Z. Angew. Math. Phys. (2022) 73:151

0044-2275/22/040001-17
published online July 2, 2022
https://doi.org/10.1007/s00033-022-01789-7

Zeitschrift für angewandte Mathematik und Physik ZAMP

A class of elliptic mixed boundary value problems with (p,q)-Laplacian: existence, comparison and optimal control

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Abstract. The paper deals with two nonlinear elliptic equations with (p,q)-Laplacian and the Dirichlet–Neumann–Dirichlet (DND) boundary conditions, and Dirichlet–Neumann–Neumann (DNN) boundary conditions, respectively. Under mild hypotheses, we prove the unique weak solvability of the elliptic mixed boundary value problems. Then, a comparison and a monotonicity results for the solutions of elliptic mixed boundary value problems are established. We examine a convergence result which shows that the solution of (DND) can be approached by the solution of (DNN). Moreover, two optimal control problems governed by (DND) and (DNN), respectively, are considered, and an existence result for optimal control problems is obtained. Finally, we provide a result on asymptotic behavior of optimal controls and system states, when a parameter tends to infinity.

Mathematics Subject Classification. 35J25, 35J66, 35J92, 49J20, 35Bxx.

Keywords. Mixed boundary value problem, Optimal control, (p, q)-Laplacian, Comparison, Sensitivity, Asymptotic behavior.

1. Introduction

Let Ω be a bounded domain in $\mathbb{R}^N$ $(N \geq 2)$ with a Lipschitz boundary $\Gamma := \partial\Omega$ which is divided into three measurable and mutually disjoint parts Γ_1, Γ_2, and Γ_3 such that Γ_1 is of positive measure. Let $1 < q < p < +\infty$, α, β, $\mu > 0$, $b > 0$ and $\theta < p^*$, where p^* is the critical exponent to p (see (2.1) in Sect. 2). Given functions $g\colon \Omega \to \mathbb{R}$, $r\colon \Gamma_2 \to \mathbb{R}$ and $l\colon \Gamma_3 \times \mathbb{R} \to \mathbb{R}$, in the paper we are interested in the investigation of the following mixed boundary value problems involving (p,q)-Laplacian operator:

Problem 1. *Find $u\colon \Omega \to \mathbb{R}$ such that*

$$\begin{aligned} & -\Delta_p u(x) - \mu\Delta_q u(x) + \beta|u(x)|^{\theta-2}u(x) = g(x) && \text{in } \Omega,\\ & u = 0 && \text{on } \Gamma_1,\\ & -\frac{\partial_{(p,q)}u}{\partial\nu} = r(x) && \text{on } \Gamma_2,\\ & u = b && \text{on } \Gamma_3, \end{aligned}$$

and

Problem 2. *Find $u\colon \Omega \to \mathbb{R}$ such that*

$$\begin{aligned} & -\Delta_p u(x) - \mu\Delta_q u(x) + \beta|u(x)|^{\theta-2}u(x) = g(x) && \text{in } \Omega,\\ & u = 0 && \text{on } \Gamma_1,\\ & -\frac{\partial_{(p,q)}u}{\partial\nu} = r(x) && \text{on } \Gamma_2,\\ & -\frac{\partial_{(p,q)}u}{\partial\nu} = \alpha\, l(x,u(x)) && \text{on } \Gamma_3, \end{aligned}$$

where Δ_p *denotes the* p*-Laplace differential operator of the form*

$$\Delta_p u = \mathrm{div}(|\nabla u|^{p-2}\nabla u) \quad \textit{for all} \quad u \in W^{1,p}(\Omega),$$

ν *is the outward unit normal at the boundary* Γ*, and*

$$\frac{\partial_{(p,q)} u}{\partial \nu} := \left(|\nabla u|^{p-2}\nabla u + \mu |\nabla u|^{q-2}\nabla u, \nu\right)_{\mathbb{R}^N}.$$

The weak solutions to Problems 1 and 2 are understood as follows.

