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Primeras Jornadas sobre Ecuaciones Diferenciales, Optimización y Análisis Numérico

Segunda Parte

Departamento
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SERIE A: CONFERENCIAS, SEMINARIOS Y TRABAJOS DE MATEMÁTICA

No. 8

PRIMERAS JORNADAS SOBRE ECUACIONES DIFERENCIALES, OPTIMIZACIÓN Y ANÁLISIS NUMÉRICO

Segunda Parte

Domingo A. Tarzia (Ed.)

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Rosario, Octubre 2004

Las Primeras Jornadas sobre Ecuaciones Diferenciales, Optimización y Análisis Numérico tuvieron lugar en el Departamento de Matemática de la FCE de la Universidad Austral, en Rosario, del 11 al 12 de Marzo de 2004. Fue realizado con el apoyo del Proyecto de Investigación Plurianual "Partial Differential Equations and Numerical Optimization with Applications" subsidiado por la Fundación Antorchas e integrado por los siguientes subproyectos:

- "Free Boundary Problems for the Heat-Diffusion Equation" (UA-UNR-UNRC-UNSa);
- "Inverse and Control Problems in the Mathematical Modeling of Phase Transitions in Shape Memory Alloys" (UNL);
- "Geophysical Scale Stratified Flows and Hydraulic Jumps" (UNC-UBA);
- "Optimization Applied to Mechanical Engineering Problems" (UNS-UNC-UNCo)

El Comité Organizador estuvo compuesto por: M.C. Maciel (Bahía Blanca), R.D. Spies (Santa Fe), D. A. Tarzia, (Rosario, Coordinador); C.V. Turner (Córdoba). La Secretaría Local estuvo compuesta por: A. C. Briozzo (Coordinador), G. G. Garguichevich, M. F. Natale, E. A. Santillan Marcus, M. C. Sanziel.

Las Jornadas estuvieron dirigidas a graduados, profesionales y estudiantes de Matemática, Física, Química, Ingeniería y ramas afines, con conocimientos básicos sobre ecuaciones diferenciales, análisis numérico y optimización.

En MAT – Serie A – 7 (2004) (primera parte) y en MAT – Serie A – 8 (2004) (segunda parte) se publican siete de las conferencias y comunicaciones presentadas. Los manuscritos fueron recibidos y aceptados en octubre de 2004.

CONFERENCIAS Y COMUNICACIONES DE LAS PRIMERAS JORNADAS SOBRE ECUACIONES DIFERENCIALES, OPTIMIZACIÓN Y ANÁLISIS NUMÉRICO

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DIFFERENTIABILITY OF THE SOLUTIONS OF A SEMILINEAR ABSTRACT CAUCHY PROBLEM WITH RESPECT TO PARAMETERS

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Abstract. The Fréchet differentiability with respect to a parameter q of the solutions $z(t;q)$ of Cauchy problems of the form $\frac{d}{dt}z(t) = A(q)z(t) + F(q,t,z(t))$ is analyzed. Sufficient conditions on the operator $A(q)$ and on F are derived and the corresponding sensitivity equations for the Fréchet derivative $D_q z(t;q)$ are found.

1. Introduction

We consider the problem of continuous dependence an differentiability with respect to a parameter q of the solutions $z(t;q)$ of the semilinear abstract Cauchy problem

$$(\mathcal{P})_q \begin{cases} \frac{d}{dt}z(t) = A(q)z(t) + F(q,t,z(t)) & z(t) \in Z, \\ z(0) = z_0 & t \in [0,T] \end{cases}$$

where Z is a Banach space, $q \in Q_{ad} \subset Q$, a normed linear space (Q_{ad} is an open subset of Q), and $A(q)$ is the infinitesimal generator of an analytic semigroup $T(t;q)$ on Z for all $q \in Q_{ad}$. Z and Q are the state space and the parameter space, respectively, while Q_{ad} is called the admissible parameter set.

Identification problems associated to system $(\mathcal{P})_q$ and other similar type of equations ([2], [5], [7]) are usually solved by direct methods such as quasilinearization. These methods require that solutions be differentiable with respect to the parameter q. In addition, their numerical implementation require an approximation to the corresponding Fréchet derivative.

Problems of the type $\frac{d}{dt}z(t) = A(q)z(t) + u(t)$, where $A(q)$ generates a strongly continuous semigroup and $A(q) = A + B(q)$ where $B(q)$ is assumed to be bounded where studied by Clark and Gibson ([4]), Brewer ([1]). Burns et al ([3]) studied problems of the type $\frac{d}{dt}z(t) = Az(t) + F(q,t,z(t))$. The parameter q here did not appear in the linear part of the equation.

Here, we prove that, under certain conditions, the solutions of the general abstract Cauchy problem $(\mathcal{P})_q$ are Fréchet differentiable with respect to q and we find the corresponding sensitivity equations.

Key words and phrases. Abstract Cauchy problem, analytic semigroup, infinitesimal generator, Fréchet differentiability, Fréchet derivative.

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2. Preliminary Results

The following standing hypotheses are considered:

H1: There exist $\varepsilon_0 > 0$ such that the type of $T(t;q)$, call it w_q, is less than or equal to $-\varepsilon_0$ for all $q \in Q_{ad}$ and there exists $C_q > 0$ such that $\|T(t;q)\| \leq C_q e^{-\varepsilon_0 t}$ for all $t \geq 0$ and $q \in Q_{ad}$. The constant C_q depends on q but it can be chosen independent of q on compact subsets of Q_{ad}.

H2: $\mathcal{D}(A(q)) = D$ is independent of q and D is a dense subspace of Z.

We shall denote by Z_δ the space $D\left((-A(q))^\delta\right)$ imbedded with the norm of the graph of $-A(q))^\delta$. Since $0 \in \rho(A(q))$ it follows that this norm is equivalent to $\|z\|_{q,\delta} \doteq \|(-A(q))^\delta z\|$. Also, there exists a constant M_q such that $\|(-A(q))^\delta T(t;q)\| \leq M_q \frac{e^{-\epsilon_0 t}}{t^\delta}$, for all $t > 0$ (see [13], Theorem 2.6.13).

H3: There exists $\delta \in (0,1)$ such that

$$\|F(q,t_1,z_1) - F(q,t_2,z_2)\|_Z \leq L(|t_1 - t_2| + \|z_1 - z_2\|_{q,\delta})$$

for $(t_i, z_i) \in U$, where L can be chosen independent of q on any compact subset of Q_{ad}.

This last regularity condition guarantees existence and uniqueness of solutions of problem $(\mathcal{P})_q$, provided that the initial condition z_0 is in Z_δ. See [12] and [11] for details.

The next results can be easily proved by using the Closed Graph Theorem.

Lemma 1: *Under hypotheses H1 and H2, for any $q_1, q_2 \in Q_{ad}$ and $\delta \in (0,1)$ we have:*
i) $A(q_1)(-A(q_2))^{-\delta}$ *is bounded on* $Z_{1-\delta}$.
ii) $A(q_1)T(\cdot;q_2) \in L^1(0,\infty : \mathcal{L}(Z))$ and $A(q_1)T(\cdot;q_2) \in L^\infty(\eta,\infty;\mathcal{L}(Z))$, for each $\eta > 0$.
iii) $T(\cdot;q_2) \in L^1(0,\infty : \mathcal{L}(Z, Z_{q_1,\delta}))$ and $T(\cdot;q_2) \in L^\infty(\eta,\infty : \mathcal{L}(Z; Z_{q_1,\delta}))$, for each $\eta > 0$.

Note: This result implies that the operator $A(q_1)T(t;q_2)$ is bounded for each $t > 0$. However, no uniform bound can be found for t near zero. For $q_1 = q_2 = q$, it implies, in particular, that the derivative $\frac{d}{dt}T(t;q)$ of the solution operator of the homogeneous equation associated with $(\mathcal{P})_q$ is integrable near $t = 0$.

We will also assume that $A(q)$ satisfies the following hypothesis:

H4: For δ as in H3 and for any $q_1, q_2 \in Q_{ad}$ there are constants $M(q_1,q_2)$ and $C(q_1,q_2)$ both depending on q_1 and q_2, such that $\|(-A(q_1))^\delta(-A(q_2))^{-\delta}\|_{\mathcal{L}(Z)} \leq M(q_1,q_2)$,
$\|A(q_1)[A(q_2)]^{-1} - I\| \leq C(q_1,q_2)$ and $C(q_1,q_2) \to 0$ as $q_1 \to q_2$.

Note: It is sufficient to request that H4 be true for $\delta = 1$.
We also consider the hypothesis:

H4': For each $q_0 \in Q_{ad}$ there exists $C = C(q_0)$ such that

$$\|(A(q) - A(q_0))z\| \leq C\|q - q_0\|\,\|A(q_0)z\| \quad z \in D, \quad q \in Q_{ad}.$$

Theorem 2: *Assume H1-H4 hold. Then for any $q_0 \in Q_{ad}$* and $\varepsilon > 0$, there exists $\widetilde{\delta} > 0$ *such that*

$$\|A(q)T(\cdot,q_0)z - A(q_0)T(\cdot,q_0)z\|_{L^1(0,\infty;Z)} \leq \varepsilon\|z\|$$

for all $z \in Z$, *and for all* $q \in Q_{ad}$ *satisfying* $\|q - q_0\| < \widetilde{\delta}$, *that is*

$$\|A(q)T(\cdot, q_0) - A(q_0)T(\cdot, q_0)\|_{L^1(0,\infty;\mathcal{L}(Z))} \leq \varepsilon,$$

or equivalently, for every fixed $q_0 \in Q_{ad}$ *the mapping from* Q into $L^1(0, \infty; \mathcal{L}(Z))$ defined by

$$q \to A(q)T(\cdot, q_0)$$

is continuous on Q_{ad} .

The proof follows immediately using Lemma 1.

3. Main results

Recall that for $z_0 \in Z_\delta$, $z(t; q)$ satisfies

$$z(t;q) = T(t;q)z_0 + \int_0^t T(t-s;q)F(q,s,z(s;q))ds \doteq T(t;q)z_0 + S(t;q), \quad t \in [0,T].$$

Consider now the following standing hypothesis concerning the q-regularity of $\frac{d}{dt}T(t;q)$.

