuming that $g \in \Delta_2$, it happens also that $\tilde{g} \in \Delta_2$. The latter is equivalent to $g \in \Delta_2 \cap \nabla_2$ (see Rao-Ren [17]).
As $g \in \Delta_2 \cap \nabla_2$, the weak* topology and the weak topology in L_g agree, that is $(L_g)^* \equiv L_{\tilde{g}}$. In this case we have that $u_h \rightharpoonup u$ in the weak topology of L_g if

$$\lim_{h\to\infty}\int_\Omega u_h v dx = \int_\Omega uvdx \qquad \forall v \in L_{\tilde{g}}.$$

We can characterize the weak* convergence in W^1L_g: $u_h \rightharpoonup u$ in $\text{w}^* - W^1L_g(\Omega\mathbb{R}^N)$ if and only if $u_h \rightharpoonup u$ and $D_iu \rightharpoonup D_iu$ for every $i = 1, \ldots, n$ in $\text{w}^* - L_g(\Omega, \mathbb{R}^N)$.

The Orlicz-Sobolev spaces have been used in the study of elliptic partial differential equations. In particular we refer to Donaldson-Trudinger [8], Gossez [11],[12],[13], Fougères [10], Talenti [18].

Let us came back to the regularity of minimizers of functionals with non standard growth.
In the following we will consider *local minimizers* of a functional $\mathcal{I}$, that is those functions $u \in W^{1,1}_{loc}(\Omega, \mathbb{R}^N)$ such that $f(\cdot, Du) \in L^1_{loc}(\Omega)$ and

$$\int_{spt(\psi)} f(x, Du)dx \le \int_{spt(\psi)} f(x, Du + D\psi)dx,$$

for every $\psi \in W^{1,1}(\Omega, \mathbb{R}^N)$ with $spt(\psi) \subset\subset \Omega$.

Let

$$g(|z|) - c_1 \leq f(x,z) \leq c_2(g(|z|)+1),$$

where g is a $N-$function. For an integrand function of this type we consider local minimizers that are functions of the Orlicz-Sobolev space $W^1L_g(\Omega)$.
In Mascolo-Papi [69], [70] it is proved that:

(1) If $g \in \Delta_2^m$ for some $m > 1$, then the local minimizers satisfy the Euler-Lagrange equations (this fact is not trivial since we are assuming no natural growth conditions and then is not possible to apply the classical method to derive the system in variation);
(2) if $g \in \Delta_2^m$, the local minimizers satisfy an inequality of *De Giorgi-type* and are locally bounded;
(3) if $g \in \Delta_2^m \cap \nabla_2^r$ then the positive local minimizers satisfy a *Harnack-type* inequality.

The Harnack inequality in the case of $p-q$ growth, with $p^* > q$ has been proved by Moscariello [72] and also results of Hölder continuity can be found in Moscariello-Nania [73] and Lieberman [63].

Moreover we have some boundedness results for local minimizers of integral functionals of the general form

$$\mathcal{I}_2(u) = \int_\Omega f(x,u,Du)dx.$$

These results can be found in Dall'Aglio-Mascolo-Papi [49].
Let us assume that the integrand function $f : (x,s,z) \in \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ satisfy the following growth conditions

$$g(|z|) - c_1 \leq f(x,s,z) \leq c_2\left[g_*(z) + g_*(s)\right]^\beta \tag{15}$$

where g is a Young function and g_* is his Sobolev conjugate, as defined above.