Definition 3. We say that
(i) a function $u\colon \Omega \to \mathbb{R}$ is a weak solution of Problem 1, if $u \in W^{1,p}(\Omega)$ is such that $u = 0$ on Γ_1, $u = b$ on Γ_3 and

$$\int_\Omega \left(|\nabla u(x)|^{p-2}\nabla u(x) + \mu|\nabla u(x)|^{q-2}\nabla u(x), \nabla v(x)\right)_{\mathbb{R}^N} \,\mathrm{d}x + \int_\Omega \beta |u(x)|^{\theta-2}u(x)v(x)\,\mathrm{d}x = \int_\Omega g(x)v(x)\,\mathrm{d}x - \int_{\Gamma_2} r(x)v(x)\,\mathrm{d}\Gamma$$

for all $v \in W^{1,p}(\Omega)$ with $v = 0$ on $\Gamma_1 \cup \Gamma_3$,
(ii) a function $u\colon \Omega \to \mathbb{R}$ is a weak solution of Problem 2, if $u \in W^{1,p}(\Omega)$ satisfies $u = 0$ on Γ_1 and

$$\int_\Omega \left(|\nabla u(x)|^{p-2}\nabla u(x) + \mu|\nabla u(x)|^{q-2}\nabla u(x), \nabla v(x)\right)_{\mathbb{R}^N} \,\mathrm{d}x + \int_\Omega \beta|u(x)|^{\theta-2}u(x)v(x)\,\mathrm{d}x + \alpha\int_{\Gamma_3} l(x,u(x))v(x)\,\mathrm{d}\Gamma = \int_\Omega g(x)v(x)\,\mathrm{d}x - \int_{\Gamma_2} r(x)v(x)\,\mathrm{d}\Gamma$$

for all $v \in W^{1,p}(\Omega)$ with $v = 0$ on Γ_1.

The main feature of our research contains two perspectives. First, we deal with problems with mixed boundary value conditions and (p,q)-Laplacian operator. Note that (p,q)-Laplace operator with $1 < q < p$ is the sum of a p-Laplacian and a q-Laplacian, so the energy functional $I(u)$ corresponding to the (p,q)-Laplace operator defined by

$$I(u) := \int_\Omega \left(\frac{\|\nabla u\|^p}{p} + \frac{\|\nabla u\|^q}{q}\right)\mathrm{d}x \quad \text{for all} \quad u \in W^{1,p}(\Omega),$$

is mainly controlled by the exponent q if $u \in B_1(0) := \{u \in W^{1,p}(\Omega) \mid \|\nabla u\|_{L^p(\Omega;\mathbb{R}^N)} \le 1\}$, or by the exponent p when $u \in W^{1,p}(\Omega) \setminus B_1(0)$. This structure impels the huge potential applications of (p,q)-Laplacian in diverse fields, for instance, it can used to describe exactly the geometry of composites made of two different materials with distinct power hardening exponents. Second perspective concerns applications in which mixed boundary value problems are a powerful mathematical tool. They have been widely applied to explain various complicated natural phenomena and to solve a lot of engineering problems, for instance, contact mechanics problems, semipermeability problems, and free boundary problems. The research of mixed boundary value problems with or without (p,q)-Laplacian can be found in Alves et al. [1], Axelsson et al. [2], Bai et al. [3], Barboteu et al. [4], Zeng et al. [36], Duvaut and Lions [8], Figueiredo [9], Gasiński and Papageorgiou [11], Gasiński and Winkert [12], Han [13], Liu et al. [16,17], Maz'ya and Rossmann [19], Zeng et al. [34], Migórski et al. [22,23], Mihailescu and Rădulescu [25], Mitrea [26], Papageorgiou et al. [29], Liu and Papageorgiou [18], Papageorgiou et al. [27,28] and Yu and Feng [32]. Results on convergence of optimal solutions in optimal control problems can be found in Denkowski and Migórski [5], Gariboldi and Tarzia [10], Denkowski and Mortola [7], Zeng et al. [37], Migórski [20,21], Denkowski et al. [6, Section 4.2], Liu et al. [15], Zeng et al. [35] and the references therein.

