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Primeras Jornadas sobre Ecuaciones Diferenciales, Optimización y Análisis Numérico

Primera Parte

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SERIE A: CONFERENCIAS, SEMINARIOS Y TRABAJOS DE MATEMÁTICA

No. 7

PRIMERAS JORNADAS SOBRE ECUACIONES DIFERENCIALES, OPTIMIZACIÓN Y ANÁLISIS NUMÉRICO

Primera Parte

Domingo A. Tarzia (Ed.)

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Rosario, Octubre 2004

Las Primeras Jornadas sobre Ecuaciones Diferenciales, Optimización y Análisis Numérico tuvieron lugar en el Departamento de Matemática de la FCE de la Universidad Austral, en Rosario, del 11 al 12 de Marzo de 2004. Fue realizado con el apoyo del Proyecto de Investigación Plurianual "Partial Differential Equations and Numerical Optimization with Applications" subsidiado por la Fundación Antorchas e integrado por los siguientes subproyectos:

- "Free Boundary Problems for the Heat-Diffusion Equation" (UA-UNR-UNRC-UNSa);
- "Inverse and Control Problems in the Mathematical Modeling of Phase Transitions in Shape Memory Alloys" (UNL);
- "Geophysical Scale Stratified Flows and Hydraulic Jumps" (UNC-UBA);
- "Optimization Applied to Mechanical Engineering Problems" (UNS-UNC-UNCo)

El Comité Organizador estuvo compuesto por: M.C. Maciel (Bahía Blanca), R.D. Spies (Santa Fe), D. A. Tarzia, (Rosario, Coordinador); C.V. Turner (Córdoba). La Secretaría Local estuvo compuesta por: A. C. Briozzo (Coordinador), G. G. Garguichevich, M. F. Natale, E. A. Santillan Marcus, M. C. Sanziel.

Las Jornadas estuvieron dirigidas a graduados, profesionales y estudiantes de Matemática, Física, Química, Ingeniería y ramas afines, con conocimientos básicos sobre ecuaciones diferenciales, análisis numérico y optimización.

Esta primera parte contiene cuatro de las conferencias y comunicaciones presentadas; las restantes se publicarán en el MAT – Serie A – 8 (2004). Los manuscritos fueron recibidos y aceptados en octubre de 2004.

CONFERENCIAS Y COMUNICACIONES DE LAS PRIMERAS JORNADAS SOBRE ECUACIONES DIFERENCIALES, OPTIMIZACIÓN Y ANÁLISIS NUMÉRICO

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A Survey of the Spectral Gradient Method[1]

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Abstract

Finite dimensional unconstrained minimization problems are generally solved by iterative methods like Newton, quasi-Newton, steepest descent, conjugate gradient methods and its variations. All of them share the same property: they are descent methods.
In 1988 Barzilai and Borwein introduce the spectral gradient method which have been analyzed by Raydán in 1991. Initially it has been developed for the quadratic case and later has been extended to the general unconstrained minimization problem by using a globalization strategy. Its main characteristic is the non-descent property. Furthermore the method requires few storage locations and very inexpensive computations.
The purpose of this work is to trace the development of the spectral gradient method, in particular when the problem is defined in a Hilbert space. The algorithms are treated in details but in dealing with them we have presented only the most general results available and we have given these a broad brush treatment.

Key words: Projected gradients, non-monotone line search, large scale problems, spectral gradient method.
AMS Subject Classification: 49M07, 49M10, 65K, 90C06, 90C20.

1 Introduction

Unconstrained minimization problems in finite dimension are generally solved by iterative methods like Newton, quasi-Newton, steepest descent and conjugate gradient methods. There are several variations of these methods and all of them share the same property: they are descent methods.

In 1991, Raydán [39, 40] introduces the spectral gradient method which is an extension to the Barzilai and Borwain method [3]. This method has been developed for the quadratic case and its main characteristic is the non-descent property. The method has also been extended to the general unconstrained minimization problem and it results to be globally convergent when a non-monotone line search is incorporated [41].

For the unconstrained problem defined in function space, methods like Newton and like conjugate gradients have been developed and well studied [2, 15, 16, 22, 23, 24, 26, 27, 28, 29, 30, 38, 44].

At the beginning of this decade some attempts to analyze the non-descent spectral gradient method for the infinite dimensional case have been done [1, 33].
The purpose of this work is to trace the development of the spectral gradient method, in particular when the problem is defined in a Hilbert space. The algorithms are treated in details but in dealing with them we have presented only the most general results available and we have

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given these a broad brush treatment.

This work is organized into sections, each dealing with a specific problem and the based-on-spectral gradient algorithm developed for them. Section 2 is devoted to the finite case. The third section to the infinite case and in the fourth the unconstrained control problem is described. The final remarks are established in section 5.

2 Finite dimensional case

Let be $f : \mathbb{R}^n \to \mathbb{R}$ a differentiable function. It is well known that any solution of the unconstrained minimization problem,

$$\min_{x\in\mathbb{R}^n} f(x) \tag{1}$$

solves the nonlinear equations problem:

$$\text{find } x_\star \in \mathbb{R}^n \text{ such that } \nabla f(x_\star) = 0. \tag{2}$$

The methods proposed to solve it are usually iterative procedures: if x_k denotes the current iterate, and if it is not a good estimator of $x_\star$, a better one, $x_{k+1} = x_k + s_k$ is required. Here s_k means the step and it can be obtained by different methods.
In many algorithms, each iteration involves the calculation of a quasi-Newton step: $s_k^{QN} = -A_k^{-1}\nabla f(x_k)$, where $A_k \in \mathbb{R}^{n\times n}$ is an approximation of the Hessian matrix of f at x_k. After each iteration, the current A_k is updated to A_{k+1}, and usually it is chosen satisfying the secant equation:

$$A_{k+1}s_k = y_k \tag{3}$$

where $y_k = \nabla f(x_{k+1}) - \nabla f(x_k)$.

If $n > 1$, this equation does not completely specify the matrix A_{k+1}, so in addition to obeying (3) there are some desirable properties for A_{k+1}. In the spectral gradient method the update is restricted to be $A_{k+1} = \alpha_{k+1} I$ where $\alpha_{k+1} \in \mathbb{R}$ solves the linear system $y_k = \alpha_{k+1} I s_k$ in the least square sense: that is, if $s_k \neq 0$

$$\alpha_{k+1} = \frac{s_k^T y_k}{s_k^T s_k}. \tag{4}$$

Taking these ideas into account, the Spectral Gradient algorithm (SG) is established as:

Algorithm 2.1 Given $x_0 \in \mathbb{R}^n, \alpha_0 \in \mathbb{R}$.
For $k = 0, 1, ...,$ repeat until convergence:

Step 1. $s_k = -\dfrac{1}{\alpha_k}\nabla f(x_k)$

Step 2. $x_{k+1} = x_k + s_k$

Step 3. $y_k = \nabla f(x_{k+1}) - \nabla f(x_k)$

Step 4. $\alpha_{k+1} = \dfrac{s_k^T y_k}{s_k^T s_k}$.

It is important to point out that in this method, the new iterate is chosen in the same direction as the Cauchy method, but another step length is used and in numerical experiments the SG algorithm is significantly faster than the classic gradient method at the same cost per iteration.

The most important features of Algorithm 2.1 are:

- Every iteration requires two inner products, one scalar-vector multiplication, two vector additions and only one gradient evaluation.
- It is a gradient method which uses information of the two previous iterates. It makes a difference with the Cauchy method, which uses only information of the previous iterate.
- It satisfies the *weak* secant equation: $s_k^T A_{k+1} s_k = s_k^T y_k$.
- The scalar α_{k+1} is a Rayleigh quotient of the matrix

$$\int_0^1 \nabla^2 f(x_k + ts_k)dt.$$

If the objective function is quadratic,

$$f(x) = \frac{1}{2}x^T Ax - b^T x + c \tag{5}$$

where A is a symmetric positive definite (SPD) matrix, (4) becomes in:

$$\alpha_{k+1} = \frac{s_k^T As_k}{s_k^T s_k}. \tag{6}$$

In this case, **Steps 3-4** of the algorithm can be changed by (6) and because of α_{k+1} is the Rayleigh quotient of A at s_k, there is not danger of dividing by zero in **Step 1.**
There is a relationship between this method and the shifted power method to approximate eigenvalues and eigenvectors which is extensively used to establish local and global convergence results [39]. This relation holds for the infinite dimensional case and more details are shown in Section 2.

Barzilai and Borwein [3] have established r-superlinear rate of convergence if $n = 2$ and A has two distinct eigenvalues. The convergence of the method when applied to the minimization of a strictly convex quadratic function was established by Raydán [39, 40] in the following theorem:

Theorem 2.1 *Let f(x) be a strictly convex quadratic function. Let $\{x_k\}$ be the sequence generated by the Algorithm 2.1 and $x_\star$ the unique minimizer of f. Then, either $x_j = x_\star$ for some index j, or the sequence $\{x_k\}$converges to $x_\star$.*

Related to the convergence rate, Raydán [39] proved the following results:

Theorem 2.2 *Let $f(x) = \frac{1}{2}x^T Ax - b^T x + c$ where A is a SPD matrix that has only two distinct eigenvalues, $\lambda_1 < \lambda_2$. Assume that for same k, α_k is sufficiently close to λ_1 or λ_2. Then the sequence $\{x_k\}$ from Algorithm 2.1 converges to $x_\star$ q-quadratically. Furthermore, if α_k is equal to either λ_1 or λ_2 then, $x_{k+3} = x_\star$.*

Corollary 2.1 *Under the assumptions of Theorem 2.2, Algorithm 2.1 has the exact r-rate of convergence of $\sqrt[5]{4}$.*

In numerical experiments as well as in the convergence analysis, it can be seen one of the most important features of this proposal: it is a non-monotone method, in some cases the error's norm and the objective function value increase.
If $n > 2$, numerical experiments show that the performance of the algorithm depends on the eigenvalues of A and its condition number $\kappa(A)$. So, in order to improve convergence, this matrix is preconditioned in such a way that either $\kappa(A)$ is reduced or the eigenvalues of A are clustered.

In the quadratic case, the basic idea is to transform the objective function (5) in

$$\hat{f}(x) = \frac{1}{2}x^T \hat{A}x - \hat{b}^T x + \hat{c}$$

where $\hat{A} = E^{-1}AE^T,\ \hat{b} = E^{-1}b,\ \hat{c} = c$ for some nonsingular matrix E.
Therefore, if Algorithm 2.1 is applied to $\hat{f}(x)$ setting $C = EE^T,\ h_k = C^{-1}\nabla f(x_k)$, the preconditioned version is stated as follows:

Algorithm 2.2 Given $x_0 \in \mathbb{R}^n, \alpha_0 \in \mathbb{R}$ and C a symmetric and positive definite matrix, set $g_0 = Ax_0 - b$.
For $k = 0, 1, ...,$ repeat until convergence:

Step 1. Solve $Ch_k = g_k$ for h_k

Step 2. Set $p_k = Ah_k$

Step 3. Set $x_{k+1} = x_k - \dfrac{1}{\alpha_k}h_k$

Step 4. Set $g_{k+1} = g_k - \dfrac{1}{\alpha_k}p_k$

Step 5. Set $\alpha_{k+1} = \dfrac{h_k^T p_k}{g_k^T h_k}$.

The matrix C is called the *preconditioning matrix* and A the *preconditioned matrix*. Numerical results and comparison with the preconditioned conjugate gradient method can be found in [39].

For the general case, the method needs to be incorporated in a globalization scheme; Raydán proposes an algorithm based on the non-monotone line search strategy, introduced by Grippo, Lampariello y Lucidi [25] and proves global convergence [41] when the iterates are generated by the following algorithm.

Algorithm 2.3 Given $x_0 \in \mathbb{R}^n, \alpha_0 \in \mathbb{R}, \delta > 0,\ 0 < \sigma_1 < \sigma_2 < 1,\ \gamma \in (0,1), 0 < \epsilon < 1$ and $M \geq 0$, an integer. Set $k = 0, g_0 = \nabla f(x_0)$.

Step 1. If $\| g_k \| = 0$, stop.

Step 2. If $\alpha_k \leq \epsilon$ or $\alpha_k \geq \dfrac{1}{\epsilon}$, set $\alpha_k = \delta$

Step 3. Set $\lambda = \dfrac{1}{\alpha_k}$

Step 4. (non-monotone line search)
If $f(x_k - \lambda g_k) \leq \max\limits_{0 \leq j \leq min(k,M)} f(x_{k-j}) - \gamma\lambda g_k^T g_k$ set $\lambda_k = \lambda$, $x_{k+1} = x_k - \lambda_k g_k$ and go to Step 6.

Step 5. Choose $\sigma \in [\sigma_1, \sigma_2]$, set $\lambda = \sigma\lambda$ and go to Step 4.

Step 6. Set $\alpha_{k+1} = -\dfrac{g_k^T y_k}{\lambda_k g_k^T g_k}$, $k = k+1$ and go to Step 1.

Dai and Liao [14] have also established r-linear convergence of the method for any dimensional strictly convex quadratics and as a consequence of this result the method is also locally r-linear convergent for general objective functions, and therefore the stepsize in the SG method will always be accepted by the non-monotone line search when the iterate is close to the solution.

