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VII Italian Latin American Conference on Industrial and Applied Mathematics

Second Part

Domingo A. Tarzia (Ed.)

Departamento de Matemática,
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INDICE

Rosario, Julio 2015

Preface

On December 17-21, 2012, ASAMACI (Asociación Argentina de Matemática Aplicada, Computacional e Industrial) co-hosted with AR-SIAM (Argentinean Section of SIAM - Society for Industrial and Applied Mathematics, USA), SIMAI (Società Italiana di Matematica Applicata e Industriale - Italy) and the Department of Mathematics of FCE-Universidad Austral Rosario, the Seventh Italian - Latin American Conference on Industrial and Applied Mathematics (VII ITLA 2012) in the city of Rosario, Argentina.

The International Scientific committee was integrated by Nicola Bellomo (Italy), Mario Primicerio (Italy), Marco Calahorrano (Ecuador), José A. Cuminato (Brazil), Fabián Flores-Bazán (Chile), Obidio Rubio (Peru), Julio Ruiz Claeyssen (Brazil), Rubén D. Spies (Argentina) and Domingo Tarzia (Argentina).

The Local Scientific Committee was integrated by Carlos D'Attellis (Univ. Favaloro, UNSAM, Buenos Aires), Javier I. Etcheverry (Tenaris - UBA, Campana - Buenos Aires), Pablo Jacovkis (UBA, Buenos Aires), Cristina Maciel (UNS, Bahía Blanca), Sergio Preidikman (CONICET-UNC, Córdoba), Diana Rubio (UNSAM, Buenos Aires), Rubén D. Spies (IMAL (CONICET-UNL), Santa Fe), Domingo A. Tarzia (CONICET-UA, Rosario), (Chairman) and Cristina V. Turner (CONICET-UNC, Córdoba).

The Local Organizing Committee was integrated by Julieta Bollati, Adriana C. Briozzo, Mariela Cirelli, Ma. Fernanda Natale, Sabrina Roscani, Eduardo A. Santillan Marcus and Jóse Semitiel.

The goals of the meeting were:
1) To provide a forum for the discussion and enhancement of Applied Mathematics in the Latin American region;
2) To promote the application of Mathematics in Science, Engineering, Computational Science, Industry and Technology;
3) To provide media for the exchange of information and ideas between latinamerican and Italian mathematicians;
4) To promote interdisciplinary research between different disciplines of Science and Engineering in areas involving Mathematics;
5) To encourage and endorse activities for the enhancement and promotion of the education of Applied Mathematics at both undergraduate and graduate levels;
6) To promote the interest in Applied Mathematics among all disciplines of Science and Engineering.

All researchers, academics, graduate students, senior undergraduate students, and post-doctoral fellows studying Mathematics, Physics, Chemistry, Economics, Finance and Engineering and other branches of Science were invited to participate in this conference.

The VII ITLA 2012 Conference was sponsored by ANPCyT (National Agency for Promotional of Science and Technology, Argentina), ICIAM (International Council for Industrial and Applied Mathematics), Secretary of Science and Technology (State of Santa Fe, Argentina) and Embassy of Italy at Buenos Aires. This financial support sponsored partially 56 participants.

ITLA is a periodic meeting between Latin American and Italian researchers since 1995, serving as a bridge between European and Latin American scenarios. In the course of successive editions, these meetings have allowed the discussion of new researches in Industrial and Applied Mathematics. Earlier editions of ITLA were held in Porto Alegre - Brazil (I ITLA, January 1995), Rome - Italy (II ITLA, January 1997), Río de Janeiro – Brazil (III ITLA - November 1999), Trujillo -Peru (III ITLA - December 2004), Florence - Italy (V ITLA - July 2007) and Quito - Ecuador (VI ITLA - September

2009). On this occasion this tradition was prolifically continued, being the first meeting held in Argentina. This year's event was very excited to welcome a number of prominent mathematicians from different countries of the world, including Argentina, Brazil, Chile, Colombia, Ecuador, Italy, Peru, Switzerland, United Kingdom and United States of America, and it was a great success with over a hundred researchers, academics, graduate students, senior undergraduate students, and post-doctoral fellows studying Applied Mathematics and other branches of Science in attendance.

The event was structured in eighty presentations: thirty-eight plenary conferences and forty-two contributed presentations, plus three short courses given by Mario Primicerio (Italy), Paolo Marcellini (Italy) and Paolo Podio-Guidugli (Italy).

The main areas addressed by ITLA included Nonlinear Analysis and Applications, Numerical Analysis, Biomathematics, Ordinary Differential Equations and Applications, Partial Differential Equations and Applications, Continuous Mechanics and Applications, Optimization, Optimal Control and Applications, and Inverse Problems and Applications.

The conferences were done by Marco Calahorrano (Ecuador), Gabriel Cárcamo (Chile), José Carcione (Italy), Julio Ruiz Claeyssen (Brazil), José A. Cuminato (Brazil), Guillermo Durán (Argentina), Javier Etcheverry (Argentina), Analía Gastón (Argentina), Pablo Jacovkis (Argentina), Rolf Jeltsch (Switzerland), Barbara Lee Keyfitz (USA), Adrián Lew (USA), Pablo Lotito (Argentina), María Cristina Maciel (Argentina), Paolo Marcellini (Italy), Elvira Mascolo (Italy), José Luis Menaldi (USA), Fabio Milner (USA), Jaime Ortega (Chile), Paolo Podio-Guidugli (Italy), Giovanni Porru (Italy), Sergio Preidikman (Argentina), Mario Primicerio (Italy), Héctor Ramirez-Cabrera (Chile), Juan C. Reginato (Argentina), Fabio Rosso (Italy), Diana Rubio (Argentina), Obidio Rubio (Peru), Juan Santos (Argentina), Andrey Sarychev (Italy), Eduardo Serrano (Argentina), Geraldo Nunes Silva (Brazil), Rubén D. Spies (Argentina), Domingo A. Tarzia (Argentina), Cristina V. Turner (Argentina), Stella Vernier Piro (Italy), Vincenzo Vespri (Italy) and Adrián Will (Argentina).

Ultimately, the organizers hoped that the event would inspire to provide the discussion and enhancement of Applied Mathematics and to promote the application of Mathematics in Science, Engineering, Computational Science, Industry and Technology in the Latin American region.

Moreover, during this Conference the first step in order to create the CLAMAI (Latin America Committee for Industrial and Applied Mathematics) and to decide the First Latin America Congress on Industrial and Applied Mathematics were taken.

Domingo A. Tarzia, Ed.

Publications:
From these activities, five new communications received during 2014 and accepted on July 2015 are published.

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SOBRE LA ESTABILIDAD DE UN SISTEMA ELÁSTICO DINÁMICO EN CRISTALES CÚBICOS

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Resumen: Probamos la existencia de solución global del sistema elástico dinámico relativo a cristales cúbicos usando la Teoría de Operadores maximales monótonos y el Teorema de Minty-Browder. Así, usando técnicas multiplicativas, resultados de Dautry R. - Lions J. L., Desigualdades integrales no lineales, y adaptando el método de F. Conrad y B. Rao en el cálculo de las estimativas, mostramos que la energía asociada al sistema decae a cero cuando $t \to +\infty$. Aquí vemos dos casos uno con tasa exponencial y la otra con tasa polinomial.

Palabras clave: *Desigualdades integrales no linelaes, estabilidad de un sistema elástico dinámico, el método de Conrad y Rao.*
2000 AMS Subject Classification: 47B44,35L90.

1. INTRODUCCIÓN

Estudiamos la existencia y la estabilidad para un sistema elástico dinámico relativo a cristales cúbicos. Basandonos en Desigualdades Integrales no lineales, desarrollados en la sección 2, presentamos la estabilidad en dos casos, una con tasa exponencial y la otra con tasa polinomial.

Lagnese en [8] obtiene resultados de decrecimiento uniforme de la energía del sistema elástico con control lineal $f(u')$ mas condiciones técnicas sobre el tensor de elasticidad, pero esto no es posible aplicar al sistema elástico homogéneo Isotrópico Lineal (SEHIL). En [9] Lagnese consigue estimativas de decrecimiento uniforme de la energía para el SEHIL bidimensional con control no lineal $f(u')$ mas un suavizante lineal. Por otro lado, Komornik en [7] mostró el resultado [9] sin el suavizante lineal para el caso N=2,3, sometidos a una fuerza que depende linealmente de la velocidad.

Para fundamentar nuestro estudio podemos citar Feynman [6] y Ciarlet [1]. También es importante citar a Dafermos [4], uno de los pioneros en el estudio del comportamiento asintótico de soluciones de ecuaciones de evolución no lineal.

Es interesante también citar los trabajos [3], [5], [10], [11] y [12], donde se abordan el comportamiento asintótico de algunos sistemas de evolución.

2. DESIGUALDADES INTEGRALES NO LINEALES

Lema 1 (Desigualdad Integral tipo Gronwall) *Sea $E : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ una función no creciente y supongamos que exista una constante $T > 0$ tal que*

$$\int_t^{+\infty} E(s)ds \leq TE(t)\,, \quad \forall t \geq 0 \tag{1}$$

entonces

$$E(t) \leq E(0)e^{1-\frac{t}{T}}\,, \quad \forall t \geq T\,. \tag{2}$$

Prueba. Si $E(0) = 0$ y como $0 \leq E(t) \leq E(0)$ para $t \geq 0$, entonces $E(t) = 0$. Antes de continuar con la prueba, daremos algunas observaciones:

Nota 1 *Se verifica fácilmente que "la desigualdad (2) también es válida para $0 \leq t < T$".*

En efecto, si $0 < s < T$, entonces $E(s) \leq E(0) \neq 0$ entonces $\frac{E(s)}{E(0)} \leq 1 = e^0 < e^{1-\frac{s}{T}}$, desde que $s < T$. Por lo tanto, $E(s) < E(0)e^{1-\frac{s}{T}}$, para $0 \leq s < T$.

Nota 2 *La desigualdad (2) implica que $E(t) \leq E(0)$, $\forall t \geq 0$.*

En efecto, como $t \geq T$ entonces $\frac{t}{T} \geq 1$, i.e. $1 - \frac{t}{T} \leq 0$, así $e^{1-\frac{t}{T}} \leq 1$ y $E(0)e^{1-\frac{t}{T}} \leq E(0)$. Usando (2), tenemos $E(t) \leq E(0)e^{1-\frac{t}{T}} \leq E(0)$.

Nota 3 *Se observa fácilmente, desde que E es no creciente, que si $0 \leq s \leq T$ entonces $E(s) \leq E(0)$.*

Prueba del Lema

Definimos

$$f(x) := e^{\frac{x}{T}} \int_x^{+\infty} E(s)ds\,.$$

Entonces de (1), tenemos

$$f(t) \leq e^{\frac{t}{T}} TE(t)\,,\ \forall t \geq 0\,. \tag{3}$$

También, f es no creciente, i.e. $f'(x) \leq 0$. En efecto,

$$f'(x) \;=\; \frac{1}{T}e^{\frac{x}{T}} \int_x^{\infty} E(s)ds - e^{\frac{x}{T}}E(x) \;=\; \frac{e^{\frac{x}{T}}}{T}\left\{\int_x^{+\infty} E(s)ds - TE(x)\right\} \leq 0$$

desde que vale (1).
Por otro lado, como f es no-creciente y vale (3) tenemos que $f(x) \leq f(0) \leq TE(0)$, $\forall x \in \mathbb{R}^+$, luego $e^{\frac{x}{T}} \int_x^{+\infty} E(s)ds \leq TE(0)$, i.e.

$$\int_x^{+\infty} E(s)ds \leq e^{-\frac{x}{T}} TE(0)\,,\ \forall x \in \mathbb{R}^+\,. \tag{4}$$

Como E es no creciente y positiva, al integrar E sobre $[x, x+T]$ tenemos que $TE(x+T) \leq \int_x^{x+T} E(s)ds$ y como $\int_x^{x+T} E(s)ds \leq \int_x^{+\infty} E(s)ds$, tenemos:

$$TE(x+T) \leq \int_x^{+\infty} E(s)ds\,. \tag{5}$$

De (4) y (5) tenemos:

$$TE(x+T) \leq Te^{-\frac{x}{T}}E(0)\,.$$

Tomando: $t = x + T$, tenemos:

$$E(t) \leq e^{1-\frac{t}{T}}E(0)\,.$$

□

Teorema 1 (Desigualdad Integral no lineal) *Sea $f : \mathbb{R}^+ \to \mathbb{R}^+$ una función no creciente y supongamos que existan $\alpha > 0$, $T > 0$ tal que*

$$\int_t^{+\infty} [f(s)]^{\alpha+1}\,ds \leq T[f(0)]^{\alpha} f(t)\,,\ \forall t \in \mathbb{R}^+ \tag{6}$$

entonces

$$f(t) \leq f(0)\left[\frac{T+\alpha t}{T+\alpha T}\right]^{-\frac{1}{\alpha}}\,,\ \forall t \geq T\,. \tag{7}$$

Prueba. Observamos que (7) implica $f(t) \leq f(0)$. Si $f(0) = 0$ entonces $f = 0$. Por otro lado si $f(0) \neq 0$, podemos definir:

$$G(t) = \frac{f(t)}{f(0)}\,.$$

Multiplicando $[f(0)]^{-(\alpha+1)}$ por (6), tenemos $\int_t^{+\infty}[G(s)]^{\alpha+1}ds \leq TG(t)\,.$

También se tiene que $G(0) = 1$. Luego todo se reduce a probar que $G(t) \leq \left[\frac{T+\alpha t}{T+\alpha T}\right]^{\frac{-1}{\alpha}}\,,\ \forall t \geq T\,.$

Esto nos permite asumir $f(0) = 1$.
Introducimos la siguiente función

$$\begin{aligned} F : \mathbb{R}^+ &\rightarrow \mathbb{R}^+ \\ t &\rightarrow F(t) = \int_t^{+\infty} [f(s)]^{\alpha+1} ds \,, \end{aligned}$$

F es no creciente y localmente absolutamente continua. Diferenciando tenemos que $F'(t) = -[f(t)]^{\alpha+1}$ y usando (6) i.e.

$$F(t) \;\leq\; T[f(0)]^{\alpha} f(t) \;\leq\; Tf(t)$$

tenemos

$$\left[\frac{F(t)}{T}\right]^{\alpha+1} \leq [f(t)]^{\alpha+1} = -F'(t)\,.$$

Así hemos demostrado que

$$-F' \geq T^{-\alpha-1} F^{\alpha+1} \text{ en c. t. p. en } (0, \infty)\,.$$

Por otro lado, diferenciando tenemos

$$(F^{-\alpha})' \;=\; -\alpha F^{-\alpha-1} F' \;=\; \alpha F^{-\alpha-1} f^{\alpha+1} \;\geq\; \alpha T^{-(\alpha+1)} \text{ en c.t.p. en } (0, B) \tag{8}$$

donde $B := \sup\{t\,,\; f(t) > 0\}$, y también se observa que $[F(t)]^{-\alpha}$ esta bien definida para $t < B$.

Integrando (8) sobre $[0, s]$ se obtiene

$$[F^{-\alpha}(s) - F^{-\alpha}(0)] \geq \alpha T^{-(\alpha+1)} s \text{ en c.t.p. en } [0, B)$$

i.e.

$$F^{-\alpha}(s) \geq F^{-\alpha}(0) + \alpha T^{-(\alpha+1)} s \text{ en c.t.p. en } [0, B)\,.$$

Es decir,

$$\frac{1}{F^{-\alpha}(0) + \alpha T^{-(\alpha+1)} s} \geq [F(s)]^{\alpha}\,.$$

Por lo tanto,

$$[F^{-\alpha}(0) + \alpha T^{-(\alpha+1)} s]^{-\frac{1}{\alpha}} \geq F(s), \text{ para cada } s \in [0, B)\,. \tag{9}$$

Desde que $F(s) = 0$ si $s \geq B$, tenemos que (9) sucede para $s \in \mathbb{R}$.