If we assume that:
(i) $g \in \Delta_2^m \cap \nabla_2^r$ where $r \geq 1$, $1 < m < \min\{r^*, n\}$ and $0 < \beta < 1$,
(ii) f satisfy one of the following hypothesis
- For a.e. $x \in \Omega$, $f(x,\cdot,\cdot)$ is convex in $\mathbb{R} \times \mathbb{R}^n$;
- For a.e. $x \in \Omega$ and for every $s \in \mathbb{R}$, $f(x,s,\cdot)$ is convex in $\mathbb{R}^n$ and there exists a constant $c_3 > 0$ such that

$$f(x,s_1,z) \leq c_3 f(x,s_2,z) \quad \forall s_1, s_2 \text{ such that } |s_1| < |s_2| \quad \text{and every } z \in \mathbb{R}^N,$$

then a local minimizer of $\mathcal{I}_2$, $u \in W^1L_g(\Omega)$, is locally bounded and the following inequality holds

$$\sup_{Q_{\frac{R}{2}}} g_*(|u|) \leq 1 + C_R \left[\int_{Q_R} [g_*(|u|)]^\beta dx\right]^{\frac{r^*-m}{m(1-\beta)}}$$

where $Q_R \subset\subset \Omega$ is a cube of side $2R$.
It is clear that the growth condition (15) is a generalization of the condition $q < p^*$, usually assumed in the case of $p-q$ growth, to general growth conditions.

REGULARITY PROBLEM: The vectorial case.

The De Giorgi regularity theorem cannot be extended to vectorial functionals, that is when $N > 1$. This fact was proved by De Giorgi in 1968. In the following example indeed he proved that there exists a linear elliptic system with regular coefficients, that has non continuous solutions.

Example (E.De Giorgi). Let $\Omega = B_1(0) \subset \mathbb{R}^n$, with $n = N \geq 3$. Let us consider the following functional in $W^{1,2}(\Omega)$

$$\int_\Omega \left[\sum_{\alpha,i=1}^{n} |D_\alpha u^i|^2 + \left(\sum_{\alpha,i=1}^{n} \left((n-2)\delta_{\alpha i} + n\frac{x_i x_\alpha}{|x|^2} \right) D_\alpha u^i \right)^2 \right] dx.$$

The corresponding Euler-Lagrange system is

$$\int_\Omega A_{ij}^{\alpha\beta}(x) D_\beta u^j D_\alpha \varphi^i dx = 0 \qquad \forall \varphi \in W_0^{1,2}(\Omega, \mathbb{R}^N),$$

where we set

$$A_{ij}^{\alpha\beta}(x) = \delta_{\alpha\beta}\delta_{ij} + \left[(n-2)\delta_{\alpha i} + n\frac{x_i x_\alpha}{|x|^2} \right] \left[(n-2)\delta_{\beta j} + n\frac{x_j x_\beta}{|x|^2} \right] \in L^\infty(\Omega).$$

This system is strongly elliptic since there are two positive constants ν, M such that $0 < \nu \leq M$ and

$$\nu|\xi|^2 \leq A_{ij}^{\alpha\beta}(x)\xi_\alpha^i \xi_\beta^j \leq M|\xi|^2.$$

It can be shown that the function

$$u(x) = \frac{x}{|x|^\gamma}, \qquad \text{where} \quad \gamma = \frac{n}{2}\left[1 - \frac{1}{\sqrt{4n^2 - 8n + 5}} \right]$$

belongs to $W^{1,2}(\Omega, \mathbb{R}^N)$ and solves the above Euler-Lagrange system. Nevertheless u is not bounded at the origin.

More counterexamples to regularity in the vectorial case are due to Giusti-Miranda [31] and Necas [37]. However, under natural growth conditions, we have regularity results: we recall the ones of Uhlenbeck [38], [39] (case of $f(z) = |z|^p$, $p \geq 2$), Giaquinta-Modica [28], [29] (growth $p \geq 2$) and Acerbi-Fusco [21] ($1 < p < 2$).
More recently Marcellini [68] has given a $C^{1,\alpha}-$regularity theorem for the minimizers of functionals under general growth conditions of the type

$$\mathcal{I}(u) = \int_\Omega g(|Du|)dx,$$

where $u : \Omega \to \mathbb{R}^N$ and g has an exponential form: $g(t) = \exp(t^\alpha)$, $\alpha \in \mathbb{R}^+$ or even $g \in \Delta_2^m$ with $m \geq 2$.