These results were extended for the minimization on convex sets [10]. Also, for the large scale non-linear optimization problem solved by a trust region strategy, at each iteration the quadratic subproblem is solved via an algorithm based on the algorithm designed for convex sets [32, 34].

3 The infinite case

In this section we analyze the behavior of the spectral gradient method when it is extended to the unconstrained quadratic problem defined in an infinite dimensional real Hilbert space H. Let A be a bounded operator defined on H and $\langle , \rangle$ the inner product on H. Given the quadratic functional $q(x) = \frac{1}{2}\langle x, Ax\rangle - \langle x, b\rangle$, let us consider the unconstrained minimization problem

$$\min_{x \in H} q(x). \tag{7}$$

Our objective is to analyze the convergence of the algorithm for a self-adjoint and strictly positive operator, which has a numerable amount of eigenvalues $\{\lambda_i\}$, its spectrum $\sigma(A)$ satisfies $\sigma(A) \subseteq [m, M]$ and the system of associate eigenvectors forms an orthonormal basis. Under these assumptions the Algorithm 1 can be sated on H as follows:

Algorithm 3.1 Given $x_0 \in H, \alpha_0 \in \mathbb{R}$,
For $k = 0, 1, ...$, repeat until convergence:

Step 1. $s_k = -\dfrac{1}{\alpha_k}\nabla q(x_k)$

Step 2. $x_{k+1} = x_k + s_k$

Step 3. $y_k = \nabla q(x_{k+1}) - \nabla q(x_k)$

Step 4. $\alpha_{k+1} = \dfrac{\langle s_k, y_k\rangle}{\langle s_k, s_k\rangle}$.

Let us denote $g_k = \nabla q(x_k)$ the first Gâteaux derivative.
Using the relations among y_k, s_k and g_k results

$$\alpha_{k+1} = \frac{\langle g_k, Ag_k\rangle}{\langle g_k, g_k\rangle},$$

the Rayleigh quotient of A evaluated at g_k. Then

$$0 < m \le \alpha_{k+1} \le M. \tag{8}$$

Since the search direction is the negative gradient, the method belongs to the class of gradient methods. The next lemma shows the relationship with the power method what is essential to prove local and global convergence.

Lemma 3.1 *Let $q(x) = \frac{1}{2}\langle x, Ax\rangle - \langle x, b\rangle$, where A is self-adjoint and strictly positive. Let x_* be the unique minimizer of q, $\{x_k\}$ the sequence generated by the algorithm and $e_k = x_\star - x_k$, for all k. Then:*

1) $Ae_k = \alpha_k s_k = -g_k$

2) $e_{k+1} = \frac{1}{\alpha_k}(\alpha_k I - A)e_k$

3) $s_{k+1} = \frac{1}{\alpha_{k+1}}(\alpha_k I - A)s_k$

4) $g_{k+1} = \frac{1}{\alpha_k}(\alpha_k I - A)g_k.$

The relations established in Lemma 3.1 allow us to conclude that $||e_k||$ goes to zero if and only if $||s_k||$ goes to zero, and it is equivalent to prove that $||g_k||$ goes to zero.

For any initial error e_0 there are constants d_i^0 such that

$$e_0 = \sum_{i=1}^{\infty} d_i^0 v_i.$$

Again, the lemma 3.1 allows us to obtain the following expression for the error

$$e_{k+1} = \sum_{i=1}^{\infty} d_i^{k+1} v_i,$$

where

$$d_i^{k+1} = \left(\frac{\alpha_k - \lambda_i}{\alpha_k}\right) d_i^k = \prod_{j=0}^{k} \left(\frac{\alpha_j - \lambda_i}{\alpha_j}\right) d_i^0.$$

The convergence properties of the sequence $\{e_k\}$ will depend on the behavior of each of the sequences $\{d_i^k\}$. If the spectrum of A satisfies the condition $M < 2m$ we can establish the following result:

Lemma 3.2 *Let $q(x) = \frac{1}{2}\langle x, Ax\rangle - \langle x, b\rangle$, where A is strictly positive such that $M < 2m$. Let $x_\star$ the unique minimizer of q. Then, the sequence $\{x_k\}$ generated by the spectral gradient method converges q-linearly to $x_\star$ and the factor of convergence $c < 1$ is $c = \frac{M-m}{m}$.*

If the restrictive condition $M < 2m$ does not hold, the sequence $\{d_i^k\}$ show a non-monotone behavior, according to the relative position of the scalars α_k and the eigenvalues λ_i in the spectrum, however this situation does not perturb the convergence of the algorithm. Assuming that the sequence of eigenvalues λ_i is increasing

$$0 < m = \lambda_1 < \lambda_2 < \lambda_3......... \tag{9}$$

and $\lambda_i \to M$, the next lemmas help us to prove the main theorem.

Lemma 3.3 *The sequence $\{d_1^k\}$ converges to zero q-linearly and the factor of convergence is $\hat{c}$=$1 - (m/M)$.*

Lemma 3.4 *If for a fixed integer l, $1 \leq l$, the sequences $\{d_1^k\}$,...,$\{d_l^k\}$ converge to zero then*

$$\liminf_{k\to\infty} |d_{l+1}^k| = 0.$$

Theorem 3.1 *Let $q(x) = \frac{1}{2}\langle x, Ax\rangle - \langle x, b\rangle$, where A is self-adjoint and strictly positive operator and the sequence of eigenvalues satisfies (9). Let $\{x_k\}$ be the sequence generated by the spectral gradient method and $x_\star$ the unique minimizer of q. Then, either $x_j = x_\star$, for some j, or the whole sequence $\{x_k\}$ converges to $x_\star$.*

The proofs of these results can be found in [33].

4 An application to control problems

The spectral gradient method has successfully been used in different areas such as Geophysics [4, 6, 12, 13], Physics [7, 37], Chemistry [18, 19, 21, 20, 45], etc. Also have been developed algorithms to solve algebraic nonlinear systems [11, 31], partial differential equations [35, 42] and other nonlinear programming problems [5, 8, 9, 17, 36].

In this section we present an application of the spectral gradient method to an infinite dimension control problem. Optimal control problems belong to a more ample class: the differential equations-constrained optimization problem. A nice review of some aspects of PDE-constrained optimization can be found in [43]. Control problems and their discretized form, viewed as minimization problems have already been solved by quasi-Newton methods. Kelley and Sachs [29] analyze the behavior of the BFGS-secant method when it is applied to the control problem.

Let us consider the general nonlinear control problem to minimize

$$F(u) = \int_0^T L(x(t), u(t), t)dt \tag{10}$$

subject to

$$\begin{aligned} \dot{x} &= f(x(t), u(t), t) \\ x(0) &= x_0, \end{aligned} \tag{11}$$

where $L : \mathbb{R}^{n+m+1} \to \mathbb{R}$ and $f : \mathbb{R}^{n+m+1} \to \mathbb{R}$.
Recall, the gradient of F is given by

$$\nabla F(u) = p(.)^T f_u(x(.), u(.), .) + L_u(x(.), u(.), .), \tag{12}$$

and $p(t)$ solves the adjoint equation

$$\begin{aligned} -\dot{p} &= p(t)^T f_x(x(t)u(t), t) + L_x(x(t), u(t), t) \\ p(T) &= 0. \end{aligned} \tag{13}$$

The second derivative of $F(u)$ is given by

$$\begin{aligned} \langle w, \nabla^2 F(u)v\rangle &= \langle \xi(w), H_{xx}(x, u, .)\xi(v)\rangle + \langle w, H_{ux}(x, u, .)\xi(v)\rangle \\ &\quad + \langle \xi(w), H_{xu}(x, u, .)v\rangle, \end{aligned}$$

where $H(x, u, t) = p(t)f(x, u, t) + L(x, u, t)$, $\xi(w)$ solves an initial value problem and $\langle ., .\rangle$ denotes the inner product defined by $\langle u, v\rangle = \int_0^T u(t)v(t)dt$ for all $u, v \in L^2[0, T]$. It is clear that each gradient evaluation involves the solution of a system of differential equations and it is necessary to compute $\nabla^2 F(u)$ if we want to apply Newton's method. These two facts suggest us to choose appropriate methods not only to solve the ordinary differential systems but also to approximate the Hessian matrix.

Let us consider the unconstrained problem

$$\min F(u), \qquad u \in H. \tag{14}$$

Recall that any solution of (14) is also solution of the nonlinear algebraic system $G(u) = \nabla F(u) = 0$.

The Hilbert space H is approximated by a finite dimensional space H^N and let us replace the functional F by the functional F^N defined on H^N and finally we consider the minimization problem

$$\min F^N(u^N), \qquad u^N \in H^N. \tag{15}$$

Any solution of (15) is also solution of $G^N(u^N) = \nabla F^N(u^N) = 0$.

The algorithm will be analyzed under the following assumptions.
Let $\{P^N\}$ denote a sequence of linear prolongation operators

$$P^N : H^N \to Z,$$

being Z a normed subspace of H, with the property that $\|.\|_Z \geq \|.\|_H$, where $\|.\|_H$ and $\|.\|_Z$ denote the norms in the Hilbert space H and the subspace Z respectively.
Let $\langle , \rangle_N$ denote the inner product on H^N and $\|.\|_N$ the induced norm by such inner product.
Let $G^N : H^N \to H^N$ be a Fréchet-differentiable operator.
A sequence $u^N \in H^N$ is said to be Z-convergent to u, and we denote $u^N \underset{Z}{\to} u \in Z$, if

$$\lim_{N\to\infty} \|P^N u^N - u\|_Z = 0.$$

Assuming,

A1) If $u^N \underset{Z}{\to} u \in Z$, $v^N \underset{Z}{\to} v \in Z$ then $\lim_{N\to\infty} \langle u^N, v^N\rangle_N = \langle u, v\rangle$.

A2) If $u^N \underset{Z}{\to} u$ then $G^N(u^N) \underset{Z}{\to} G(u)$

A3) If $u^N \underset{Z}{\to} u$ then $\lim_{N\to\infty} F^N(u^N) = F(u)$.

The assumptions A1) and A2) are the same as the established by Kelley and Sachs [29]. In order to analyze the spectral gradient algorithm we need the assumption A3).

The discretized function $F^N : H^N \to \mathbb{R}$ from the control problem is not strictly convex. However, it is interesting to analyze the behavior of the algorithm when F^N is strictly convex because of the properties of the finite dimensional algorithm.
We state the following results, whose proof can be found in [1].

Theorem 4.1 *Let $G : H \to H$ and $G^N : H^N \to H^N$ be under the assumptions (A1) and (A2). If for all N, the sequence $\{u_k^N\}_{k\in\mathbf{N}}$ satisfies:*

i) For all $N \in \mathbb{N}$, $\lim_{k\to\infty} \|G^N(u_k^N)\|_N = 0$.

ii) For all $k \in \mathbb{N}$, $u_k^N \underset{Z}{\rightarrow} u_k \in Z$,

then $\lim_{k\to\infty} \|G(u_k)\|_H = 0$.

Lemma 4.1 *Let* $G : H \to H$ *and* $G^N : H^N \to H^N$ *be under assumptions (A1) and (A2). If* $u_0 \in H$ *and* $\alpha_0 \in \mathbb{R}$, *let the sequence* $\{u_0^N\}_{N\in\mathbf{N}}$ *satisfies* $u_0^N \underset{Z}{\rightarrow} u_0$ *and* $\alpha_0^N = \alpha_0$ *for all N. If* $\{u_k^N\}_{k\in\mathbf{N}}$ *is the sequence generated by algorithm 2.1, starting with* u_0^N *and* α_0^N *for each* $N \in \mathbb{N}$, *then for all* $k \in \mathbb{N}$,

$$u_k^N \underset{Z}{\rightarrow} u_k, \quad \alpha_k^N \underset{N\to\infty}{\longrightarrow} \alpha_k$$

Corollary 4.1 *Let* $G : H \to H$ *and* $G^N : H^N \to H^N$ *be under the conditions (A1) and (A2) and assume that* F^N *satisfies the conditions required by the convergence theorem of the spectral gradient method applied to the finite case. Let* $N \in \mathbb{N}$, $\{u_k^N\}_{k\in\mathbf{N}} \subset H^N$ *be the sequence generated by algorithm 2.1. Then*

a) For all $N \in \mathbb{N}$, $\lim_{k\to\infty} \|G^N(u_k^N)\|_N = 0$.

b) For all $k \in \mathbb{N}$, $u_k^N \underset{Z}{\rightarrow} u_k \in Z$.

From the corollary it is possible to conclude that the sequence $\{u_k^N\}$ generated by the Algorithm 2.1 satisfies the assumptions of Theorem 4.1. It means that the sequence generated by the algorithm for the finite dimensional case approximates to the solution of the infinite dimensional problem for a discretization with N large enough.

If the functional $F^N : H^N \to \mathbb{R}$ is a non-linear function not necessarily strictly convex, a globalization strategy is incorporated to the algorithm [41]. Even though global convergence results are known for this case, it is not possible to extend the proof to the infinite dimensional case. When the BFGS method is applied, Kelley and Sachs [29] use the rate of convergence of the algorithm in Hilbert space.