The purpose of this paper is fourfold. The first goal is to prove the unique weak solvability of Problems 1 and 2 by applying a surjectivity theorem for pseudomonotone operators. The second purpose is to establish a comparison principle and a monotonicity result for solutions of Problems 1 and 2. The third aim is to deliver a convergence result which shows that the solution of Problem 1 can be approached by the solution of Problem 2, as $\alpha \to \infty$. Moreover, our last intention is to investigate two optimal control problems, Problems 9 and 10, and to examine the asymptotic behavior of optimal solutions (i.e., control-state pairs) and of minimal values for Problem 10, when parameter α in the boundary condition, representing for instance a heat transfer coefficient, tends to infinity.

The rest of the paper is organized as follows. In Sect. 2, we recall basic notation and collect the necessary preliminary material. Section 3 is devoted to the proof of existence and uniqueness of solutions to Problems 1 and 2, and to discuss a comparison principle as well as a convergence result to Problems1 and 2. Finally, in Sect. 4, we introduce two optimal control problems governed by Problems 1 and Problem 2, respectively, and explore the asymptotic behavior of the optimal controls and system states to Problem 10.

2. Mathematical background

In this section, we review some basic notation, definitions and the necessary preliminary material, which will be used in next sections. More details can be found, for instance, in [6,24,30,31].

Let $(Y, \|\cdot\|_Y)$ be a Banach space and Y^* stand for the dual space to Y. We denote by $\langle\cdot,\cdot\rangle$ the duality brackets for the pair of Y^* and Y. Everywhere below, the symbols $\xrightarrow{w}$ and $\to$ represent the weak and strong convergences, respectively. We say that a mapping $F\colon Y \to Y^*$ is

(i) monotone, if
$$\langle Fu - Fv, u - v\rangle \geq 0 \ \text{ for all } \ u, v \in Y,$$
(ii) strictly monotone, if
$$\langle Fu - Fv, u - v\rangle > 0 \ \text{ for all } \ u, v \in Y \text{ with } u \neq v,$$
(iii) of type $(S)_+$ (or F satisfies (S_+)–property), if for any sequence $\{u_n\} \subset Y$ with $u_n \xrightarrow{w} u$ in Y as $n \to \infty$ for some $u \in Y$ and $\limsup_{n\to\infty}\langle Fu_n, u_n - u\rangle \leq 0$, then the sequence $\{u_n\}$ converges strongly to u in Y,

(iv) pseudomonotone, if it is bounded and for every sequence $\{u_n\} \subseteq Y$ converging weakly to $u \in Y$ with $\limsup_{n\to\infty}\langle Fu_n, u_n - u\rangle \leq 0$, then
$$\langle Fu, u - v\rangle \leq \liminf_{n\to\infty}\langle Fu_n, u_n - v\rangle \ \text{ for all } \ v \in Y,$$
(v) coercive, if
$$\lim_{\|v\|_Y\to\infty} \frac{\langle Fv, v\rangle}{\|v\|_Y} = +\infty.$$

It is not difficult to see that if F is of type $(S)_+$, then F is pseudomonotone as well. Note that the operator $F\colon Y \to Y^*$ is pseudomonotone if and only if it is bounded and $y_n \to y$ weakly in Y with $\limsup_{n\to\infty}\langle Fy_n, y_n - y\rangle \leq 0$ entails $\lim_{n\to\infty}\langle Fy_n, y_n - y\rangle = 0$ and $Fy_n \to Fy$ weakly in Y^*. Furthermore, if $F \in \mathcal{L}(Y, Y^*)$ (the class of linear and bounded operators) is nonnegative, then it is pseudomonotone.

Theorem 4. *Let Y be a Banach space, and F, $G\colon Y \to Y^*$. Then, we have*

(i) *if F is bounded, hemicontinuous, and monotone, then F is pseudomonotone,*

(ii) *if F and G are pseudomonotone, then $F + G$ is also pseudomonotone.*

The class of pseudomonotone and coercive operators enjoys the well-known surjectivity property.