Recently Dall'Aglio-Mascolo [50], study the boundedness of local minimizers of vectorial functionals of the type

$$\mathcal{I}_3(u) = \int_\Omega g(x, |Du|)dx$$

where $u : \Omega \subset \mathbb{R}^n \to \mathbb{R}^N$, by introducing the concept of a $\Omega - N-$function.
We say that $g : \Omega \times \mathbb{R}^+ \to \mathbb{R}^+$ is a $(\Omega - N)-$function if:

- $g(\cdot, t)$ is measurable in Ω for every $t > 0$;
- $g(x, \cdot)$ is a $N-$function in $\mathbb{R}^+$ for a.e. $x \in \Omega$.

For instance $g(x,t) = a(x)h(t)$ where a is a bounded function in Ω and h is a $N-$function is a function of this kind, as well as $g(x,t) = [h(t)]^{\alpha(x)}$, with α bounded and h a $N-$function.

Integrands of the form $g(x,t) = t^{\alpha(x)}$ appears for the first time in some papers of Zhikov [77],[78].
Acerbi-Fusco in [41] study the case where α is a discontinuous function, assuming only two values p, q in two subsets of Ω separated by a regular surface.

In the scalar case, the functional $g(x,t) = t^{\alpha(x)}$ has been studied by Mascolo-Papi [69], Chiadò Piat-Coscia [46], Fan [53], with $\alpha \in L^\infty$ and $p \le \alpha(x) \le q < p^*$.

Then the functional $\mathcal{I}_3$ is defined in the most suitable functional class i.e. the Orlicz class associated to the $(\Omega - N)-$function $g(x,t)$.
Let

$$K_g(\Omega, \mathbb{R}^N) = \left\{ u : \int_\Omega g(x,u)dx < \infty \right\}.$$

K_g is a vector space if and only if $g \in \Delta_2^m$ that is if and only if

$$g(x, \lambda t) \le \lambda^m g(x,t), \quad \forall t \ge 0, \lambda > 1 \text{ a.e. in } \Omega.$$

To get the regularity result some further assumptions on g are given.
Assume that:

- g is a $\Omega - N-$function of class Δ_2, $g_t(x,t)$ is a Carathéodory function and $g(x,1) \in L^\infty(\Omega)$;
- $g(x,t)$ has weak derivatives with respect to x in L^1_{loc}, $g_{x_i}(x,t)$ is a Carathéodory function and there exists a function $\gamma \in L^s(\Omega)$, with $s > nm$, such that

$$|g_x(x,t)| \le \gamma(x) g(x,t) t^\delta \qquad \forall t > 0, \text{ a.e. in } \Omega$$

with $\delta > 0$ small enough.

Under the previous assumptions it is proved that
(i) If $u \in W^{1,1}_{loc}(\Omega, \mathbb{R}^N)$ and $g(x, |Du|) \in L^1_{loc}(\Omega)$, then $g^{\frac{n}{n-1}}(x, |u|) \in L^1_{loc}(\Omega)$.
(ii) The local minimizers of $\mathcal{I}_3$ in $W^1_{loc} L_g(\Omega, \mathbb{R}^N)$ are locally bounded.

The result obtained is new also in the case of $f(z) = g(|z|)$ with a $N-$function $g \in \Delta_2$.
In particular, the function $g(x,t) = [h(t)]^{\alpha(x)}$ with $\alpha \in W^{1,s}$ and $s > nm$ ed h a $N-$function of class Δ_2 satisfies the above hypothesis.
Observe that in the case of $g(x,t) = t^{\alpha(x)}$ with $\alpha \in W^{1,s}$ $(s > n)$ and $p \le \alpha(x) < n$, the following embedding result holds:

$$W^1 L_g \equiv W^1 L_\alpha \hookrightarrow L^{\alpha^*}$$

where $\alpha^* = \frac{n\alpha(x)}{n - \alpha(x)}$.