If the globalized spectral gradient method is applied, the difficulty appears when the convergence of $\{\alpha_k^N\}$ to α_k has to be proved. In this case the value α_k^N varies when the line search is used. The number of inner iterations requires by the line search can change with the dimension of the problem. This argument is illustrated in the tables.

Since in the optimal control problem the Hilbert space is $H = L^2[0,T]$, let $N \in \mathbb{N}$ be the discretization parameter, the time interval $[0,T]$ is divided into N subinterval of equal length $h = \frac{T}{N}$. The approximate finite space is $H^N = \mathbb{R}^N$, with the inner product $\langle,\rangle_N$ defined by the composite Simpson's rule.

At each iteration the state and the adjoint equations must be solved to evaluate the functions F^N and G^N. The solutions $x(t)$ and $p(t)$ of the differential equations (11) y (13) are approximated by the fourth order Runge Kutta and Hermite interpolation. The evaluation of $F^N(u^N)$ is made by the composite Simpson's rule. The function $G^N = {p^N}^T f_x^N + L_x^N$ approximates the gradient G in the equation (12) is not, in general, the gradient of the scalar valued function F^N. The following lemma shows that, in spite of this, the directions chosen are descent directions.

Lemma 4.2 *Assuming the assumptions (A2) and (A3), let* $G^N = {p^N}^T f_x^N + L_x^N$ *be a discretization of* G, *the gradient of* $F(u)$. *Let* $\{u^N\}_{N\in\mathbf{N}}$ *be a sequence such that* $u^N \underset{Z}{\rightarrow} u$. *If* $\|G^N(u^N)\| \neq 0$ *then* $-G^N(u^N)$ *is a descent direction of* F^N *from* u^N, *for all* $N \geq N_0$, *for some* $N_0 \geq 0$.

With these choices of the inner product, the functions F^N and G^N, the assumptions (A1)-(A3) are verified.

Different problems have been tested with different discretizations [1]. The low cost of the spectral gradient method allowed us to use high discretizations obtaining very good approximations to the solutions. The following example represents a simple production inventory model. It has beeen solved by Kelley and Sachs [29] by using the BFGS secant method.
The description is the following:

$$F(u) = \int_0^T \frac{e^{-\rho t}}{2} \left(d(x - a(t))^2 + c(u - b(t))^2 \right) dt \tag{16}$$

subject to

$$\begin{aligned} \dot{x} &= u(t) - s(t) \\ x(0) &= x_0. \end{aligned}$$

The parameters were chosen as follows:

$$T = 0.3, \rho = 1,\ d = c = 1,\ a = 15,\ b = 30,\ s(t) = t^2,\ x_0 = 10.$$

Using Pontryagin maximum principle, it is possible to compute the optimal control $u_\star$ for the infinite dimensional problem. It is of the form

$$u_\star(t) = \alpha_1 \lambda_1 e^{\lambda_1 t} + \alpha_2 \lambda_2 e^{\lambda_2 t} + t^2 - 2t + 4$$

where $\lambda_{1/2} = (1 \pm \sqrt{5})/2$ and α_i are other constants. More details about them can be found in [29].

Following the framework proposed by Kelly and Sachs, we use different discretization to solve the problems. The following table report the number of iterates necessary to achieve the tolerance in the norm of the gradient, for different values of N.

Number of iterations		
N	$u_0 = u_\star + 100$	$u_0 = \sqrt{u_\star + 100}$
400	14	13
800	14	13
1600	15	13
3200	15	14
6400	16	16
12800	18	18
25600	26	23

We have already said that the inductive proof of the convergence of the algorithm does not hold when the non monotone line search is added. It occurs because the number of line searches per iteration is independent of the discretization. This fact can be observed in the following table where each iteration of problem is detailed by using a discretization with $N = 12800$ and starting point $u_0 = u_\star + 100$.

Iter	F	$\|G^N\|$	LS
1	1309.91610867154	102.660152979672	3
2	1005.86083986541	87.9911473130837	3
3	772.951943556889	75.4104396549632	2
4	438.939972885863	53.7843562176199	1
5	114.728359380689	27.6994895434872	0
6	2.05149541312622	4.75685850055365	0
7	0.951154993329865	0.745660189477803	0
8	0.934393423093945	0.110252930849182	0
9	0.934121964482021	2.149098455769760E-002	0
10	0.934115833756604	2.579978101994840E-003	0
11	0.934115741378638	4.558961104628168E-004	0
12	0.934115739509478	8.442761702198709E-005	0
13	0.934115739453236	4.945139366441254E-006	0
14	0.934115739453054	1.620860793138945E-006	0
15	0.934115739453047	1.300917098490117E-007	0
16	0.934115739453041	4.037641332388375E-009	0
17	0.934115739453054	1.043105157805257E-009	0
18	0.934115739453049	9.858780458671390E-012	

The following table shows the number of line search at each iteration para first iterations of problem, starting with $u_0 = exp(u_\star + 100)$, and using 11 different discretizations.

Iterate n^o	Number of points in the discretization										
	1600	3200	6400	12800	16000	19200	22400	24200	25000	25600	26000
1	99	100	102	103	103	104	104	104	104	104	104
2	0	0	1	3	3	4	4	4	4	5	5
3	0	0	0	3	3	4	4	4	4	4	4
4	0	0	0	2	2	4	4	4	4	4	4
5	0	0	0	1	1	3	4	4	4	4	4
6	0	0	0	0	0	3	3	3	4	4	4
7	0	0	0	0	0	2	3	3	3	4	4
8	0	0	0	0	0	1	2	3	3	3	3
9	0	0	0	0	0	0	2	2	2	3	3
10	0	0	0	0	0	0	0	1	2	2	2
11	0	0	0	0	0	0	0	0	1	2	2
12	0	0	0	0	0	0	0	0	0	1	1
13	0	0	0	0	0	0	0	0	0	0	0
⋮	⋮										⋮
⋮	⋮										⋮
Total iter.	64	64	67	68	68	72	72	72	74	75	75

5 Conclusions

We have presented a survey of the spectral gradient method for the unconstrained case. It is a relatively novel non-descent method, appropriate to large scale optimization problems and competitive with the traditional conjugate gradient method.

The extension of the method to an infinite Hilbert space has been analyzed for the quadratic case when the Hessian operator is self-adjoint. In this case the extension is straightforward. The compact operator case is currently being analyzed.

It is important to remark that for the infinite dimensional nonlinear case was not possible to prove the convergence of the method because the global convergence results for the finite case can not be extended. In spite of this fact, the algorithm has been applied to a well known control problem and the numerical results are much better that the obtained by a secant method. These facts encourage us to continue analyzing the spectral gradient method for the infinite dimensional optimization problems.

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AN INTEGRAL EQUATION IN ORDER TO SOLVE A ONE-PHASE STEFAN PROBLEM WITH NONLINEAR THERMAL CONDUCTIVITY *

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Abstract

We study a one-phase Stefan problem for a semi-infinite material with temperature-dependent thermal conductivity with a constant temperature or a heat flux condition of the type $-q_0/\sqrt{t}$ $(q_0 > 0)$ at the fixed face $x = 0$. We obtain in both cases sufficient conditions for data in order to have a parametric representation of the solution of the similarity type for $t \geq t_0 > 0$ with t_0 an arbitrary positive time. These explicit solutions are obtained through the unique solution of an integral equation with the time as a parameter.

Key words : Stefan problem, free boundary problem, moving boundary problem, phase-change process, nonlinear thermal conductivity, fusion, solidification, similarity solution.

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I. Introduction. We will consider a phase-change problem (Stefan problem) for a non-linear heat conduction equation for a semi-infinite region $x > 0$ with a nonlinear thermal conductivity $k(\theta)$ given by

$$k(\theta) = \frac{\rho c}{(a + b\theta)^2} \tag{1}$$

and phase change temperature θ_f. This kind of thermal conductivity or diffusion coefficient was considered in [3, 5, 6, 14, 16, 18, 21, 23, 28]. The modeling of this type of systems is a great mathematical and industrial significance problem. Phase-change problems appear frequently in industrial processes and other problems of technological interest [1, 7, 8, 9, 11, 12, 13, 15, 17]. A recent large bibliography on the subject was given recently in [27].

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The mathematical formulation of our free boundary (fusion process) problem consists in determining the evolution of the moving phase separation $x = s(t)$ and the temperature distribution $\theta = \theta(x,t)$ satisfying the conditions

$$\rho c \frac{\partial \theta}{\partial t} = \frac{\partial}{\partial x}\left(k(\theta)\frac{\partial \theta}{\partial x}\right) \ , \ 0 < x < s(t) \ , \ t > 0 \tag{2}$$

$$k(\theta(0,t))\frac{\partial \theta}{\partial x}(0,t) = -\frac{q_0}{\sqrt{t}} \ , \ q_0 > 0 \ , \ t > 0 \tag{3}$$

$$k\left(\theta\left(s(t),t\right)\right)\frac{\partial \theta}{\partial x}\left(s(t),t\right) = -\rho l \ \dot{s}(t) \ , \ t > 0 \tag{4}$$

$$\theta\left(s(t),t\right) = \theta_f \ , \ t > 0 \tag{5}$$

$$s(0) = 0 \tag{6}$$

where $a + b\theta_f > 0$, in order to guarantee that k is well defined. Here $-q_0/\sqrt{t}$ denotes the prescribed heat flux on the boundary $x = 0$ which is of the type imposed in [26]. This kind of heat flux condition (3) was also considered in numerous papers, e.g. [2, 10, 22]. Other problems in this subject are [4, 19, 23, 24].

The free boundary problem (2) – (6) with $k(\theta)$ defined by (1) is a particular case of one studied in [20, 25] by taking the parameter $d = 0$ for the following equation

$$\rho c \frac{\partial \theta}{\partial t} = \frac{\partial}{\partial x}\left(k(\theta)\frac{\partial \theta}{\partial x}\right) - v(\theta)\frac{\partial \theta}{\partial x} \ , \ 0 < x < s(t) \ , \ t > 0 \tag{7}$$

where the thermal conductivity $k(\theta)$ and the velocity term $v(\theta)$ are given by (1) and

$$v(\theta) = \rho c \frac{d}{2\left(a + b\theta\right)^2} \tag{8}$$

respectively, and c, ρ and l are the specific heat, the density and the latent heat of fusion of the medium respectively, all of them are assumed to be constant with positive parameters a, b and d.

In those papers temperature and flux type conditions on the fixed face $x = 0$ were studied. Furthermore, necessary and sufficient conditions for the existence of an explicit solution was found in [20]. Here we study the case without the velocity term, i.e. $d = 0$ in the differential equation (7) which cannot be obtained from what it was previously done in [20, 25] for the case $d \neq 0$. In those papers it was defined the transformation

$$y = \frac{2}{d}\left[(1 + dx)^{\frac{1}{2}} - 1\right] \tag{9}$$

which is the identity if we take $d \to 0$ since

$$\lim_{d \to 0} \frac{2}{d}\left[(1 + dx)^{\frac{1}{2}} - 1\right] = x \ , \ \forall \ x > 0.$$

Then, the case $d = 0$ must be solved by using other techniques which will be the goal of this study.

In Section II we prove the existence and uniqueness of an explicit solution of the similarity type of the free boundary problem $(2)-(6)$ for $t \geq t_0 > 0$ with t_0 an arbitrary positive time when data satisfy condition $a + b\theta_f \geq bl/c$. The solution is explicitly given by $(41), (47)$ and (48), and by $(50), (51)$ for the cases $a + b\theta_f > bl/c$ and $a + b\theta_f = bl/c$ respectively. The explicit solution for the two cases is obtained through the unique solution of an integral equation in which time is a parameter.

Besides, there does not exist any solution of the similarity type to the free boundary problem $(2)-(6)$ for the case $a + b\theta_f < bl/c$.