H5: The mapping $q \to A(q)T(\cdot; q_0)$ from Q into $L^1(0, \infty; \mathcal{L}(Z))$ is Fréchet differentiable at q_0 for all $q_0 \in Q_{ad}$ (under H1-H4, we already know that this mapping is continuous, by virtue of Theorem 2).

Theorem 3: *Suppose H1-H5 hold. It follows that*

i) The mapping $q \to T(\cdot; q)$ *from* $Q \to L^\infty(0, \infty; \mathcal{L}(Z))$ *is Fréchet differentiable at* q_0, *for each* $q_0 \in Q_{ad}$. *Moreover, for any* $t > 0$ *and* $h \in Q_{ad}$ *the q-Fréchet derivative of* $T(t; q)$ *evaluated at* $q_0 \in Q_{ad}$ *and applied to* $h \in Q$, *i.e.* $[D_q T(t; q_0)]h$, *is the solution* $v_h(t)$ *of the following linear IVP, the so called "sensitivity equation" for* $T(t; q)$, *in* $\mathcal{L}(Z)$

$$(S_1): \begin{cases} \frac{d}{dt}v_h(t) = A(q_0)v_h(t) + \left[D_q A(q)T(t;q_0)|_{q=q_0}\right] h \\ v_h(0) = 0, \end{cases}$$

and ii) for every $q_0 \in Q_{ad}$, $D_q T(\cdot; q_0) = D_q T(\cdot; q)|_{q=q_0} \in L^\infty(0, \infty; \mathcal{L}(Q; \mathcal{L}(Z)))$.

Proof: For $q_0 \in Q_{ad}$ we have

$$(2) \qquad [D_q T(t;q_0)z_0](\cdot) = \int_0^t T(t-s;q_0)\left[D_q A(q)T(s;q_0)z_0|_{q=q_0}\right](\cdot)\, ds.$$

It remains to show the Fréchet differentiability of the mapping $q \to T(\cdot; q)$ when viewed as a mapping form Q into $L^\infty(0, \infty; \mathcal{L}(Z))$, i.e. in the stronger $L^\infty(0, \infty; \mathcal{L}(Z))$ norm. Let $\varepsilon > 0$, $t > 0$ and $q_0 \in Q_{ad}$. First note that for any $h \in Q$ with $\|h\| < \tilde{\delta}$, ($\tilde{\delta}$ as in Theorem 2) we have

$$\begin{aligned} \frac{d}{dt}[T(t;q_0+h)z_0 - T(t;q_0)z_0] &= A(q_0+h)T(t;q_0+h)z_0 - A(q_0)T(t;q_0)z_0 \\ &= A(q_0+h)[T(t;q_0+h)z_0 - T(t;q_0)z_0] + (A(q_0+h) - A(q_0))T(t;q_0)z_0. \end{aligned}$$

From Theorem 2 and [13] (Corollary 2.2) it follows that

$$(3) \quad T(t;q_0+h)z_0 - T(t;q_0)z_0 = \int_0^t T(t-s;q_0+h)\left(A(q_0+h) - A(q_0)\right)T(s;q_0)z_0\, ds.$$

and therefore for all $h \in Q$ with $\|h\| < \tilde{\delta}$, we have

$$\begin{aligned}\|T(t;q_0+h)z_0 - T(t;q_0)z_0\|_Z &\le \int_0^t M_{q_0+h}e^{-\varepsilon_0(t-s)}\,\|(A(q_0+h)T(s;q_0) - A(q_0)T(s;q_0))z_0\|_Z\,ds\\ &\le C\,\|(A(q_0+h)T(\cdot;q_0) - A(q_0)T(\cdot;q_0))z_0\|_{L^1(0,\infty;Z)}\\ &\le C\varepsilon\|z_0\|_Z,\end{aligned}$$

Thus for $t>0$

$$\|T(t;q_0+h) - T(t;q_0)\|_{\mathcal{L}(Z)} \le C\varepsilon, \quad \text{for } \|h\| < \tilde{\delta}, \tag{4}$$

and, since the constant C above does not depend on t,

$$\|T(\cdot;q_0+h) - T(\cdot;q_0)\|_{L^\infty(0,\infty;\mathcal{L}(Z))} \le C\varepsilon, \quad \text{for } \|h\| < \tilde{\delta}.$$

The following estimate then follows

$$\begin{aligned}&\left\|T(t;q_0+h) - T(t;q_0) - \int_0^t T(t-s;q_0)\left[D_qA(q)T(s;q_0)|_{q=q_0}\right]\,h\,ds\right\|_{\mathcal{L}(Z)}\\ &\quad\le (\varepsilon+1)C\,\|A(q_0+h)T(\cdot;q_0) - A(q_0)T(\cdot;q_0) - [D_qA(q)T(\cdot;q_0)|_{q=q_0}]\,h\|_{L^1(0,\infty;\mathcal{L}(Z))}\\ &\qquad + \varepsilon C\,\|[D_qA(q)T(\cdot;q_0)|_{q=q_0}]\,h\|_{L^1(0,\infty;\mathcal{L}(Z))}\,.\end{aligned} \tag{5}$$

Now by (H5) for the given $\varepsilon>0$ there exists $\xi>0$ such that

$$\|A(q_0+h)T(\cdot;q_0) - A(q_0)T(\cdot;q_0) - [D_qA(q)T(\cdot;q_0)|_{q=q_0}]\,h\|_{L^1(0,\infty;\mathcal{L}(Z))} \le \varepsilon\|h\| \tag{6}$$

for $\|h\|<\xi$, and since $D_qA(q)T(\cdot;q_0)|_{q=q_0} \in \mathcal{L}\left(Q, L^1(0,\infty;\mathcal{L}(Z))\right)$ there exists M, $0<M<\infty$ such that

$$\|D_qA(q)T(\cdot,q_0)|_{q=q_0}\|_{\mathcal{L}(Q,L^1(0,\infty,\mathcal{L}(Z)))} \le M. \tag{7}$$

Now, employing (6) and (7) in (5) we get that for $\|h\| < \min(\tilde{\delta},\xi)$

$$\begin{aligned}&\left\|T(t;q_0+h) - T(t;q_0) - \int_0^t T(t-s;q_0)\left[D_qA(q)T(s;q_0)|_{q=q_0}\right]h\,ds\right\|_{\mathcal{L}(Z)}\\ &\quad\le (\varepsilon+1)C\varepsilon\|h\| + \varepsilon CM\|h\| \le K\varepsilon\|h\|.\end{aligned}$$

Hence the mapping from Q into $L^\infty(0,\infty;\mathcal{L}(Z))$ defined by

$$q \to T(\cdot;q)$$

is Fréchet q-differentiable at q_0 and

$$\left[D_qT(t;q_0)\right](\cdot) = \int_0^t T(t-s;q_0)\left[D_qA(q)T(s;q_0)|_{q=q_0}\right](\cdot)\,ds. \tag{8}$$

Since $q_0 \in Q_{ad}$ is arbitrary, part (i) of the Theorem follows. To prove (ii) we first note that by H5, for $q_0 \in Q_{ad}$, there exists $C=C(q_0)$ such that for $h\in Q$

$$\|D_qA(q)T(\cdot;q_0)|_{q=q_0}\,h\|_{L^1(0,\infty;\mathcal{L}(Z))} \le C(q_0)\,\|h\|. \tag{9}$$

Now, it follows from (8) that for $t > 0$, $q_0 \in Q_{ad}$ and $h \in Q$, one has $\| [D_qT(t; q_0)]\ h\|_{\mathcal{L}(Z)} \leq \tilde{C}(q_0)\,\|h\|$. Thus $\|D_qT(t; q_0)\|_{\mathcal{L}(Q;\mathcal{L}(Z))} \leq \tilde{C}(q_0)$, and since $\tilde{C}(q_0)$ does not depend on $t > 0$, it follows that $D_qT(\cdot; q_0) \in L^\infty(0,\infty;\mathcal{L}(Q;\mathcal{L}(Z)))$.

■

Under slightly stronger assumptions on the mapping $q \to A(q)T(\cdot; q_0)$, it is possible to obtain the Lipschitz continuity of the mapping $q \to D_qT(\cdot; q_0)$ as a mapping from Q into $L^\infty(0,\infty;\mathcal{L}(Q;\mathcal{L}(Z)))$ and from Q into $L^\infty(0,\infty;\mathcal{L}(Q;\mathcal{L}(Z, Z_\delta)))$. In fact, consider the following hypothesis.

H6: The mapping $q \to D_qA(q)T(\cdot; q_0)$ from Q into $L^1(0,\infty;\mathcal{L}(Q;\mathcal{L}(Z)))$ is locally Lipschitz continuous at q_0, for all $q_0 \in Q_{ad}$.

THEOREM 4: *Let $q_0 \in Q_{ad}$ and assume hypotheses H1-H6 hold. Then the mapping $q \to D_qT(\cdot; q_0)$ from Q into $L^\infty(0,\infty;\mathcal{L}(Q;\mathcal{L}(Z)))$ is locally Lipschitz continuous at q_0.*

PROOF: Choose $h \in Q$ such that $\|h\| < \tilde{\delta}$ ($\tilde{\delta}$ as in Theorem 2) and denote $G_q(t; q_0)(\cdot) = D_qA(q)T(t; q_0)|_{q=q_0}(\cdot) \in \mathcal{L}(Q, \mathcal{L}(Z))$. Theorem 3 together with the appropriate choice of $\alpha(h)$, $0 \leq |\alpha(h)| \leq 1$, yield

$$\begin{aligned}
&\|D_qT(t; q_0 + h)(\cdot) - D_qT(t; q_0)(\cdot)\|_{\mathcal{L}(Q;\mathcal{L}(Z))}\\
&\leq M_{q_0+h}\,\|G_q(\cdot; q_0 + h) - G_q(\cdot; q_0)\|_{L^1(0,\infty;\mathcal{L}(Q;\mathcal{L}(Z)))}\\
&\quad + \|D_qT(\cdot; q_0 + \alpha(h)h)\|_{L^\infty(0,\infty;\mathcal{L}(Q;\mathcal{L}(Z)))}\,\|G_q(\cdot; q_0)\|_{L^1(0,\infty;\mathcal{L}(Q;\mathcal{L}(Z)))}\,\|h\|\\
&\leq C\|h\|.
\end{aligned}$$

The last inequality follows from H6 and Theorem 3(ii), and by the fact that $G_q(\cdot, q_0) \in L^1(0,\infty;\mathcal{L}(Q;\mathcal{L}(Z)))$, which is a result of H6. ■

In order to obtain the q-Frechét differentiability of $S(\cdot; q)$, we will need the local Lipschitz continuity of the mapping $q \to D_qT(\cdot; q_0)$ when viewed as a mapping from Q into $L^\infty(0,\infty;\mathcal{L}(Q;\mathcal{L}(Z; Z_\delta)))$. This can be achieved by requiring the following hypothesis.