Moreover, the assumption $\alpha \in W^{1,s}$, with $s > n$ is sharp in some sense. Indeed it is possible to find a function $\alpha \in W^{1,r}$ with $r < n$, such that the immersion result is no longer true.

Let us consider the case of $f(x,z) = f(z)$, $z \in \mathbb{R}^{nN}$, assuming that we have $p-q$ growth conditions:

$$|z|^p \leq f(z) \leq c(1+|z|^q) \qquad p < q.$$

In [59] Leonetti-Mascolo-Siepe study the regularity of local minimizers when $1 < p < q \leq 2$ (for the case of natural growths, $p = q < 2$ one can refer to Acerbi-Fusco [21]).
The higher summability of the gradient of minimizer is obtained using the method of *a-priori estimates*.
First one obtains a higher integrability result for minimizers of a suitable perturbated functional, and then, by mean of a double approximation argument, the same result is proved for the minimizers of the original functional.
For a general integrand function $f = f(z)$ it is needed that $p > \frac{2n}{n+2}$.
When we consider a particular class of integrands, i.e. $f(z) = g(|z|)$, with g a $N-$function of class Δ_2, then the same regularity result is proved without conditions on p and q, taking in account the boundedness result of [50].

In Mingione-Siepe [71], it is proved the everywhere $C^{1,\alpha}$-regularity of minimizers of the model functional

$$\int_\Omega |Du| \log(1+|Du|)dx,$$

that grows in a nearly linear way. In particular the result is proved in the vectorial case under the special assumption that the integrand function depends of the modulus of the gradient, and in the scalar case without assuming any special structure for the functional.
The method used here consists of a classical *Moser-type iteration technique*, together with standard methods that allow to prove $C^{1,\alpha}$-regularity. To avoid the problem of the growth $(1, 1+\varepsilon)$ of such a functional is used a technique close to the one described above to prove the regularity result of [59].

Finally, Leonetti-Mascolo-Siepe, in a paper [60] wich will appear soon, prove $C^{1,\alpha}$ regularity for minimizers of general functionals of the type

$$\mathcal{I}(u) = \int_\Omega g(|Du|)dx$$

where $g \in \Delta_2^m$ and $1 < m < 2$.

References

N-functions, Orlicz spaces, Orlicz-Sobolev spaces and applications to partial differential equations.

[1] R.A. Adams, *Sobolev Spaces*, Academic Press, New York, 1975.

[2] R.A. Adams, *On the Orlicz-Sobolev imbedding theorem*, J. Funct. Anal. **24** (1977), 241–257.

[3] T. Andò, *On some properties of convex functions*, Bull. Acad. Polon. Sci. Ser. Mat. **8**, 413–418.

[4] A. Cianchi, *Continuity properties of functions from Orlicz-Sobolev spaces and embedding theorem*, Ann. Sc. Norm. Sup. Pisa, **IV,23** (1996), 575–608.

[5] A. Cianchi, *A sharp embedding theorem for Orlicz-Sobolev spaces*, Indiana Univ. Math. J. **45** (1996), 39–65.

[6] de la Vallèe Poussin, *Sur l'integrale de Lebesgue*, Trans. Amer. Math. Soc. **16**, 435–501.

[7] T.K. Donaldson, *Non linear elliptic boundary value problems in Orlicz-Sobolev spaces*, J. Diff. Eq. **10** (1971), 507–528.

[8] T.K. Donaldson & N.S. Trudinger, *Orlicz-Sobolev spaces and imbedding theorems*, J. Funct. Anal. **8** (1971), 52–75.

[9] D.E. Edmunds, B. Opic & L. Pick, *Poincaré and Friedrichs inequalities in abstract Sobolev spaces*, Math. Proc. Camb. Phil. Soc. **113** (1993), 355–378.

[10] A. Fougères, *Operateurs elliptiques du calcul des variations a coefficients très fortement non lineaires*, C. R. Acad. Sc. Paris Eq. **274** (1972).