II. Existence and uniqueness of solution of the free boundary problem with flux boundary condition on the fixed face.

We consider the free boundary problem $(2)-(6)$ with the parameters a, b and the coefficients l, c satisfy the following condition

$$a + b\theta_f > \frac{bl}{c} \quad . \tag{10}$$

If we define

$$\Theta = \frac{1}{a + b\theta} \ , \tag{11}$$

the problem $(2)-(6)$ becomes

$$\frac{\partial \Theta}{\partial t} = \Theta^2 \frac{\partial^2 \Theta}{\partial x^2} \ , \ 0 < x < s(t) \ , \ t > 0 \tag{12}$$

$$\frac{\partial \Theta}{\partial x}(0,t) = \frac{w}{\sqrt{t}} \ , \ t > 0 \tag{13}$$

$$\frac{\partial \Theta}{\partial x}(s(t),t) = \frac{bl}{c} \overset{\bullet}{s}(t) \ , \ t > 0 \tag{14}$$

$$\Theta(s(t),t) = \frac{1}{a + b\theta_f} \ , \ t > 0 \tag{15}$$

$$s(0) = 0 \tag{16}$$

where w is a constant defined by

$$w = \frac{bq_0}{\rho c}. \tag{17}$$

Let us perform the transformation

$$\begin{aligned} \chi(x,t) &= \int_0^x \frac{d\eta}{\Theta(\eta,t)} \\ \Psi(\chi,t) &= \Theta(x,t) \end{aligned} \tag{18}$$

and

$$S(t) = \chi(s(t),t) \quad . \tag{19}$$

The problem $(12)-(16)$ becomes

$$\frac{\partial \Psi}{\partial t} = \frac{\partial^2 \Psi}{\partial \chi^2} - \frac{w}{\sqrt{t}} \frac{\partial \Psi}{\partial \chi} \ , \ 0 < \chi < S(t) \ , \ t > 0 \tag{20}$$

$$\frac{\partial \Psi}{\partial \chi}(0,t) = \frac{w}{\sqrt{t}}\Psi\left(0,t\right)\ ,\ t>0 \tag{21}$$

$$\frac{\partial \Psi}{\partial \chi}(S(t),t) = \frac{1}{(a+b\theta_f)\left(\frac{c}{bl}\left(a+b\theta_f\right)-1\right)}\left(\overset{\bullet}{S}\left(t\right)-\frac{w}{\sqrt{t}}\right)\ ,\ t>0 \tag{22}$$

$$\Psi(S(t),t) = \frac{1}{a+b\theta_f}\ ,\ t>0 \tag{23}$$

$$S(0)=0 \tag{24}$$

where

$$\overset{\bullet}{S}\left(t\right) = \left(a+b\theta_f-\frac{bl}{c}\right)\overset{\bullet}{s}\left(t\right)+\frac{w}{\sqrt{t}}\ . \tag{25}$$

If we introduce the similarity variable

$$\xi = \frac{\chi}{2\sqrt{t}}\qquad , \tag{26}$$

and the solution is sought of type

$$\Psi\left(\chi,t\right) = \varphi\left(\xi\right) = \varphi\left(\frac{\chi}{2\sqrt{t}}\right) \tag{27}$$

then the free boundary $S(t)$ of the problem $(20)-(24)$ must be of the type

$$S(t) = 2\Lambda_1\sqrt{t}\ ,\ t>0 \tag{28}$$

with $\Lambda_1 > 0$ an unknown coefficient to be determined and the problem $(20)-(24)$ yields

$$\varphi''\left(\xi\right)+2\varphi'\left(\xi\right)\left(\xi-w\right)=0\ ,\ 0<\xi<\Lambda_1 \tag{29}$$

$$\varphi'\left(0\right) = 2w\varphi\left(0\right) \tag{30}$$

$$\varphi\left(\Lambda_1\right) = \frac{1}{a+b\theta_f} \tag{31}$$

$$\varphi'\left(\Lambda_1\right) = \frac{2}{(a+b\theta_f)\left(\frac{c}{bl}\left(a+b\theta_f\right)-1\right)}\left(\Lambda_1-w\right). \tag{32}$$

Taking into account the expression (25) we have

$$s(t) = 2\lambda_1\sqrt{t} \tag{33}$$

with

$$\lambda_1 = \frac{\Lambda_1 - w}{a+b\theta_f-\frac{bl}{c}}\quad . \tag{34}$$

If we integrate (29) we obtain

$$\varphi(\xi) = D_1 \operatorname{erf}(\xi - w) + C_1 \tag{35}$$

where D_1 and C_1 are two constants of integration which can be determined from (30) and (31)

$$D_1 = \frac{\sqrt{\pi} w \exp(w^2)}{(a + b\theta_f)\left[1 + \sqrt{\pi} w \exp(w^2)\left(\operatorname{erf}(\Lambda_1 - w) + \operatorname{erf}(w)\right)\right]} \tag{36}$$

$$C_1 = \frac{1 + \sqrt{\pi} w \exp(w^2) \operatorname{erf}(w)}{(a + b\theta_f)\left(1 + \sqrt{\pi} w \exp(w^2)\left(\operatorname{erf}(\Lambda_1 - w) + \operatorname{erf}(w)\right)\right)} \tag{37}$$

Now, we have to consider here the condition (32) which implies that Λ_1 must be the solution of the following equation

$$W_1(x) = W_2(x) \qquad , \qquad x > w \tag{38}$$

where

$$W_1(x) = \frac{w \exp(w^2) \exp\left[-(x - w)^2\right]}{1 + w \exp(w^2)\sqrt{\pi}\left(\operatorname{erf}(x - w) + \operatorname{erf}(w)\right)} \tag{39}$$

and

$$W_2(x) = \frac{bl}{c(a + b\theta_f) - bl}(x - w) \quad . \tag{40}$$

It is easy to prove that $W_1(0) = w > 0, W_1(+\infty) = 0$, and W_1 is a decreasing function, and $W_2(w) = 0, W_2(+\infty) = +\infty$ and W_2 is an increasing function because condition (10). So, there exists a unique solution Λ_1 of the equation (38) and then we have the following theorem.

Theorem 1.- Let us consider the hypothesis (10).

(i) If (Θ, s) is a solution of the free boundary problem $(12) - (16)$ then $\Theta = \Theta(x,t)$ is a solution, in variable x, of the integral equation:

$$\Theta(x,t) = C_1 + D_1 \operatorname{erf}\left(\frac{\int_0^x \frac{d\eta}{\Theta(\eta,t)}}{2\sqrt{t}} - w\right) \ , \ 0 \le x \le s(t) \ , \tag{41}$$

where $t > 0$ is a parameter and w, D_1 and C_1 are defined by (17), (36) and (37) respectively, and $s(t)$ is given by (33) and Λ_1 is the unique solution of the Eq. (38). Moreover, function $Y(x,t)$ defined by

$$Y(x,t) = \frac{1}{2\sqrt{t}} \int_0^x \frac{d\eta}{\Theta(\eta,t)} - w \qquad , \qquad 0 \le x \le s(t) \ , \ t > 0 \tag{42}$$

satisfies the conditions

$$\frac{\partial Y}{\partial x}(x,t) = \frac{1}{2\sqrt{t}} \frac{1}{\Theta(x,t)} \ , \ 0 < x < s(t) \ , \ t > 0 \tag{43}$$

$$Y(0,t) = -w \ , \ t > 0 \tag{44}$$

$$\frac{\partial Y}{\partial t}(x,t) = -\frac{1}{2t}\left(Y(x,t) + \frac{D_1}{\sqrt{\pi}}\frac{\exp\left(-Y^2(x,t)\right)}{\Theta(x,t)}\right) \ , \ 0 < x < s(t) \ , \ t > 0 \tag{45}$$

$$Y\left(s(t),t\right) = \Lambda_1 - w \ , \ t > 0 \tag{46}$$

(ii) Conversely, if Θ is a solution of the integral equation (41) with s given by (33) and function Y, defined by (42) satisfies the conditions $(43) - (46)$, and w, D_1 and C_1 are defined by $(17), (36)$ and (37) respectively, and Λ_1 is the unique solution of the Eq. (38) then (Θ, s) is a solution of the free boundary problem $(12) - (16)$.

(iii) The integral equation (41) has a unique solution for $t \geq t_0 > 0$ with t_0 is an arbitrary positive time.

(iv) The free boundary problem $(2) - (6)$ satisfying the hypothesis (10) has a unique similarity type solution (θ, s) for $t \geq t_0 > 0$ (with t_0 an arbitrary positive time) which is given by

$$\theta(x,t) = \frac{1}{b}\left[\frac{1}{\Theta(x,t)} - a\right], \quad 0 < x < s(t), \quad t \geq t_0 > 0 \tag{47}$$

$$s(t) = \frac{2\left(\Lambda_1 - w\right)}{a + b\theta_f - \dfrac{bl}{c}}\sqrt{t} \ , \ t \geq t_0 > 0 \tag{48}$$

where Θ is the unique solution of the integral Eq. (41) where Λ_1 is the unique solution of the Eq. (38), and w, D_1 and C_1 are defined by $(17), (36)$ and (37) respectively.

Proof.

(i) From the previous computation we have

$$\Theta(x,t) = \varphi(\xi) = C_1 + D_1 \operatorname{erf}\left(\xi - w\right) = C_1 + D_1 \operatorname{erf}\left(\frac{\int_0^x \frac{d\eta}{\Theta(\eta,t)}}{2\sqrt{t}} - w\right)$$

that is Θ is a solution of the integral equation (41). Function Y, defined by (42), satisfies the conditions $(43), (44)$ by elementary computations, and

$$\frac{\partial Y}{\partial t}(x,t) = -\frac{1}{4t\sqrt{t}}\int_0^x \frac{d\eta}{\Theta(\eta,t)} - \frac{1}{2\sqrt{t}}\int_0^x \Theta_{xx}(\eta,t)d\eta =$$

$$= -\frac{1}{2\sqrt{t}}\left(\frac{Y(x,t)}{\sqrt{t}} + \Theta_x(x,t)\right) = -\frac{1}{2\sqrt{t}}\left(\frac{Y(x,t)}{\sqrt{t}} + \frac{D_1}{\sqrt{\pi t}}\frac{\exp\left(-Y^2(x,t)\right)}{\Theta(x,t)}\right)$$

that is (45). Finally we get

$$Y\left(s(t),t\right) = \frac{1}{2\sqrt{t}}\int_0^{s(t)} \frac{d\eta}{\Theta(\eta,t)} - w = \frac{\chi\left(s(t),t\right)}{2\sqrt{t}} - w = \frac{S\left(t\right)}{2\sqrt{t}} - w = \Lambda_1 - w$$

that is (46).

(ii) In order to proof that (Θ, s) is a solution of the free boundary problem $(12) - (16)$ we get:

a)

$$\Theta_{xx}(x,t) = \left(\frac{D_1}{\sqrt{\pi t}}\frac{\exp\left(-Y^2(x,t)\right)}{\Theta(x,t)}\right)_x =$$

$$= -\frac{D_1}{\sqrt{\pi t}}\frac{\exp\left(-Y^2(x,t)\right)}{\Theta^2(x,t)}\left(Y(x,t) + \frac{D_1}{\sqrt{\pi}}\frac{\exp\left(-Y^2(x,t)\right)}{\Theta(x,t)}\right);$$

b)

$$\Theta_t(x,t) = \frac{2D_1}{\sqrt{\pi}}\exp\left(-Y^2(x,t)\right)Y_t(x,t) =$$

$$= -\frac{D_1}{\sqrt{\pi}t}\exp\left(-Y^2(x,t)\right)\left(Y(x,t) + \frac{D_1}{\sqrt{\pi}}\frac{\exp\left(-Y^2(x,t)\right)}{\Theta(x,t)}\right)$$

that is Eq. (12);

c)

$$\Theta(0,t) = C_1 - D_1\,\mathrm{erf}(w) = \frac{D_1}{\sqrt{\pi}w\exp(w^2)};$$

d)

$$\Theta_x(0,t) = \frac{D_1}{\sqrt{\pi t}}\frac{\exp\left(-Y^2(0,t)\right)}{\Theta(0,t)} = \frac{w}{\sqrt{t}}\,, \text{ that is } (13);$$

e)

$$\Theta(s(t),t) = C_1 + D_1\,\mathrm{erf}(\Lambda_1 - w) = \frac{1}{a + b\theta_f}\,, \text{ that is } (15);$$

f)

$$\Theta_x(s(t),t) = \frac{D_1}{\sqrt{\pi t}}\frac{\exp\left(-Y^2(s(t),t)\right)}{\Theta(s(t),t)} = \frac{(a+b\theta_f)\,D_1}{\sqrt{\pi t}}\exp\left(-(\Lambda_1 - w)^2\right) =$$

$$= \frac{1}{\sqrt{t}}W_1(\Lambda_1) = \frac{1}{\sqrt{t}}W_2(\Lambda_1) =$$

$$= \frac{1}{\sqrt{t}}\frac{bl}{c(a+b\theta_f) - bl}(\Lambda_1 - w) = \frac{bl\lambda_1}{c\sqrt{t}} = \frac{bl}{c}\dot{s}(t)\,, \text{ that is } (14)$$

(iii) Now in order to complete the proof, we just have to proof the existence of a solution of the integral equation (41). If we define $Y(x,t)$ by (42) then, Eq. (41) is equivalent to the following Cauchy differential problem

$$\frac{\partial Y}{\partial x}(x,t) = \frac{1}{2\sqrt{t}}\frac{1}{\left(C_1 + D_1\,\mathrm{erf}\left(Y(x,t)\right)\right)} \equiv G_1\left(x,t,Y(x,t)\right) \quad , \quad 0 < x < s(t)\,,\ t > 0$$
$$Y(0,t) = -w\,, \tag{49}$$

with a positive parameter $t > 0$. We have $\left|\frac{\partial G_1}{\partial Y}\right| \leq \frac{D_1}{C_1^2\sqrt{\pi t}}$ which is bounded for all

$t \geqslant t_0 > 0$, $0 \leq x \leq s(t)$, for an arbitrary positive time t_0. Then, problem (49) (i.e. the integral Eq. (41)) has a unique solution for $t \geq t_0 > 0$, for an arbitrary positive time t_0.

(iv) It follows from elementary but tedious computation.■

Remark 1. $Y(x,t)$ dos not possess a limit at $(0,0)$ because $Y(0,t) = -w = -\frac{bq_0}{\rho c} < 0$ for $t > 0$ and $\lim\limits_{t \mapsto 0} Y(s(t),t) = \Lambda_1 - w > 0$ for all $t > 0$.