H7: For every $q_0 \in Q_{ad}$, $D_qA(q)T(\cdot; q_0)|_{q=q_0} \in L^1(0,\infty;\mathcal{L}(Q;\mathcal{L}(Z; Z_\delta)))$ and the mapping $q \to D_qA(q)T(\cdot; q_0)$ from Q into $L^1(0,\infty;\mathcal{L}(Q;\mathcal{L}(Z; Z_\delta)))$ is locally Lipschitz continuous at q_0, for all $q_0 \in Q_{ad}$.

Clearly H7 implies H6 (since the Z_δ-norm is stronger that the Z-norm).

THEOREM 5: *Assume H1-H5 and H7 hold. Then, for all $q_0 \in Q_{ad}$, $D_qT(\cdot; q_0) \in L^\infty(0,\infty;\mathcal{L}(Q;\mathcal{L}(Z; Z_\delta)))$ and, the the mapping $q \to D_qT(\cdot; q)$ from Q into the space $L^\infty(0,\infty;\mathcal{L}(Q;\mathcal{L}(Z; Z_\delta)))$ is locally Lipschitz continuous at q_0.*

PROOF: For $t > 0$, $z \in Z$, $h \in Q$, it follows that

$$\begin{aligned}
\|\,[D_qT(t; q_0)h]\ z\|_{Z_\delta} &= \left\|(-A(q_0))^\delta\,([D_qT(t; q_0)]\ h)\,z\right\|_Z\\
&= M_{q_0}\|h\|\,\|z\|_Z\,\|D_qA(q)T(\cdot; q_0)|_{q=q_0}\|_{L^1(0,t;\mathcal{L}(Q;\mathcal{L}(Z;Z_\delta)))}\\
&\leq C(q_0)\,\|h\|\,\|z\|_Z \qquad \text{(by virtue of H7)}
\end{aligned}$$

Hence, $\|D_qT(t; q_0)\,h\|_{\mathcal{L}(Z;Z_\delta)} \leq C(q_0)\,\|h\|$, and $\|D_qT(t; q_0)\|_{\mathcal{L}(Q;\mathcal{L}(Z;Z_\delta))} \leq C(q_0)$.

Since $C(q_0)$ does not depend on $t > 0$, it follows that $D_qT(\cdot;q_0) \in L^\infty(0,\infty;\mathcal{L}(Q;\mathcal{L}(Z;Z_\delta)))$. The Lipschitz continuity of this mapping is obtained following the same steps as in Theorem 4. ■

This result implies that $q \to T(\cdot;q)$ is Fréchet differentiable as a mapping from Q into $L^\infty(0,\infty;\mathcal{L}(Q;\mathcal{L}(Z;Z_\delta)))$. In fact, THEOREM 6: *Under the same hypotheses of Theorem*

5, $T(\cdot;q)$ is Fréchet differentiable at q_0, for each $q_0 \in Q_{ad}$, when viewed as a mapping from Q into $L^\infty(0,\infty;\mathcal{L}(Z;Z_\delta))$.

PROOF: For $h \in Q$ with $\|h\| < \tilde{\delta}$ so that $q_0 + \alpha h \in Q_{ad}$, α satisfying $|\alpha| \leq 1$, $\beta(h)$ appropriately chosen, $0 \leq |\beta(h)| \leq 1$, and any $t > 0$ it follows that

$$\begin{aligned}
&\|T(t;q_0+h) - T(t;q_0) - [D_qT(t;q_0)]\,h\|_{\mathcal{L}(Z;Z_\delta)} \\
&\leq C(q_0)\,\|\beta(h)h)\|\;\|h\| \\
&\leq C(q_0)\,\epsilon\,\|h\|, \qquad\qquad \text{for } \|h\| < \epsilon, \text{ for all } \epsilon \text{ such that } 0 < \epsilon \leq \tilde{\delta}.
\end{aligned}$$

■

Note that Theorems 3 and 6 imply that the solution $z_h(t;q)$ of the linear homogeneous problem associated to $(\mathcal{P})_q$ is Fréchet differentiable with respect to q, both as a mapping into Z and into Z_δ, respectively. Theorems 4 and 5 imply, moreover, that the corresponding Fréchet derivatives are locally Lipschitz continuous.

The following generalization of Growall's Lemma for singular kernels will be needed later. Its proof can be found in [6], Lemma 7.1.1.

LEMMA 7: *Let L, T, δ be positive constants, $\delta < 1$, $a(t)$ a real valued, nonnegative, locally integrable function on $[0,T]$ and $\mu(t)$ a real-valued function on $[0,T]$ satisfying*

$$\mu(t) \leq a(t) + L\int_0^t \frac{\mu(s)}{(t-s)^\delta}\,ds, \quad t \in [0,T].$$

Then, there exists a constant K depending only on δ such that

$$\mu(t) \leq a(t) + KL\int_0^t \frac{a(s)}{(t-s)^\delta}\,ds, \quad t \in [0,T].$$

Before proving the Fréchet differentiability of the mapping $q \to S(\cdot;q)$ from $Q \to L^\infty(0,T;Z_\delta)$, we will show that if $F(q,t,z)$ satisfies appropriate regularity properties, such a mapping is locally Lipschitz continuous at q_0, for all $q_0 \in Q_{ad}$. This result will be needed later.

Consider the hypothesis:

H8: The mapping $q \to F(q,\cdot;z)$ from Q into $L^\infty(0,T;Z)$ is locally Lipschitz continuous for all $z \in Z_\delta$ with Lipschitz constant independent of z on Z_δ-bounded sets.

THEOREM 8: *Let $q_0 \in Q_{ad}, z_0 \in D_\delta$ and assume H1-H5, H7 and H8 hold. Then the mapping $q \to S(\cdot;q)$ from $Q \to L^\infty(0,T;Z_\delta)$ is locally Lipschitz continuous at q_0.*

PROOF: Unsing Theorem 3 we write

$$\begin{aligned}
&S(t;q_0+h)-S(t;q_0)=\\
&\quad=\int_0^t T(t-s;q_0+h)\left[F(q_0+h,s,z(s;q_0+h))-F(q_0,s,z(s;q_0+h))\right]ds\\
&\quad+\int_0^t T(t-s;q_0+h)\left[F(q_0,s,z(s;q_0+h))-F(q_0,s,z(s;q_0))\right]ds\\
&\quad+\int_0^t D_qT(t-s;q_0+\beta(h)h)\,hF(q_0,s,z(s;q_0))\,ds,
\end{aligned}$$

where $q_0+h\in Q_{ad}$ for all $\|h\|\le\gamma_1$ and $\beta(h)$ is an appropriately selected constant satisfying $0\le|\beta(h)|\le 1$.
Using H8, H3 and Theorem 5 it then follows that

$$\begin{aligned}
&\|S(t;q_0+h)-S(t;q_0)\|_\delta\\
&\le\int_0^t\frac{M_{q_0+h}e^{-\varepsilon_0(t-s)}}{(t-s)^\delta}C_1\|h\|\,ds+\int_0^t\frac{M_{q_0+h}e^{-\varepsilon_0(t-s)}}{(t-s)^\delta}L\,\|z(s;q_0+h)-z(s;q_0)\|_\delta+C_2\|h\|\\
&\le C_3\|h\|\;+\;C_4\int_0^t\frac{\|\left[D_qT\left(s;q_0+\beta(h)h\right)h\right]z_0+S(s;q_0+h)-S(s;q_0)\|_\delta}{(t-s)^\delta}\,ds\\
&\le C_5\|h\|\;+\;C_4\int_0^t\frac{\|S(s;q_0+h)-S(s;q_0)\|_\delta}{(t-s)^\delta}\,ds.
\end{aligned}$$

Hence, by Lemma 7, there exist a constant K such that

$$\|S(t;q_0+h)-S(t;q_0)\|_\delta\le C_5\|h\|+KC_4C_5\|h\|\int_0^T\frac{1}{(t-s)^\delta}\,ds\;\doteq\;C_6\|h\|,\quad t\in[0,T],$$

provided $\|h\|\le\gamma_1$. The Theorem follows. ■

It is appropriate to note at this point that this result together with Theorem 6 imply that the mapping $q\to z(\cdot;q)$ from Q into $L^\infty(0,T;Z_\delta)$ is locally Lipschitz continuous at q_0. We proceed now to prove the Fréchet differentiability of the mapping $q\to S(t;q)$, corresponding to the nonlinear part of problem $(\mathcal{P})_q$. For that purpose, we consider the following hypothesis.

H9: The mapping $(q,z(\cdot))\to F(q,\cdot,z(\cdot))$ from $Q_{ad}\times L^1(0,T;Z_\delta)$ into $L^\infty(0,T;Z)$ is Fréchet differentiable in both variables, the mapping $(q,z(\cdot))\to F_q(q,\cdot,z(\cdot))$ from $Q\times L^\infty(0,T;Z_\delta)$ into $L^\infty(0,T:\mathcal{L}(Q;Z_\delta))$ is locally Lipschitz continuous with respect to q and z, with Lipschitz constant independent of z on Z_δ-bounded sets and $F_z(q,\cdot,z(\cdot;q))\in L^\infty(0,T;\mathcal{L}(Z;Z_\delta))$.