Theorem 5. *Let Y be a Banach space and $F\colon Y \to Y^*$ be pseudomonotone and coercive. Then F is surjective, i.e., for any $f \in Y^*$, there is at least one solution to the equation $Fu = f$.*

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain such that its Lipschitz boundary $\Gamma = \partial\Omega$ is divided into three measurable and mutually disjoint parts Γ_1, Γ_2, and Γ_3, and Γ_1 has a positive measure. Let $1 < p < +\infty$ and $p' > 1$ be the conjugate exponent of p, i.e., $\frac{1}{p} + \frac{1}{p'} = 1$. In the sequel, we denote by p^* the critical exponent to p given by

$$p^* = \begin{cases} \frac{Np}{N-p} & \text{if } \ p < N, \\ +\infty & \text{if } \ p \geq N. \end{cases} \tag{2.1}$$

Throughout the paper, the norms of the Lebesgue space $L^p(\Omega)$ and Sobolev space $W^{1,p}(\Omega)$ are defined by

$$\|u\|_{L^p(\Omega)} := \left(\int\limits_\Omega |u(x)|^p \, \mathrm{d}x \right)^{\frac{1}{p}} \quad \text{for all } \ u \in L^p(\Omega),$$

and

$$\|u\|_{W^{1,p}(\Omega)} := \left(\|u\|^p_{L^p(\Omega)} + \|\nabla u\|^p_{L^p(\Omega;\mathbb{R}^N)} \right)^{1/p} \quad \text{for all } \ u \in W^{1,p}(\Omega),$$

respectively. We introduce a subspace V of $W^{1,p}(\Omega)$ given by

$$V := \{ u \in W^{1,p}(\Omega) \ | \ u = 0 \text{ on } \Gamma_1 \}.$$

From the fact that Γ_1 has a positive measure and by the Poincaré inequality, it follows that V endowed with the norm

$$\|u\|_V := \left(\int\limits_\Omega |\nabla u|^p \, \mathrm{d}x \right)^{\frac{1}{p}} \quad \text{for all } \ u \in V$$

is a reflexive Banach space. Further, we consider the subsets K and K_0 of V defined by

$$K := \{ u \in V \mid u = b \ \text{ on } \ \Gamma_3 \}, \tag{2.2}$$

$$K_0 := \{ u \in V \mid u = 0 \ \text{ on } \ \Gamma_3 \}, \tag{2.3}$$

respectively, where $b > 0$ is given in Problem 1.

We end the section with the nonlinear operator $A\colon V \to V^*$ defined by

$$\langle Au, v \rangle = \int\limits_\Omega \left(|\nabla u(x)|^{p-2} \nabla u(x) + \mu \, |\nabla u(x)|^{q-2} \nabla u(x), \nabla v(x) \right)_{\mathbb{R}^N} \, \mathrm{d}x \tag{2.4}$$

for all u, $v \in V$. The following result summarizes the main properties of this map (see, e.g., [14, Chapter 3, Example 1.7, p. 303]).

Proposition 6. *Let $\mu > 0$ and $1 < q < p < +\infty$. Then, the operator $A\colon V \to V^*$ defined by (2.4) is bounded, continuous, strictly monotone (hence maximal monotone) and of type (S_+).*

3. Existence, uniqueness and convergence results

This section is devoted to study the unique solvability of Problems 1 and 2. We discuss a comparison principle which reveals the essential relations between the unique weak solutions of Problems 1 and 2 as well as the constant $b > 0$. We also establish a monotonicity property of solution to Problem 2 with respect to the parameter α, and obtain a convergence result which shows that the unique solution to Problem 1 can be approached by the unique solution to Problem 2 when the parameter α tends to infinity.