If Θ is the solution of the integral equation (41) then Θ is strictly monotone in variable x. We obtain that $\theta(x,t) = (1/\Theta(x,t) - a)/b$ does not have limit when $(x,t) \to (0,0)$ but $\theta(x,t)$ is bounded in a neighborhood of $(0,0)$ checking that

$$\begin{aligned} \theta_f &= \lim_{(\eta,\tau) \mapsto (0,0)} \inf \theta(\eta,\tau) \leq \theta(x,t) \leq \lim_{(\eta,\tau) \mapsto (0,0)} \sup \theta(\eta,\tau) = \\ &= \theta_f + \frac{a + b\theta_f}{b} \sqrt{\pi} w \exp(w^2) \left(\operatorname{erf}(w) + \operatorname{erf}(\Lambda_1 - w) \right) \end{aligned}$$

When the hypothesis (10) is not satisfied we can follow an analogous method to the one described before in order to obtain the following result.

Theorem 2.

(i) The result of the Theorem 1 is also true if we replace the condition (10) by $a + b\theta_f = \frac{bl}{c}$. Furthermore, in this case, the solution of the free boundary problem $(2)-(6)$ is given by

$$\theta(x,t) = \frac{1}{b}\left[\frac{1}{\Theta(x,t)} - a\right] \quad , \quad s(t) = 2D_0\sqrt{\frac{t}{\pi}} \tag{50}$$

where Θ is the unique solution of the following integral equation

$$\Theta(x,t) = D_0 \operatorname{erf}\left(\frac{\int_0^x \frac{d\eta}{\Theta(\eta,t)}}{2\sqrt{t}} - w \right) + \frac{c}{bl} \; , \; 0 \leq x \leq s(t) \; , \tag{51}$$

with

$$D_0 = \frac{q_0 \sqrt{\pi} \exp(w^2)}{\rho l \left(1 + \sqrt{\pi} w \exp(w^2) \operatorname{erf}(w)\right)}$$

for $t \geq t_0 > 0$, $0 \leq x \leq s(t)$ for any arbitrary positive time t_0 and w defined by (17).

(ii) There does not exist any solution to the free boundary problem $(2)-(6)$ for the case $a + b\theta_f < \frac{bl}{c}$.■

A more complete version of these results and the corresponding study for the analogous problem with a temperature condition on the fixed face $x = 0$ instead of the heat flux condition (3) will be given in a forthcoming paper.

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OPTIMIZATION OF THE HEAT FLUX IN A MIXED ELLIPTIC PROBLEM WITH TEMPERATURE CONSTRAINTS

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Abstract

For a steady-state heat conduction problem in a poligonal domain $\Omega \subset \mathbb{R}^n$, with heat flux condition in a portion of the boundary, Γ_2, and a Fourier type condition in the rest of the boundary, Γ_1, we obtain the minimum total heat flux on Γ_2, so that the whole material is in the solid phase. For this purpose we use the finite element method in order to convert the optimization problem into a linear programming problem.

Key Words: Mixed Elliptic Problem, Steady-State Stefan Problem, Finite Element Method, Linear Programming Problem.

AMS Subject Classification: 65K10, 49K20, 35J85, 65N30

I. Introduction

We consider a steady-state heat conduction problem in a material Ω which occupies a poligonal bounded domain in $\mathbb{R}^n$, with boundary $\Gamma = \Gamma_1 \cup \Gamma_2$ (with meas$(\Gamma_1) > 0$ and meas$(\Gamma_2) > 0$). We impose a Newton law with a transfer coefficient $\alpha > 0$ and an exterior temperature $b > 0$ on Γ_1, and an outcoming heat flux $q > 0$ on Γ_2. We assume, without loss of generality, that the phase-change temperature of the material is 0°C.

This problem was studied in [TaTa] and it was stablished that if the heat flux q is between a minimum flux q_m and a maximum flux q_M, which are functions of the coefficient α and the temperature b, then there is a steady-state two phase Stefan Problem, that is the temperature is of non-constant sign in Ω.

In [GoTa1] a thermic flux optimization problem was solved: the maximization of the output heat flux on a portion of the boundary domain, Γ_2, while on the other portion, Γ_1, the distribution of the temperature was fixed. The maximization was carried out under the condition that there is no phase change.

In [GoTa2] the maximum heat total flux on Γ_2 was found, such that the temperature is positive in the whole domain Ω considering a boundary Fourier type condition on Γ_1.

Following the ideas of these papers, the goal of the present work is to minimize the total heat flux on the boundary Γ_2 so that all the material is in the solid phase. In order to solve this minimization problem we will use the finite element method and we will obtain a linear programming problem.

We remark that all the results of this work are still valid if we consider that the boundary Γ of the domain Ω is the union of three portions, Γ_1, Γ_2 and Γ_3, such that on Γ_1 and Γ_2 there exist the same conditions stated above, and Γ_3 is a wall impermeable to heat.

In Section II we present the mathematical model of the minimization problem and in Section III we discretize it and a linear programming problem must be solved in order to obtain the solution.

II. Mathematical Model of the Problem

If θ represents the temperature in Ω and we define the function u = $k_l \theta^+ - k_s\theta^-$, where k_i is the conductivity of the phase i ($i = l$ for the liquid and $i =$ s for the solid), then the following equations represent the mathematical model of the corresponding steady-state heat conduction problem [Du, Ta]:

$$\begin{array}{ll} \Delta \mathbf{u} = 0 & \text{in } \Omega \\ -\left.\frac{\partial \mathbf{u}}{\partial n}\right|_{\Gamma_1} = \alpha(\mathbf{u} - B), & -\left.\frac{\partial \mathbf{u}}{\partial n}\right|_{\Gamma_2} = q \end{array} \tag{1}$$

where B = k_lb > 0 and b is the exterior temperature.

We want to minimize the total heat flux on Γ_2 with the constraint that the whole material is in the solid phase. In other words, the problem is

$$\textit{Find } \mathbf{q}^* \textit{ such that } \mathbf{J(q^*)} = \mathop{\mathit{Inf}}_{u \,\le\, 0 \text{ in } \Omega} \mathbf{J(q)} \tag{2}$$

where

$$\mathbf{J(q)} = \int_{\Gamma_2} q \, d\gamma \ , \ q \in L^2(\Gamma_2)$$

The variational formulation of the problem (1) is given by

$$a_\alpha(u,v) = L_\alpha(v) , \qquad \forall v \in V , u \in V \tag{3}$$

where

$$a_\alpha(u,v) = a(u,v) + \alpha\int_{\Gamma_1} uv\, d\gamma \quad , \qquad L_\alpha(v) = L(v) + \alpha B\int_{\Gamma_1} v\, d\gamma$$

$$a(u,v) = \int_\Omega \nabla u\, \nabla v\, dx \ , \qquad L(v) = -\int_{\Gamma_2} q\, v\, d\gamma \ , V = H^1(\Omega).$$

III. The Discrete Problem and the Linear Programming Problem

We construct a regular triangulation τ_h, of the poligonal domain Ω with Lagrange triangles of type 1, with afin equivalent finit elements of class C^o, and we approach the space V by [Ci1]:

$$V_h = \left\{ v_h \in C^0(\bar{\Omega}) \Big/ v_h|_T \in P_1(T), \forall T \in \tau_h \right\} ,$$

where P_1 is the set of the polynomials of degree ≤ 1.

The approximate variational problem consists in finding $u_{h\alpha} \in V_h$ so that

$$a_\alpha(u_{h\alpha},v_h) = L_\alpha(v_h) \ , \ \forall v_h \in V_h \ . \tag{4}$$

We call ***N*** the total number of nodes of the triangulation, r is the number of nodes on the portion of the boundary Γ_1 and p is the number of nodes on Γ_2.

Let $\{\omega_i\}_{i=1}^N$ a basis of the space V_h. We can think the basis as

$$\{\omega_i\}_{i=1}^N = \{\omega_i^1\}_{i=1}^r \cup \{\omega_i^\Omega\}_{i=r+1}^{N-p} \cup \{\omega_i^2\}_{i=N-p+1}^N$$

where we denote ω_i^j the function whose value is 1 on the node N_i, of the boundary Γ_1 if j = 1, of the boundary Γ_2 if j = 2 or of the interior of the domain Ω if j = Ω, and whose values are zero in the other nodes. Then we have

$$u_{h\alpha} = \sum_{i=1}^r u_i^1 \omega_i^1 + \sum_{i=r+1}^{N-p} u_i^\Omega \omega_i^\Omega + \sum_{i=N-p+1}^{N} u_i^2 \omega_i^2 \tag{5}$$

where u_i^1 (i = 1, . . ., r), u_i^Ω (i = r + 1, . . ., $N-$ p) and u_i^2 (i = $N-$ p + 1, . . ., N) are the real unknown values at the corresponding nodes.

With the expresion (5) and considering $v_h = \omega_i^j$ in (4), we get the following system of linear equations

$$\begin{aligned} \boldsymbol{A}_1\mathbf{u}^1 + \boldsymbol{A}_2\mathbf{u}^{\Omega}+\boldsymbol{A}_3\mathbf{u}^2 &= \mathbf{b}^{\boldsymbol{\alpha}} \\ \boldsymbol{A}_4\mathbf{u}^1 + \boldsymbol{A}_5\mathbf{u}^{\Omega}+\boldsymbol{A}_6\mathbf{u}^2 &= \mathbf{0} \\ \boldsymbol{A}_7\mathbf{u}^1 + \boldsymbol{A}_8\mathbf{u}^{\Omega}+\boldsymbol{A}_9\mathbf{u}^2 &= \mathbf{b}(\boldsymbol{q}) \end{aligned} \tag{6}$$

where the matrices $\boldsymbol{A}_1, ..., \boldsymbol{A}_9$ are given by:

$\boldsymbol{A}_1 = (a^1_{ij}) \in \boldsymbol{R}^{r\,x\,r}$, $a^1_{ij} = a_\alpha(\omega^1_j, \omega^1_i)$, $\boldsymbol{A}_2 = (a^2_{ij}) \in \boldsymbol{R}^{r\,x\,N-(p+r)}$, $a^2_{ij} = a(\omega^\Omega_j, \omega^1_i)$,

$\boldsymbol{A}_3 = (a^3_{ij}) \in \boldsymbol{R}^{r\,x\,p}$, $a^3_{ij} = a(\omega^2_j, \omega^1_i)$, $\boldsymbol{A}_4 = (a^4_{ij}) \in \boldsymbol{R}^{N-(p+r)\,x\,r}$, $a^4_{ij} = a(\omega^1_j, \omega^\Omega_i)$,

$\boldsymbol{A}_5 = (a^5_{ij}) \in \boldsymbol{R}^{N-(p+r)\,x\,N-(p+r)}$, $a^5_{ij} = a(\omega^\Omega_j, \omega^\Omega_i)$,

$\boldsymbol{A}_6 = (a^6_{ij}) \in \boldsymbol{R}^{N-(p+r)\,x\,p}$, $a^6_{ij} = a(\omega^2_j, \omega^\Omega_i)$, $\boldsymbol{A}_7 = (a^7_{ij}) \in \boldsymbol{R}^{p\,x\,r}$, $a^7_{ij} = a(\omega^1_j, \omega^2_i)$,

$\boldsymbol{A}_8 = (a^8_{ij}) \in \boldsymbol{R}^{p\,x\,r}$, $a^8_{ij} = a(\omega^\Omega_j, \omega^2_i)$, $\boldsymbol{A}_9 = (a^9_{ij}) \in \boldsymbol{R}^{p\,x\,p}$, $a^9_{ij} = a(\omega^2_j, \omega^2_i)$,

$\mathbf{u}^1 \in \boldsymbol{R}^r$, $\mathbf{u}^2 \in \boldsymbol{R}^p$, $\mathbf{u}^\Omega \in \boldsymbol{R}^{N-(p+r)}$

and $\mathbf{b}^{\boldsymbol{\alpha}} = (b^\alpha_i)^{r}_{i=1} \in \boldsymbol{R}^r$, $\mathbf{b}(\boldsymbol{q}) = (b_i(\boldsymbol{q}))^{N}_{i=N-p+1} \in \boldsymbol{R}^p$,

with

$$b^\alpha_i = \alpha B \int_{\Gamma_1} \omega^1_i \, d\gamma \,, \qquad b_i(\boldsymbol{q}) = -\int_{\Gamma_2} q\, \omega^2_i \, d\gamma \,.$$

The system (6) can be expressed as follows

$$\boldsymbol{A}\mathbf{u} = \boldsymbol{b}, \tag{7}$$

where $\boldsymbol{A} \in \boldsymbol{R}^{N\,x\,N}$ is the finite element matrix, $\boldsymbol{A} = \begin{pmatrix} \boldsymbol{A}_1 & \boldsymbol{A}_2 & \boldsymbol{A}_3 \\ \boldsymbol{A}_4 & \boldsymbol{A}_5 & \boldsymbol{A}_6 \\ \boldsymbol{A}_7 & \boldsymbol{A}_8 & \boldsymbol{A}_9 \end{pmatrix}$

and

$$\mathbf{u} = (\mathbf{u}^1, \mathbf{u}^\Omega, \mathbf{u}^2) \in \boldsymbol{R}^N, \qquad \boldsymbol{b} = (\mathbf{b}^{\boldsymbol{\alpha}}, 0, \mathbf{b}(\boldsymbol{q})) \in \boldsymbol{R}^N.$$

We obtain the elements b^α_i and $b_i(\boldsymbol{q})$ through a numerical computation of the corresponding integrals, for example by using the trapezoidal rule. For this purpose we consider that the curve Γ_i $(i = 1, 2)$ can be decomposed by

$$\Gamma_i = \bigcup_{j=1}^{s} \Gamma_i^j, \qquad \text{with } s = \text{p} - 1 \text{ if } \Gamma_i \text{ is open or} \qquad s = \text{p if } \Gamma_i \text{ is closed}$$

and

$$\int_{\Gamma_i} f \, d\gamma = \sum_{j=1}^{s} \int_{\Gamma_i^j} f \, d\gamma \approx \sum_{j=1}^{s} \frac{1}{2} \left|\Gamma_i^j\right| \Big(f(\mathcal{N}_j) + f(\mathcal{N}_{j+1}) \Big),$$

where we have denoted with Γ_i^j the portion of the curve Γ_i whose limit points are the nodes $\mathcal{N}_j$ and $\mathcal{N}_{j+1}$ and we have denoted with $\left|\Gamma_i^j\right|$ the measure of that portion of the curve.