Clearly H9 is stronger than H8.
Theorem 9: *Let $q_0\in Q_{ad}$, $z_0\in D_\delta$ and suppose H1-H5, H7 and H9 hold. Then the mapping $q\to S(t;q)=\int_0^t T(t-s;q)F(q,s,z(s;q))\,ds$ from $Q\to L^\infty(0,T;Z_\delta)$ is Fréchet differentiable at q_0. Moreover, for any $t\in[0,T]$, and any $h\in Q_{ad}$, $[D_qS(t;q_0)]h\doteq w_h(t)$ satisfies the integral equation*

$$w_h(t) = \int_0^t \Big\{ T(t-s;q_0)\Big[F_q(q_0,s,z(s;q_0))h + F_z\left(q_0,s,z(s;q_0)\right)\left[D_qT(s;q_0)z_0\right]h$$

(10)

$$+ F_z(q_0,s,z(s;q_0))w_h(s)\Big] + \left[D_qT(t-s;q_0)F(q_0,s,z(s;q_0))\right]\,h \Big\}\, ds,$$

and $w_h(t)$ is the solution of the following non-homogeneous linear IVP, the so-called "sensitivity equation" for $S(t;q)$, in Z:

$$(S_2)\begin{cases} \frac{d}{dt}w_h(t) = \big(A(q_0)+F_z(q_0,t,z(t;q_0))\big)w_h(t)F_q(q_0,t,z(t;q_0))h + \\ \qquad + F_z(q_0,t,z(t;q_0))[D_qT(t;q_0)z_0]h + \int_0^t D_qA(q)T(t-s;q_0)\big|_{q=q_0} h\, F\left(q_0,s,z(s;q_0)\right)\, ds \\ w_h(0) = 0. \end{cases}$$

PROOF: That the solution $w_h(t)$ of(S_2) satisfies (10)follows immediately (S_1) in Theorem 2 and the fact that $[D_qT(0;q_0)z]h = 0$ for $z \in Z$ and $h \in Q$.

We write

$$\begin{aligned} &S(t;q_0+h) - S(t;q_0) - w_h(t) = \\ &= \int_0^t T(t-s;q_0)\big[F(q_0+h,s,z(s;q_0)) - F(q_0,s,z(s;q_0)) - F_q(q_0,s,z(s;q_0))h\big]\, ds \\ &\quad + \int_0^t T(t-s;q_0)\Big[F(q_0,s,z(s;q_0+h)) - F(q_0,s,z(s;q_0)) \\ &\qquad\qquad - F_z(q_0,s,z(s;q_0))\big(z(s;q_0+h)-z(s;q_0)\big)\Big]\, ds \\ &\quad + \int_0^t T(t-s;q_0)F_z(q_0,s,z(s;q_0))\big[S(s;q_0+h) - S(s;q_0) - w_h(s)\big]\, ds \\ &\quad + \int_0^t T(t-s;q_0)F_z(q_0,z(s;q_0))\Big[\left[D_qT(s;q_0+\alpha(h)h)z_0\right]h - \left[D_qT(s;q_0)z_0\right]h\Big]\, ds \\ &\quad + \int_0^t \Big\{T(t-s;q_0+h)F(q_0,s,z(s;q_0)) - T(t-s;q_0)F(q_0,s,z(s;q_0)) \\ &\qquad\qquad - \big[D_qT(t-s;q_0)F(q_0,s,z(s;q_0))\big]h\Big\}\, ds \\ &\quad + \int_0^t T(t-s;q_0+h)\big[F(q_0+h,s,z(s;q_0+h)) - F(q_0,s,z(s;q_0))\big]\, ds \\ &\quad - \int_0^t T(t-s;q_0)\big[F(q_0+h,s,z(s;q_0)) - 2F(q_0,s,z(s;q_0)) + F(q_0,s,z(s;q_0+h))\big]\, ds \\ &\doteq \sum_{i=1}^7 I_i, \end{aligned}$$

where I_i is the i^{th} term in the expression written above. Here, $\alpha(h)$ is an appropriately chosen constant satisfying $0 \le |\alpha(h)| \le 1$.

In what follows, C_i will denote a generic finite positive constant depending on q_0. Let $\gamma_1 > 0$ be such that $q_0 + \eta \in Q_{ad}$ for all $\eta \in Q$ satisfying $\|\eta\| < \gamma_1$. Then for any $h \in Q_{ad}$

with $\|h\| < \gamma_1$ it follows, by virtue of Theorem 8 and hypothesis H9 that there exist positive constants C_1, C_2 and L, such that:

$$\begin{aligned}\| I_6 + I_7\|_\delta &\leq C_1\|h\|^2 + \int_0^t \frac{L}{(t-s)^\delta}\Big(|\alpha_1(h) - \alpha_3(h)|\,\|h\| + \|z(s;q_0+h) - z(s;q_0)\|_\delta\Big)\|h\|\,ds \\ &+ \int_0^t \frac{C_2}{(t-s)^\delta}\,\|z(s;q_0+h) - z(s;q_0)\|_\delta\,\|h\|\,ds \\ &\leq C_3 \parallel h \parallel^2, \quad \text{provided} \ \parallel h \parallel \leq \gamma_1, \end{aligned} \tag{11}$$

where the last inequality follows from the Lipschitz continuity of the mapping $q \to z(\cdot;q)$ from Q into $L^\infty(0,T;Z_\delta)$ at q_0. Now let ε be a fixed positive constant. It follows from H9 that there exist $\gamma_2 > 0$ and $\gamma_3 > 0$ such that

$$\|I_1\|_\delta \leq \int_0^t \frac{C_4}{(t-s)^\delta}\,\varepsilon\|h\|\,ds \leq C_5\,\varepsilon\,\|h\|, \tag{12}$$

provided $\|h\| \leq \gamma_2$, and also

$$\begin{aligned}\|I_2\|_\delta &\leq \int_0^t \frac{C_6\,\varepsilon}{(t-s)^\delta}\,\|z(s;q_0+h) - z(s;q_0)\|_Z\,ds \\ &\leq C_7\,\varepsilon\,\|h\|, \quad \text{provided}\|h\| \leq \gamma_3.\end{aligned} \tag{13}$$

Also, by using H9 we have that $F_z\,(q_0,\cdot,z(\cdot;q_0)) \in L^\infty(0,T;\mathcal{L}(Z;Z_\delta))$, and therefore there exists a constant C_8 such that

$$\|I_3\|_\delta \leq C_8 \int_0^t \frac{\big\|S(s;q_0+h) - S(s;q_0) - w_h(s)\big\|_\delta}{(t-s)^\delta}\,ds. \tag{14}$$

On the other hand, the local Lipschitz continuity of $D_qT(\cdot\,;q_0)$ (Theorem 4), implies the existence of two finite positive constants C_9 and γ_4 such that

$$\|I_4\|_\delta \leq \int_0^t \frac{C_9}{(t-s)^\delta}|\alpha(h)|\,\|h\|^2\,ds \leq C_{10}\|h\|^2, \ \text{provided} \ \|h\| \leq \gamma_4. \tag{15}$$

Finally from Theorem 6 and H9, there are two finite positive constants C_{10} and γ_5 such that

$$\|I_5\|_\delta \leq C(q_0)\varepsilon\|h\| \int_0^t \|F(q_0,s,z(s;q_0))\|_Z\,ds \leq C_{10}\,\varepsilon\,\|h\|, \tag{16}$$

provided $\|h\| \leq \gamma_5$.

From (11)-(16) we conclude that there exist finite positive constants C_{11}, C_{12}, and γ such that for $t \in [0,T]$ and $h \in Q_{ad}$ with $\|h\| \leq \gamma$

$$\begin{aligned}\|S(t;q_0+h) - S(t;q_0) - w_h(t)\|_\delta &\leq C_{11}\,\varepsilon\|h\| + \\ &+ C_{12}\int_0^t \frac{\|S(s;q_0+h) - S(s;q_0) - w_h(s)\|_\delta}{(t-s)^\delta}\,ds.\end{aligned}$$

Hence, applying Lemma 7 we conclude that

$$\|S(t;q_0+h)-S(t;q_0)-w_h(t)\|_\delta \le C_{11}\,\varepsilon\|h\| + KC_{12}C_{11}\varepsilon\|h\|\int_0^t \frac{1}{(t-s)^\delta}\,ds$$
$$\le C_{13}\varepsilon\|h\|, \quad t\in[0,T], \quad \|h\|\le\gamma.$$

hence the mapping $q\to S(\cdot;q)$ from $Q\to L^\infty(0,T;Z_\delta)$ is Fréchet differentiable at q_0 and $w_h(t)$ is the Fréchet derivative of $S(t;q)$ at q_0, i.e. $D_qS(t;q_0)=w_h(t)$. ■

Theorem 10: *Under the same hypotheses of Theorem 9, the mapping $q\to z(\cdot;q)$ from the admissible parameter set Q_{ad} into the solution space $L^\infty(0,T;Z_\delta)$, is Fréchet differentiable at q_0. Moreover, for any $h\in Q$, $t\in[0,T]$, the q-Fréchet derivative of $z(t;q)$ evaluated at q_0 and applied to h, i.e. $[D_qz(t;q_0)]h$ is the solution $v_h(t)$ of the following linear non-homogeneous initial value problem in Z, the sensitivity equation for $z(t;q)$*

$$(S)\begin{cases} \frac{d}{dt}v_h(t) = \big(A(q_0)+F_z(q_0,t,z(t;q_0))\big)\,v_h(t) \;+\; F_q(q_0,t,z(t;q_0))h\,+ \\ \qquad + D_qA(q)T(t;q_0)z_0\big|_{q=q_0}h \;+\; \int_0^t D_qA(q)T(t-s;q_0)\big|_{q=q_0}h\,F\left(q_0,s,z(s;q_0)\right)\,ds \\ \\ v_h(0)=0. \end{cases}$$

Proof: The Fréchet differentiability of $z(t;q)=T(t;q)z_0+S(t;q)$ follows immediately from Theorems 6 and 9 and the sensitivity equation is readily obtained by combining the sensitivity equations (S_1) and (S_2). ■

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An Explicit Solution for a Two-Phase Stefan Problem with a Similarity Exponential Heat Sources

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Abstract

A two-phase Stefan problem with heat source terms in both liquid and solid phases for a semi-infinite phase-change material is considered. The internal heat source functions are given by $g_j(x,t) = (-1)^{j+1}\dfrac{\rho l}{t}\exp\left(-(\frac{x}{2a_j\sqrt{t}}+d_j)^2\right)$ ($j=1$ solid phase; $j=2$ liquid phase), ρ is the mass density, l is the fusion latent heat by unit of mass, a_j^2 is the diffusion coefficient, x is spatial variable, t is the temporal variable and $d_j \in \mathbb{R}$. A similarity solution is obtained for any data when a temperature boundary condition is imposed at the fixed face $x=0$; when a flux condition of the type $-q_0/\sqrt{t}$ ($q_0>0$) is imposed on $x=0$ then there exists a similarity solution if and only if a restriction on q_0 is satisfied.