De (6) tenemos que $F(0) \leq T$ y el lado izquierdo de (9) esta mayorada por

$$\begin{aligned} [F^{-\alpha}(0) + \alpha T^{-(\alpha+1)} s]^{-\frac{1}{\alpha}} &\leq [T^{-\alpha} + \alpha T^{-(\alpha+1)} s]^{-\frac{1}{\alpha}} \;=\; [T^{-(\alpha+1)}[T + \alpha s]]^{-\frac{1}{\alpha}} \\ &= T^{\frac{\alpha+1}{\alpha}} [T + \alpha s]^{-\frac{1}{\alpha}}\,. \end{aligned} \tag{10}$$

Siendo f no negativo y no creciente, el lado derecho de (9) puede ser estimado,

$$\begin{aligned} F(s) &= \int_s^{+\infty} [f(t)]^{\alpha+1} dt \;\geq\; \int_s^{T+(\alpha+1)s} [f(t)]^{\alpha+1} dt \\ &\geq (T + \alpha s).[f(T + (\alpha+1)s)]^{\alpha+1}\,. \end{aligned} \tag{11}$$

Por transitividad, usando (11) y (10) en (9) obtenemos

$$(T + \alpha s).[f(T + (\alpha+1)s)]^{\alpha+1} \leq T^{\frac{\alpha+1}{\alpha}} [T + \alpha s]^{-\frac{1}{\alpha}}$$

esto es,

$$[f(T + (\alpha+1)s)]^{\alpha+1} \leq T^{\frac{\alpha+1}{\alpha}} [T + \alpha s]^{-(\frac{\alpha+1}{\alpha})}$$

i.e. $\forall s \geq 0$ se tiene

$$f(T+(\alpha+1)s) \;\leq\; T^{\frac{1}{\alpha}}[T+\alpha s]^{-\frac{1}{\alpha}} = \left(\frac{T+\alpha s}{T}\right)^{-\frac{1}{\alpha}} = \left(1+\frac{\alpha}{T}s\right)^{-\frac{1}{\alpha}} . \tag{12}$$

Haciendo $t := T+(\alpha+1)s$ tenemos $s = \frac{t-T}{\alpha+1}$ y sustituyendo en (12) obtenemos

$$f(t) \;\leq\; \left[1+\frac{\alpha}{T}\cdot\frac{t-T}{\alpha+1}\right]^{-\frac{1}{\alpha}} = \left[\frac{T+\alpha t}{T+\alpha T}\right]^{-\frac{1}{\alpha}}$$

para $t \geq T$. □

3. Sistema elástico dinámico y existencia de solución

Sea Ω un dominio acotado de $\mathbb{R}^N$ de clase C^1, $\Gamma = \partial\Omega$, $\nu = (\nu_1, \ldots, \nu_n)$ vector normal unitario sobre Γ.
Sea Γ_0, Γ_1 una partición de la frontera Γ tal que $\overline{\Gamma_0} \cap \overline{\Gamma_1} = \emptyset$. Las constantes positivas: λ, μ y η, son los parámetros del tensor de elasticidad. Definimos $\gamma := \eta - \lambda - 2\mu$.
Sean $a, l : \Gamma_1 \to \mathbb{R}^+$ funciones de clase C^1 y supondremos que $a \not\equiv 0$ si $\Gamma_0 = \emptyset$. También, consideramos $g : \mathbb{R} \to \mathbb{R}$ una función continua no decreciente que se anula en el cero (i.e. $g(0) = 0$) tal que $\forall x \in \mathbb{R}$, $|g(x)| \leq C(1+|x|)$, donde $C > 0$.

Nota 4 *Se observa que* $xg(x) \geq 0$ *,* $\forall x \in \mathbb{R}$.

Consideremos el sistema de evolución $(\mathcal{P})$

$$u_i'' - \mu\Delta u_i - (\lambda+\mu)\frac{\partial}{\partial x_i}(div\, u) - \gamma\frac{\partial^2 u_i}{\partial x_i} = 0 \text{ en } \Omega \times \mathbb{R}^+ \tag{13}$$

$$u_i = 0 \text{ en } \Gamma_0 \times \mathbb{R}^+ \tag{14}$$

$$\mu\partial_\nu u_i + (\lambda+\mu)(divu)\nu_i + \gamma\frac{\partial u_i}{\partial x_i}\nu_i + au_i + lg(u_i') = 0 \text{ en } \Gamma_1 \times \mathbb{R}^+ \tag{15}$$

$$u_i(0) = u_i^o \quad u_i'(0) = u_i^1 \text{ en } \Omega \tag{16}$$

para $i = 1, \ldots, N$.
Usando resultados estandares de Teoría de Operadores Maximales Monótonos obtenemos el siguiente resultado de existencia y regularidad de solución.

Teorema 2 *Asumamos:* $\eta > \lambda + \mu$.

1. *Dado* $(u^o, u^1) \in H^1_{\Gamma_0}(\Omega)^N \times L^2(\Omega)^N$. *El Problema* $(\mathcal{P})$ *admite una única solución débil*

$$u \in C(\mathbb{R}^+, H^1_{\Gamma_0}(\Omega)^N) \cap C^1(\mathbb{R}^+, L^2(\Omega)^N) . \tag{17}$$

Además, la aplicación : $(u^o, u^1) \longrightarrow u$ *es continua con respecto a dichas topologías; y la energía de* u *definida por*

$$E(t) \;:=\; \frac{1}{2}\int_\Omega u'^2 + \mu|\nabla u|^2 + (\lambda+\mu)[div(u)]^2 + \gamma\sum_{i=1}^N\left(\frac{\partial u_i}{\partial x_i}\right)^2 + \frac{1}{2}\int_{\Gamma_1} au^2 \tag{18}$$

es una función no creciente.

2. *Además, si* g *es globalmente Lipschitziana y si* u^o, u^1 *verifican la siguiente condición fuerte:*

$$\begin{cases} \forall i = 1, \ldots, N \\ u^o \in H^2(\Omega)^N \cap H^1_{\Gamma_o}(\Omega)^N , \quad u^1 \in H^1_{\Gamma_o}(\Omega)^N \\ \mu\partial_\nu u_i^o + (\lambda+\mu)(divu^o)\nu_i + \gamma\frac{\partial u_i^o}{\partial x_i}\nu_i + au_i^o + lg(u_i^1) = 0 \end{cases} \tag{19}$$

Entonces la solución u del problema $\mathcal{P}$ posee **la propiedad de regularidad mas fuerte**

$$u \in L^\infty(\mathbb{R}^+, H^2(\Omega)^N) \tag{20}$$
$$u' \in L^\infty(\mathbb{R}^+, H^1_{\Gamma_o}(\Omega)^N) \tag{21}$$
$$u'' \in L^\infty(\mathbb{R}^+, L^2(\Omega)^N) \tag{22}$$

En este caso u es llamada Solución Fuerte del problema $\mathcal{P}$.

A seguir presentamos algunos cristales cúbicos que verifican la condición $0 < \eta - \lambda - \mu$, hipótesis considerada en el Teorema 2.

	η	λ	2μ	$\gamma = \eta - \lambda - 2\mu$	$0 < \eta - \lambda - \mu$
	c_{xxxx}	c_{xxyy}	c_{xyxy}		
Na	0,055	0,042	0,049	$-0{,}036$	$-0{,}0115$
K	0,046	0,037	0,026	$-0{,}017$	$-0{,}004$
Fe	2,37	1,41	1,16	$-0{,}2$	0,38
$Diamond$	10,76	1,25	5,76	3,75	✓
Al	1,08	0,62	0,28	0,18	✓
LiF	1,19	0,54	0,53	0,12	✓
$NaCl$	0,486	0,127	0,128	0,231	✓
KCl	0,40	0,062	0,062	0,276	✓
$NaBr$	0,33	0,13	0,13	0,07	✓
KI	0,27	0,043	0,042	0,185	✓
$AgCl$	0,60	0,36	0,062	0,178	✓

4. Estabilidad de la energía asociada al sistema

4.1. Tasa exponencial

Sabemos que el Problema $\mathcal{P}$ es disipativo. Supongamos satisfechas las siguientes condiciones:

a) $N \geq 3$

b) Sea $x_0 \in \mathbb{R}^N$, definimos $m_{x_0} = m(x) := x - x_0,$ tal que

$$m(x) \cdot \nu(x) \leq 0 \text{ sobre } \Gamma_0 \tag{23}$$
$$m(x) \cdot \nu(x) > 0 \text{ sobre } \Gamma_1 \tag{24}$$

c) $R := \sup_{x \in \overline{\Omega}} |m(x)|$

d) λ, μ , η satisfacen la desigualdad:

$$\eta > \lambda + \mu\,, \tag{25}$$

luego $\mu + \gamma > 0$, desde que $\gamma := \eta - \lambda - 2\mu$ y $\mu + \gamma = \eta - \lambda - \mu$.

Nota 5 *El caso homogéneo Isotrópico corresponde a ($\gamma = 0$) $\eta = \lambda + 2\mu$.*

e)

$$\mathcal{E} := \text{ínf}\{\mu\,,\, \mu + \gamma\} = \text{ínf}\{\mu\,,\, \eta - \lambda - \mu\}\,. \tag{26}$$

Por lo tanto $0 < \mathcal{E}$.

f) Sean α, β: $\Gamma_1 \longrightarrow \mathbb{R}_+$ funciones continuas y estrictamente positivas.

Definimos las siguientes funciones: a , l: $\Gamma_1 \longrightarrow \mathbb{R}$, mediante:

$$\left.\begin{array}{lcl} a(x) & := & \alpha(x) m(x) \cdot \nu(x) \\ l(x) & := & \beta(x) m(x) \cdot \nu(x) \end{array}\right\} \tag{27}$$

Teorema 3 *Supongamos que se verifican (23), (24) y (25). Si existen las constantes positivas c y c′ tal que*

$$c|x| \leq |g(x)| \leq c'|x|\,,\ \forall x \in \mathbb{R}\,. \tag{28}$$

Nota 6 *Si $g(x) = x$ entonces $c = c' = 1$.*

Entonces, existe $w > 0$ tal que, la solución débil de $(\mathcal{P})$ satisface la siguiente estimativa,
Decaimiento exponencial de la Energía*:*

$$E(t) \leq E(0)e^{1-wt}\,,\ \forall t > 0\,. \tag{29}$$

Además*, si definimos a y l de la siguiente forma:*

$$a = \widetilde{a} := \frac{N-1}{2R^2}\,\xi\, m\cdot\nu \text{ y } l = \widetilde{l} := \frac{\sqrt{\xi}}{Rc'}m\cdot\nu\,. \tag{30}$$

Entonces, la energía de la solución débil de (P) satisface:
Decaimiento Exponencial*:*

$$E(t) \leq E(0)e^{1-\frac{c\sqrt{\xi}}{c+c'R}t}\,,\ \forall t \geq 0\,. \tag{31}$$

Prueba. Nos basamos en la identidad del Lema 3. Previamente enunciamos el siguiente resultado.

Lema 2

$$E(S) - E(T) = \int_S^T\int_{\Gamma_1} lu_i'g(u_i')\,ds\,dt\,,\ \forall\, 0 \leq S \leq T < +\infty\,. \tag{32}$$

Nota 7 *La igualdad (32) implica que E es absolutamente continua entonces es derivable en casi todo punto y $E'(t) = -\int_{\Gamma_1} lu_i'g(u_i')\,ds$.*

Para simplificar los cálculos, consideremos el vector $M(u) \in \mathbb{R}^N$, que tiene por componentes:

$$M(u)_i = M(u_i) \;:=\; 2m_k u_{i,k} + (N-1)u_i = 2\sum_{k=1}^{N}(m_k u_{i,k}) + (N-1)u_i \tag{33}$$

donde $m = (m_1.\ldots,m_N)$.
Para el caso, cuando g satisface (28), basta tomar $p = 1$ en el siguiente resultado.

Lema 3 *Para todo $0 \leq S < T < +\infty$ tenemos:*

$$\begin{aligned}
&2\int_S^T E(t)^{\frac{p+1}{2}}\,dt = \frac{p-1}{2}\int_S^T E(t)^{\frac{p-3}{2}}E'\int_\Omega u'M(u)\,dx\,dt - \left[E^{\frac{p-1}{2}}\int_\Omega u'\cdot M(u)\,dx\right]_S^T + \\
&\int_S^T E^{\frac{p-1}{2}}\int_\Gamma(\mu\partial_\nu u_i + (\lambda+\mu)divu\nu_i + \gamma u_{i,i}\nu_i)\cdot M(u_i) + au^2\,ds\,dt + \\
&\int_S^T E^{\frac{p-1}{2}}\int_\Gamma m\cdot\nu(u'^2 - \mu|\nabla u|^2 - (\lambda+\mu)(divu)^2 - \gamma u_{i,i}^2)ds\,dt\,.
\end{aligned} \tag{34}$$

Ahora vamos a mayorar la identidad (34), usando las condiciones de frontera sobre Γ_0 y sobre Γ_1. Esto lo conseguimos en los siguientes Lemas.

Lema 4

$$\begin{aligned}
J:\;=\;&\int_{\Gamma_0}\underbrace{(\mu\partial_\nu u_i + (\lambda+\mu)divu\nu_i + \gamma u_{i,i}\nu_i)\cdot M(u_i)}_{J_1:=} + \underbrace{au^2}_{=0}\,ds \\
&+\int_{\Gamma_0} m\cdot\nu(u'^2 - \mu|\nabla u|^2 - (\lambda+\mu)(divu)^2 - \gamma u_{i,i}^2)\,ds \;\leq\; 0\,.
\end{aligned} \tag{35}$$

Supongamos que g satisface (28) y a y l estan definidas por (30), entonces se obtienen las siguientes estimativas.

Lema 5

$$\int_{\Gamma_1}(\mu\partial_\nu u_i+(\lambda+\mu)divu\nu_i+\gamma u_{i,i}\nu_i)\cdot M(u_i)+\widetilde{a}u^2\,ds$$
$$+\int_{\Gamma_1} m\cdot\nu(u'^2-\mu|\nabla u|^2-(\lambda+\mu)(divu)^2-\gamma u_{i,i}^2)\,ds \;\le\; \frac{2R}{\sqrt{\xi}}\frac{c'}{c}\int_{\Gamma_1}\widetilde{l}u_i'g(u_i')\,ds\,. \tag{36}$$

Lema 6 *Se satisface la siguiente desigualdad:*

$$|\int_\Omega u'\cdot M(u)\,dx|\le\frac{2R}{\sqrt{\xi}}E(t)\,. \tag{37}$$

Finalmente, usando las estimativas (35), (36) y (37) y eligiendo $p=1$ en la desigualdad (34) obtenemos:

$$2\int_S^T E(r)\,dr \;\le\; \frac{2R}{\sqrt{\xi}}\left(1+\frac{c'}{c}\right)E(S)+\frac{2R}{\sqrt{\xi}}\left(1-\frac{c'}{c}\right)E(T)$$
$$\int_S^\infty E(r)\,dr \;\le\; \frac{R}{\sqrt{\xi}}\left(1+\frac{c'}{c}\right)E(S)\,,\quad \forall S\ge 0\,.$$

Aplicando el Lema 1 obtenemos (31), i.e. $E(S)\le E(0)e^{1-\frac{c\sqrt{\xi}}{(c+c')R}S}\,,\quad \forall S\ge 0\,.$
Por otro lado, supongamos que g satisface (28) y a y l estan definidas en (27), entonces

Lema 7 *Existe una constante $C>0$ tal que*

$$\int_{\Gamma_1}(u'^2-\mu|\nabla u|^2-(\lambda+\mu)(div\,u)^2-\gamma u_{i,i}^2)m\cdot\nu$$
$$+\int_{\Gamma_1}(\mu\partial_\nu u_i+(\lambda+\mu)div\,u\,\nu_i+\gamma u_{i,i}\nu_i)\cdot M(u_i)+au^2 \;\le\; C\int_{\Gamma_1} lu_i'g(u_i')+au_iu_i\,, \tag{38}$$

$$|\int_\Omega u'\cdot M(u)|\le CE(t)\,. \tag{39}$$

Utilizando (35), (38) y (39) obtenemos

$$2\int_S^T E(\tau)d\tau\le 2CE(S)+C\int_S^T\int_{\Gamma_1} au^2\,. \tag{40}$$

El siguiente resultado es una adaptación del método de F. Conrad y B. Rao [2].