[11] J.P. Gossez, *Nonlinear elliptic boundary value problems for equations with rapidly (or slowly) increasing coefficients*, Trans. Amer. Math. Soc. **190** (1974).

[12] J.P. Gossez, *Un probleme de Dirichlet fortement non linéaire*, Boll. UMI **14-A** (1977), 118–125.

[13] J.P. Gossez, *Some approximation properties in Orlicz-Sobolev spaces*, Studia Mat. **74** (1982), 17–24.

[14] M.A. Krasnosel'skii & Y.A. Rutickii, *Convex functions and Orlicz spaces*, Noordhof, Groningen (1961).

[15] A. Kufner, O. John & S. Fucik, *Function spaces*, Acdemia Praga (1977).

[16] W.A.J. Luxemburg, *Banach function spaces*, Ph.D. Dissertation (1955).

[17] M.M. Rao & Z.D. Ren, *Theory of Orlicz spaces*, Marcel Dekker, New York, (1991).

[18] G. Talenti, *Non linear elliptic equations, rearrangements of functions and Orlicz spaces*, Ann. Mat. Pura e Appl. **4-120**, 159–184.

[19] N.S. Trudinger, *On embedding into Orlicz spaces and some applications*, J. Math. and Mech. **17-5** (1967), 473–483.

[20] E. Acerbi & N. Fusco, *A regularity theorem for minimizers of quasiconvex integrals*, Arch. Rat. Mech. and Anal. **99** (1987), 261–281.

[21] E. Acerbi & N. Fusco, *Regularity for minimizers of non-quadratic functionals. The case* $1 < p < 2$, J. Math. Anal. Appl. **140** (1989), 115–135.

[22] E. De Giorgi, *Sulla differenziabilità e l'analiticità delle estremali degli integrali multipli regolari*, Mem. Accad. Sci. Torino (Classe di Sci. mat., fis. e nat.) **3(3)** (1957), 25–43.

[23] E. Di Benedetto & N.S. Trudinger, *Harnack inequality for quasi minima of variational integrals*, Ann. Inst. H. Poincaré, Analyse non lineaire **1** (1984), 295–308.

[24] M. Giaquinta, *Multiple integrals in the calculus of variations and non linear elliptic systems*, vol. **105**, Annals of Math. Studies, Princeton Univ. Press, 1983.

[25] M. Giaquinta & E. Giusti, *On the regularity of the minima of variational integrals*, Acta Math. **148** (1982), 31–46.

[26] M. Giaquinta & E. Giusti, *Differentiability of the minima of non differentiable functionals*, Invent. Math. **72** (1983), 285–298.

[27] M. Giaquinta & E. Giusti, *Quasi-minima*, Ann. Inst. H. Poincaré (Analyse non linéaire) **1** (1984), 79–107.

[28] M. Giaquinta & G. Modica, *Partial regularity of minimizers of quasiconvex integrals*, Ann. Inst. H. Poincaré (Analyse non linéaire) **3** (1986), 185–208.

[29] M. Giaquinta & G. Modica, *Remarks on the regularity of the minimizers of certain degenerate functionals*, Manus. Math. **57** (1986), 55–99.

[30] D. Gilbarg & N.S. Trudinger, *Elliptic partial differential equations of the second order*, Springer, New York (1983).

[31] E. Giusti & M. Miranda, *Un esempio di soluzioni discontinue per un problema di minimo relativo ad un integrale regolare del calcolo delle variazioni*, Boll. UMI **2** (1968), 1–8.

[32] E. Giusti, *Equazioni ellittiche del secondo ordine*, vol. **6**, Quaderni U.M.I., Pitagora Ed., 1978.

[33] E. Giusti, *Metodi diretti nel calcolo delle variazioni*, U.M.I., 1994.

[34] O. Ladyzhenskaya & N. Ural'tseva, *Linear and quasilinear elliptic equations*, vol. **46**, Academic Press New York, 1968.

[35] C.B. Morrey, *Multiple integrals in the calculus of variations*, Springer, Hidelberg and New York, 1966.