Key words : Stefan problem, free boundary problem, Lamé-Clapeyron solution, Neumann solution, phase-change process, fusion, sublimation-dehydration process, heat source, similarity solution.

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I. Introduction.

Sublimation-dehydration, which is commonly known as freeze-dying, is used as a method for removing moisture from biological materials, such as food, pharmaceutical, and biochemical products. Some of the advantages of sublimation-dehydration over evaporative drying are that the structural integrity of the material is maintained and product degradation is minimized [1], [13]. The major disadvantage of the freeze-drying process is that it is generally slow, and consequently, the process is economically unfeasible for certain materials. One means of alleviating this problem is through the use of microwave energy. Several mathematical models have been proposed to describe the freeze-drying process without microwave heating [6], [8]. Only a few studies have also included a microwave heat source in the model [1].

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In [9] the one-phase Lamé-Clapeyron-Stefan problem [7] with a particular type of sources was studied and a generalized Lamé-Clapeyron explicit solution was obtained. Moreover, necessary and sufficient conditions were given in order to characterize the source term which provides a unique solution.

Several applied papers give us the significance in order to consider source terms in the interior of the material which can undergo a change of phase, e.g. [3], [5], [10], [14]. Phase-change problems appear frequently in industrial processes; a large bibliography on the subject was given recently in [16].

In [14] there is a mathematical model for sublimation-dehydration with volumetric heating was presented from which analytical solutions for dimensionless temperature, vapor concentration, and pressure were obtained for two different temperature boundary conditions. It was considered a semi-infinite frozen porous medium with constant thermal properties subject to a sublimation-dehydration process involving a volumetric heat source of the type $g(x,t) = \dfrac{const.}{t} \exp\left(-(x+d)^2\right)$, and, a sensitivity study was also conducted in which the effects of the material properties inherent in these solutions were analyzed.

In this paper a semi-infinite homogeneous phase-change material initially in solid phase at the uniform temperature $-C < 0$, with a volumetric heat source, is considered. A mathematical description for the temperature within the material is given by

$$\frac{\partial T_2}{\partial t}(x,t) = a_2^2 \frac{\partial^2 T_2}{\partial x^2}(x,t) + \frac{1}{\rho c_2} g_2(x,t), \qquad 0 < x < s(t), \qquad t > 0; \tag{1}$$

$$\frac{\partial T_1}{\partial t}(x,t) = a_1^2 \frac{\partial^2 T_1}{\partial x^2}(x,t) + \frac{1}{\rho c_1} g_1(x,t), \qquad x > s(t), \qquad t > 0; \tag{2}$$

for two given internal source functions ([9], [14]) given by

$$g_j = g_j(x,t) = (-1)^{j+1} \frac{\rho l}{t} \exp\left(-(\frac{x}{2a_j\sqrt{t}} + d_j)^2\right) \quad j = 1, 2, \tag{3}$$

ρ is the mass density, l is the fusion latent heat per unit of mass, a_j^2 is the diffusion coefficient, c_j is the specified heat per unit of mass and k_j is the thermal conductivity, for $j = 1$ (solid phase), 2 (liquid phase).

The initial temperature and the temperature as $x \to \infty$ are assumed to be constant

$$T_1(x,0) = T_1(+\infty, t) = -C < 0, \qquad x > 0, \qquad t > 0. \tag{4}$$

At $x = 0$, two different temperature boundary conditions are considered, the first is a constant temperature condition

$$T_2(0,t) = B > 0, \qquad t > 0 \tag{5}$$

which is studied in Section II, and the second is an assumed heat flux of the form

$$k_2 \frac{\partial T_2}{\partial x}(0,t) = \frac{-q_0}{\sqrt{t}}, \qquad t > 0 \tag{6}$$

which is studied in Section III. This kind of heat flux condition was also considered in several papers, e.g. [2], [11], [12] and [15].

The phase-change interface condition is determined from an energy balance at the free boundary $x = s(t)$:

$$k_1 \frac{\partial T_1}{\partial x}(s(t),t) - k_2 \frac{\partial T_2}{\partial x}(s(t),t) = \rho l \dot{s}(t), \qquad t > 0, \tag{7}$$

where the temperature conditions at the interface are assumed to be constant:

$$T_1(s(t),t) = T_2(s(t),t) = 0, \qquad t > 0. \tag{8}$$

Moreover, the initial position of the free boundary is

$$s(0) = 0. \tag{9}$$

In section II we obtain an explicit solution for the problem (1)-(5),(7)-(9) for internal heat sources given by (3).

In Section III we solve the same free boundary problem but with the heat flux condition of the type $-\frac{q_0}{\sqrt{t}}$ $(q_0 > 0)$ prescribed on the fixed face $x = 0$, and we obtain an explicit solution to this problem if the coefficient q_0 satisfies an appropriate inequality (48) or (49); this restriction on q_0 is new with respect to [14].

II. Solution of the free boundary problem with temperature boundary condition at x=0.

Applying the immobilization domain method (see [4]), we are looking for solutions of the type

$$T_j(x,t) = \theta_j(y) \qquad j = 1, 2, \tag{10}$$

where the new independent spatial variable y is defined by

$$y = \frac{x}{s(t)}. \tag{11}$$

Then, the condition (7) is transformed in

$$k_1\theta_1'(1) - k_2\theta_2'(1) = \rho l s(t)\dot{s}(t), \tag{12}$$

and we must have necessarily that $s(t)\dot{s}(t)$ =const. i.e.,

$$s(t) = 2a_2\lambda\sqrt{t}, \tag{13}$$

where the dimensionless parameter $\lambda > 0$ is unknown.

Next, we define

$$R_j(\eta) = \theta_j\left(\frac{\eta}{\lambda}\right), \qquad j = 1, 2, \qquad \eta = \lambda y, \tag{14}$$

then the problem (1)-(5),(7)-(9) is equivalent to the following one:

$$R_2''(\eta) + 2\eta R_2'(\eta) = \frac{4l}{c_2}\exp\left(-(\eta + d_2)^2\right), \qquad 0 < \eta < \lambda; \tag{15}$$

$$R_1''(\eta) + 2\frac{a_2^2}{a_1^2}\eta R_1'(\eta) = -\frac{4a_2^2 l}{a_1^2 c_2}\exp\left(-\left(\frac{a_2}{a_1}\eta + d_1\right)^2\right), \qquad \eta > \lambda; \tag{16}$$

$$R_1(\lambda) = R_2(\lambda) = 0; \tag{17}$$

$$k_1 R_1'(\lambda) - k_2 R_2'(\lambda) = 2\rho l \lambda a_{2;}^2 \tag{18}$$

$$R_1(+\infty) = -C; \tag{19}$$

$$R_2(0) = B. \tag{20}$$

After some elementary computations, from (15), (17) and (20) we obtain

$$R_2(\eta) = B - (B + \varphi_2(\lambda)) \frac{\operatorname{erf}(\eta)}{\operatorname{erf}(\lambda)} + \varphi_2(\eta), 0 < \eta < \lambda\ , \tag{21}$$

$$\varphi_2(\eta) = \frac{-l\sqrt{\pi}}{c_2 d_2} \left[\operatorname{erf}(\eta + d_2) - \operatorname{erf}(d_2) - \operatorname{erf}(\eta) \exp\left(-d_2^2\right)\right],\ if\ d_2 \neq 0 \tag{22}$$

$$\varphi_2(\eta) = \frac{2l}{c_2}\left[1 - \exp\left(-\eta^2\right)\right],\ if\ d_2 = 0. \tag{23}$$

and, from (16), (17) and (19), we have

$$R_1(\eta) = -\frac{(C + \varphi_1(+\infty))}{\operatorname{erf} c\left(\frac{a_2}{a_1}\lambda\right)} \frac{2}{\sqrt{\pi}} \int_{\frac{a_2}{a_1}\lambda}^{\frac{a_2}{a_1}\eta} \exp(-u^2)du + \varphi_1(\eta), \quad \eta > \lambda, \tag{24}$$

where

$$\begin{aligned} \varphi_1(\eta) \quad &= \quad \frac{l\sqrt{\pi}}{c_1 d_1} \exp\left(-d_1^2\right) [\exp(-2\frac{a_2}{a_1}\lambda d_1) \left(\operatorname{erf}(\frac{a_2}{a_1}\lambda) - \operatorname{erf}(\frac{a_2}{a_1}\eta)\right) + \\ &+ \exp\left(d_1^2\right) \left(\operatorname{erf}(\frac{a_2}{a_1}\eta + d_1) - \operatorname{erf}(\frac{a_2}{a_1}\lambda + d_1)\right)],\ \text{if}\ d_1 \neq 0 \end{aligned} \tag{25}$$

$$\varphi_1(+\infty) = \tfrac{l\sqrt{\pi}}{c_1 d_1} \exp\left(-d_1^2\right) [\exp\left(d_1^2\right) \operatorname{erf} c(\tfrac{a_2}{a_1}\lambda + d_1) - \exp(-2\tfrac{a_2}{a_1}\lambda d_1) \operatorname{erf} c(\tfrac{a_2}{a_1}\lambda)],\ \text{if}\ d_1 \neq 0 \tag{26}$$