Lema 8 *Existe una constante $C>0$ tal que, para todo $\epsilon>0$ vale*

$$\int_S^T\int_{\Gamma_1} au^2dsdt\le\frac{C}{\epsilon}E(s)+\epsilon\int_S^T E(\tau)\,d\tau\,. \tag{41}$$

En la desigualdad (41) consideramos ϵ suficientemente pequeño de modo que la desigualdad (40), nos conduce a $\int_S^\infty E(\tau)d\tau\le C_5E(S)$. Luego por Lema 1, obtenemos (29) , i.e. $E(S)\le E(0)e^{1-\frac{1}{C_5}S}$, $\forall S\ge 0$. □

4.2. TASA POLINOMIAL

Teorema 4 *Supongamos que se verifican (23) , (24) , (25) y supongamos que existan las constantes:* $p > 1$, $c_1 > 0$, $c_2 > 0$, $c_3 > 0$, $c_4 > 0$ *tal que*

$$c_1|x|^p \leq |g(x)| \leq c_2|x|^{\frac{1}{p}} \qquad si\ |x| \leq 1 \tag{42}$$

$$c_3|x| \leq |g(x)| \leq c_4|x| \qquad si\ |x| > 1\,. \tag{43}$$

Entonces, dado $(u^0, u^1) \in H^1_{\Gamma_0}(\Omega)^N \times L^2(\Omega)^N$ *(i.e. Debido al Teorema 2 existe la solución débil* u *tal que* $u \in C(\mathbb{R}_+, V) \cap C^1(\mathbb{R}_+, H)$ *y* $E = \frac{1}{2}\|(u, u')\|^2_{V\times H}$ *es decreciente), la solución de* $(\mathcal{P})$ *satisface la siguiente estimativa,*
Decaimiento Polinomial de la Energía:

$$E(t) \leq Ct^{\frac{-2}{p-1}}\,,\ \forall t > 0 \tag{44}$$

donde C *es una constante que depende de la energía inicial* $E(0)$ *y del modo continuo.*

Prueba. Para el caso g satisfaciendo (42), (43) y a y l definidos por (27), del Lema 3 obtenemos la siguiente estimativa

$$\exists C > 0 \ \text{tal que} \int_S^{+\infty} E^{\frac{p+1}{2}}(r)dr \leq CE(S)\,. \tag{45}$$

Luego, de (45) y la desigualdad integral no lineal Teorema 1 con $\alpha = \frac{p-1}{2}$, podemos concluir que

$$E(t) \leq C(E(0))t^{\frac{-2}{p-1}}\,,\ \forall t \geq 0\,.$$

□

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UN ENFOQUE BAYESIANO PARA LA ESTIMACIÓN DE HUMEDAD DEL SUELO A PARTIR DE DATOS SAR POLARIMÉTRICOS EN BANDA L

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Resumen: Estimar la humedad del suelo a partir de datos de un radar de apertura sintética (SAR) polarimétrico presenta varios desafíos. Es necesario dar cuenta del scattering que produce un suelo rugoso, para lo que se dispone en la literatura de modelos analíticos aproximados y modelos semiempíricos basados en datos experimentales. Al mismo tiempo en las imágenes SAR la presencia de ruido speckle suele dificultar los procedimientos de inversión basados en minimizaciones directas. En este trabajo se presentan algunas estrategias basadas en modelos bayesianos y diferentes modelos directos conocidos. El enfoque propuesto permite tomar en cuenta el efecto del speckle y permite incorporar información a priori sobre la rugosidad que es posible obtener mediante otros sistemas. Se muestran los resultados de la aplicación de estas metodologías a imágenes obtenidas con el sistema aerotransportado SARAT de CONAE, y se contrastan los resultados con medicions in-situ de humedad volumétrica.

Palabras clave: *humedad del suelo, problema inverso, SAR, métodos bayesianos.*
2000 AMS Subject Classification: 86A22, 62P12

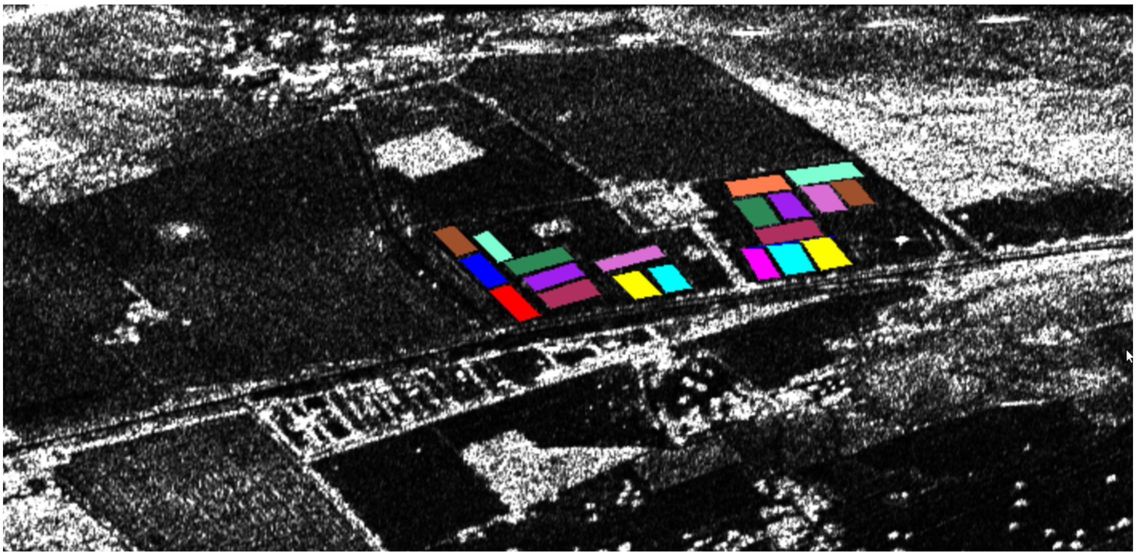


Imagen SARAT del Centro Espacial Teófilo Tabanera (CETT).

En este trabajo estudiamos el problema de estimar la humedad del suelo a partir de datos de un radar, que es la principal aplicación planteada para la misión SAOCOM. Más formalmente, se trata de un problema inverso para la constante dieléctrica. Estudiamos este problema utilizando como modelos directos diferentes modelos aproximados disponibles en la literatura.

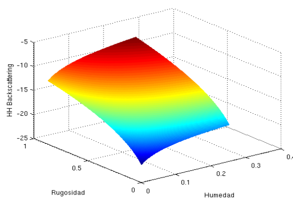
El principal desafío para resolver este problema consiste en que la señal recibida depende tanto de la constante dieléctrica como de la geometría (rugosidad) de la superficie, que no es conocida de antemano. Como veremos más adelante, la sensibilidad a la rugosidad es tan significativa que sin información sobre ella probablemente sería imposible resolver el problema con la precisión requerida.

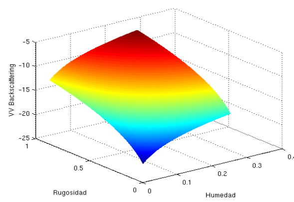
Por otra parte, en el estudio de problemas inversos los datos simulados pueden ayudar a entender mejor algunos aspectos del problema, pero las verdaderas dificultades aparecen con el uso de datos reales. En ese sentido, se dispone de datos SAR adquiridos por el sistema aerotransportado SARAT de CONAE, y mediciones de humedad in-situ, así como de información sobre la rugosidad de diferentes parecelas. Compararemos los datos de humedad contra el resultado de la inversión de los datos de radar hechos con este esquema de inferencia, lo que nos permitirá analizar la factibilidad de realizar estimaciones de humedad del suelo con un radar, así como estimar el peso relativo entre los diferentes factores que limitan la precisión de las estimaciones en la etapa actual.

1. Análisis de sensitividad del problema directo.

El modelo de Oh [5, 6], utilizado para modelar el scattering en microondas de un suelo descubierto, es el siguiente:

$$
\begin{aligned}
HV =& 0{,}11 mv^{0{,}7} \cos(\theta)^{2{,}2}(1 - e^{-0{,}32 ks^{1{,}8}}) \\
VV =& \frac{HV}{0{,}095 * (0{,}13 + \sin(1{,}5\theta)^{1{,}4}(1 - e^{-1{,}3ks^{0{,}9}})} \\
HH =& VV * (1 - \left(\frac{2\theta}{\pi}\right)^{0{,}35 mv^{-0{,}65}} e^{-0{,}4ks^{1{,}4}})
\end{aligned}
$$





En la inversión de datos reales - que no fueron generados por el modelo - hay que tener en cuenta el efecto que produce el 'ruido' en la inversión. En el caso de los problemas inversos lineales, una herramienta que permite caracterizar estos efectos [3] es la descomposición en valores singulares (SVD). Por lo tanto, en primer lugar linealizamos el problema y utilizamos esta herramienta.

Propagación de las incertezas en los datos a la solución Analicemos por ejemplo el modelo semiempírico de Oh linealizado.

Si tenemos $Kf = d + \epsilon$, la solución a $Kx = d$ será

$$x = f + \sum_k \frac{u_k^t \epsilon}{\sigma_k} v_k$$

En nuestro caso la SVD de la matriz es $K = U\Sigma V$, con

$$\Sigma = \begin{pmatrix} 0{,}0760 & 0 \\ 0 & 0{,}0046 \\ 0 & 0 \end{pmatrix}, V = \begin{pmatrix} -0{,}4883 & -0{,}8727 \\ -0{,}8727 & 0{,}4883 \end{pmatrix}$$

Al invertir este modelo, el 'ruido' se proyecta más hacia la dirección de la humedad que de la rugosidad. Este es un inconveniente que habrá que superar, pero antes veamos que tipo de ruido está presente en las imágenes de radar.

2. Incertezas

2.1. Rudio Speckle

Consideremos en principio un tipo de ruido clásico presente en todas las imágenes SAR, conocido como 'Speckle'.

En una aplicación dada, determinar qué es el 'ruido' y que es 'la señal' es algo que le corresponde determinar al investigador. La causa del 'ruido' llamado Speckle es que cada píxel de una imagen SAR (típicamente del orden de 10 m^2) es, en los hechos, distinto de los otros por más que para el aplicador se trate de una misma zona homogénea. Veamos: es correcto considerar la geometría de un medio natural (por ejemplo una superficie) como un proceso aleatorio. Ese proceso está caracterizado por parámetros (ej. la desviación estándard de las alturas), y son éstos parámetros los que pueden considerarse constantes a lo largo de un área, que para el aplicador se tratará de un área constante ('un suelo de rugosidad x'). Pero sin embargo, al ser sistemas de alta resolución y muy sensibles a las características geométricas, estaremos viendo las diferencias entre las distintas 'realizaciones' de ese proceso.

2.2. Caminatas al azar

El modelo más común para el speckle es el de la caminata al azar. Esto surge de la siguente manera: podemos pensar que la energía que vemos en cada pixel de la imagen está compuesta por la suma de una cantidad de dispersores independientes. Es decir

$$z = \sum_{i}^{N} A_i e^{-2\pi i\theta_i}$$

Donde A_i y θ_i son variables aleatorias, la primera típicamente normal y la segunda uniforme. Esto constituye una caminata al azar en el plano complejo.

Veremos que la intensidad (módulo) sigue una distribución Rayleigh bajo condiciones bastante generales. En la práctica se usa la distribución gamma, que 'generaliza' a la Rayleigh y tiene dos parámetros libres.

$$\sum_{i=1}^{N} \Gamma(k_i, \theta) \sim \Gamma(\sum_{i=1}^{N} k_i, \theta)$$

$$c\Gamma(k, \theta) = \Gamma(k, c\theta)$$

Existen expresiones simples para el promedio de gammas, que se utilizan para disminuir la incerteza a costa de perder resolución espacial.

Modelo Multiplicativo: El scattering se modela como $Z = X.Y$, donde Y es el ruido (de media uno y la varianza un parámetro libre), y X la predicción de los modelos directos. La varianza del speckle se representa con un parámetro N que se conoce en la literatura de radar como el 'número de looks', que vendría a indicar cuantas adquisiciones fueron promediadas para obtener el valor de cada pixel.

2.3. Cómo surge la distribución Rayleigh

La deducción rigurosa a partir de un modelo físico - por más que contenga simplificaciones importantes - de una distribución estadística para los valores de una imagen SAR es de gran valor porque permite interpretar información contenida en las mismas que de otro modo no sería posible. De hecho, un problema recurrente en el modelado de imágenes SAR consiste en que las distribuciones que ajustan bien los datos observados, obtenidas mediante 'generalizaciónes', agregan parámetros libres que no tienen un correlato físico claro, y por lo tanto tienen un significado muy vago.

Es interesante entonces el estudio probabilístico de 'dinámicas' (modelos físicos simples), que den lugar a diferentes distribuciones de probabilidad para los datos y que al mismo tiempo permitan asignar un significado concreto a los parámetros involucrados. En esta sección mostraremos una versión moderna del resultado clásico antes citado de la distribución Rayleigh.

De la caminata al azar a la que nos referimos, la magnitud que mide el sistema es la intensidad (módulo al cuadrado), y es posible deducir su distribución de diversas formas. En Beckmann-Spizziccino [7], esto se hace trabajosamente mediante el teorema central de límite, y llama la atención una nota al pie sobre cómo había hecho esta cuenta originalmente Lord Rayleigh en el siglo XIX:

> Rayleigh, who was not familiar with today's routine methods of probability theory, used an ingenious method of adding one more vector to the sum and transforming the resulting difference equation to a PDE, wich turns out to be the equation of heat conduction (...) [Rayleigh, 1896, Section 42a]

Esto no puede dejar de llamarle la atención a cualquier persona familizarizada con la teoría moderna de difusiones y el cálculo de Itô.

Siguiendo a [2] (e indirectamente a Rayleigh), podemos usar la fómula de Itô y el teorema de Feynmann-Kac, para obtener una PDE para la intensidad, cuya solución estacionaria resulta

$$P(z) = \frac{\alpha}{2} exp\left(\frac{-\alpha z}{2}\right)$$

2.4. Gammas multivariadas

En el caso de un sistema polarimétrico, los datos resultarán multivariados.

En dimensión 2 (que utilizaremos para tratar datos co-polarizados ignorando los cross-polarizados), hay una expresión analítica para una de tales distribuciones, conocida como la gamma de Kibble

$$pdf(x,y) = \frac{(\lambda_1\lambda_2)^v}{(1-\rho)\Gamma(v)}\left(\frac{xy}{\rho\lambda_1\lambda_2}\right)^{\frac{v-1}{2}} exp\left(-\frac{\lambda_1 x + \lambda_2 y}{\rho\lambda_1\lambda_2}\right) I_{v-1}\left(\frac{2\sqrt{\rho\lambda_1\lambda_2 xy}}{1-\rho}\right)$$

(con I_{v-1} la función modificada de Bessel de tipo 1)

Si n>2, no es posible obtener expresiones analíticas para la distrubución, por lo menos para generalizaciones de la gamma de Kibble. Únicamente se puede escribir esta distribución como una transformada inversa de Laplace.

2.5. Desbalances y cross-talk

En un sistema SAR hay otras clases de ruido además del Speckle, debido a desbalances y cross-talk, que los identificaremos como el error de ingeniería. Se pueden modelar según [10], como

$$\mathbf{M} = A\begin{pmatrix}1 & \delta_1\\ \delta_2 & f_1\end{pmatrix}\begin{pmatrix}S_{hh} & S_{vh}\\ S_{hv} & S_{vv}\end{pmatrix}\begin{pmatrix}1 & \delta_3\\ \delta_4 & f_2\end{pmatrix} + \begin{pmatrix}n_{hh} & n_{vh}\\ n_{hv} & n_{vv}\end{pmatrix}$$

Los valores de los parámetros se obtienen con una calibración con reflectores, y a partir de los errores residuales se estima el error de ingeniería. Sin embargo, no disponemos actualmente de una función de error caracterizada en términos estadísticos.

3. Estimadores Bayesianos

A continuación estudiaremos la resolución óptima del sistema y veremos que a altas resoluciones el ruido Speckle genera incertezas demasiado altas en la obtención de humedad. A menores resoluciones (obtenidas mediante multi-looking), se justifica mediante el Teorema Centra del límite considerar al Speckle como una variable normal. Así, podemos considerar al error total como una variable normal con una covarianza que es la suma de la covarianza del Speckle, más parámetros que caracterizarían el error de ingeniería.

Entonces, la función de verosimilitud podemos modelarla como:

$$\text{Likelihood}(Z|p) = \text{Norm}(Z - f(p), \Sigma)$$

Z Mediciones
p Parámetros del Modelo
f Modelo Directo
Σ Covarianza del speckle + error de ingeniería

La distribución a posteriori, via el teorema de Bayes, resulta entonces:

$$\text{Post}(p/Z) = \frac{\text{Likelihood}(Z/p)\text{Prior}(p)}{\int_{p\in P}\text{Likelihood}(Z/p)}$$

Y aquí hay dos estimadores a considerar: la media de la distribución a posteriori, y el máximo de la misma. El primer método es el estimador óptimo en la teoría bayesiana y el segundo está muy emparentado con los métodos de minimización.