[36] J. Moser, *A new proof of De Giorgi's theorem concerning the regularity problem for elliptic differential equations*, Comm. Pure Appl. Math. **14** (1961), 577–591.

[37] J. Necas, *Example of an irregular solution to a nonlinear elliptic system with analytic coefficients and conditions for regularity*, Abhand Akad. der Wiss (1977).

[38] K. Uhlenbeck, *Regularity for a class of non-linear elliptic systems*, Acta Math. **138** (1977), 219–240.

[39] K. Uhlenbeck, *Harmonic maps: a direct method in the calculus of variations*, Bull. Amer. Math. Soc. **76** (1970), 1082–1087.

[40] E. Acerbi & N. Fusco, *Partial regularity under anisotropic (p,q) growth conditions*, J. Diff. Eq. **107** (1994), 46–67.

[41] E. Acerbi & N. Fusco, *A trasmission problem in the calculus of variations*, Calc. of Variat. **1** (1994), 1–16.

[42] T. Battacharya & F. Leonetti, *A new Poincaré inequality and its applications to the regularity of minimizers of integral functionals with nonstandard growth*, Nonlinear Anal. **17,9** (1991), 833–839.

[43] T. Battacharya & F. Leonetti, *Some remarks on the regularity of minimizers of integrals with anisotropic growth*, Comment. Math. Univ. Carolinae **34,4** (1993), 597–611.

[44] L. Boccardo, P. Marcellini & C. Sbordone, *L^∞-regularity for variational problems with sharp non standard growth conditions*, Boll. U.M.I. **4-A** (1990), 219–225.

[45] A. Cianchi & N. Fusco, *Gradient regularity for minimizers under general growth conditions*, To appear on J. Reine Ang. Math.

[46] V. Chiadò-Piat & A. Coscia, *Hölder continuity for minimizers of functionals with variable growth exponent*, Manus. Math. **93** (1997), 283–299.

[47] H.J. Choe, *Regularity for the minimizers of certain functionals with non standard growth conditions*, Comm. Part. Diff. Eq. **16** (1991), 363–372.

[48] H.J. Choe, *Interior behaviour of minimizers for certain functionals with non standard growth conditions*, Nonlin. Anal. TMA **19,10** (1992), 933–945.

[49] A. Dall'Aglio, E. Mascolo & G. Papi, *Regularity for local minima of functionals with non standard growth*, Rend. Mat. **7,18** (1998), 305–326.

[50] A. Dall'Aglio & E. Mascolo, *Regularity for a class of non linear elliptic systems with non standard growth conditions*, To appear.

[51] L. Esposito, F. Leonetti & G. Mingione, *Regularity for minimizers of functionals with $p-q$ growth conditions*, To appear on Nonl. Diff. Eq. and Appl..

[52] L. Esposito, F. Leonetti & G. Mingione, *Higher integrability for minimizers of integral functionals with (p,q) growth*, To appear on J. Diff. Eq..

[53] X. Fan, *A class of De Giorgi type and Hölder continuity of minimizers of integrals with $m(x)-$growth conditions*, Preprint Lanzhou Univ., China (1995).

[54] N. Fusco & C. Sbordone, *Local boundedness of minimizers in a limit case*, Manus. Math. **69** (1990), 19–25.

[55] N. Fusco & C. Sbordone, *Higher integrability of the gradient of minimizers of functionals with nonstandard growth conditions*, Comm. Pure Appl. Math. **43** (1990), 673–683.

[56] N. Fusco & C. Sbordone, *Some remarks on the regularity of minima of anisotropic integrals*, Comm. Part. Diff. Eq. **18** (1993), 153–167.

[57] M. Giaquinta, *Growth conditions and regularity, a counterexample*, Manus. Math. **59** (1987), 245–248.

[58] M.-C. Hong, *Some remarks on the minimizers of variational integrals with non standard growth conditions*, Boll. U.M.I. **6-A** (1992), 91–101.