or

$$\begin{aligned} \varphi_1(\eta) &= \frac{2l\sqrt{\pi}}{c_1}[\frac{a_2}{a_1}\lambda \left(\operatorname{erf}(\frac{a_2}{a_1}\eta) - \operatorname{erf}(\frac{a_2}{a_1}\lambda)\right) + \\ &+ \frac{1}{\sqrt{\pi}}\left(\exp\left(-\left(\frac{a_2}{a_1}\eta\right)^2\right) - \exp\left(-\left(\frac{a_2}{a_1}\lambda\right)^2\right)\right)]\ ,\ \text{if}\ d_1 = 0 \end{aligned} \tag{27}$$

$$\varphi_1(+\infty) = \frac{2l\sqrt{\pi}}{c_1}[\frac{a_2}{a_1}\lambda \operatorname{erf} c(\frac{a_2}{a_1}\lambda) + -\frac{1}{\sqrt{\pi}} \exp\left(-\left(\frac{a_2}{a_1}\lambda\right)^2\right)]\ ,\ \text{if}\ d_1 = 0 \tag{28}$$

where λ is the unknown coefficient which must verify the condition (18). Furthermore, the Eq.18 for λ is equivalent to the following equation

$$f_1(x) = f_2(x)\ ,\quad x > 0. \tag{29}$$

where

$$f_1(x) = F_0(x) h_1(x)\ \ ,\ \ f_2(x) = Q\left(\frac{a_2}{a_1}x\right)\ h_2(x) \tag{30}$$

with

$$Q(x) = \sqrt{\pi} x \exp(x^2) \operatorname{erf} c(x) \ , \quad F_0(x) = x \operatorname{erf}(x) \exp(x^2), \tag{31}$$

$$h_1(x) = Ste_1 - \frac{\sqrt{\pi}\exp(-d_1^2)}{d_1}\left[\exp(-\tfrac{2d_1a_2}{a_1}x)\operatorname{erf} c(\tfrac{a_2}{a_1}x) - \exp(d_1^2)\operatorname{erf} c(\tfrac{a_2}{a_1}x + d_1)\right] \tag{32}$$

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x \exp(-u^2)du, \quad \operatorname{erf} c(x) = 1 - \operatorname{erf}(x) \ , \ h_2(x) = \frac{Ste_2}{\sqrt{\pi}} - F(x), \tag{33}$$

with

$$F(x) = F_0(x) + \frac{\exp(-d_2^2)}{d_2}\left[\exp(d_2^2)\left(\operatorname{erf}(x+d_2) - \operatorname{erf}(d_2)\right) - \exp(-2xd_2)\operatorname{erf}(x)\right], \tag{34}$$

and

$$Ste_1 = \frac{Cc_1}{l} \ , \ Ste_2 = \frac{Bc_2}{l} \tag{35}$$

are the Stefan number for phase $j = 1$ and $j = 2$ respectively.

Theorem 1 *The Eq.29 has a unique solution $\lambda > 0$. Moreover, the free boundary problem with heat source terms (1)-(5),(7)-(9) has an explicit solution given by*

$$\begin{aligned} &T_1(x,t) = \frac{-\left(C + \varphi_1(+\infty)\right)}{\operatorname{erf} c\left(\frac{a_2}{a_1}\lambda\right)}\left[\operatorname{erf}\left(\frac{x}{2a_1\sqrt{t}}\right) - \operatorname{erf}\left(\frac{a_2}{a_1}\lambda\right)\right] + \varphi_1(\frac{x}{2a_2\sqrt{t}}), \\ &\textit{for} \ \ x > s(t), \qquad t > 0; \\ &T_2(x,t) = B - \left(B + \varphi_2(\lambda)\right)\frac{\operatorname{erf}(\frac{x}{2a_2\sqrt{t}})}{\operatorname{erf}(\lambda)} + \varphi_2(\tfrac{x}{2a_2\sqrt{t}}) \ , \\ &\textit{for} \ \ 0 < x < s(t), \qquad t > 0; \end{aligned} \tag{36}$$

*where $\varphi_1(\eta)$ and $\varphi_2(\eta)$ are defined in $(25) - (28)$ and $(22) - (23)$ respectively. The free boundary $s(t)$ is given by (13) where the coefficient λ is the unique solution of Eq.*29.

Proof. Taking into account the Lemma 2 (see below) we can prove that Eq.29 has a unique solution $\lambda > 0$. We invert the relations (14), (10) and (11) in order to obtain an explicit solution of problem (1)-(5),(7)-(9) with the source terms g_j defined by (3).■

Lemma 2 *A) Functions $Q(x), F_0(x)$ and $F(x)$ satisfy the following properties:*

(*i*) $Q(0) = 0, \ Q(+\infty) = 1, \ Q'(x) > 0, \ \forall x > 0, Q'(0) = \sqrt{\pi}.$

(*ii*) $F_0(0) = 0 \ , \ F_0(+\infty) = +\infty \ , \ F_0'(x) > 0, \ \forall x > 0.$

(*iii*) $F(0) = 0 \ , \ F(+\infty) = +\infty \ , \ \dfrac{\partial F}{\partial x}(x) > 0, \ \forall x > 0.$

B) (a) Function $f_1(x)$, satisfies the following properties:

(*i*) $f_1(0^+) = 0 \ , \ $ (*ii*) $f_1(+\infty) = +\infty,$

(iii) if condition (47) *is verified then* $f_1(x) > 0,\ \forall x > 0,$
$\dfrac{\partial f_1}{\partial x}(x) > 0$ *and* $\dfrac{\partial f_1}{\partial x}(0^+) = 0^+,$
(iv) if conditions (47) *is not verified then* $f_1(\xi_1) = 0$ *and* $f_1(x)$ *is negative in* $(0, \xi_1)$, *and is positive in* $(\xi_1, +\infty)$; *then there exists* $x_1 \in (0, \xi_1)$ *such that* $\dfrac{\partial f_1}{\partial x}(x_1) = 0$. *Moreover we have* $\dfrac{\partial f_1}{\partial x}(x) > 0$ $\forall\, x > \xi_1$.

(b) Function $f_2(x)$ *satisfies the following properties:*

$(i)\ f_2(0^+) = 0\ ,\ \ (ii)\ f_2(+\infty) = -\infty\ ,\ \ (iii)\ f_2(\xi_2) = 0,$

$$(iii)\ \frac{\partial f_2}{\partial x}(x) = \frac{a_2}{a_1} Q'\left(\tfrac{a_2}{a_1}x\right)\ h_2(x) + Q\left(\tfrac{a_2}{a_1}x\right)\ \frac{\partial h_2}{\partial x}(x),$$

$$(iv)\ \frac{\partial f_2}{\partial x}(0^+) = \frac{a_2}{a_1} Ste_2 > 0,$$

(v) there exists $x_2 \in (0, \xi_2)$ *such that* $\dfrac{\partial f_2}{\partial x}(x_2) = 0,$

$$(vi)\ \frac{\partial f_2}{\partial x}(x) < 0\ ,\ \forall\, x > \xi_2.$$

C) *Function* $W(x)$ *satisfies the following properties:*

$$(i)\ \ W(0) = \frac{a_1}{a_2\sqrt{\pi}}[Ste_1 - 2\sqrt{\pi}P(d_1)] \text{ if } d_1 \neq 0,\ \textit{where } P \textit{ is defined by}$$
$$P(x) = \frac{\exp(-x^2) - \operatorname{erf} c(x)}{2x},$$

$$(ii)\ \ W(0) = \frac{a_1}{a_2\sqrt{\pi}}[Ste_1 - 2] \text{ if } d_1 = 0, \tag{37}$$

$$(iii)\ W(+\infty) = +\infty,$$

(iv) if condition (47)*is verified then* $W(0) \geq 0$ *and* $\dfrac{\partial W}{\partial x}(x) > 0,\ \forall x > 0.$

D) *Function* $V(x)$ *satisfies the following properties:*

$$(i)\ \ V(0) = \frac{q_0}{\rho l a_2}\ ,\ \ (ii)\ V(+\infty) = -\infty\ \ ,\ (iii)\ \frac{\partial V}{\partial x}(x) < 0,\ \forall x > 0.$$

III. Solution of the free boundary problem with a heat flux condition on the fixed face x=0.

In this section we consider the problem (1)-(5),(7)-(9), but condition (5) will be replaced by condition (6) (see [12], [15]). We can define the same transformations (10),(11) and (14) as were done for the previous problem, and we obtain (15)-(19) and

$$R_2'(0) = \frac{-2q_0}{\rho c_2 a_2} \tag{38}$$

It is easy to see that the free boundary must be of the type $s(t) = 2a_2\mu\sqrt{t}$ where μ is a dimensionless constant to be determined. The solution to the problem (15)-(19) and (38) is given by

$$R_1(\eta) = -\frac{(C+\varphi_3(+\infty))}{\operatorname{erf} c\left(\frac{a_2}{a_1}\mu\right)}\left[\operatorname{erf}\left(\frac{a_2}{a_1}\eta\right) - \operatorname{erf}\left(\frac{a_2}{a_1}\mu\right)\right] + \varphi_3(\eta), \quad \eta > \mu, \tag{39}$$

where

$$\begin{aligned} \varphi_3(\eta) &= \frac{l\sqrt{\pi}}{c_1 d_1}\exp\left(-d_1^2\right)[\exp(-2\frac{a_2}{a_1}\mu d_1)\left(\operatorname{erf}(\frac{a_2}{a_1}\mu) - \operatorname{erf}(\frac{a_2}{a_1}\eta)\right) + \\ &\quad + \exp\left(d_1^2\right)\left(\operatorname{erf}(\frac{a_2}{a_1}\eta + d_1) - \operatorname{erf}(\frac{a_2}{a_1}\mu + d_1)\right)], \text{ if } d_1 \neq 0 \end{aligned} \tag{40}$$

$$\varphi_3(+\infty) = \frac{l\sqrt{\pi}}{c_1 d_1}\exp\left(-d_1^2\right)[\exp\left(d_1^2\right)\operatorname{erf} c(\tfrac{a_2}{a_1}\mu + d_1) - \exp(-2\tfrac{a_2}{a_1}\mu d_1)\operatorname{erf} c(\tfrac{a_2}{a_1}\mu)], \text{ if } d_1 \neq 0 \tag{41}$$