Calcular los momentos de la distribución a posteriori involucra integrales de dimensión igual al número de parámetros ($dim(p)$), lo que puede resultar un proceso complejo en caso de que esta dimensión sea alta. En el caso del modelo de Oh y del IEM [9, 8] con $\frac{s}{l}$ parametrizado, esta dimensión es 2, por lo que estos momentos pueden ser calculados fácilmente mediante integración numérica. En el presente trabajo implementamos una regla de gauss de orden 100 en cada dimensión con muy buenos resultados, y notamos que los nodos y pesos necesitan ser calculados por única vez, pudiendo ser reutilizados todas las veces que fuera necesario.

3.1. Análisis de la varianza y resolución óptima

La humedad la podemos considerar como un campo aleatorio. Su heterogeneidad depende de la escala espacial y del clima, entre otros factores.

Según Famiglietti [4], la varianza de la humedad en un terreno depende de la media según

$$\sigma = k_1 \cdot \mu \cdot exp^{-k_2 \mu}$$

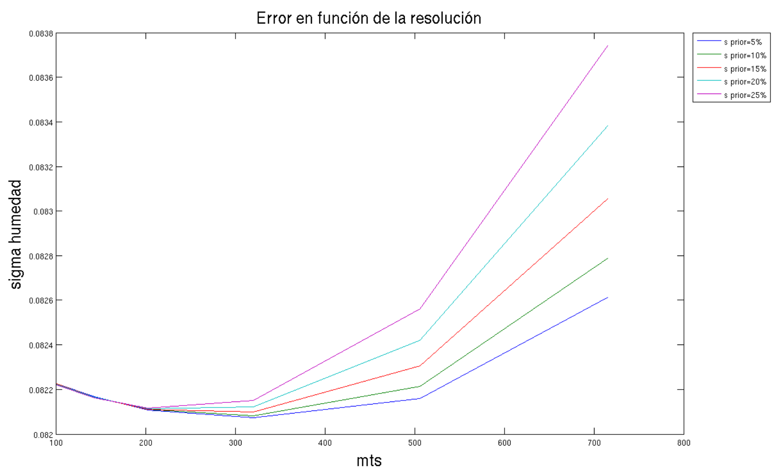
donde k_1 y k_2 dependen de la escala espacial

escala	k_1	k_2
2.5 m	0.7803	9.0607
16 m	0.7287	7.3796
100 m	0.8941	8.0774
800 m	0.8840	5.8070
1.6 km	1.2070	7.1128
50 km	1.0429	5.2212

En nuestro modelo, hacemos retrieval de una 'constante efectiva', que será menos representativa si la varianza del campo aumenta. Al mismo tiempo, a escalas mayores disminuye el speckle por el proceso de multilooking (tomar promedios).

Suponiendo que la distribución a posteriori de la constante efectiva es independiente de la heterogeneidad, la varianza se suma linealmente.

$$Var(H) = Var(C_{ef}) + Var(Het)$$



4. Método bayesiano para el problema inverso

El enfoque estadístico (bayesiano) con el que trabajamos provee un marco que permite incorporar información a priori sobre la rugosidad, que no es sólo una ventaja sino una necesidad para la aplicación propuesta. Al mismo tiempo es posible incorporar las incertezas inherentes a cualquier sistema SAR, y analizar fácilmente el efecto de las mismas en las estimaciones de humedad. A partir de este análisis, es posible determinar una resolución espacial óptima para el sistema. Por último, para el caso en que sean pocos los parámetros a obtener, como es nuestro caso, estos estimadores se pueden implementar muy eficientemente mediante integración numérica.

5. Inversión de datos SARAT

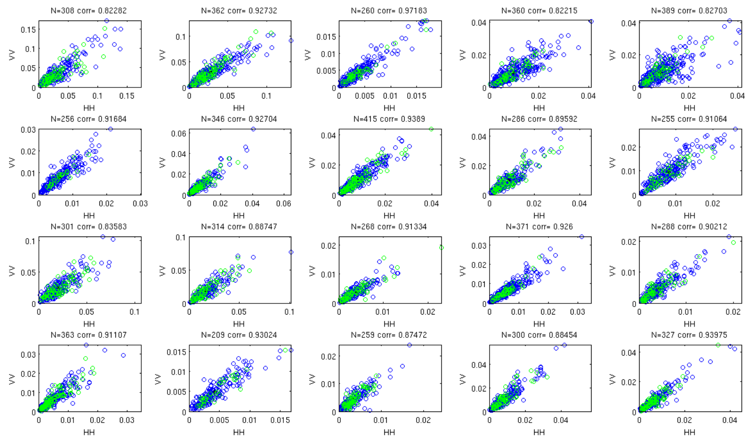
La implementación del esquema de inversión es la siguiente.

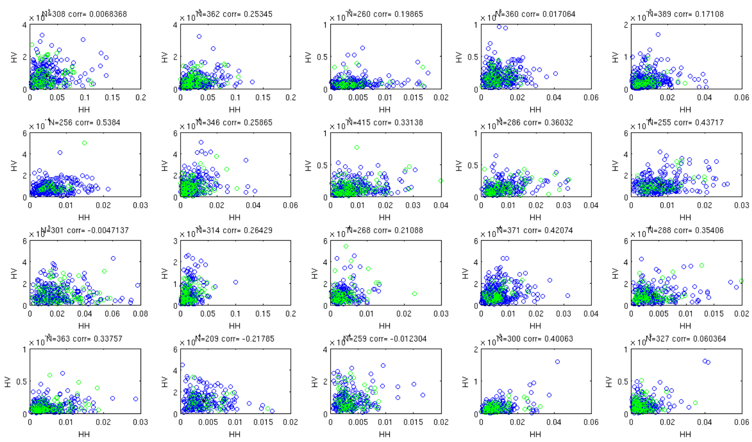
En principio, dado que los datos tienen una correlación espacial, se procede a decorrelacionarlos de una manear eficiente, que es tomar promedios en regiones de tamaño creciente, hasta que la auto-correlación haya disminuido suficientemente. Los datos iniciales se muestran en azul, y los decorrelacionados en verde.

Posteriormente se procede a estimar los parámetros del ruido speckle. Como estamos en valores grandes de N, puede utilizarse una distribución normal.

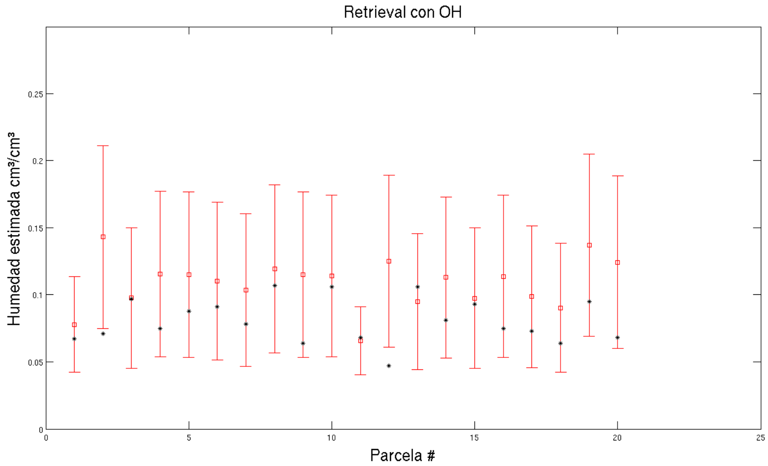
Por último se construyen los likelihoods sumando el error de medición propiamente dicho, y se calculan las integrales para los diferentes modelos directos.

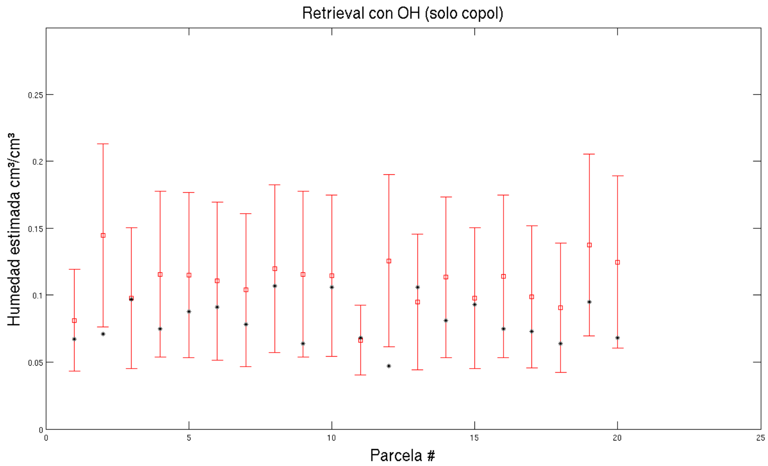
5.1. Datos

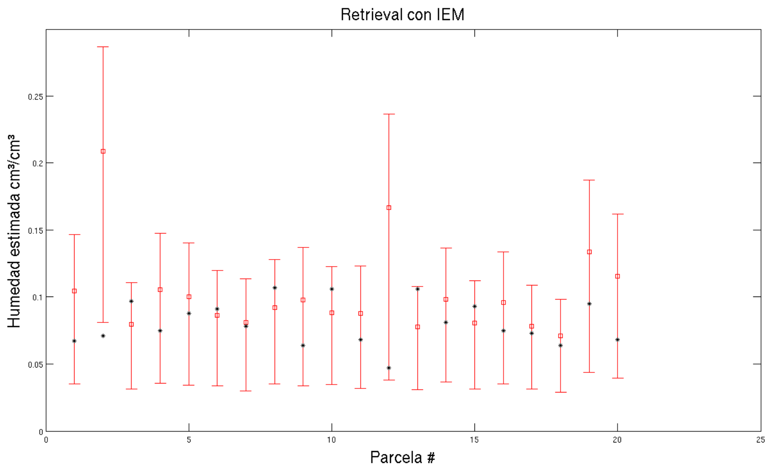




5.2. Resultados







6. Conclusiones y perspectivas

- Con información sobre la rugosidad, *es posible obtener buenas estimaciones de humedad con un radar*, al menos para suelos secos.

- También parece posible obtener buenas estimaciones *utilizando solamente datos co-pol* .

- Al mismo se analizaron los errores y se exploró *la resolución óptima* del sistema.

En principio, si bien no parece afectar al retrieval considerablemente, en los datos se observa que polarizaciones (HH y VV) son prácticamente iguales, mientras que los modelos predicen VV $>$ HH. Falta explicar cuál es el efecto que causa esta anomalía. Además, se tiene conocimiento de que esta anomalía es aún más pronunciada en longitudes de onda más pequeñas, donde puede ocurrir HH $>$ VV.

Esto marca la dificultad de un problema de gran interés, que es obtener información sobre la rugosidad con banda X (en la que opera el sistema italiano CosmoSkymed, parte de la misma constelación que el SAOCOM). En este caso, el problema de scattering es más desafiante. Por lo tanto, el esfuerzo futuro se centrará en resolver estos problemas de scattering con métodos numéricos eficientes.

A su vez, los modelos de back-scattering desarrollados en la literatura no dan cuenta de la correlación entre canales, por lo que este observable ha tenido que ser tomado como un parámetro libre en la distribución de speckle, sin ninguna relación con características de la superficie, lo que indica otro de los posibles caminos de mejora de dichos modelos.

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ESTRATEGIA DEL CÁLCULO PARA LA RESOLUCIÓN DE LA PRIMERA ETAPA DEL PROCESO DE FREÍDO POR INMERSIÓN.

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Resumen: El proceso de freído por inmersión profunda involucra la transferencia simultánea de calor y materia, en el que tiene lugar la desorsión de humedad y la entrada de aceite a través de la superficie del sólido a freír. En el caso específico de papa natural, la primera etapa del mismo, denominada de burbujeo vigoroso, tiene lugar aproximadamente, entre los 10 y 110 seg de iniciado el mismo. La descripción matemática de tal proceso, resulta expresada mediante un sistema de ecuaciones que involucran ambos fenómenos, los que se encuentran acoplados entre sí. En este trabajo, se presenta una estrategia de cálculo, que permite la resolución del problema antedicho, mediante la introducción de un parámetro oportuno en el modelo descriptivo original, encontrándose la solución para el comportamiento dinámico del frente de desorción de humedad libre. Se comparan los resultados obtenidos mediante este modelo con los provenientes de experiencias realizadas con anterioridad, encontrándose un acuerdo adecuado entre ambos.

Abstract: In the immersion frying process, simultaneous heat and mass transfer occur, involving water desorption and oil input through the solid surface subjected to frying. In the specific case of natural potato, the first step during the process named as vigorous bubble, takes place along the time interval 10 – 110 sec. The descriptive model for the process is a mathematical system consisting in an initial - boundary problem associated to the corresponding coupled heat and mass transfer differential equations. In this paper, a calculation strategy, in order to solve the problem, is presented. To this respect, an opportune parameter is introduced in the original model, in order to find the solution for the dynamic behavior for the free – water desorption front. Good agreements emerge for the comparison upon the experimental results.

Palabras Claves: freído por inmersión, desorción de humedad, transferencia de calor y materia
Key – words: immersion frying, water desorption, heat and mass transfer,
2000 AMS Subjects Classification: 35K20.

1. INTRODUCCIÓN

El freído por inmersión profunda de papa natural involucra la transferencia simultánea de calor y materia. El inicio del proceso, consiste en el precalentamiento de la muestra de papa cuya duración es del orden de 10 seg. A continuación, tiene lugar la denominada primera etapa (comprendida aproximadamente entre 10 y 110seg), durante la cual se produce un burbujeo vigoroso, por vaporización del agua libre desorbida en la muestra.

Diversos autores [2], [3], han realizado aportes respecto a la problemática de esta etapa, para lo cual postularon distintas hipótesis en cuanto al planteo del problema. Asimismo, en un artículo previo [5] se han reportado conclusiones interesantes en lo referente al análisis de un modelo matemático consistente en un problema de valor inicial (PVI) y de frontera libre descriptivo del proceso simultáneo de transferencia de calor y materia, que tiene lugar en la primera etapa del freído por inmersión de porciones de papa natural en aceite caliente.

En un trabajo posterior, se resolvió numéricamente el problema mediante un modelo matemático simplificado del original propuesto con anterioridad, lo que permitió transformar el PVI y de frontera libre, en otro de PVI y frontera móvil [6]. Se confrontaron también los resultados obtenidos con datos experimentales disponibles, encontrándose un buen acuerdo.

En la Figura1, se ilustra esquemáticamente el proceso físico que tiene lugar durante dicha etapa del proceso de freído, motivo de los trabajos anteriores y del presente. Se ha trabajado con bastones de papa natural, cortados en forma de prisma.

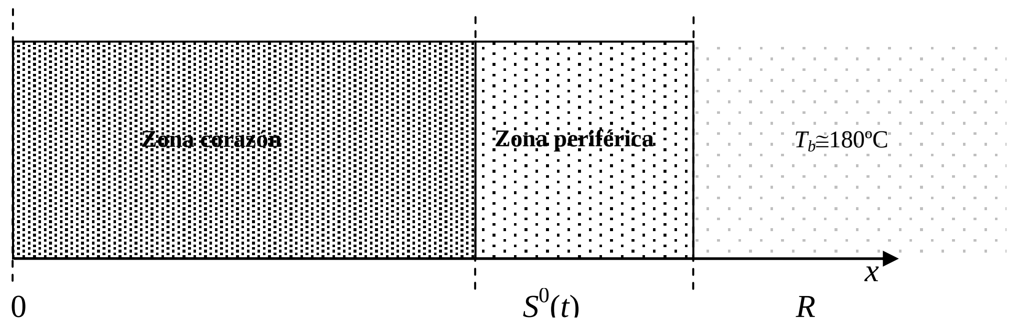

Figura 1: Esquema de la etapa de burbujeo vigoroso del freído ($t>t_1$)

En la Figura 1, t_1 denota el tiempo de precalentamiento necesario para llevar la superficie de la porción de papa, a la temperatura de ebullición del agua, este tiempo es de aproximadamente 10 seg; R el semiespesor del prisma de papa natural sometido a freído, T_b la temperatura del baño de aceite donde se sumerge la muestra a freír. La función real $S^0=S^0(t)$, de la variable independiente t, da la posición instantánea del frente de desorsión de humedad libre de la muestra, para cada $t>t_1$. La muestra empieza a desorber en la superficie de contacto $x=R$, entre el prisma y el aceite una vez que ha transcurrido el tiempo t_1, por tanto, $S^0(t_1)=R$.

Se ha observado experimentalmente [4] que el frente de desorsión de humedad libre alcanza el centro de la porción prismática, en forma simétrica desde ambos laterales, en aproximadamente 110 seg.