Let us consider the nonlinear operators B, $L\colon V \to V^*$ defined by

$$\langle Bu, v\rangle := \int_\Omega \beta|u(x)|^{\theta-2}u(x)v(x)\,\mathrm{d}x \quad \text{for all} \quad u, v \in V, \tag{3.1}$$

and

$$\langle Lu, v\rangle := \int_{\Gamma_3} l(x, u(x))v(x)\,\mathrm{d}\Gamma \quad \text{for all} \quad u, v \in V, \tag{3.2}$$

respectively. By the Riesz representation theorem, we introduce the function $f \in V^*$ defined by

$$\langle f, v\rangle = \int_\Omega g(x)v(x)\,\mathrm{d}x - \int_{\Gamma_2} r(x)v(x)\,\mathrm{d}\Gamma \quad \text{for all} \quad v \in V. \tag{3.3}$$

Using the notation above, it is not difficult to see that Definition 3 can be equivalently rewritten as follows:

(i)$'$ a function $u_\alpha \in V$ is called to be a weak solution of Problem 2 associated with $\alpha > 0$, if it satisfies

$$\langle Au_\alpha + Bu_\alpha, v\rangle + \alpha\,\langle Lu_\alpha, v\rangle = \langle f, v\rangle \quad \text{for all} \quad v \in V, \tag{3.4}$$

(ii)$'$ a function $u_\infty \in K$ is called to be a weak solution of Problem 1, if

$$\langle Au_\infty + Bu_\infty, v\rangle = \langle f, v\rangle \quad \text{for all} \quad v \in K_0, \tag{3.5}$$

where K and K_0 are given by (2.2) and (2.3), respectively.

We assume that the function l in the operator L satisfies the following hypotheses.

H(l): $l\colon \Gamma_3 \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function (i.e., for all $s \in \mathbb{R}$, the function $x \mapsto l(x, s)$ is measurable, and for a.e. $x \in \Gamma_3$, $s \mapsto l(x, s)$ is continuous) such that

(i) there exist $a_l \in L_+^{p'}(\Gamma_3)$ and $b_l > 0$ satisfying

$$|l(x, s)| \le a_l(x) + b_l\,(1 + |s|^{p-1})$$

for all $s \in \mathbb{R}$ and a.e. $x \in \Gamma_3$,

(ii) for a.e. $x \in \Gamma_3$, $s \mapsto l(x, s)$ is nondecreasing, i.e., it satisfies

$$(l(x, s_1) - l(x, s_2))(s_1 - s_2) \ge 0$$

for all $s_1, s_2 \in \mathbb{R}$ and a.e. $x \in \Gamma_3$,

(iii) for a.e. $x \in \Gamma_3$, $l(x, s) = 0$ if and only if $s = b$.

We next give a concrete example for function l which satisfies hypotheses $H(l)$.

Example 7. Let $b > 0$ be given in Problem 1 and $\mathrm{sgn}\colon \mathbb{R} \to \{-1, 0, 1\}$ be the sign function, namely,

$$\mathrm{sgn}(s) := \begin{cases} 1 & \text{if} s > 0, \\ 0 & \text{if} s = 0, \\ -1 & \text{if} s < 0. \end{cases}$$

Also, let $1 < p < +\infty$ and $\omega \in L^\infty(\Gamma_3)$, $\omega > 0$. Then, the function $l\colon \Gamma_3 \times \mathbb{R} \to \mathbb{R}$ defined by

$$l(x, s) = \omega(x)\,\mathrm{sgn}(s - b)\,|s - b|^{p-1} \quad \text{for all} \quad s \in \mathbb{R} \quad \text{and} \quad x \in \Gamma_3,$$

satisfies hypotheses $H(l)$.

Besides, we need the following assumption.

H(0): $g \in L^{p'}(\Omega)$ with $g \le 0$ in Ω, $r \in L^{p'}(\Gamma_2)$ with $r \ge 0$ on Γ_2, and $b > 0$.

The main results on existence, uniqueness, comparison, monotonicity and convergence to Problem 2 are provided in the following theorem.