or

$$\begin{aligned} \varphi_3(\eta) &= \frac{2l\sqrt{\pi}}{c_1}[\frac{a_2}{a_1}\mu\left(\operatorname{erf}(\frac{a_2}{a_1}\eta) - \operatorname{erf}(\frac{a_2}{a_1}\mu)\right) + \\ &\quad + \frac{1}{\sqrt{\pi}}\left(\exp\left(-\left(\frac{a_2}{a_1}\eta\right)^2\right) - \exp\left(-\left(\frac{a_2}{a_1}\mu\right)^2\right)\right)], \text{ if } d_1 = 0 \end{aligned} \tag{42}$$

$$\varphi_3(+\infty) = \frac{2l\sqrt{\pi}}{c_1}[\frac{a_2}{a_1}\mu\operatorname{erf} c(\frac{a_2}{a_1}\mu) - \frac{1}{\sqrt{\pi}}\exp\left(-\left(\frac{a_2}{a_1}\mu\right)^2\right)], \text{ if } d_1 = 0 \tag{43}$$

and

$$R_2(\eta) = \frac{q_0\sqrt{\pi}}{\rho c_2 a_2}(\operatorname{erf}(\mu) - \operatorname{erf}(\eta)) + \varphi_2(\eta) - \varphi_2(\mu), 0 < \eta < \mu \tag{44}$$

where φ_2 was defined in (22)-(23) and the unknown μ must satisfy the following equation

$$W(x) = V(x) \quad , \quad x > 0 \tag{45}$$

where

$$W(x) = \frac{x\exp(x^2)}{Q\left(\frac{a_2}{a_1}x\right)}[Ste_1 - \frac{\sqrt{\pi}\exp(-d_1^2)}{d_1}\left(\exp\left(-\frac{2a_2}{a_1}xd_1\right)\operatorname{erf} c\left(\frac{a_2}{a_1}x\right) - \exp(d_1^2)\operatorname{erf} c\left(\frac{a_2}{a_1}x + d_1\right)\right)]$$

if $d_1 \neq 0$,

$$W(x) = \frac{x\exp(x^2)\exp\left(-\left(\frac{a_2}{a_1}x\right)^2\right)}{Q\left(\frac{a_2}{a_1}x\right)}[Ste_1\exp\left(\frac{a_2}{a_1}x\right)^2 + 2Q\left(\frac{a_2}{a_1}x\right) - 2], \text{ if } d_1 = 0,$$

and

$$V(x) = \frac{q_0}{\rho l a_2} - x\exp(x^2) + \frac{\exp(-d_2^2)}{d_2}(\exp(-2d_2x) - 1), \text{ if } d_2 \neq 0, \tag{46}$$

$$V(x) = \frac{q_0}{\rho l a_2} - x\exp(x^2) - 2x, \text{ if } d_2 = 0.$$

Theorem 3 *(a) If*

$$Ste_1 \geqslant 2\,,\ \text{if}\, d_1 \geqslant 0 \quad or \quad Ste_1 \geqslant 2\sqrt{\pi}P(d_1),\ \text{if}\, d_1 < 0 \tag{47}$$

then Eq.45 has a unique solution $\mu > 0$ *if and only if* q_0 *satisfies the following inequality*

$$q_0 \ \geq \ 2a_1\rho l\left[\frac{Ste_1}{2\sqrt{\pi}} - P(d_1)\right] \ \text{if}\, d_1 \neq 0, \tag{48}$$

$$q_0 \ \geq \ \frac{a_1\rho l}{\sqrt{\pi}}\left[Ste_1 - 2\right] \ \ \text{if}\, d_1 = 0, \tag{49}$$

where P *was defined in* $(37i)$ *.*

(b) The free boundary problem with sources term (1)-(4), (6)-(9) has an explicit solution given by

$$T_1(x,t) = \frac{-\left(C + \varphi_3(+\infty)\right)}{\operatorname{erf} c\left(\frac{a_2}{a_1}\mu\right)}\left[\operatorname{erf}\left(\frac{x}{2a_1\sqrt{t}}\right) - \operatorname{erf}\left(\frac{a_2}{a_1}\mu\right)\right] + \varphi_3\left(\frac{x}{2a_2\sqrt{t}}\right) \tag{50}$$
for $x > s(t), \qquad t > 0$

$$T_2(x,t) = \frac{q_0\sqrt{\pi}}{\rho c_2 a_2}\left[\operatorname{erf}(\mu) - \operatorname{erf}\left(\frac{x}{2a_2\sqrt{t}}\right)\right] + \varphi_2\left(\frac{x}{2a_2\sqrt{t}}\right) - \varphi_2\left(\mu\right) \tag{51}$$
for $0 < x < s(t), \qquad t > 0;$

where φ_3 and φ_2 are defined in (40)-(43) and (22)-(23) respectively, the free boundary is given by

$$s(t) = 2a_2\mu\sqrt{t},$$

and μ is the unique solution given in (a).
Proof. To prove (a) we use the definitions of the functions W and V, and Lemma 2. We invert the relations (14), (10) and (11) in order to obtain (50)-(51).■

A more general case for internal heat sources of the non-exponential type will be given in a forthcoming paper.

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An Explicit Solution for a Two-Phase Unidimensional Stefan Problem with a Convective Boundary Condition at the Fixed Face*

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Abstract

In this paper we do the mathematical analysis of the problem which was analysed in S.M. Zubair - M.A. Chaudhry, Wärme- und Stoffübertragung, 30 (1994), 77-81. We consider the solidification of a semi-infinite material which is initially at its liquid phase at a uniform temperature T_i. Suddenly at time $t > 0$ the fixed face $x = 0$ is submitted to a convective cooling condition with a time-dependent heat transfer coefficient of the type $h(t) = h_0 t^{-1/2}$ $(h_0 > 0)$. The bulk temperature of the liquid at a large distance from the solid-liquid interface is T_∞, a constant temperature such that $T_\infty < T_f < T_i$ where T_f is the freezing temperature. The density jump between the two phases are neglected.

We obtain that the corresponding phase-change process has an explicit solution of a similarity type for the solid-liquid interface and the temperature of both phases if an only if the coefficient h_0 is large enough, that is $h_0 > \frac{k_l}{\sqrt{\pi\alpha_l}}\frac{T_i - T_f}{T_i - T_\infty}$ where k_l and α_l are the conductivity and diffusion coefficients of the initial liquid phase.

Key words : Stefan problem, free boundary problem, Neumann solution, phase-change process, solidification process, similarity solution.

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I. Introduction.

Heat transfer problems involving a change of phase due to melting or freezing processes are very important in science and technology [5], [6], [9], [13], [14]. This kind of problems are generally refered as moving-free boundary problems which have been the subject of numerous theoretical, numerical and experimental investigations, e.g. we can see the large bibliography on the subject given in [18].

We consider the solidification of a semi-infinite material which is initially at its liquid phase at a uniform temperature T_i. Suddenly at time $t > 0$ the fixed face $x = 0$ is submitted to a convective cooling condition due to a sudden drop in the ambient temperature. The bulk temperature of the liquid at a large distance from the solid- liquid interface is

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T_∞, a constant such that $T_\infty < T_f < T_i$ where T_f is the freezing temperature. The density jump between the two phases are neglected.

In order to solve the phase-change process with a convective condition at the fixed face $x = 0$, approximate method were used, for example in [2], [8], [10], [12]. In [3], [4] a convective condition is considered after a transformation in order to solve a free boundary problem for a nonlinear absorption model of mixed saturated-unsaturated flow with a nonlinear soil water diffusivity.

In [19] the problem was analyzed and a closed-form expression for the solid-liquid interface and both temperatures were found when the heat transfer coefficient h is time-dependent and proportional to $t^{-\frac{1}{2}}$. The solution is obtain graphically.

The goal of this paper is to give the mathematical analysis of this problem, that is the solidification of a semi-infinite material which is initially at the constant temperature T_i and a convective cooling condition is impossed at the fixed boundary $x = 0$ for a time-dependent heat transfer coefficient of the type

$$h(t) = \frac{h_0}{\sqrt{t}}\ , \ h_0 > 0,\ t > 0. \tag{1}$$

We prove that there exists an instantaneous phase-change process if and only if the coefficient h_0 is large enough, that is

$$h_0 > \frac{k_l}{\sqrt{\pi \alpha_l}} \frac{T_i - T_f}{T_i - T_\infty} \tag{2}$$

where k_l and α_l are the conductivity and diffusion coefficients of the initial liquid phase. Moreover we can obtain the explicit expression for the solid-liquid interface $s(t)$ and the temperatures of the solid $T_s(x,t)$ and liquid $T_l(x,t)$ phases respectively.

The plan is the following: in Section II we solve the heat conduction problem for a semi-infinite material which is initially at a constant temperature T_i and a convective cooling condition of the type (1) is imposed at $x = 0$. The solution can be obtained explicitly and we can conclude that inequality (2) must hold if an instantaneous solidification process occurs.

In Section III we solve the corresponding phase-change problem; we get that the explicit solution for the solid-liquid interface and the temperature of both phases can be obtained if and only if the inequality (2) is verified for the coefficient h_0 which characterizes the dependent-time heat transfer coefficient $h(t)$ given by (1).

II. Heat conduction problem for a semi-infinite material with a convective condition at x=0.

We consider the heat conduction problem for the liquid phase which is initially at the constant temperature T_i and a convective cooling condition at $x = 0$ is imposed, that is

$$T_t = \alpha T_{xx},\ x > 0,\ t > 0 \tag{3}$$

$$T(x,0) = T(+\infty,t) = T_i,\ x > 0,\ t > 0 \tag{4}$$

$$kT_x(x,0) = \frac{h_0}{\sqrt{t}}\left(T(0,t) - T_\infty\right),\ t > 0 \tag{5}$$

where ρ, c, k and $\alpha = \frac{k}{\rho c}$ are the density mass, heat capacity, heat conductivity and diffusion coefficients of the liquid phase (we must consider the subscript l when it is necessary).