2. Modelo Matemático Descriptivo de la Primera Etapa

Como se ha señalado en [5] para tal etapa se ha formulado el modelo descriptivo del proceso, el que está sujeto a las siguientes ecuaciones (1) a (11). Ellas modelan la evolución de la temperatura y la concentración, ambos fenómenos acoplados por la ecuación (3). Para resolver la dinámica del frente de desorción de humedad libre no es necesario obtener ni el perfil de concentración, ni el de temperatura, en el dominio de interés ($S^0(t)\leq x\leq R$); en consecuencia tampoco es necesario tener en cuenta el acople entre ambas funciones. Asimismo, la ecuación (4) expresa la condición inicial antes de que comience la etapa de desorción de humedad libre.

$$\rho_s C_s \frac{\partial T}{\partial t} = k_s \frac{\partial^2 T}{\partial x^2} \qquad t > t_1 \qquad 0 < x < R \tag{1}$$

$$\frac{\partial T}{\partial x}(0,t) = 0 \qquad t > t_1 \tag{2}$$

$$k_s \frac{\partial T}{\partial x}(R,t) - \Delta H \cdot D \frac{\partial C}{\partial x}(R,t) = h\left[T_b - T(R,t)\right] \qquad t > t_1 \tag{3}$$

$$T(x,t_1) = Q(x) \qquad 0 \leq x \leq R \tag{4}$$

$$C(x,t) = C_0 \qquad t > t_1 \qquad 0 \leq x \leq S^0(t) \tag{5}$$

$$C(x,t_1) = C_0 \qquad 0 \leq x \leq R \tag{6}$$

$$\frac{\partial C}{\partial t} = D \frac{\partial^2 C}{\partial x^2} \qquad t > t_1 \qquad S^0(t) \leq x \leq R \tag{7}$$

$$C(S^0(t),t) = C_0 \qquad t > t_1 \tag{8}$$

$$-AD \frac{\partial C}{\partial x}(R,t) = \omega_v \qquad t > t_1 \tag{9}$$

$$(C^0 - C_0)\frac{dS^0}{dt} = -D \frac{\partial C}{\partial x}(S^0_+(t),t) \qquad t > t_1 \tag{10}$$

$$S^0(t_1) = R \tag{11}$$

donde: A es el área de la superficie lateral (m^2); C_s, la capacidad calorífica efectiva del sólido papa (J/kg K); C,C_0 yC^0 las concentraciones volumétricas de humedad: libre, inicial libre e inicial total en la papa, respectivamente (todas en kg/m^3); D el coeficiente global de difusividad de humedad (m^2/s); k_s la conductividad térmica de la papa (W/m K); R el semi-espesor de la porción de papa (m); ΔH, el calor de vaporización del agua a 100ºC (J/kg); ρ_s la densidad de la papa (kg/m^3); h el coeficiente convectivo global de transferencia de calor (W/m^2 K); S^0 la posición del frente de desorción de humedad (m); ω_υ la velocidad de vaporización; T_b la temperatura del baño de aceite y t_1=10seg, el tiempo en que comienza la etapa de burbujeo.

3. FORMULACIÓN DEL PVI PARA EL FRENTE DE DESORCIÓN.

Denominando $M=M(t)$ al contenido en peso de humedad libre desorbida en el frente de desorción: $x=S^0(t)$ con sede en la zona periférica, ($S^0(t)\leq x\leq R$), al tiempo t, se encuentra que está dado por la expresión (12)

$$M(t)=4L\left[R\int_{S^0(t))}^{R}C(x,t)\,dx+S^0(t)\int_{S^0(t))}^{R}C(x,t)\,dx\right] \tag{12}$$

Siendo, R el semiespesor y L el largo del prisma de papa. Teniendo presente las ecuaciones (5) a (11), a partir de (12) se obtiene

$$\frac{dM(t)}{dt}=4L\left[-C_0\left(R+S^0(t))\right)+\int_{S^0(t))}^{R}C(x,t)\,dx\right]\frac{dS^0(t)}{dt} \tag{13}$$

La superficie de contacto $x=R$, entre el prisma de papa y el aceite caliente, constituye el sumidero de materia por vaporización para $M(t)$. Por otra parte, teniendo presente la ecuación (9) y realizando el balance de materia se tienen las ecuaciones (14) a (16)

$$\frac{dM(t)}{dt}=4LD\left(R+S^0(t)\right)\frac{\partial C}{\partial x}\left(S_+^0(t),t\right)-AD\frac{\partial C}{\partial x}(R,t) \tag{14}$$

$$-D\frac{\partial C(S_+^0(t),t)}{\partial x}=\left(C^0-C_0\right)\frac{dS^0}{dt} \tag{15}$$

$$\omega_V(t)=4R^2L\left(\rho_s-C^0\right)\frac{K_x}{60}\left(X_0-X_e\right)\exp\left(-K_xt/60\right) \tag{16}$$

Siendo $\omega_v(t)$, la velocidad de vaporización de humedad libre al tiempo t [3]. Llevando (9) y (15) a (14) se obtiene la ecuación (17)

$$\frac{dM(t)}{dt}=-4L\left(R+S^0(t)\right)\left(C^0-C_0\right)\frac{dS^0}{dt}-\omega_v(t) \tag{17}$$

Reemplazando la expresión de dM/dt, dada por (13) y de $\omega_v(t)$ dada por (16), a partir de (17) resulta la expresión del PVI buscado para la posición del frente de desorción de humedad libre, dada por la ecuación (18)

$$\begin{cases}\dfrac{dS^0}{dt}=\dfrac{-4R^2L\left(\rho_s-C^0\right)\left(K_x/60\right)\left(X_0-X_e\right)\exp\left(-K_xt/60\right)}{\left(C^0-2C_0\right)\left(S^0(t)+R\right)+\displaystyle\int_{S^0(t)}^{R}C(x,t)dx}\\[2ex] S^0(t_1)=R\end{cases} \tag{18}$$

Donde X_0, denota el contenido inicial de humedad en la muestra, el que se obtiene experimentalmente; X_e, el contenido de humedad cuando se alcanza el equilibrio y K_x es la constante cinética de velocidad de pérdida de humedad libre. Los coeficientes X_e y K_x se toman de lo reportado por [3], para un proceso de freído en condiciones similares.

Introduciendo los parámetros a, b, d y l; dados por:

$$a=R^2\left(\rho_s-C^0\right)b\left(X_0-X_e\right)\qquad b=\left(K_x/60\right) \tag{19}$$

$$d = \left(C^0 - 2C_0\right) \qquad\qquad l = Rd \tag{20}$$

Remplazando tales parámetros, en el PVI planteado en (18), este resulta expresado por la ecuación (21)

$$\begin{cases} \dfrac{dS^0}{dt} = \dfrac{-a\,exp(-bt)}{dS^0(t) + l + \int_{S^0(t)}^{R} C(x,t)dx} = f(x,t,S^0(t)) \\ \\ S^0(t_1) = R \end{cases} \tag{21}$$

Denotando ahora con $\varphi = \varphi(t)$ a la función definida como:

$$\varphi(t) = \int_{S^0(t))}^{R} C(x,t)dx \tag{22}$$

De aplicar la regla de Leibniz a (22), asumiendo igualdad de flujos sobre el frente de desorción y el borde externo $x=L$, se tiene la expresión (23):

$$\frac{d\varphi}{dt} = \frac{-1}{8LR} 4R^2 L\left(\rho_s - C^0\right)\left(K_x/60\right)\left(X_0 - X_e\right)\exp\left(-K_x t/60\right) - C(S^0(t),t)\,\frac{dS^0}{dt} \tag{23}$$

Entonces de (22) y (23) se deduce el siguiente PVI expresado por (24), que constituye el modelo descriptivo de la dinámica del frente de desorción.

$$\begin{cases} \dfrac{dv(t)}{dt} = \left[-50 + \dfrac{p}{d(-0.00009t + 0.0009) + v + l}\right] a\,exp(-bt) \\ \\ v(0) = 0 \end{cases} \tag{24}$$

4. Resolución del PVI

Al abordar la resolución numérica de (24) surge el hecho que los rangos de las funciones frente de desorción y la dada por la ecuación (22), en el dominio (0, 110 seg) para la variable independiente tiempo son muy diferentes. En vista de tal situación se procedió a re escalar la función definida oportunamente por (22). Entonces resulta finalmente el siguiente PVI dado por (25) en el que el frente de desorción de humedad libre es la función incógnita de interés principal

$$\begin{cases} \dfrac{dS^0}{dt} = \dfrac{-a\exp(-bt)}{dS^0(t) + 250u + l} \\ \\ \dfrac{du}{dt} = \dfrac{1}{250}\left[\dfrac{p}{dS^0(t) + 250u + l} - 50\right] a\exp(-bt) \\ \\ S^0(10) = 0.01 \qquad u(10) = 0 \end{cases} \tag{25}$$

El PVI dado por (25) se resolvió numéricamente para los siguientes valores de los parámetros $a = 0.00144$; $b = 0.001$; $d = 436$; $l = 4.7$; $p = 896.6$; los que provienen de datos experimentales para el tipo de papa utilizada [1] y los coeficientes de transporte en el proceso de freído [3]. En la Figura 2, se ilustra gráficamente la solución numérica de (25); observándose el buen acuerdo con los valores experimentales obtenidos para le frente e de desorción de humedad libre.

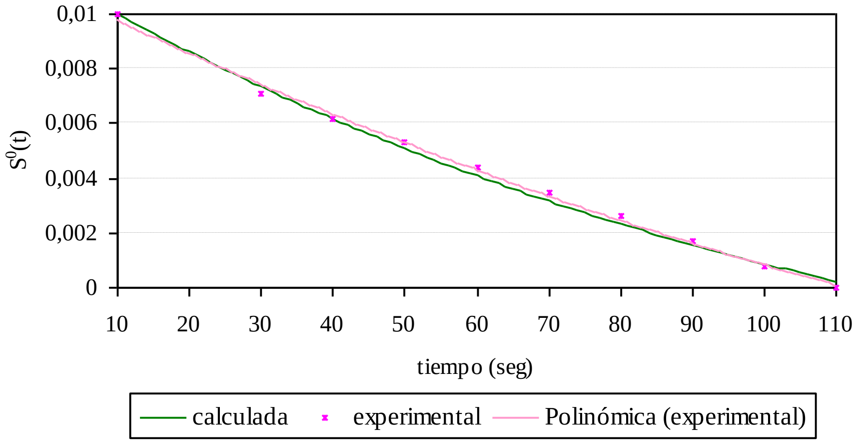

Figura 2: Comparación de la solución numérica y los datos experimentales.

5. CONCLUSIONES

Se presenta, en este trabajo, una contribución al campo de conocimiento relativo a la primera etapa (burbujeo vigoroso) del proceso de freído de papa natural por inmersión en aceite caliente. En efecto, tomando como base la formulación del modelo descriptivo para la etapa de interés, provista en trabajos precedentes, se ha logrado precisamente simular la dinámica del frente de desorción de humedad libre, mediante un modelo consistente en un problema de valor inicial para un sistema acoplado de dos ecuaciones diferenciales ordinarias no lineales: una para la función frente de desorción propiamente dicho y otra para la función definida como la integral del perfil de concentración de materia (agua libre vaporizada) en la zona en que tiene presencia el vapor de agua libre de la papa.

Para el procesamiento matemático se utilizó el programa *Mathematica,* [7] que presenta un alto nivel de cálculo, resuelve rápidamente y es fácil de implementar. Al poseer rutinas y sub rutinas contenidas en él, se evitan posibles errores de programación, lo que lo hace altamente confiable. Se observa que el acuerdo con los datos experimentales y los calculados es bueno.

AGRADECIMIENTOS

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GLOBAL MODELS AND APPLIED MATHEMATICS

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Abstract: During the 1960s and, above all, during the 1970s, large global mathematical dynamical computer models were implemented, where “global” means that the models simulate the entire world (or a large portion of it) and that several different interrelated subjects are included (e.g. economics, ecology, demography, natural resources), and “dynamical” means that the models evolve over time trying to forecast the future, considering an horizon of at least some decades. Global models continued to be developed of course also after the 1970s, but their focus started to evidence an increasing bias towards environmental matters, or at least the public impact of models with an environmental approach started to be greater than the impact of those that privilege other approaches.

Key words: global models, environmental models, complex systems
2000 AMS Subjects Classification: 93A15, 93A30

1. INTRODUCTION

During the 1960s and, above all, during the 1970s, large global mathematical dynamical computer models were implemented, where “global” means that the models simulate the entire world (or a large portion of it) and that several different interrelated subjects are included (e.g. economics, ecology, demography, natural resources), and “dynamical” means that the models evolve over time trying to forecast the future, considering an horizon of at least some decades. Global models continued to be developed of course also after the 1970s, but their focus started to evidence an increasing bias towards environmental matters, or at least the public impact of models with an environmental approach started to be greater than the impact of those that privilege other approaches.

In a sense, for social scientists interested in this kind of models, the computer is their laboratory. Models including social dynamics must either have a strong random structure, or be prepared to analyze several very different scenarios based on a wide range of assumptions, or both. In this context, interestingly, (at least) two questions appear: on the one hand, how much is it possible to characterize developed and underdeveloped countries using as a parameter the “degree of randomness” of the forecasts for any given region? That is, is randomness a characteristic of underdeveloped (or developing) countries? Clearly not: Argentina has a more developed structure than, say, Namibia, but in spite of that Namibia is more “predictable” than Argentina. On the other hand, how sophisticated must be the mathematics employed to formulate a global model in order to offer acceptable levels of confidence? In other words, is there a correlation between the mathematical complexity behind a global model and the accuracy of its forecasting power? In this case (to the applied mathematicians' disappointment) the answer is also no, or at least not necessarily: as an extreme example, two outstanding works predicting the fall of the Soviet Union, namely Andrei Amalrik's 1970 book [1] and Emmanuel Todd's 1976 [2], had been published many years before its actual dissolution; none of them used mathematical models, and it seems that global modelers did not paid attention to them.[1] Yet, when used carefully, global models are very powerful tools; in fact, we think that not using this kind of models is a complete mistake.

2. THE ORIGINS OF GLOBAL MODELS

[1] It is appropriate to remark here that preparing global mathematical models cannot be considered exclusively a work for applied mathematicians, computer scientists, engineers, economists and sociologists. Although the usefulness of “political mathematical models” (see later) is extremely arguable, if the global model gets sufficiently ambitious political scientists and in general people from cultural environments and all professions)should be also consulted. This approach makes sense at least if one believes that political circumstances may change dramatically not only the economic and social structure of a country, but also the international relations and the relative power of the different nations, and that the likelihood of these changes are not reflected in the available data (be it because data are insufficient, biased, or deliberately hidden). For instance, in [3] (see later) no reference is made to political works, with the exception perhaps of [4] and [5], which are anyway writings of a general character, with no mention to actual facts susceptible to be accounted for into a mathematical model.

In late 1950s, the brilliant engineer turned specialist in management Jay Wright Forrester, already well known because of his participation in the building of an aircraft air simulator -that eventually became the Whirlwind digital computer, and then became the Semi-Automatic Ground Environment (SAGE) air defense system for North America- and for having developed the "Multi-coordinate digitally information storage device", forerunner of today's RAM, created Systems Dynamics, a discipline whose goal is to understand the dynamic behavior of complex socio-natural systems using mathematical tools and computer simulations. His book [6] studied industrial business cycles using Systems Dynamics as a convenient tool. In 1969, after many discussions with former Boston mayor John F. Collins he published a book [7] modeling social urban problems. His work was extremely important in the process of modeling complex social, industrial and political situations and in formulating current and future feasible scenarios through mathematical models.

But Forrester did not apply his technique to entire countries –at least during the 1960s, let alone large regions that include several countries, or continents, or the entire world. It seems that the first application of such ambitious approach is due to E.P. Holland (see [8]). As Holland tells in the preface of this book, after trying unsuccessfully to implement a model of these characteristics in an *analog* computer, he eventually prepared the model to be run on the IBM 704 computer of the MIT Computational Center, using the simulation language DYNAMO, which Forrester and his group were just completing at the MIT School of Industrial Management (moreover, in the process of translation the model became more elaborated: perhaps one can consider the failure with the analog model and the success -and improvement- with the digital one as a symbolic demonstration of the triumph of digital computation). By the way, Holland and his group were helped in the programming by A. L. Pugh, III, and P. A. Fox, designers of DYNAMO (see [9]). Therefore, one can see Forrester's influence, though Forrester himself did not participate in this project. In this work the economy of a plausible underdeveloped country -that had some of the characteristics of India- was simulated (Holland used many data from India, but some considerations important for India were omitted or grossly simplified). The model was dynamic, i.e., with fixed parameters and state variables evolving over time, it showed the supposed changes of the economy of the "India-like" underdeveloped country. It contained around 250 equations (for technical reasons, DYNAMO needed around 400 equations); simulation experiments were performed during 1960. According to Varsavsky and Calcagno [10], "the model did not pretend to have practical use, and there was not yet a clear idea of the characteristics of the approach, its possibilities and its difficulties, but it was a typical example, sufficient to suggest its potential".