[59] F. Leonetti, E. Mascolo & F. Siepe, *Gradient regularity for minimizers of functionals under $p-q$ subquadratic growth*, Preprint Dip. Mat. "U. Dini", Firenze.

[60] F. Leonetti, E. Mascolo & F. Siepe, *Everywhere regularity for a class of elliptic systems with non standard growth conditions*, To appear.

[61] G.M. Lieberman, *The natural generalization of the natural growth conditions of Ladyzhenskaya and Ural'tseva for elliptic equations*, Comm. Part. Diff. Eq. **16** (1991), 311–361.

[62] G.M. Lieberman, *On the regularity of the minimizers of a functional with exponents growth*, Comm. Math. Univ. Carolinae **33** (1992), 45–49.

[63] G.M. Lieberman, *Gradient estimate for a class of elliptic systems*, Ann. Mat. Pura e Appl. **164** (1993), 103–120.

[64] P. Marcellini, *Un example de solution discontinue d'un problème variationel dans le cas scalaire*, Preprint Dip. Mat. "U.Dini", Firenze (1987).

[65] P. Marcellini, *Regularity of minimizers of integrals in the calculus of variations with non standard growth conditions*, Arch. Rat. Mech. and Anal. **105** (1989), 267–284.

[66] P. Marcellini, *Regularity and existence of solutions of elliptic equations with p,q-growth conditions*, J. Diff. Eq. **90** (1991), 1–30.

[67] P. Marcellini, *Regularity for elliptic equations with general growth*, J. Diff. Eq. **105** (1993), 296–333.

[68] P. Marcellini, *Everywhere regularity for a class of elliptic systems without growth conditions*, Ann. Sc. Norm. Sup. Pisa **23** (1996), 1–25.

[69] E. Mascolo & G. Papi, *Local boundedness of minimizers of integrals of the calculus of variations*, Ann. Mat. Pura Appl. **167** (1994), 323–339.

[70] E. Mascolo & G. Papi, *Harnack inequality for minimizers of integral functionals with general growth conditions*, NoDEA **3** (1996), 231–244.

[71] G. Mingione & F. Siepe, *Full $C^{1,\alpha}$-regularity for minimizers of integral functionals with $L\log L$-growth*, Preprint Dip. Mat. "U. Dini", Firenze **23** (1998).

[72] G. Moscariello, *Regularity results for quasi minima of functionals with non polynomial growth*, J. Math. Anal. Appl. **168** (1992), 500–512.

[73] G. Moscariello & L. Nania, *Hölder continuity of minimizers of functionals with non standard growth conditions*, Ric. Mat. **152** (1991), 259–273.

[74] F. Siepe, *Regularity for minimizers of anisotropic integral functionals in the calculus of variations*, Ph.D. Dissertation (1999).

[75] G. Talenti, *Boundedness of minimizers*, Hokkaido Math. J. **19** (1990), 259–279.

[76] Q. Tang, *Regularity of minimizers of non isotropic integrals in the calculus of variations*, Ann. Mat. Pura e App. **164** (1993), 77–87.

[77] V.V. Zhikov, *Averaging of functionals of the calculus of variations and elasticity theory*, Math. USSR Izvestiya **29** (1987), 33–66.

[78] V.V. Zhikov, *Laurentiev phenomenon and homogeneization for some variational problem*, Proc. Workshop "Composite materials and homogeneization theory", Trieste (1995).

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[79] E. Acerbi & G. Dal Maso, *New lower semicontinuity results for polyconvex integral case*, Preprint SISSA, Trieste **52-M** (1993).

[80] E. Acerbi & N. Fusco, *Semicontinuity problems in the calculus of variations*, Arch. Rat. Mech. and Anal. **86** (1984), 25–145.

[81] J.M. Ball, *Convexity conditions and existence theorems in non linear elasticity*, Arch. Rat. Mech. and Anal. **63** (1977), 1103–1122.

[82] J.M. Ball & F. Murat, *$W^{1,p}$–quasiconvexity and variational problems for multiple integrals*, J. Funct. Anal. **58** (1984), 225–253.