Theorem 1 *(i) The explicit solution to the problem* $(3)-(5)$ *is given by*

$$T(x,t) = \frac{T_\infty + \frac{kT_i}{h_0\sqrt{\pi\alpha}}}{1+\frac{k}{h_0\sqrt{\pi\alpha}}} + \frac{T_i - T_\infty}{1+\frac{k}{h_0\sqrt{\pi\alpha}}} \operatorname{erf}\left(\frac{x}{2\sqrt{\alpha t}}\right), \quad x>0,\ t>0, \tag{6}$$

where the error function erf *is defined by*

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x \exp\left(-z^2\right) dz. \tag{7}$$

(ii) The temperature at the fixed face $x=0$ *is constant for all* $t>0$ *and it is given by*

$$T(0,t) = \frac{T_\infty + \frac{kT_i}{h_0\sqrt{\pi\alpha}}}{1+\frac{k}{h_0\sqrt{\pi\alpha}}}, \quad \forall t>0. \tag{8}$$

(iii) The material will undergo an instantaneous phase-change process when the coefficient h_0 *verifies the condition* (2).

Proof.

(i) By using the similarity method [1], [7], [11], [17], we get that a solution of the Eq. (3) is given by

$$T(x,t) = A + B \operatorname{erf}\left(\frac{x}{2\sqrt{\alpha t}}\right)$$

where coefficients A and B must be determined by impossing the two boundary conditions (4) and (5). From these conditions we obtain the following system of equations

$$A + B = T_i \tag{9}$$

$$kB = h_0 (A - T_\infty) \sqrt{\pi\alpha} \tag{10}$$

which solution is given by

$$A = \frac{T_\infty + \frac{kT_i}{h_0\sqrt{\pi\alpha}}}{1+\frac{k}{h_0\sqrt{\pi\alpha}}}, \qquad B = \frac{T_i - T_\infty}{1+\frac{k}{h_0\sqrt{\pi\alpha}}} \tag{11}$$

that is the expression (6) holds.

(ii) It follows by taking $x=0$ in expression (6).

(iii) The material will undergo an instantaneous phase-change process if the constant temperature at the fixed face $T(0,t)$ is less than the freezing temperature T_f, that is

$$\frac{T_\infty + \frac{kT_i}{h_0\sqrt{\pi\alpha}}}{1+\frac{k}{h_0\sqrt{\pi\alpha}}} < T_f \Longleftrightarrow T_\infty + \frac{kT_i}{h_0\sqrt{\pi\alpha}} < T_f + \frac{T_f k}{h_0\sqrt{\pi\alpha}}$$

$$\Longleftrightarrow \frac{k}{h_0\sqrt{\pi\alpha}}(T_i - T_f) < T_f - T_\infty \Longleftrightarrow \textit{condition } (2). \blacksquare$$

Remark 1 *The method utilized in the previous proof follows [15], [16]; it is useful in order to give us the necessary condition* (2) *for the coefficient* h_0 *but it does not give us the explicit solution for the solid-liquid interface and temperatures for the liquid and solid phases which will be the goal of the following Section III.*

III. Instantaneous phase-change process and its corresponding explicit solution.

We consider the following free boundary problem: find the solid-liquid interface $x = s(t)$ and the temperature $T(x,t)$ defined by

$$T(x,t) = \begin{cases} T_s(x,t) & \text{if } 0 < x < s(t), \quad t > 0 \\ T_f & \text{if } x = s(t), \quad t > 0 \\ T_l(x,t) & \text{if } x > s(t), \quad t > 0 \end{cases}$$

which satisfy the following equations and boundary conditions

$$T_{s_t} = \alpha_s T_{s_{xx}}, \ 0 < x < s(t), \ t > 0 \tag{12}$$

$$T_{l_t} = \alpha_l T_{l_{xx}}, \ x > s(t), \ t > 0 \tag{13}$$

$$T_s(s(t),t) = T_l(s(t),t) = T_f, \ x = s(t), \ t > 0 \tag{14}$$

$$T_l(x,0) = T_l(+\infty,t) = T_i, \ x > 0, \ t > 0 \tag{15}$$

$$k_s T_{s_x}(0,t) = \frac{h_0}{\sqrt{t}}\left(T_s(0,t) - T_\infty\right), \ t > 0 \tag{16}$$

$$k_s T_{s_x}(s(t),t) - k_l T_{l_x}(s(t),t) = \rho l \, \dot{s}(t), \ t > 0 \tag{17}$$

$$s(0) = 0 \tag{18}$$

where the subscripts s and l represent the solid and liquid phases respectively, ρ is the common density of mass and l is the latent heat of fusion, and $T_\infty < T_f < T_i$.

We obtain the following results:

Theorem 2 *(i) If the coefficient* h_0 *verifies the inequality* (2) *then the free boundary problem* (12)—(18) *has the explicit solution of a similarity type given by*

$$s(t) = 2\lambda\sqrt{\alpha_l t} \tag{19}$$

$$T_s(x,t) = T_\infty + \frac{(T_f - T_\infty)\left[1 + \frac{h_0\sqrt{\pi\alpha_s}}{k_s}\operatorname{erf}\left(\frac{x}{2\sqrt{\alpha_s t}}\right)\right]}{1 + \frac{h_0\sqrt{\pi\alpha_s}}{k_s}\operatorname{erf}\left(\lambda\sqrt{\frac{\alpha_l}{\alpha_s}}\right)} \tag{20}$$

$$T_l(x,t) = T_i - (T_i - T_f)\frac{\operatorname{erf}c\left(\frac{x}{2\sqrt{\alpha_l t}}\right)}{\operatorname{erf}c(\lambda)} \tag{21}$$

where $\operatorname{erf}c$ *is the complementary error function defined by* $\operatorname{erf}c(z) = 1 - \operatorname{erf}(z)$, $\forall z \geqslant 0$*; and the dimensionless parameter* $\lambda > 0$ *satisfies the following equation*

$$F(x) = x, \ x > 0 \tag{22}$$

where function F *and the* $b's$ *coefficients are given by*

$$F(x) = b_1\frac{\exp(-bx^2)}{1 + b_2\operatorname{erf}\left(x\sqrt{b}\right)} - b_3\frac{\exp(-x^2)}{\operatorname{erf}c(x)} \tag{23}$$

$$b = \frac{\alpha_l}{\alpha_s} > 0; \qquad b_1 = \frac{h_0 \left(T_f - T_\infty\right)}{\rho l \sqrt{\alpha_l}} > 0 \tag{24}$$

$$b_2 = \frac{h_0}{h_s} \sqrt{\pi \alpha_s} > 0; \qquad b_3 = \frac{c_l \left(T_i - T_f\right)}{l \sqrt{\pi}} > 0. \tag{25}$$

(ii) The Eq. (22) has a unique solution if and only if the coefficient h_0 satisfies the inequality (2). In this case, there exists an instantaneous solidification process.

Proof.

Following the Neumann's method [6], [7], [17], the solution of the free boundary problem (12)-(18) is given by

$$T_s(x,t) = A + B \operatorname{erf}\left(\frac{x}{2\sqrt{\alpha_s t}}\right) \tag{26}$$

$$T_l(x,t) = C + D \operatorname{erf}\left(\frac{x}{2\sqrt{\alpha_l t}}\right) \tag{27}$$

$$s(t) = 2\lambda \sqrt{\alpha_l t} \tag{28}$$

where the coefficients A, B, C, D and λ must be determinated by imposing conditions $(14) - (17)$. We obtain

$$A = \frac{T_f + T_\infty \frac{h_0 \sqrt{\pi \alpha_s}}{k_s} \operatorname{erf}\left(\lambda \sqrt{\frac{\alpha_l}{\alpha_s}}\right)}{1 + \frac{h_0 \sqrt{\pi \alpha_s}}{k_s} \operatorname{erf}\left(\lambda \sqrt{\frac{\alpha_l}{\alpha_s}}\right)} \tag{29}$$

$$B = \frac{h_0 \sqrt{\pi \alpha_s}}{k_s} \frac{T_f - T_\infty}{1 + \frac{h_0 \sqrt{\pi \alpha_s}}{k_s} \operatorname{erf}\left(\lambda \sqrt{\frac{\alpha_l}{\alpha_s}}\right)} \tag{30}$$

$$C = \frac{T_f - T_i \operatorname{erf}(\lambda)}{\operatorname{erf} c(\lambda)}, \qquad D = \frac{T_i - T_f}{\operatorname{erf} c(\lambda)} \tag{31}$$

and coefficient λ must satisfy the Eq. (22).

Function F has the following properties:

$$F\left(0^+\right) = b_1 - b_3 = \frac{h_0 \left(T_f - T_\infty\right)}{\rho_l \sqrt{\alpha_l}} - \frac{c_l \left(T_i - T_f\right)}{l \sqrt{\pi}} \tag{32}$$

$$F(+\infty) = -\infty, \qquad F'(x) < 0, \ \forall x > 0. \tag{33}$$

Therefore, there exists a unique solution $\lambda > 0$ of the Eq. (22) if and only if $F\left(0^+\right) > 0$, that is inequality (2) holds.■

Remark 2 *(i) We note that the temperature at the fixed face $x = 0$ is given by $T_s(0,t) = A < T_f$ because*

$$T_f - A = T_f - \frac{T_f + T_\infty \frac{h_0 \sqrt{\pi \alpha_s}}{k_s} \operatorname{erf}\left(\lambda \sqrt{\frac{\alpha_l}{\alpha_s}}\right)}{1 + \frac{h_0 \sqrt{\pi \alpha_s}}{k_s} \operatorname{erf}\left(\lambda \sqrt{\frac{\alpha_l}{\alpha_s}}\right)}$$

$$= \frac{\left(T_f - T_\infty\right) \frac{h_0 \sqrt{\pi \alpha_s}}{k_s} \operatorname{erf}\left(\lambda \sqrt{\frac{\alpha_l}{\alpha_s}}\right)}{1 + \frac{h_0 \sqrt{\pi \alpha_s}}{k_s} \operatorname{erf}\left(\lambda \sqrt{\frac{\alpha_l}{\alpha_s}}\right)} > 0.$$

(ii) We note that the inequality (2) *for the coefficient* h_0 *which characterizes the time-dependent heat transfer is of the type that it was obtained in [16] when a time-dependent heat flux condition on the fixed face is impossed.*

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