Now, it is necessary to clarify why we don't mention as antecedents of Holland and Gillespie's model (and of Forrester's models, by the way) some very important economic models of the United States, such as the Klein-Goldberger [11] and Leontiev [12] models. The reason is simple: the Klein-Goldberger model is econometric (while we are interested in models that take also into account other social -or political- variables), and Leontiev's is a static input-output model (while we are interested in dynamical models). In these senses, the type of models which are object of this article use a very different approach.

Finally, it is interesting to remark that Holland's interest was the economic development of an *underdeveloped* nation, as he clearly states in his book; that is, he tried to use his global model as a tool for the development of a country. This was exactly Oscar Varsavsky's approach, as we shall see in the following section.

3. Early Latin American applications

The beginning of the 1960s was an interesting time in Latin America. Science was considered a fundamental tool for development, and a considerable number of intellectuals and scientists were enthusiastic about the region's future: some of them became very impressed by the Cuban Revolution and wanted to put science and technology "in the service of Revolution", while others were optimistic about the success of democratic institutional changes (remember that several military dictatorships had just been replaced by constitutionally-elected governments). For instance, we may mention the role of FLACSO (Latin American Faculty of Social Sciences), located in Santiago, Chile, whose Latin American School of Sociology, directed between 1960 and 1965 by the Swiss sociologist Peter Heintz, was extremely influent in the development of a generation (perhaps the first) of Latin American sociologists with a global vision of sociopolitical problems in the region along with a training in formalization, modeling and simulation of such problematics. Some of these sociologists were therefore "ready" to be influenced by mathematical models. Only a few years later, an impressive experience took place in Chile during the Allende administration, which was interrupted by the coup-d'état that overthrew Allende in 1973: the Cybersyn project (or Synco, in Spanish, see http://www.cybersyn.cl/) was an attempt to manage through computers the Chilean economy in real time (even including daring ideas about direct democracy through the participation of people from their "interconnected households"). Cybersyn may be considered technically as a sort of practical mixture of Stafford Beer's management cybernetics (through his Viable System Model, VSM [13]) with Forrester's System Dynamics[2]. The models

[2] On the one hand, Cybersin was an intellectual creation of Beer (subject of course to dynamical changes during its implementation, due to the active participation of Chilean professionals in charge of the project); on the other hand, in some

envisioned had the spirit of global models (this time applied at a national scale) in the sense that they tried to capture concurrently interactions among several heterogeneous "layers" in which a socioeconomic reality may be decomposed.

Perhaps the country where confidence in democratic changes was greatest was Venezuela. On the one hand, the military government of General Marcos Pérez Jiménez had been overthrown (January 23, 1958) and a constitutional civil administration governed since 1959. On the other hand, oil royalties for an important oil exporter such as Venezuela allowed significant investments (compared to the Latin American standards) in higher education and science. Besides, in those years "planning" was considered by many economists at the Economic Commission for Latin America and the Caribbean (ECLAC) a powerful tool for Latin American economic development. That was the political climate in which Holland (and Varsavsky in several opportunities) worked in Venezuela.[3]

As Oscar Varsavsky and Alfredo E. Calcagno tell in their book, Holland traveled to Caracas, Venezuela, in 1961, to try to apply his model for planning an actual economy; in fact, he even wrote an article for a Venezuelan journal [18]; eventually he began to work in 1963 with a small group of collaborators and almost three years later produced the model V-2, with which he reproduced the Venezuelan economic history from 1950 to 1962, taking into account two economic sectors: oil and all the others. Varsavsky met Holland in Caracas, became impressed by the power this type of models could have in social sciences, and organized, in 1962, a group to prepare socio-economic models at the Instituto de Cálculo (a sort of Institute of Applied Mathematics) of the University of Buenos Aires. In 1963 the MEIC-0 economic model (Economic Model-0 of the Instituto de Cálculo) was already working (Varsavsky [1965]). Varsavsky and his colleagues prepared several economic, demographic, political and educational models in Buenos Aires, Caracas and Santiago de Chile (there was also an application in Bolivia) during the 1960s. We can mention, among others, short-term models of economic policies in Bolivia and Chile and analysis of results of different styles of development in Venezuela. A mathematical model of Thomas Moore's *Utopia* was also prepared (see [19]).

Varsavsky's idea was that several models were "modules" of a larger one, and outputs of one model should be inputs for other(s). This vision of modularity applied to socioeconomic models was conceptually very advanced for its time. The evolution of modeling for computer simulation until our days has shown that the concept of building models of complex systems by means of the coupling of simpler models has progressed enormously in the engineering domains (mainly concerned with building physical devices, e.g., spaceships, bridges, computers), but is very underexplored in the domain of social and natural sciences, leading to knowledge islands of deep specialization too often difficult to interconnect.

Varsavsky collaborated with and directed many groups in all those countries, and for some years a Latin American network actively worked in those subjects. Some models then implemented were described in Varsavsky and Calcagno's book, but we may also mention national models related to Brazil [20]. In sum, there was an important activity in Latin America of preparation of models which, although referred each of them to a single country, had many ideas which would be applied in the following decade for the construction of global models. In those years, in the mathematical modeling of society, Latin American science was outstanding.

By the way, it is interesting to remark that this "Latin American network" was very unusual. Latin American scientists not always work together; more often than not (and especially in those times) their contacts originate in the United States or in Europe and the center of their network is (or was) an American or European University.

4. THE FIRST WORLD MODELS

Meanwhile, Forrester continued working in his models. During the 1960s, Dick Bennett collaborated with Forrester and prepared a compiler for Industrial Dynamic models (SIMPLE: Simulation of Industrial Management Problems with Lots of Equations) and Jack Pugh extended the compiler into the DYNAMO series; we have already mentioned the use of DYNAMO by Holland. DYNAMO may be considered one of the first continuous time simulation languages, and certainly the first Systems Dynamics simulation language (i.e., the Forrester's approach). An impressive bibliography exists about both continuous time and discrete events simulation languages, and the 1960s was perhaps the decade when simulation models became very popular: on the one hand, many mathematical problems had not an explicit solution (or, at least, a technically feasible explicit solution) and, on the other hand, computers were already sufficiently developed to treat significantly complex problems (from the point of view of the size of data and number of equations).

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parts of the project Forrester's influence may be detected. The DYNAMO programming language was used e.g. to develop the CHECO (CHilean ECOnomy) module, a real time monitoring dashboard and mid- to long-term forecasting panel. The story may be consulted in [14] or [15]. For a detailed biography of Beer, including especially his contribution to Cybersyn, see [16]; for a theoretical approach to the VSM/SD mixing see [17].

3 In fact, in this context the Center of Studies for the Development (Centro de Estudios del Desarrollo, CENDES) was created in 1961 at the Central University of Venezuela in Caracas. CENDES was (and still is) an influential institution in Venezuela: some of Varsavsky's collaborators and colleagues were affiliated with it.

After *Urban Dynamics* was published, Forrester began to be known in the circles of urban planners, and, in general, of people interested in prospective scenarios. In 1970, Aurelio Peccei and Eduard Pestel, who had founded the Club of Rome in 1968 as a global think tank, contacted Forrester. After that, Forrester published *World Dynamics* [21] and Meadows *et al. The Limits to Growth* [3], both with very simple equations. *World Dynamics* initially used five worldwide sectors and assigned one state variable to each of them: Population, Natural Resources, Capital Investment (industrialization), Pollution and Capital Investment in Agriculture Fraction (food production). *The Limits to Growth* extended the model to 13 sectors represented by 41 state variables. The mathematical models (named World2 and World3 in 1971 and 1972, respectively) were written in DYNAMO.

The impact of the book by Meadows and collaborators was very strong. In fact, [22] and [23] are two updates of their book, both maintaining the original pessimism. What they said, in a sense, could be thought of as a kind of "neo-Malthusianism", that is, the world was growing much faster than the natural resources available to feed this growth, and so the only solution was to stop the growth. Independently of the merits or flaws of the World3 model, it is evident that for people in the upper classes all over the world the consequences of stopping growth would be very different than the consequences for people in the lower classes, and above all for people in the lower classes in underdeveloped countries -or, as it is more politically correct to say, in developing countries. Interestingly, it was in one developing country, in Latin America, where a model was prepared to challenge the "doomsday scenario" anticipated by the World3 model: an interdisciplinary group in Bariloche, Argentina, designed and implemented the Latin American World Model, usually better known as the Bariloche Model (see [24]). The Bariloche Group preferred a model which would outline the path to sustainable development rather than predict doomsday. The Bariloche Model was a scientific success, and its influence, exposing rigorously and explicitly the pre-assumed points-of-view of researchers from a developing country, was significant: suffice it to say that UNESCO used an interactive version of the model in courses in planning in Paris. Besides, it was an impressive exercise in interdisciplinary work in Argentina, where scientists from different areas seldom endeavor into interdisciplinary projects.

The Bariloche model was officially presented in 1974 to the IIASA (International Institute for Applied System Analysis). The same year another model was also presented, as the second report to the Club of Rome, but which supposedly corrected the mistakes of which the World3 Model was accused: the Mesarovic and Pestel's World Interdependence Model (WIM) [25].

The objections –in general very sound– to Meadows' model (and to Mesarovic and Pestel's model) spanned through the decades, are well known, and are not only of political but also of technical nature, but this is not the place to analyze them (see, for instance [26], [27], [28], [29][4]).

Very recently, fresh words started to be heard about those models and the conclusions offered in *The Limits to Growth*, mainly due to its 40th anniversary (see for instance Bardi [32]). Interestingly, a somewhat non-traditional reinterpretation of what that work really was (and what its real message was really intended to be) comes from one of its very authors, Jorgen Randers [33]. These recent retrospective thoughts are more concerned with the non-technical criticisms received by Meadows' World3 model (i.e. the interpretations of "the message"). In the words of Randers: "…[the book] points to the need for a solution to three fundamental and legitimate problems: poverty, unemployment, and old age insecurity that underlie the global fascination with growth. These problems must be solved in a way which is compatible with planned reduction of the human ecological footprint." Here, "planned reduction" instead of 'growth stop", and "ecological footprint" instead of "economic growth", represent alone big conceptual departures from the originally popularized message of the book. Moreover, Randers continues: "… Most likely this [the reduction of ecological footprint] will ultimately require *equitable allocation of finite global resources on a per capita basis*", a sentence suggesting the egalitarian distribution of global wealth among all human societies perhaps not too easy to claim loud in the United States by the time the book was published.

All in all, it is fair to say also that Meadows' model had a great virtue: although Forrester's model can be considered the first world model, Meadows' model was the first global model that became famous; it had an extraordinary impact in the press and in the public opinion, and for that reason it is very valuable, because it compelled many people to study the characteristics that a global model should have to be reliable. Anyway, as the controversial judge Alex Kozinski [34] remarks, sometimes predictive models reflect, not reality, but the preconceptions of the model's creators. There was an excessive "doomsday scenario" but the work was challenging: perhaps the doomsday scenario is too pessimistic, but it is really important to count on a scientific community becoming used to analyze projections of global plausible possible futures.

A myriad of global models appeared after the ones mentioned above, and it is not necessary to mention them here. For example, Brecke [35, 36] discusses the history of many of them, including a not very known Soviet one [37];

4 Anyway, it is interesting to comment on Kalman's [29] paper. His approach is more general from the point of view of the "theory of models", with a touch of epistemological flavor. His reference to Meadows' model is very short; he attacks it from a "pure system-theoretic" angle, and he mentions Vermeulen and de Jongh [30], who observe that "...the mathematical model basic to the results of *The Limits to Growth* has been found to be very sensitive to small perturbations". On the sensitivity of Meadows' model to changes in parameters and initial conditions, and on the issue of model's validation against historical data see [31].

regarding this one, it is curious that so few Western modelers investigated what the Soviets were doing: it should have been obvious that, in spite of political constraints and control, a country that invented central planning by means of five-year plans probably should be (as in fact was) interested in this methodology. Lazarevic [38] shows some opinions about global models at the end of the 1970s. During the 1970s this kind of global models was very popular; moreover, the already mentioned IIASA - housed in the Blauer Hof Palace in Laxenburg, near Vienna- was created in 1972 with the main purpose of analyzing and confronting the models.

5. SOME REMARKS ON GLOBAL MODELS

In what sense these global models could be considered "forecasting" models? The models neither predict the fall of the Soviet Union or the war in Yugoslavia, nor took into account a scenario in which these events could happen. Meanwhile, runs of the GLOBUS model in the 1980s predicted the political system in Czechoslovakia, East Germany and Poland to be more stable than that of Western Europe.[5] It can be argued that for that kind of predictions, a model dealing with *political* variables should have been used, and we are very far from implementing mathematical models of political events (Domingo and Varsavsky prepared a mathematical model of Utopia, not of a real political situation). Nevertheless, in many cases changes in *economic* variables (e.g., the oil price) are not easy to forecast. Besides, other difficulties appear that should not be overlooked: the models may be too sensitive to parameters whose values are arguable, and the variables not modeled explicitly must be accounted for by the introduction of some "randomness". Of course a modeler can never introduce in a global model *all* the variables involved: on the one hand, one does not even *know* all of them, and on the other hand, the relationships among the known ones are too complicated for all of them to be explicitly described. Therefore, as in many other scientific models, "hidden variables" should be perhaps replaced by random variables. Moreover, randomness *does* exist: had it been feasible in January, 1914, to prepare a global model of the world, or at least of Europe, with an horizon of, say, twenty years, it is extremely implausible that the model had taken into account that seven months later the most important countries in Europe would be immersed in a horrific war (war that, by the way, most planners in the different General Staffs supposed would be short). Unfortunately, lack of predictability cannot be associated with lack of data (so that the model seems "more random"); if one assumes that when a country is more developed then more data exist, and thus more reliable the data are, it could be argued that the more developed a country is, the more predictable or robust the model becomes. Unfortunately, it is not so: "robustness" or "predictability" of a global model is not necessarily related to the degree of development of the country or region studied: it is likely that an Argentinean model should be less robust or less predictable than a Namibian model. And sometimes population dynamics may be unpredictable: population size may run amok [40].

6. COMPLEX SYSTEMS

During the 1980s the mass media and public opinion became less and less interested in global models of this kind. For the Latin American groups, the explanation is simple: the optimism, that we already mentioned, of the 1960s, disappeared completely in the 1970s: in Argentina there was a military dictatorship from 1966 to 1973, but scientists interested in political, social and economic problems who abandoned the country or the research at the universities could continue working in other Latin American countries (the "Varsavsky network" is an example) or in non-governmental organizations (as shown by the Bariloche Foundation that funded the Bariloche model).

In the 1970s the political situation was unhealthy in most South American countries, and global models were not a priority for scientists exiled or silenced. In the developed world the explanation is not so easy: it is likely that pessimism related with doomsday scenarios began to decrease, given that it was obvious that in the 1980s the western powers – especially the United States – were winning the competition against the Soviet Union, and that "normal" scenarios are less attractive, from the mass-media point-of-view, than doomsday scenarios. Anyway, it was impossible *not* to use this kind of models, or similar ones: since the appearance of the computers at the end of the 1940s, that meant the possibility of making more and more computations with more and more variables in less and less time; it was obvious that complex systems, impossible to work with prior to the computer era, should –and would– be mathematically and computationally treated.

Continued increasing attention was paid to the study of more general complex systems, not necessarily related to socio-economic worldwide prospective analyses. The study of complex systems is an interdisciplinary field of science, which studies the structure and characteristics of complex interactions among systems in nature and society [41]. As complex systems deal with many, many variables, with many, many attributes, subject to many, many relationships which in turn might represent many possible causes, it was impossible to try to model them mathematically before the

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[5] For the GLOBUS model, see for instance [39].

computer. Global models are a class of complex systems that analyze some parts of the society related with different countries, of geographic regions. What was instead observed was a more intense interest in complex models in general, and in "complexity science". Simultaneously, several institutes -similar in a sense to the IIASA- were created in different places, for instance the Santa Fe Institute, founded in 1984, and the New England Complex Systems Institute (NECSI), founded in 1996. All these approaches are necessarily interdisciplinary. NECSI, for instance, defines Complex Systems as "a new field of science studying how parts of a system give rise to the collective behaviors of the system, and how the system interacts with its environment" (http://www.necsi.edu).