In this way the linear system (7) becomes in

$$\boldsymbol{A}\mathbf{u} = \boldsymbol{M}\widetilde{\boldsymbol{q}} + \widetilde{\boldsymbol{b}}$$

where :

- $\widetilde{\boldsymbol{q}} \in \boldsymbol{R}^{\boldsymbol{p}}$ and $\widetilde{\boldsymbol{q}}_{\boldsymbol{i}}$ is the value of the flux $\boldsymbol{q}$ in the node $\mathcal{N}_{N-\text{p+i}}$,
- $\widetilde{\boldsymbol{b}} \in \boldsymbol{R}^{\boldsymbol{N}}$, $\widetilde{\boldsymbol{b}}_{\boldsymbol{i}}$ approaches the value of b_i^{α} for i = 1 . . .r and $\widetilde{\boldsymbol{b}}_{\boldsymbol{i}} = 0$ for i = r+1, . . ., N,
- $\boldsymbol{M} = \begin{pmatrix} \boldsymbol{M}^{\mathbf{1}} \\ \boldsymbol{M}^{\mathbf{2}} \end{pmatrix} \in \boldsymbol{R}^{\boldsymbol{N} \times \boldsymbol{p}}$, $\quad \boldsymbol{M}^{\mathbf{1}} \in \boldsymbol{R}^{(\boldsymbol{N}-\boldsymbol{p}) \times \boldsymbol{p}}$ is a zero matrix,

$\boldsymbol{M}^{\mathbf{2}} = (m_{ii})_{i=N-p+1}^{N} \in \boldsymbol{R}^{\boldsymbol{p} \times \boldsymbol{p}}$ is a diagonal matrix , with

$$m_{ii} = \begin{cases} -\frac{1}{2}\left|\Gamma_2^i\right| & \text{if } i = N-p+1 \text{ or } i = N, \\ -\frac{1}{2}\left(\left|\Gamma_2^{i-1}\right| + \left|\Gamma_2^i\right|\right) & \text{if } i = N-p+2, \ldots, N-1, \end{cases}$$

if Γ_2 is an open curve, or

$$m_{ii} = -\frac{1}{2}\left(\left|\Gamma_2^{i-1}\right| + \left|\Gamma_2^i\right|\right) \qquad \text{if } i = N-p+1, \ldots, N,$$

if Γ_2 is a closed curve, (here we have considered $\Gamma_2^{N-p} = \Gamma_2^N$), and the i-th component of the vector $\boldsymbol{M}\widetilde{\boldsymbol{q}}$ approaches the value of $b_i(\boldsymbol{q})$ i.e. $(\boldsymbol{M}\widetilde{\boldsymbol{q}})_i \approx b_i(\boldsymbol{q})$.

After all these considerations, the optimization problem (2) is transformed into the following linear programming problem :

$$\underset{\mathrm{u} \in \boldsymbol{U}}{\text{Minimize}} \quad \hat{F}(\widetilde{\boldsymbol{q}}) = \langle \boldsymbol{T}_{\Gamma_2}, \widetilde{\boldsymbol{q}} \rangle_{\boldsymbol{R}^{\boldsymbol{p}}} \tag{8}$$

where

$$\boldsymbol{T}_{\Gamma_2} = \begin{cases} \frac{1}{2}\left(\left|\boldsymbol{\Gamma}_2^{\mathbf{1}}\right|, \left(\left|\Gamma_2^1\right| + \left|\Gamma_2^2\right|\right), \ldots, \left(\left|\Gamma_2^{\text{p}-2}\right| + \left|\Gamma_2^{\text{p-1}}\right|\right), \left|\Gamma_2^{\text{p-1}}\right|\right) & \text{if } \Gamma_2 \text{ is open,} \\ \frac{1}{2}\left(\left(\left|\Gamma_2^{\text{p}}\right| + \left|\boldsymbol{\Gamma}_2^{\mathbf{1}}\right|\right), \left(\left|\Gamma_2^1\right| + \left|\Gamma_2^2\right|\right), \ldots, \left(\left|\Gamma_2^{\text{p-1}}\right| + \left|\Gamma_2^{\text{p}}\right|\right)\right) & \text{if } \Gamma_2 \text{ is closed,} \end{cases}$$

$< . , . >_{R^p}$ is the usual inner product in $\boldsymbol{R^p}$ and the set $\boldsymbol{U}$ is defined by

$$\boldsymbol{U} = \{ \tilde{\boldsymbol{q}} \in \boldsymbol{R^p} : \boldsymbol{C}\tilde{\boldsymbol{q}} \leq \boldsymbol{d} , \boldsymbol{C} \in \boldsymbol{R^{N \times p}}, \boldsymbol{d} \in \boldsymbol{R^N} \},$$

with $\boldsymbol{C} = \boldsymbol{A}^{-1}\boldsymbol{M}$ and $\quad \boldsymbol{d} = -\boldsymbol{A}^{-1}\tilde{\boldsymbol{b}}$.

Taking into account that $\hat{F}(\tilde{\boldsymbol{q}}) \geq 0, \ \forall \ \tilde{\boldsymbol{q}} \geq 0$ it results that the linear programming problem (8) admits at least one solution [Ci2].

We construct a programm in MATLAB and we get the solution of the problem (8) for some different domains. From these numerical results we can guess that the minimum optimum flux is given by $q^* = -\left.\frac{\partial \mathbf{u}^*}{\partial n}\right|_{\Gamma_2}$ where $\mathbf{u}^*$ is the solution of the following elliptic problem

$$\Delta \mathbf{u}^* = 0 \qquad \text{in } \Omega$$

$$\left.\mathbf{u}^*\right|_{\Gamma_1} = 0 \ , \qquad \left.\frac{\partial \mathbf{u}^*}{\partial n}\right|_{\Gamma_1} = \alpha B.$$

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A NEW PROOF OF THE CONVERGENCE OF DISTRIBUTED OPTIMAL CONTROLS ON THE INTERNAL ENERGY IN MIXED ELLIPTIC PROBLEMS *

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Abstract

We consider two steady-state heat conduction problems P and P_α (for each $\alpha > 0$) with mixed boundary conditions for the same Poisson equation. The difference between both problems is that on the boundary portion Γ_1 a Dirichlet condition is verified for P and a Newton condition with transfer coefficient α is verified for P_α. We formulate distributed optimal control problems, for a suitable cost function, over the internal energy g in the material. We make a new proof with respect to the one given in C.M. Gariboldi - D.A. Tarzia, App. Math. Optim. 47 (2003), 213-230 on the strongly convergence when $\alpha \to \infty$ of the optimal control g_{op_α}, the system state $u_{g_{op_\alpha}\alpha}$ and the adjoint state $p_{g_{op_\alpha}\alpha}$ to the optimal control g_{op}, system state $u_{g_{op}}$ and adjoint state $p_{g_{op}}$ corresponding to P_α and P respectively. For this proof we eliminate the restriction on the constant of coerciveness of the bilinear form, and we use properties of the cost function and the theory of variational equalities instead of the fixed point theorem.

Key words: Variational Inequality, Distributed Optimal Control, Mixed Elliptic Problem, Adjoint State, Steady-State Stefan Problem, Optimality Condition.
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1 Introduction

We consider a bounded domain Ω in $\mathbb{R}^n$ whose regular boundary Γ consists of the union of two disjoint portions Γ_1 y Γ_2 with meas$(\Gamma_1)>0$ and meas$(\Gamma_2)>0$. We denote with meas (Γ) the (n-1)-dimensional Lebesgue measure of Γ.

We consider the following two steady-state heat conduction problems P and P_α (for each parameter $\alpha > 0$) respectively with mixed boundary conditions:

$$-\Delta u = g \text{ in } \Omega \qquad u|_{\Gamma_1} = B \qquad -\frac{\partial u}{\partial n}|_{\Gamma_2} = q \tag{1}$$

and

$$-\Delta u = g \text{ in } \Omega \qquad -\frac{\partial u}{\partial n}|_{\Gamma_1} = \alpha(u-B) \qquad -\frac{\partial u}{\partial n}|_{\Gamma_2} = q \tag{2}$$

where g is the internal energy in Ω, B is the temperature on Γ_1 for (1) and the temperature of the external neighboord of Γ_1 for (2), q is the heat flux on Γ_2 and $\alpha > 0$ is the heat transfer coefficient of Γ_1 (Newton's law on Γ_1), that satisfy the following assumptions:

$$g \in H = L^2(\Omega)\,, \qquad q \in L^2(\Gamma_2)\,, \qquad B \in H^{\frac{1}{2}}(\Gamma_1). \tag{3}$$

Problems (1) and (2) can be considered as the steady-state Stefan problem for suitable data q, g and B [5], [8], [11], [18], [19] and [21] .

Let u_g and $u_{g\alpha}$ be the unique solutions of the mixed elliptic problems (1) and (2) respectively whose variational equalities are given by [15], [19]:

$$a(u_g, v) = L_g(v),\ \forall v \in V_0,\ u_g \in K \tag{4}$$

and

$$a_\alpha(u_{g\alpha}, v) = L_{g\alpha}(v),\ \forall v \in V,\ u_{g\alpha} \in V \tag{5}$$

where

$$V = H^1(\Omega)\,; \qquad V_0 = \{v \in V /\, v|_{\Gamma_1} = 0\}\,;$$

$$K = v_0 + V_0; \qquad (g,h) = (g,h)_H = \int_\Omega gh\, dx; \tag{6}$$

$$a(u,v) = \int_\Omega \nabla u.\nabla v\ dx\ ; \qquad a_\alpha(u,v) = a(u,v) + \alpha \int_{\Gamma_1} Bv\ d\gamma$$

$$L_g(v) = (g,v)_H - \int_{\Gamma_2} qv\, d\gamma\,; \quad L_{g\alpha}(v) = L_g(v) + \alpha \int_{\Gamma_1} Bv\ d\gamma$$

for a given $v_0 \in V$, $v_0|_{\Gamma_1} = B$.

We consider g as a control variable for the cost functionals $J : H \to \mathbb{R}_0^+$ and $J_\alpha : H \to \mathbb{R}_0^+$ respectively given by:

$$J(g) = \frac{1}{2} \|u_g - z_d\|_H^2 + \frac{M}{2} \|g\|_H^2 \tag{7}$$

and

$$J_\alpha(g) = \frac{1}{2} \|u_{g\alpha} - z_d\|_H^2 + \frac{M}{2} \|g\|_H^2 \tag{8}$$

where $z_d \in H$ is given and $M =$const.> 0.

Then we can formulate the following distributed optimal control problems [7], [9], [10], [12] and [16]:

$$\text{Find } g_{op} \in H \text{ such that } J(g_{op}) = \min_{g \in H} J(g) \tag{9}$$

and

$$\text{Find } g_{op_\alpha} \in H \text{ such that } J_\alpha(g_{op_\alpha}) = \min_{g \in H} J_\alpha(g) \tag{10}$$

respectively.

The use of the variational inequality theory in connection with optimal control problems was done, for example, in [1], [2], [3], [4], [6], [14] and [17]. In [13] an optimization problem corresponding to (1) is studied in order to avoid a change phase process.

In Section 2 we get that the functional J is coercive and Gâteaux differentiable on H, J' is a lipschitzian and strictly monotone application on H. We also obtain the existence and uniqueness of the distributed optimal control problem (9). Similary, in Section 3 we get that the functional J_α is coercive and Gâteaux differentiable on H, J'_α is a lipschitzian and strictly monotone application on H for all $\alpha > 0$. We also obtain the existence and uniqueness of the distributed optimal control problem (10) and strongly convergence (when $\alpha \to \infty$) of the states system (2) and the corresponding adjoint states to the respectives of the system (1), for all $g \in H$. Sections 2 and 3 follow [12].