[83] J.M. Ball & K.W. Zhang, *Lower semicontinuity of multiple integrals and biting lemma*, Proc. Roy. Soc. Edimburgh **114-A** (1990), 367–379.

[84] B. Dacorogna, *Direct methods in the calculus of variations*, vol. **78**, Appl. Math. Sci. Springer Verlag, Heidelberg and New York, 1989.

[85] B. Dacorogna & P. Marcellini, *Semicontinuitè pour des intègrandes polyconvexes sans continuitè des dèterminants*, C.R. Acad. Sci. Paris **311** (1990), 393–396.

[86] G. Dal Maso & C. Sbordone, *Weak lower semicontinuity of polyconvex integrals: a borderline case*, Preprint SISSA, Trieste **45-M** (1993).

[87] E. De Giorgi, *Teoremi di semicontinuità nel calcolo delle variazioni*, Appunti INdAM (1968-69), 309–338.

[88] I. Ekeland & R. Temam, *Convex analysis and variational problems*, North Holland, Amsterdam (1976).

[89] M. Focardi, *Semicontinuità di funzionali vettoriali nella classe di Orlicz-Sobolev*, Tesi di Laurea, Univ. Firenze (1996).

[90] M. Focardi, *Semicontinuity of vectorial functionals in Orlicz-Sobolev spaces*, Rend. Mat. Univ. Trieste (1998), 141–161.

[91] M. Focardi & E. Mascolo, *Lower semicontinuity for integral functionals under non standard growth conditions*, To appear.

[92] I. Fonseca & J. Malý, *Relaxation of multiple integrals in Sobolev spaces below the growth exponent for the energy density*, Ann. Inst. H. Poincaré (Analyse Non Linéaire) **14,3** (1997), 309–338.

[93] I. Fonseca & P. Marcellini, *Relaxation of multiple integrals in subcritical Sobolev spaces*, To appear on J. Geom. Anal..

[94] N. Fusco, *Quasiconvessità e semicontinuità per integrali multipli di ordine superiore*, Ricerche di Mat. **29** (1980), 307–323.

[95] N. Fusco & J. Hutchinson, *A direct proof for lower semicontinuity of polyconvex functionals*, Manus. Math. **87** (1995), 35–50.

[96] J. Kristensen, *Lower semicontinuity in Sobolev spaces below the growth exponent of integrand*, Proc. Roy. Soc. Edimburgh **127-A** (1997), 797–817.

[97] J. Malý, *Weak lower semicontinuity of polyconvex and quasiconvex integrals*, Vortragsreihe, Bonn (1993).

[98] J. Malý, *Lower semicontinuity of quasiconvex integrals*, Manus. Math. **85** (1994), 419–428.

[99] P. Marcellini, *Approximation of quasiconvex functions and lower semicontinuity of multiple integrals*, Manus. Math. **51** (1986), 1–28.

[100] P. Marcellini, *On the definition and the lower semicontinuity of certain quasi-convex integrals*, Ann. Inst. H. Poincaré (Analyse Non Lineaire) **3** (1986), 391–409.

[101] P. Marcellini & C. Sbordone, *On the existence of minima of multiple integrals in the calculus of variations*, J. Math. Pures Appl. **62** (1983), 1–10.

[102] N.G. Meyers, *Quasiconvexity and semicontinuity of multiple integrals of any order*, Trans. Amer. Math. Soc. **119** (1965), 125–149.

[103] C.B. Morrey Jr., *Quasiconvexity and lower semicontinuity of multiple integrals*, Pacific J. Math. **2** (1952), 25–53.

[104] V. Šverák, *Quasiconvex functions with subquadratic growth*, Proc. Roy. Soc., London **433A** (1991), 723–725.

[105] K.W. Zhang, *A construction of quasiconvex functions with linear growth at infinity*, Ann. Sc. Norm. Sup. Pisa **19** (1992), 313–326.