The applications are varied: NECSI indicates among its applications economics, multiscale methods, ethnic violence, networks, health care, evolution and ecology, management, systems biology, education, engineering, military conflict, development, negotiation. Some of these fields are a priori not supposed to be easily modeled; others may be thought as descendants of game theory models, tracing back to the impressive effort that von Neumann and Morgenstern [42] invested in their famous book, perhaps one of the first (very simple) attempts to model complex situations under competition. It is amusing to see that almost all the book analyzes scenarios with up to 3 or 4 participants: it was written before the appearance of the modern computer.

The areas of interest of the Santa Fe Institute (http://www.santafe.edu/) are physics of complex systems; emergence, innovation, and robustness in evolutionary systems; information processing and computation in complex systems; dynamic and quantitative studies of human behavior; and emergence, organization and dynamics of living systems. As it can be seen, a lot of systems may be labeled "complex" and tried to be mathematically modeled. Obviously, multi-disciplinary collaboration is absolutely necessary in this kind of approach. Another institute, the Bath Institute for Complex Systems -this one in Britain (http://www.bath.ac.uk/math-sci/bics/)- has research lines related to modeling and analysis of multiscale problems; structure of biologic populations; deterministic properties of highly disordered systems; numerical methods for multiscale problems; and highly structured stochastic systems. Systems on which these institutes work seem perhaps more difficult to model than systems studied by global models; but, for the above mentioned reasons (among many others), interest gradually changed in the 1980s from global models to complex systems, understood in the sense these three Institutes (and many others) approached them. It could be well regarded as a symptom of the prevalence of domain-oriented specialization over cross-domain analysis at least from the application point of view (even when interdisciplinarity remains a key feature).

Many valuable and interesting papers have been written on complex systems, as well as dull and uninteresting ones (as in all fields of science, of course). As the definition of complex systems is very broad, a lot of important contributions can be classified as contributions to complex systems. Anyway, many contributions on complex systems are theoretical, so that they cannot be compared with works on global models. Perhaps the most mature organized body of knowledge related with complex systems is the Springer Complexity interdisciplinary publishing program (www.springer.com/physics/complexity) which stems from physics and includes research and teaching-level texts on fundamental and applied aspects of complex systems.

7. ENVIRONMENTAL MODELS

In the last years, an interest arose on other kind of models, environmental models, which have many common traits with the traditional global models, although more focused, naturally, in environmental problems than in development problems. Global models in general assumed that what they simulated were feasible situations originated in human decisions on economics, policy, military affairs, fight against poverty, and so on. In current environmental models, on the other hand, human decisions refer especially to their contributions to global warming or climate change; according to most experts, global warming is due mostly to anthropic actions; according to a minority of experts, the anthropic influence in global warming is negligible. In both cases, besides anthropic action there exist natural causes, which are also included in the models. Most complex systems models are not especially interested in the future state of the world: on the other hand, the main interest of global models is to forecast the future state of the world, or parts of it, using (perhaps many) mathematical equations and assumptions, and this is also the main interest of environmental models. In both types of models not all the assumptions are explicit; some are hidden in the equations. Environmental models are more "technical", in the sense that they use less political or economic assumptions, which are always subject to discussion (although the fact that a minority of experts challenge the approach of most scientists indicates that, in spite of being "technical", they are not immune to discussions). Taking this into account it is probably sound to affirm that, in a sense, environmental models are somehow the heirs of the 1970s global models. This is the feeling one has, for instance, when looking at the IIASA website (http://www.iiasa.ac.at/), and realizes how much this institute, where thirty years ago many global models were presented, is now interested in climate change. It is interesting to remark that environmental models can be regarded as "heirs" of global models, but not "descendants" of them: environmental models evolved from the forecast numerical models that John von Neumann and Jule Charney proposed at the end of the 1940s and in the early 1950s [43], and then from the atmospheric general circulation models from 1955 on [44].

8. CONCLUSIONS

From all the discussions on global models, and also on environmental models, some words of caution are necessary: equations are sometimes controversial (not only the empirical equations, but also the supposedly "conceptual" ones); parameters must be calibrated when feasible, sometimes a difficult task; they may be extremely sensitive to changes (although sometimes we do not know their actual values); initial conditions (state of the system at the beginning of the simulation time) must be given, and sometimes, if they are difficult to obtain, they must be found through a process of "warming up" of the model which does not always converges to the true simulated beginning of the simulation; boundary conditions must be given (sometimes they are difficult to obtain, and when using a model for predicting a future state of a system one has often to invent them); if partial differential equations are used, numerical methods must be robust enough; in socio-economic models, equilibrium points may be unstable.

On the other hand, if used sensibly and cautiously, they enrich the knowledge of the system under study; a huge number of experiments with different parameters, boundary conditions and initial conditions must be performed before inferring unquestionable conclusions; some experiments must be performed with variables randomly perturbed; a global model must be developed dynamically and iteratively, that is, it must be constantly criticized, modified and improved (besides, a global model must be dynamic in the sense that it evolves in time and usually does not tend to an equilibrium point); sometimes (and perhaps this is really difficult), if the results of the model are not what we expect, we must change our expectations, not the model.

In spite of this, modelers often back their models more than they should. More caveats should be included in all models made public, more alternative results should be showed changing some input data (or parameters) not sufficiently well known, more scenarios should be included, and also the main assumptions embedded in the structure of models should be made clearly explicit.

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CONVERGENCE OF THE SOLUTION OF THE ONE-PHASE STEFAN PROBLEM WITH RESPECT TWO PARAMETERS

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Abstract: A one-phase unidimensional Stefan problem with a convective boundary condition at the fixed face $x = 0$, with a heat transfer coefficient $h > 0$ (proportional to the Biot number) and an initial position of the free boundary $b = s(0) > 0$ is considered. We study the limit of the temperature $\theta = \theta_{b,h}$ and the free boundary $s = s_{b,h}$ when $b \to 0^+$ (for all $h > 0$) and we also obtain an order of convergence. Moreover, we study the limit of the temperature $\theta_{b,h}$ and the free boundary $s_{b,h}$ when $(b, h) \to (0^+, 0^+)$.

Keywords: *Stefan problem, free boundary problem, phase-change process, convective boundary condiction.*
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1 INTRODUCTION

In this paper, we consider the unidimensional free boundary problem (one-phase Stefan problem) with a convective boundary condition at the fixed boundary $\xi = 0$. It consists in determining the temperature $\theta = \theta(\xi, \tau)$ and the free boundary $\xi = s(\tau)$ which satisfy the following conditions

$$\begin{cases} (i)\ \rho c\theta_\tau - k\theta_{\xi\xi} = 0, & 0 < \xi < s(\tau), \tau > 0, \\ (ii)\ k\theta_\xi(0,\tau) = h\left[\theta(0,\tau) - f(\tau)\right], & \tau > 0, \\ (iii)\ \theta(s(\tau),\tau) = 0, & \tau > 0, \\ (iv)\ k\theta_\xi(s(\tau),\tau) = -\rho l \frac{ds}{d\tau}(\tau), & \tau > 0, \\ (v)\ \theta(\xi,0) = \varphi(\xi), & 0 \leqslant \xi \leqslant b \\ (vi)\ s(0) = b\ (b > 0) \end{cases} \tag{1}$$

where $b > 0$ is the initial position of the free boundary, $h > 0$ is the thermal transfer coefficient, $\varphi(\xi) \geq 0,\ 0 \leqslant \xi \leqslant b$, is the initial temperature, $f = f(\tau) \geq 0,\ \tau > 0$ is the temperature of the external fluid and the compatibility conditions $k\varphi'(0) = h(\varphi(0) - f(0))$ and $\varphi(b) = 0$ are assumed. The goal of this paper is to study the mathematical behavior of the solution $\theta = \theta_{b,h}(\xi,\tau),\ s = s_{b,h}(\xi,\tau)$ of problem (1) when $b \to 0^+$ (for each $h > 0$) and when $(b,h) \to (0^+, 0^+)$. The Stefan problem was studied in the last decades, see for example, [1], [3], [6], [9], [10], [11] and a large bibliography on the subject was given in [16].

Existence and uniqueness of solution to problem (1) is given in [7]. In [17] the behavior of the solution of the free boundary problem (1) with respect to the heat transfer coefficient h in the one-phase case was studied. A generalization of this result for the two-phase problem was considered in [18]. There, it was proved that the asymptotic behavior when $t \to \infty$ of the one-phase free boundary problem with a convective boundary condition at the fixed face is the same that for the case where the temperature boundary condition, which is depending on time, is given on $x = 0$. Asymptotic behavior for the one-phase problem with temperature boundary condition on the fixed face was given by [4, 5]. For the particular case $f(\tau) = Const > 0$, for the multidimensional case, the study of the asymptotic behavior when $h \to \infty$ is obtained by using the variational inequality [14, 15] and for the one-dimensional case in [13]. In [18] the monotone dependence of the solution with respect to the data and with respect to the thermal transfer coefficient is proved for the two phase Stefan problem. In [12], the classical one-phase Stefan problem is presented in dimensionless form with a time-varying-heat-power boundary condition. The asymptotic behavior of the solution for the generalized form of the Biot number $Bi \to 0$ was studied from a physical point of view. In [2] the mathematical analysis of this asymptotic behavior of the solution with respect to the heat transfer coefficient was considered and an order of convergence was also obtained.

The goal of this paper is to analize the asymptotic behavior of the solution of the problem (1) when $b \to 0^+$ (for h fixed), and the double asymptotic behavior when $(b,h) \to (0^+, 0^+)$.

We will make the following assumptions on the initial and boundary data:

(i) Let $\varphi = \varphi(\xi)$ be a positive and piecewise continuous function, with $\varphi'(\xi) \leq 0$.

(ii) Let $f = f(\tau)$ a positive bounded piecewise continuous function, with $f'(\tau) \geq 0$

(iii) Compatibility conditions: $f(0) > \varphi(\xi), \forall \xi \in (0,b),\ k\varphi'(0) = h(\varphi(0) - f(0))$and $\varphi(b) = 0$.

If we define the following transformation

$$u(x,t) = \frac{c}{l}\theta(\xi,\tau),\ x = \frac{\xi}{b_0}\ ,\ t = \frac{\alpha}{b_0^2}\tau \tag{2}$$

where $\alpha = \frac{k}{\rho c}$ is the diffusion coefficient and b_0 is a space reference scale, then the free boundary problem (1) becomes

$$\begin{cases} (i)\ u_t - u_{xx} = 0, & 0 < x < S(t),\ t > 0, \\ (ii)\ u_x(0,t) = \frac{hb_0}{k}\left[u(0,t) - F(t)\right], & t > 0, \\ (iii)\ u(S(t),t) = 0, & t > 0, \\ (iv)\ u_x(S(t),t) = -\dot{S}(t), & t > 0, \\ (v)\ u(x,0) = \chi(x) \geq 0, & 0 \leqslant x \leqslant \frac{b}{b_0} \\ (vi)\ S(0) = \frac{b}{b_0} & \end{cases} \tag{3}$$

where

$$F(t) = \frac{c}{l}f(\frac{b_0^2 t}{\alpha}) \geq 0,\ H = b_0\frac{h}{k} > 0 \text{ (the Biot number)}, \tag{4}$$

$$\chi(x) = \frac{c}{l}\varphi(b_0 x),\ S(t) = \frac{1}{b_0}s(\frac{b_0^2 t}{\alpha}). \tag{5}$$

In Section 2, we enunciate some preliminary results about of the solution to problem (3). In Section 3 we study the convergence for the solution to problem (3) when $b \to 0^+$ and we give an order of convergence for the corresponding temperature and free boundary. In Section 4, we study the convergence for the solution to problem (3) when $(b,h) \to (0^+, 0^+)$ and we give an order of convergence for the corresponding temperature and the free boundary.

2 Properties of the solution to problem (3)

Under the condition stated in Introduction we have the following results:

Lemma 1 ([*13*], [*17*], [*18*]) *The solution $u = u_{bh}(x,t),\ S = S_{bh}(t)$ to problem (3) has the integral representation given by the following expressions*

$$u_{bh}(x,t) = \int_0^{\frac{b}{b_0}} N(x,t;\xi,0)\chi(\xi)d\xi + \int_0^t N(x,t;S_{bh}(\tau),\tau)V_{bh}(\tau)d\tau \tag{6}$$

$$-\frac{hb_0}{k}\int_0^t N(x,t;0,\tau)v_{bh}(\tau)d\tau + \frac{hb_0}{k}\int_0^t N(x,t;0,\tau)F(\tau)d\tau,$$

$$S_{bh}(t) = \frac{b}{b_0} - \int_0^t V_{bh}(\tau)d\tau \tag{7}$$

where the functions $V_{bh} = V_{bh}(t)$ and $v_{bh} = v_{bh}(t)$, defined as

$$\begin{cases} V_{bh}(t) = u_{bh_x}(S_{bh}(t),t),\ t > 0, \\ v_{bh}(t) = u_{bh}(0,t),\ t > 0, \end{cases} \tag{8}$$

are the solutions of the following system of integral equations:

$$V_{bh}(t) = 2\int_0^{\frac{b}{b_0}} \chi'(\xi)\, G(S_{bh}(t),t,\xi,0)d\xi - 2\int_0^t \frac{hb_0}{k}[v_{bh}(\tau) - F(\tau)]N_x(S_{bh}(t),t,0,\tau)d\tau$$

$$+2\int_0^t V_{bh}(\tau)N_x(S_{bh}(t),t,S_{bh}(\tau),\tau)d\tau, \tag{9}$$

$$v_{bh}(t) = \int_0^{\frac{b}{b_0}} \chi(\xi)\, N(0,t,\xi,0)d\xi - \int_0^t \frac{b_0 h}{k}[v_{bh}(\tau) - F(\tau)]N(0,t,0,\tau)d\tau \tag{10}$$

$$+\int_0^t V_{bh}(\tau)N(0,t,S_{bh}(\tau),\tau)d\tau,$$

for $0 < x < S_{bh}(t),\ 0 < t < T$, *where G and N are the Green and Neumann functions defined by:*

$$G(x,t,\xi,\tau) = K(x,t,\xi,\tau) - K(-x,t,\xi,\tau) \tag{11}$$

$$N(x,t,\xi,\tau) = K(x,t,\xi,\tau) + K(-x,t,\xi,\tau) \tag{12}$$

with

$$K(x,t,\xi,\tau) = \begin{cases} \frac{1}{2\sqrt{\pi(t-\tau)}}\exp\left(-\frac{(x-\xi)^2}{4(t-\tau)}\right) & t > \tau \\ 0 & t \le \tau. \end{cases} \tag{13}$$

For simplicity of notation, in what follows we denote $u = u_b$ and $S = S_b$ when we analize the dependence of the solution to problem (3) with respect to b. We denote $u = u_h$ and $S = S_h$ when we analize the dependence of the solution to problem (3) with respect to h.

Lemma 2 ([*13*], [*17*], [*18*])*The solution* $u = u_{bh}(x,t),\ S = S_{bh}(t)$ *to problem (3) satisfies the following inequalities:*

(a) $0 \le u_{bh}(x,t) \le F(t)$;
(b) $h_1 < h_2 \Rightarrow u_{h_1}(x,t) \le u_{h_2}(x,t)$;
(c) $h_1 < h_2 \Rightarrow S_{h_1}(t) \le S_{h_2}(t)$;
(d) $u_{bh}(x,t) \ge 0\ ,\ u_{bh_x} \le 0,\ u_{bh_t} = u_{bh_{xx}} \ge 0$
(e) $0 \le \dot{S}_{bh}(t) \le H.F(t)$

(f) for $0 < x < S_{bh}(t),\ t > 0$ *we have the following integral relations:*

$$S_{bh}(t) = \frac{b}{b_0} + \int_0^{\frac{b}{b_0}} \chi(x)dx - \frac{hb_0}{k}\int_0^t [u_{bh}(0,\tau) - F(\tau)]\,d\tau - \int_0^{S_{bh}(t)} u_{bh}(x,t)dx, \tag{14}$$

$$S_{bh}^2(t) = \frac{b^2}{b_0^2} - 2\int_0^{S_{bh}(t)} x\, u_{bh}(x,t)dx + 2\int_0^{\frac{b}{b_0}} x\,\chi(x)dx + 2\int_0^t u_{bh}(0,\tau)d\tau, \tag{15}$$

$$\int_0^{S_{bh}(t)} u_{bh}^2(x,t)dx - \int_0^{\frac{b}{b_0}} \chi^2(x)dx + 2\int_0^t\int_0^{S_{bh}(\tau)} u_{bh_x}^2(x,\tau)dxd\tau \le \frac{hb_0}{k}\int_0^t F^2(\tau)d\tau. \tag{16}$$

Lemma 3 *If* $b_1 < b_2$ *we have* $S_{b_1}(t) < S_{b_2}(t)$ *and* $u_{b_1}(x,t) < u_{b_2}(x,t)$, *for all* $t > 0, 0 < x < S_{b_1}(t)$.