In Section 4 we study the convergence when $\alpha \to \infty$ of the optimal control problem (10) corresponding to the state system (2). We prove that the optimal state system $u_{g_{op_\alpha}\alpha}$ and the optimal adjoint system $p_{g_{op_\alpha}\alpha}$ of problem (10) are strongly convergent in V to the corresponding $u_{g_{op}}$ and $p_{g_{op}}$ for problem (9) respectively when $\alpha \to \infty$. Finally the strong convergence in H of the optimal control g_{op_α} of problem (10) to the optimal control g_{op} of problem (9) is also proved when $\alpha \to \infty$. This proof is new with respect to the one given in [12]. We have eliminated the restriction on the constant of coerciveness of the bilinear form a and we use the variational equality theory and the optimal control problem instead of the fixed point theory.

2 Problem P and its Corresponding Optimal Control Problem

Let $C : H \to V_0$ be the application such that:

$$C(g) = u_g - u_0 \tag{11}$$

where u_0 is the solution of problem (4) for $g = 0$ whose variational equality is given by:

$$a(u_0, v) = L_0(v), \ \forall v \in V_0, \ u_0 \in K \tag{12}$$

with

$$L_0(v) = -\int_{\Gamma_2} qv \, d\gamma.$$

Let $\Pi : H \times H \to \mathbb{R}$ and L: $H \to \mathbb{R}$ be defined by the following expressions:

$$\Pi(g,h) = (C(g), C(h)) + M(g,h), \quad \forall \ g,h \in H \tag{13}$$

$$L(g) = (C(g), z_d - u_0), \quad \forall \ g \in H.$$

We have that a is a bilinear, continuous and symmetric form on V and coercive on V_0, that is [15], [19]:

$$\exists \ \lambda > 0 \, \text{such that} \ \ a(v,v) \geq \lambda \left\| v \right\|_V^2, \forall \, v \in V_0. \tag{14}$$

Lemma 2.1. (i) C is a linear and continuous application, Π is a linear, continuous, symmetric and coercive form on H, that is:

$$\Pi(g,g) \geq M \left\| g \right\|_H^2, \forall \, g \in H \tag{15}$$

and L is linear and continuous on H.

(ii) J can be also written as:

$$J(g) = \frac{1}{2}\Pi(g,h) - L(g) + \frac{1}{2} \left\| u_0 - z_d \right\|_H^2, \forall \, g \in H. \tag{16}$$

(iii) There exists a unique optimal control $g_{op} \in H$ such that:

$$J(g_{op}) = \min_{g \in H} J(g) \tag{17}$$

(iv) The application $g \in H \to u_g \in V$ is lipschitzian, that is:

$$\left\| u_{g_2} - u_{g_1} \right\|_V \leq \frac{1}{\lambda} \left\| g_2 - g_1 \right\|_H, \forall \, g_1, g_2 \in H. \tag{18}$$

(v) J is a Gâteaux differentiable functional and J' is given by:

$$\langle J'(g), h \rangle = (u_g - z_d, C(h)) + M(g,h) = \Pi(g,h) - L(g) \,, \quad \forall \ g,h \in H \,. \tag{19}$$

(vi) The Gâteaux derivative of J can be written as:

$$J'(g) = p_g + Mg, \forall \, g \in H. \tag{20}$$

where the adjoint state p_g corresponding to problem (1) or (4), for each $g \in H$, is the unique solution of the following mixed elliptic problem:

$$-\Delta p_g = u_g - z_d \ \text{ in } \Omega; \ \ p_g|_{\Gamma_1} = 0 \, ; \ \frac{\partial p_g}{\partial n}|_{\Gamma_2} = 0 \tag{21}$$

whose variational formulation is given by:

$$a(p_g, v) = (u_g - z_d, v), \forall v \in V_0, \ p_g \in V_0. \tag{22}$$

Moreover, the adjoint state p_g satisfy the following equalities:

$$(p_g, h) = (u_g - z_d, C(h)) = a(p_g, C(h)) \ \ \forall \ g, h \in H\,. \tag{23}$$

(vii) The optimality condition for the problem (9) is given by $J'(g_{op}) = 0$ in H, that is:

$$p_{g_{op}} + M g_{op} = 0 \ \text{ in } H \tag{24}$$

(viii) We have the following inequality:

$$\|p_{g_2} - p_{g_1}\|_V \leq \frac{1}{\lambda} \|u_{g_2} - u_{g_1}\|_H \ \ \forall \, g_1, g_2 \in H \tag{25}$$

Proof. (i)-(iv) See [12].

(v)-(viii) The mean ideas of the proof are the following expressions:

$$\begin{aligned} a) \ \frac{1}{t}[J(g + t(f-g)) - J(g)] \ &= \ \frac{t}{2}(u_f - u_g, u_f - u_g) + (u_g - z_d, u_f - u_g) \\ &+M(g, f-g) + \frac{Mt}{2}(f-g, f-g) \end{aligned}$$

$$b) \ a(p_g, C(h)) = a(p_g, u_h - u_0) = a(p_g, u_h) - a(p_g, u_0) = (p_g, h)$$

$$c) \ \lambda \|p_{g_2} - p_{g_1}\|_V^2 \leq a(p_{g_2} - p_{g_1}, p_{g_2} - p_{g_1}) \leq \|u_{g_2} - u_{g_1}\|_H \|p_{g_2} - p_{g_1}\|_H$$

and the details are given in [12]. ■

Now, we are in conditions for obtaining other properties of the functional J.

Lemma 2.2. (i) The application $g \in H \to p_g \in V_0$ is strictly monotone. Moreover, we have:

$$(p_{g_2} - p_{g_1}, g_2 - g_1) = \|u_{g_2} - u_{g_1}\|_H^2 \geq 0, \quad \forall \, g_1, g_2 \in H. \tag{26}$$

(ii) J is coercive or H-elliptic, that is:

$$(1-t)J(g_2) + tJ(g_1) - J((1-t)g_2 + tg_1) \geq \frac{Mt(1-t)}{2} \|g_2 - g_1\|_H^2 \ , \ \ \forall g_1, g_2 \in H, \quad \forall \, t \in [0,1]. \tag{27}$$

(iii) J' is a Lipschitzian and strictly monotone application, that is:

$$\|J'(g_2) - J'(g_1)\|_H \leq (M + \frac{1}{\lambda^2}) \ \|g_1 - g_2\|_H \tag{28}$$

and

$$\langle J'(g_2) - J'(g_1), g_2 - g_1 \rangle = \|u_{g_2} - u_{g_1}\|_H^2 + M \|g_2 - g_1\|_H^2 \geq M \|g_2 - g_1\|_H^2 \ , \ \ \forall g_1, g_2 \in H. \tag{29}$$

Proof. See [12] ■

3 Problem P_α and Its Corresponding Optimal Control Problem

Let $\Pi_\alpha : H \times H \to \mathbb{R}$, $L_\alpha : H \to \mathbb{R}$ and $C_\alpha : H \to V$ be defined by:

$$\Pi_\alpha(g,h) = (C_\alpha(g), C_\alpha(h)) + M(g,h)\,, \ \ \forall\, g,\, h \in H$$

$$L_\alpha(g) = (C_\alpha(g), z_d - u_{0\alpha}), \ \ \ \forall\, g \in H \tag{30}$$

$$C_\alpha(g) = u_{g\alpha} - u_{0\alpha}, \ \ \forall\, g \in H$$

where $u_{g\alpha}$ is the unique solution of the variational equality (5), $u_{0\alpha}$ is the unique solution of (5) for $g = 0$ whose variational equality is given by:

$$a_\alpha(u_{0\alpha}, v) = L_{0\alpha}(v), \forall v \in V, u_{0\alpha} \in V \tag{31}$$

with

$$L_{0\alpha}(v) = \alpha \int_{\Gamma_1} Bv\; d\gamma - \int_{\Gamma_2} qv\, d\gamma \tag{32}$$

and a_α is a bilinear, continuous, symmetric and coercive form on V, that is:

$$a_\alpha(v,v) \geq \lambda_\alpha \left\|v\right\|_V^2, \forall v \in V. \tag{33}$$

where $\lambda_\alpha = \lambda_1 \min(1,\alpha) > 0$ for all $\alpha > 0$ and λ_1 is the coerciveness constant for the bilinear form a_1 [20].

We can obtain similaries properties to Lemma 2.1, following [13], [15], [16] and [19] which proof is omitted.

Lemma 3.1. Let $\alpha > 0$ be. (i) C_α is a linear and continuous application, Π_α is linear, continuous, symmetric and coercive on H, that is:

$$\Pi_\alpha(g,g) \geq M \left\|g\right\|_H^2, \forall g \in H. \tag{34}$$

and L_α is linear and continuous on H.

(ii) J_α can be also written as:

$$J_\alpha(g) = \frac{1}{2}\Pi_\alpha(g,h) - L_\alpha(g) + \frac{1}{2}\left\|u_{0\alpha} - z_d\right\|_H^2, \ \forall\, g \in H. \tag{35}$$

(iii) There exists a unique optimal control $g_{op_\alpha} \in H$ such that:

$$J_\alpha(g_{op_\alpha}) = \min_{g \in H} J_\alpha(g). \tag{36}$$

(iv) The application $g \in H \to u_{g\alpha} \in V$ is lipschitzian, that is:

$$\left\|u_{g_2\alpha} - u_{g_1\alpha}\right\|_V \leq \frac{1}{\lambda_\alpha}\left\|g_2 - g_1\right\|_H, \forall g_1, g_2 \in H. \tag{37}$$

(v) J_α is Gâteaux differentiable functional and J'_α is given by:

$$\langle J'_\alpha(g), h\rangle = (u_{g\alpha} - z_d, C_\alpha(h)) + M(g,h) = \Pi_\alpha(g,h) - L_\alpha(g),\ \forall\ g, h \in H \tag{38}$$

(vi) The Gâteaux derivative of J_α can be written as:

$$J'_\alpha(g) = p_{g\alpha} + Mg, \forall\, g \in H \tag{39}$$

where the adjoint state $p_{g\alpha}$ is the unique solution of the following mixed elliptic problem corresponding to (2) or (5), for each $g \in H$ and $\alpha > 0$:

$$-\Delta p_{g\alpha} = u_{g\alpha} - z_d \text{ in } \Omega; \quad -\frac{\partial p_{g\alpha}}{\partial n}|_{\Gamma_1} = \alpha p_{g\alpha}; \ \frac{\partial p_{g\alpha}}{\partial n}|_{\Gamma_2} = 0 \tag{40}$$

whose variational formulation is given by:

$$a_\alpha(p_{g\alpha}, v) = (u_{g\alpha} - z_d, v), \forall v \in V, p_{g\alpha} \in V. \tag{41}$$

where $u_{g\alpha}$ is the unique solution of (5). Moreover, the adjoint state $p_{g\alpha}$ satisfies the following equalities:

$$(p_{g\alpha}, h) = (u_{g\alpha} - z_\alpha,\, C_\alpha(h)) = a_\alpha(p_{g\alpha},\, C_\alpha(h)),\, \forall\, g, h \in H. \tag{42}$$

(vii) The optimality condition for problem (10) is given by $J'_\alpha(g_{op_\alpha}) = 0$ in H, that is:

$$p_{g_{op_\alpha}\alpha} + Mg_{op_\alpha} = 0 \text{ in } H. \tag{43}$$

(viii) We have the following property:

$$\|p_{g_{2\alpha}} - p_{g_{1\alpha}}\|_V \le \frac{1}{\lambda_\alpha} \|u_{g_{2\alpha}} - u_{g_{1\alpha}}\|_H \ \ \forall\, g_{1\alpha}, g_{2\alpha} \in H \tag{44}$$

Proof. See [12] ■

Remark 1. We note the double dependence on the parameter α for the optimal state system $u_{g_{op_\alpha}\alpha}$ and the adjoint state $p_{g_{op_\alpha}\alpha}$.

Lemma 3.2. (i) The operator $g \in H \to p_{g\alpha} \in V$ is strictly monotone, that is:

$$(p_{g_2\alpha} - p_{g_1\alpha}, g_2 - g_1) = \|u_{g_2\alpha} - u_{g_1\alpha}\|^2_H \ge 0, \forall\, g_1, g_2 \in H. \tag{45}$$

(ii) J_α is coercive or H-elliptic, that is:

$$(1-t)J_\alpha(g_2)+tJ_\alpha(g_1)-J_\alpha((1-t)g_2+tg_1) \ge \frac{Mt(1-t)}{2} \|g_2 - g_1\|\, a^2_H, \forall g_1, g_2 \in H; \forall t \in [0,1]. \tag{46}$$

(iii) J'_α is a Lipschitzian and strictly monotone operator, that is:

$$\|J'_\alpha(g_2) - J'_\alpha(g_1)\|_H \le (M + \frac{1}{\lambda^2_\alpha}) \|g_1 - g_2\|_H, \forall\, g_1, g_2 \in H \tag{47}$$

and

$$\langle J'_\alpha(g_2) - J'_\alpha(g_1), g_2 - g_1\rangle \ge M \|g_2 - g_1\|^2_H, \forall g_1, g_2 \in H. \tag{48}$$

Proof. See [12] ■

Now, we will prove the following result of convergence when $\alpha \to \infty$.