Proof. If $b_1 < b_2$ we have $S_{b_1}(0) < S_{b_2}(0)$. We suppose that the assertion of Lemma 3 is false, that is there exists $t_0 > 0$ such that

$$S_{b_1}(t) < S_{b_2}(t),\ \forall\ 0 < t < t_0; \quad S_{b_1}(t_0) = S_{b_2}(t_0). \tag{17}$$

If we define

$$v(x,t) = u_{b_1}(x,t) - u_{b_2}(x,t),\ 0 < x < S_{b_1}(t),\ 0 < t < t_0$$

we have the following properties:

$$\begin{cases} (i)\ v_t - v_{xx} = 0,\ 0 < x < S_{b_1}(t), 0 < t < t_0 \\ (ii)\ v_x(0,t) = \frac{hb_0}{k}\left[u_{b_1}(0,t) - u_{b_2}(0,t)\right],\ 0 < t < t_0 \\ (iii)\ v(S_{b_1}(t),t) = -u_{b_2}(S_{b_1}(t),t) < 0, 0 < t < t_0 \\ (iv)\ v_x(S_{b_1}(t),t) = u_{b_2 x}(S_{b_1}(t),t) - \dot{S}_{b_1}(t), 0 < t < t_0 \\ (v)\ S_{b_1}(0) = 0 \end{cases} \tag{18}$$

and

$$v(S_{b_1}(t_0), t_0) = 0.$$

If $v(0,t) = u_{b_1}(0,t) - u_{b_2}(0,t) > 0, 0 < t < t_0$ by the condition (iii) we deduce that $v_x(0,t) < 0$ which is a contradiction by using the condition (ii), then $v(0,t) \leq 0, 0 < t < t_0$.
Therefore we have a maximum value $v(S_{b_1}(t_0), t_0) = 0$ and then we get $v_x(S_{b_1}(t_0), t_0) \geq 0$. But in other hand we have $v_x(S_{b_1}(t_0), t_0) < 0$ which is a contradiction.
Hence $S_{b_1}(t) < S_{b_2}(t)$ and by maximum principle we have $u_{b_1}(x,t) < u_{b_2}(x,t)$, for all $t > 0, 0 < x < S_{b_1}(t)$.

□

3 ASYMPTOTIC BEHAVIOR OF THE SOLUTION (u_b, s_b) WHEN $b \to 0^+$.

In this section we will study the behavior of the solution $u = u_{bh}(x,t),\ S = S_{bh}(t)$ to problem (3) when $b \to 0^+$. We will prove that the solution to problem (3) converges to the solution of the following parabolic free boundary problem (19) :

$$\begin{cases} (i)\ u_{0h_t} - u_{0h_{xx}} = 0, & 0 < x < S_{0h}(t),\ t > 0, \\ (ii)\ u_{0h_x}(0,t) = \frac{hb_0}{k}\left[u_{0h}(0,t) - F(t)\right], & t > 0, \\ (iii)\ u_{0h}(S_{0h}(t),t) = 0, & t > 0, \\ (iv)\ u_{0h_x}(S_{0h}(t),t) = -\dot{S}_{0h}(t), & t > 0, \\ (v)\ S_{0h}(0) = 0 \end{cases} \tag{19}$$

when $b \to 0^+$. The problem (19) has the following integral representation

$$u_{0h}(x,t) = \int_0^t N(x,t;S_{0h}(\tau),\tau) u_{0h_x}(S_{0h}(\tau),\tau) d\tau - \frac{b_0 h}{k} \int_0^t [u_{0h}(0,\tau) - F(\tau)] N(x,t,0,\tau) d\tau. \tag{20}$$

We will use some integral relations satisfied by the solutions $u = u_{bh}(x,t),\ S = S_{bh}(t)$, and $u = u_{0h}(x,t),\ S = S_{0h}(t)$ to problems (3) and (19) respectively.

Lemma 4 *For problem (*19*), we have the following integral relations:*

$$S_{0h}(t) = -\int_0^{S_{0h}(t)} u_0(x,t)dx - \frac{hb_0}{k}\int_0^t \left[u_0(0,\tau) - F(\tau)\right] d\tau,\ 0 < x < S_0(t),\ t > 0, \tag{21}$$

$$S_{0h}^2(t) = -2\int_0^{S_{0h}(t)} x\ u_{0h}(x,t)dx + 2\int_0^t u_{0h}(0,\tau)d\tau, \tag{22}$$
$$0 < x < S_{0h}(t),\ t > 0,$$

$$\int_0^{S_{0h}(t)} u_{0h}^2(x,t)dx + 2\int_0^t \int_0^{S_{0h}(\tau)} u_{0h_x}^2(x,\tau)dxd\tau \leq \frac{hb_0}{k}\int_0^t F^2(\tau)d\tau; \tag{23}$$
$$0 < x < S_{0h}(t),\ t > 0.$$

Proof. See [3], [8], [18] . □

Lemma 5 *We have* $S_{0h}(t) < S_{bh}(t)$ *and* $u_{0h}(0,t) < u_{bh}(0,t)$, *for all* $t > 0,\ b > 0$.

Proof. Following the same way in Lemma 3, we suppose that the assertion of the Lemma 5 is false, that is there exists $t_1 > 0$ such that

$$S_{0h}(t) < S_{bh}(t),\ \forall\, 0 < t < t_1, \quad S_{0h}(t_1) = S_{bh}(t_1). \tag{24}$$

If we define

$$w(x,t) = u_{bh}(x,t) - u_{0h}(x,t),\ 0 \le x \le S_{0h}(t),\ 0 < t < t_1$$

we have the following properties:

$$\begin{cases} (i)\ w_t - w_{xx} = 0,\ 0 < x < S_{0h}(t), 0 < t < t_1 \\ (ii)\ w_x(0,t) = u_{bh_x}(0,t) - u_{0h_x}(0,t) = \frac{hb_0}{k}\left[u_{bh}(0,t) - u_{00}(0,t)\right],\ 0 < t < t_1 \\ (iii)\ w(S_{0h}(t),t) = u_{bh}(S_{0h}(t),t) \ge 0, 0 < t < t_1 \\ (iv)\ w_x(S_{0h}(t),t) = u_{bh_x}(S_{0h}(t),t) + \dot{S}_{0h}(t), 0 < t < t_1 \\ (v)\ S_{0h}(0) = 0 \end{cases} \tag{25}$$

and

$$w(S_{0h}(t_1), t_1) = 0.$$

If $w(0,t) = u_{bh}(0,t) - u_{0h}(0,t) < 0, 0 < t < t_1$ we deduce that the minimum is attained on $x = 0$ and $w_x(0,t) \ge 0$ which is a contradiction by using the condition (25)(ii). Then $w(0,t) \ge 0, 0 < t < t_1$. Therefore we have a minimum value $w(S_0(t_1), t_1) = 0$ and then we get $w_x(S_{0h}(t_1), t_1) < 0$. In other hand we have

$$w_x(S_{0h}(t_1), t_1) = u_{bh_x}(S_{0h}(t_1), t_1) - u_{0h_x}(S_{0h}(t_1), t_1) = -\dot{S}_{bh}(t_1) + \dot{S}_{0h}(t_1) \ge 0$$

which is a contradiction. □

Lemma 6 *We have*

$$u_{bh}(x,t) \ge u_{0h}(x,t),\ \textit{for all}\ 0 < x < S_{0h}(t),\ t > 0. \tag{26}$$

Proof. If we consider

$$w(x,t) = u_{bh}(x,t) - u_{0h}(x,t),\ \ 0 \le x \le S_{0h}(t),\ t > 0$$

then w satisfies the following properties:

$$\begin{cases} (i)\ w_t - w_{xx} = 0, 0 < x < S_{0h}(t),\ t > 0 \\ (ii)\ w_x(0,t) = u_{bh_x}(0,t) - u_{0h_x}(0,t) = \frac{hb_0}{k}\left[u_{bh}(0,t) - u_{0h}(0,t)\right] > 0,\ t > 0 \\ (iii)\ w(S_{0h}(t),t) = u_{bh}(S_{0h}(t),t) \ge 0,\ t > 0 \\ (iv)\ w_x(S_{0h}(t),t) = -u_x(S_{0h}(t),t) + \dot{S}_{0h}(t),\ t > 0 \\ (vi)\ S_{0h}(0) = 0. \end{cases} \tag{27}$$

By the maximum principle we obtain that $w(x,t) \ge 0$ and attains its minimum on the parabolic boundary. Then we get (26).

□

Lemma 7 *We have*

$$\lim_{b \to 0^+} S_{bh}(t) = S_{0h}(t) \tag{28}$$

for each t *belongs to a compact set in* $\mathbb{R}^+$, *with the following order of convergence given by:*

$$0 \le S_{bh}(t) - S_{0h}(t) \le \frac{b}{b_0} + \int_0^{\frac{b}{b_0}} \chi(x)dx,\ \ h > 0, t > 0 \tag{29}$$

Proof. According to Lemma 2 it follows

$$S_{bh}(t)-S_{0h}(t)=\frac{b}{b_0}+\int_0^{\frac{b}{b_0}}\chi(x)dx+\int_0^{S_{0h}(t)}u_{0h}(x,t)dx-\int_0^{S_{bh}(t)}u_{bh}(x,t)dx$$

$$-\frac{b_0h}{k}\int_0^t\left[u_{bh}(0,\tau)-u_{0h}(0,\tau)\right]d\tau$$

$$=\frac{b}{b_0}+\int_0^{\frac{b}{b_0}}\chi(x)dx+\int_0^{S_{bh}(t)}\left[\widetilde{u}_{0h}(x,t)-u_{bh}(x,t)\right]dx+\frac{hb_0}{k}\int_0^t\left[u_{0h}(0,\tau)-u_{bh}(0,\tau)\right]d\tau$$

where $\widetilde{u}_{0h}$ is defined as an extension of u_{0h} by 0 as follows:

$$\widetilde{u}_{0h}(x,t)=\begin{cases} u_{0h}(x,t),\ 0\le x\le S_{0h}(t),\ t>0\\ 0,\ \ S_{0h}(t)<x\le S_{bh}(t),\ t>0.\end{cases}$$

Since $\widetilde{u}_{0h}(x,t)-u_{bh}(x,t)\le 0$ and $u_{0h}(0,t)-u_{bh}(0,t)\le 0$ the thesis holds. □

Theorem 1 *We have*

$$\lim_{b\to 0^+}u_{bh}(x,t)=u_{0h}(x,t),$$

for all compact set in the domain $0<x<S_{0h}(t),\ t>0$ *and the following estimation holds*

$$0\le\int_0^{S_{0h}(t)}\left[u_{bh}(x,t)-u_{0h}(x,t)\right]\ dx\le\frac{b}{b_0}+\frac{b}{b_0}\left\|\chi\right\|.$$

Proof. Taking into account Lemma 2 we have

$$\int_0^{S_{0h}(t)}\left[u_{bh}(x,t)-u_{0h}(x,t)\right]\ dx=\frac{b}{b_0}+\int_0^{\frac{b}{b_0}}\chi(x)dx-\int_{S_{0h}(t)}^{S_{bh}(t)}u_{bh}(x,t)dx$$

$$-\frac{hb_0}{k}\int_0^t\left[u_{bh}(0,\tau)-u_0(0,\tau)\right]d\tau-\left[S_{bh}(t)-S_{0h}(t)\right].$$

By using, Lemmas 1, 3 and 4, we have

$$0\le\int_0^{S_{0h}(t)}\left[u_{bh}(x,t)-u_{0h}(x,t)\right]\ dx\le\frac{b}{b_0}+\int_0^{\frac{b}{b_0}}\chi(x)\ dx$$

$$\le\frac{b}{b_0}+\frac{b}{b_0}\left\|\chi\right\| \tag{30}$$

and the thesis holds. □

4 ASYMPTOTIC BEHAVIOR OF THE SOLUTION $(u_{bh},\ S_{bh}\ )$ WHEN $(b,h)\to(0^+,0^+)$.

In this section we will prove the convergence of u_{bh} and S_{bh} when $(b,h)\to$ to $(0^+,0^+)$.

Lemma 8 *If* $\int_0^{+\infty}F(\tau)d\tau<\infty$ *then we have the following limit:*

$$\lim_{(b,h)\to(0^+,0^+)}S_{bh}(t)=0 \tag{31}$$

for each t *in a compact set in* $\mathbb{R}^+$*, with the following order of convergence given by:*

$$0\le S_{bh}(t)\le\frac{b}{b_0}(1+\|\chi\|)+\frac{hb_0}{k}\int_0^t F(\tau)d\tau. \tag{32}$$

Proof. Taking into account (14) and Lemma 2, we have:

$$0 \leq S_{bh}(t) = \frac{b}{b_0} + \int_0^{\frac{b}{b_0}} \chi(x)dx - \frac{hb_0}{k}\int_0^t \left[u_{bh}(0,\tau) - F(\tau)\right] d\tau - \int_0^{S_{bh}(t)} u_{bh}(x,t)dx$$

$$\leq \frac{b}{b_0} + \int_0^{\frac{b}{b_0}} \chi(x)dx + \frac{hb_0}{k}\int_0^t F(\tau)d\tau.$$

If $\int_0^{+\infty} F(\tau)d\tau < \infty$ then the thesis holds. □

Theorem 2 *If* $\int_0^{+\infty} F(\tau)d\tau < \infty$ *then we have*

$$\lim_{(b,h)\to(0^+,0^+)} u_{bh}(x,t) = 0,$$

and the following order of convergence

$$0 \leq u_{bh}(x,t) \leq \frac{b}{b_0\sqrt{\pi t}}\|\chi\| + \frac{2hb_0}{k\sqrt{\pi}}\|F\|\sqrt{t},\ \ t > 0.$$

Proof. We consider the integral representation of $u_{b,h}$ given by (6). Taking into account Lemma 2 we have:

$$0 \leq u_{bh}(x,t) \leq \int_0^{\frac{b}{b_0}} N(x,t;\xi,0)\chi(\xi)d\xi + \frac{hb_0}{k}\int_0^t N(x,t;0,\tau)F(\tau)d\tau, 0 < x < S_{bh}(t), t > 0$$

Moreover

$$N(x,t;\xi,0) = K\left(x,t,\xi,0\right) + K\left(-x,t,\xi,0\right)$$

$$= \frac{1}{2\sqrt{\pi t}}\exp\left(-\frac{(x-\xi)^2}{4t}\right) + \frac{1}{2\sqrt{\pi t}}\exp\left(-\frac{(x+\xi)^2}{4t}\right) \leq \frac{1}{\sqrt{\pi t}}$$

and

$$N(x,t;0,\tau) = K\left(x,t,0,\tau\right) + K\left(-x,t,0,\tau\right)$$

$$= \frac{1}{\sqrt{\pi(t-\tau)}}\exp\left(-\frac{x^2}{4(t-\tau)}\right).$$

Then

$$\int_0^{\frac{b}{b_0}} N(x,t;\xi,0)\chi(\xi)d\xi \leq \frac{b}{b_0\sqrt{\pi t}}\|\chi\|$$

and

$$\frac{hb_0}{k}\int_0^t N(x,t;0,\tau)F(\tau)d\tau \leq \frac{2hb_0}{k\sqrt{\pi}}\|F\|\sqrt{t}.$$

Therefore

$$0 \leq u_{bh}(x,t) \leq \frac{b}{b_0\sqrt{\pi t}}\|\chi\| + \frac{2hb_0}{k\sqrt{\pi}}\|F\|\sqrt{t}$$

and

$$u_{bh}(x,t) \to 0,\ when\,(b,h) \to (0^+,0^+).$$

□

5 Conclusions

The asymptotic behavior of the solution to the Stefan problem with a convective boundary condition at the fixed face when the heat transfer coefficient (proportional to the Biot number) and the initial position of the free boundary go to zero has been obtained with an order of convergence.

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