Lemma 3.3. For all $\alpha > 0$, $q \in L^2(\Gamma_2)$, $B \in H^{\frac{1}{2}}(\Gamma_1)$, we have the following limits:

$$\begin{aligned} \text{i)} \lim_{\alpha\to\infty} \|u_{g\alpha} - u_g\|_V &= 0, \forall g \in H \\ \text{ii)} \lim_{\alpha\to\infty} \|u_{0\alpha} - u_0\|_V &= 0 \\ \text{iii)} \lim_{\alpha\to\infty} \|p_{g\alpha} - p_g\|_V &= 0, \forall g \in H. \end{aligned} \tag{49}$$

Proof. (i) Taking into account [12] and following [19] and [20] we obtain that there exists C_1, a constant independient of α, such that for large α:

$$\|u_{g\alpha} - u_g\|_V^2 \le \frac{C_1}{\lambda_1} \quad , \quad (\alpha - 1)\int_{\Gamma_1} (u_{g\alpha} - u_g)^2 d\gamma \le \frac{(C_1)^2}{\lambda_1} \tag{50}$$

and we deduce that there exists $w_g \in K$ such that:

$$a(w_g, v) = L_g(v), \forall v \in V_0, w_g \in K \tag{51}$$

and by uniqueness, we have $w_g = u_g$. Therefore, $u_{g\alpha} \to u_g$ strongly in V as $\alpha \to \infty$ because the following inequality:

$$\lambda_1 \|u_{g\alpha} - u_g\|_V^2 \le L_g(u_{g\alpha} - u_g) - a(u_{g\alpha}, u_{g\alpha} - u_g).$$

For the case (ii) we take $g = 0$ in the case (i).

(iii) We prove that there exists C_2 a constant independent of α, for large α, such that:

$$\|p_{g\alpha} - p_g\|_V^2 \le \frac{C_2}{\lambda_1} \quad , \quad (\alpha - 1)\int_{\Gamma_1} (p_{g\alpha} - p_g)^2 d\gamma \le \frac{(C_2)^2}{\lambda_1} \tag{52}$$

and we deduce that there exists $\xi_g \in V_0$ such that:

$$a(\xi_g, v) = (u_g - z_d, v), \forall v \in V_0, \xi_g \in V_0. \tag{53}$$

and by uniqueness, we obtain $\xi_g = p_g$. Therefore, taking into account the following inequality:

$$\lambda_1 \|p_{g\alpha} - p_g\|_V^2 \le (u_{g\alpha} - z_d, p_{g\alpha} - p_g) - a(p_g, p_{g\alpha} - p_g)$$

we get that $p_{g\alpha} \to p_g$ strongly in V. ■

4 Convergence of Problem P_α and its Corresponding Optimal Control as $\alpha \to \infty$

In this section we will make a new proof with respect to the one given in [12] of the strongly convergence of the optimal control g_{op_α} of problem (10) and its corresponding adjoint state $p_{g_{op_\alpha}\alpha}$ (41) to the optimal control g_{op} of problem (9) and its corresponding adjoint state $p_{g_{op}}$ (22) respectively when the parameter α (heat transfer coefficient on Γ_1) goes to infinity. We will eliminate the restriction on the constant of coerciveness of the bilinear form a and we will use properties of the cost function and the variational equality theory instead of the fixed point theorem.

Theorem 4.1. (i) If $p_{g_{op}}$ and $p_{g_{op_\alpha}\alpha}$ are the corresponding adjoint state of the problem (9) and problem (10) respectively, then:

$$\lim_{\alpha\to\infty} \left\| p_{g_{op_\alpha}\alpha} - p_{g_{op}} \right\|_V = 0 \tag{54}$$

(ii) If g_{op} and g_{op_α} are the solutions of the problem (9) and problem (10) respectively, then:

$$\lim_{\alpha\to\infty} \left\| g_{op_\alpha} - g_{op} \right\|_H = 0. \tag{55}$$

(iii) If $u_{g_{op}}$ and $u_{g_{op_\alpha}\alpha}$ are the corresponding solutions of the problem P and problem P_α respectively, then:

$$\lim_{\alpha\to\infty} \left\| u_{g_{op_\alpha}\alpha} - u_{g_{op}} \right\|_V = 0. \tag{56}$$

Proof. First we will prove some preliminary results. Since g_{op_α} is the solution of the problem (10), we have the following inequality:

$$\frac{1}{2}\left\| u_{g_{op_\alpha}\alpha} - z_d \right\|_H^2 + \frac{M}{2}\left\| g_{op_\alpha} \right\|_H^2 \le \frac{1}{2}\left\| u_{g\alpha} - z_d \right\|_H^2 + \frac{M}{2}\left\| g \right\|_H^2, \ \forall g \in H,$$

then, taking $g = 0$, we have:

$$\frac{1}{2}\left\| u_{g_{op_\alpha}\alpha} - z_d \right\|_H^2 + \frac{M}{2}\left\| g_{op_\alpha} \right\|_H^2 \le \frac{1}{2}\left\| u_{0\alpha} - z_d \right\|_H^2 \le C_3, \forall \alpha > 0,$$

where C_3 is a constant independent of parameter α because $u_{0\alpha}$ is convergent when $\alpha \to \infty$. Therefore

$$\left\| g_{op_\alpha} \right\|_H \le C_4 \text{ and } \left\| u_{g_{op_\alpha}\alpha} \right\|_H \le C_5 \tag{57}$$

where C_4 and C_5 are constants independent of α. Now, if we take $v = u_{g_{op_\alpha}\alpha} - u_{g_{op}}$ in the variational equality (5), following [19] and [20] we obtain for $\alpha > 1$:

$$\begin{aligned} \lambda_1 \left\| u_{g_{op_\alpha}\alpha} - u_{g_{op}} \right\|_V^2 + (\alpha - 1)\int_{\Gamma_1} (u_{g_{op_\alpha}\alpha} - u_{g_{op}})^2 d\gamma \ &\le \ a_\alpha(u_{g_{op_\alpha}\alpha} - u_{g_{op}}, u_{g_{op_\alpha}\alpha} - u_{g_{op}}) \\ &\le \ C_6 \left\| u_{g_{op_\alpha}\alpha} - u_{g_{op}} \right\|_V \end{aligned}$$

where $C_6 = C_6(g_{op}, q, u_{g_{op}})$ is independent of α. Next, we have:

$$\left\| u_{g_{op_\alpha}\alpha} - u_{g_{op}} \right\|_V^2 \le \frac{C_6}{\lambda_1} \qquad , \qquad (\alpha - 1)\int_{\Gamma_1} (u_{g_{op_\alpha}\alpha} - u_{g_{op}})^2 d\gamma \le \frac{(C_6)^2}{\lambda_1} \tag{58}$$

and therefore we deduce that:

$$\exists\, \eta \in V \text{ sucht that } u_{g_{op_\alpha}\alpha} \rightharpoonup \eta \text{ weakly in } V, \tag{59}$$

and because the following inequalities:

$$0 \le \int_{\Gamma_1} (\eta - u_{g_{op}})^2 d\gamma \le \lim_{\alpha\to\infty}\inf \int_{\Gamma_1} (u_{g_{op_\alpha}\alpha} - u_{g_{op}})^2 d\gamma = 0,$$

we obtain that $\eta \in K$. Next, if we take $v = p_{g_{op_\alpha}\alpha} - p_{g_{op}}$ in the variational equality (41) we get:

$$\begin{aligned} \lambda_1 \left\| p_{g_{op_\alpha}\alpha} - p_{g_{op}} \right\|_V^2 + (\alpha - 1) \int_{\Gamma_1} (p_{g_{op_\alpha}\alpha} - p_{g_{op}})^2 d\gamma &\leq a_\alpha(p_{g_{op_\alpha}\alpha} - p_{g_{op}}, p_{g_{op_\alpha}\alpha} - p_{g_{op}}) \\ &\leq C_7 \left\| p_{g_{op_\alpha}\alpha} - p_{g_{op}} \right\|_V , \end{aligned}$$

with $C_7 = C_7(C_5, p_{g_{op}})$. Next, we obtain:

$$\left\| p_{g_{op_\alpha}\alpha} - p_{g_{op}} \right\|_V^2 \leq \frac{C_7}{\lambda_1} \quad , \quad (\alpha - 1) \int_{\Gamma_1} (p_{g_{op_\alpha}\alpha} - p_{g_{op}})^2 d\gamma \leq \frac{(C_7)^2}{\lambda_1} \tag{60}$$

and therefore we deduce that:

$$\exists \xi \in V \text{such that } p_{g_{op_\alpha}\alpha} \rightharpoonup \xi \text{ weakly in } V \tag{61}$$

and by the following inequality:

$$0 \leq \int_{\Gamma_1} (\xi - p_{g_{op}})^2 d\gamma \leq \liminf_{\alpha\to\infty} \int_{\Gamma_1} (p_{g_{op_\alpha}\alpha} - p_{g_{op}})^2 d\gamma = 0$$

we obtain $\xi \in V_0$. Now, we consider $v \in V_0$ and taking into acount (59) and (61), from the variational equality (41) we have:

$$a(\xi, v) = (\eta - z_d, v), \forall v \in V_0, \xi \in V_0. \tag{62}$$

Next, from (57) we deduce that there exists $f \in H$ such that $g_{op_\alpha} \rightharpoonup f$ weakly in H. Therefore if we put $v \in V_0$ in the variational equality (5) and we pass to the limit $\alpha \to \infty$, we obtain:

$$a(\eta, v) = (f, v) - \int_{\Gamma_2} qv d\gamma, \forall v \in V_0, \eta \in K. \tag{63}$$

Now,

$$a(\eta, v) = L_f(v), \forall v \in V_0, \eta \in K \tag{64}$$

and from the uniqueness of solution of the variational equality (4), we have:

$$\eta = u_f. \tag{65}$$

On the oder hand, from (62), (65) and the uniqueness of solution of the variational equality (22), it results that:

$$\xi = p_f$$

Now,

$$J_\alpha(g_{op_\alpha}) \leq J_\alpha(f^*), \forall f^* \in H$$

next,

$$J(f) = J_\alpha(f) \leq \liminf_{\alpha\to\infty} J_\alpha(g_{op_\alpha}) \leq \liminf_{\alpha\to\infty} J_\alpha(f^*) = \lim_{\alpha\to\infty} J_\alpha(f^*) = J(f^*)$$

and from the uniqueness of the optimal control we obtain that $f = g_{op}$. Therefore $\eta = u_f = u_{g_{op}}$ and $\xi = p_f = p_{g_{op}}$.

Moreover, from (61) and the following computation:

$$\begin{aligned}
\lambda_1 \left\| p_{g_{op_\alpha}\alpha} - p_{g_{op}} \right\|_V^2 &\leq a_\alpha(p_{g_{op_\alpha}\alpha} - p_{g_{op}}, p_{g_{op_\alpha}\alpha} - p_{g_{op}}) \\
&= a_\alpha(p_{g_{op_\alpha}\alpha}, p_{g_{op_\alpha}\alpha} - p_{g_{op}}) - a(p_{g_{op}}, p_{g_{op_\alpha}\alpha} - p_{g_{op}}) \\
&= (u_{g_{op_\alpha}\alpha} - z_d, p_{g_{op_\alpha}\alpha} - p_{g_{op}}) - a(p_{g_{op}}, p_{g_{op_\alpha}\alpha} - p_{g_{op}})
\end{aligned}$$

we have (54). From the optimality condition (24) it results that:

$$\left\| g_{op_\alpha} - g_{op} \right\|_H = \frac{1}{M} \left\| p_{g_{op}} - p_{g_{op_\alpha}\alpha} \right\|_H \leq \frac{1}{M} \left\| p_{g_{op}} - p_{g_{op_\alpha}\alpha} \right\|_V$$

and therefore (55) holds. Now, we have:

$$\begin{aligned}
\lambda_1 \left\| u_{g_{op_\alpha}\alpha} - u_{g_{op}} \right\|_V^2 &\leq a_\alpha(u_{g_{op_\alpha}\alpha} - u_{g_{op}}, u_{g_{op_\alpha}\alpha} - u_{g_{op}}) \\
&= a_\alpha(u_{g_{op_\alpha}\alpha}, u_{g_{op_\alpha}\alpha} - u_{g_{op}}) - a_\alpha(u_{g_{op}}, u_{g_{op_\alpha}\alpha} - u_{g_{op}}) \\
&= L_{g_{op_\alpha}\alpha}(u_{g_{op_\alpha}\alpha} - u_{g_{op}}) - a(u_{g_{op}}, u_{g_{op_\alpha}\alpha} - u_{g_{op}}) - \alpha \int_{\Gamma_1} b(u_{g_{op_\alpha}\alpha} - b)\, d\gamma \\
&= a(u_{g_{op_\alpha}}, u_{g_{op_\alpha}\alpha} - u_{g_{op}}) - a(u_{g_{op}}, u_{g_{op_\alpha}\alpha} - u_{g_{op}}) \\
&= a(u_{g_{op_\alpha}} - u_{g_{op}}, u_{g_{op_\alpha}\alpha} - u_{g_{op}})
\end{aligned}$$

and taking into account (55) and the fact that $u_{g_{op_\alpha}} \to u_{g_{op}}$ strongly in V when $\alpha \to \infty$ because (18), we get (56). ■

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