Theorem 8. *Assume that $H(l)$ and $H(0)$ are fulfilled. Then, we have*

(i) *Problem 1 has a unique solution* $u_\infty \in K$,
(ii) *for every* $\alpha > 0$, *Problem 2 has a unique solution* $u_\alpha \in V$,
(iii) $u_\infty \le b$ *in* Ω,
(iv) *for every* $\alpha > 0$, *it holds* $u_\alpha \le b$ *in* Ω *and* $u_\alpha \le b$ *on* Γ_3,
(v) *for every* $\alpha > 0$, *it holds* $u_\alpha \le u_\infty$ *in* Ω,
(vi) *if* $0 < \alpha_1 \le \alpha_2$, *then* $u_{\alpha_1} \le u_{\alpha_2}$ *in* Ω,
(vii) *if a sequence* $\{\alpha_n\}$ *is such that* $\alpha_n > 0$ *for all* $n \in \mathbb{N}$ *with* $\alpha_n \to \infty$ *as* $n \to \infty$, *then* $u_{\alpha_n} \to u_\infty$ *in* V *as* $n \to \infty$.

Proof. (i) It is a direct consequence of [33, Lemma 6].

(ii) From (3.4), we can observe that $u \in V$ is a weak solution to Problem 2 if and only if it solves the following abstract operator equation: find $u \in V$ such that

$$Au + Bu + \alpha Lu = f \quad \text{in} \quad V^*. \tag{3.6}$$

By Proposition 6, we know that A is a bounded, continuous, strictly monotone (hence maximal monotone) operator, and of type (S_+). Also, we can obtain

$$\|Au\|_{V^*} \le \|u\|_V^{p-1} + \mu \|\nabla u\|^{q-1}_{L^{(q-1)p'}(\Omega;\mathbb{R}^N)} \quad \text{for all} \quad u \in V. \tag{3.7}$$

Employing [24, Theorem 3.69], we deduce that A is a pseudomonotone operator. As concerns operator B, it is monotone and continuous, and satisfies

$$\|Bu\|_{V^*} \le c_1 \|u\|_V^{\theta-1} \quad \text{for all} \quad u \in V \tag{3.8}$$

with some $c_1 > 0$. The latter combined with the compactness of the embedding of V to $L^\theta(\Omega)$ (due to $\theta < p^*$) implies that B is completely continuous, so, it is also pseudomonotone. For any $u \in V$, from hypotheses $H(l)$ and the Hölder inequality, we have

$$\begin{aligned}
\|Lu\|_{L^{p'}(\Omega)} &= \sup_{v \in L^p(\Gamma_3), \|v\|_{L^p(\Gamma_3)}=1} \langle Lu, v\rangle_{L^{p'}(\Gamma_3)\times L^p(\Gamma_3)} \\
&\le \sup_{v \in L^p(\Gamma_3), \|v\|_{L^p(\Gamma_3)}=1} \int_{\Gamma_3} |l(x,u(x))v(x)| \,\mathrm{d}\Gamma \\
&\le \sup_{v \in L^p(\Gamma_3), \|v\|_{L^p(\Gamma_3)}=1} \int_{\Gamma_3} (a_l(x) + b_l(1+|u(x)|^{p-1}))|v(x)| \,\mathrm{d}\Gamma \\
&\le \sup_{v \in L^p(\Gamma_3), \|v\|_{L^p(\Gamma_3)}=1} \left(\|a_l\|_{L^{p'}(\Gamma_3)} + b_l|\Gamma_3|^{\frac{1}{p'}} + b_1\|u\|^{p-1}_{L^p(\Gamma_3)} \right) \|v\|_{L^p(\Gamma_3)} \\
&\le \|a_l\|_{L^{p'}(\Gamma_3)} + b_l\,|\Gamma_3|^{\frac{1}{p'}} + b_1\,\|u\|^{p-1}_{L^p(\Gamma_3)}.
\end{aligned}$$

Hence, $L: V \to V^*$ is well-defined. From the compactness of the trace operator $\gamma: V \to L^p(\Gamma_3)$ and the definition of L, we can also see that L is continuous. Besides, we use condition $H(l)$(ii) to infer that L is monotone, that is,

$$\langle Lu - Lv, u - v\rangle = \int_{\Gamma_3} (l(x,u(x)) - l(x,v(x)))(u(x) - v(x)) \,\mathrm{d}\Gamma \ge 0$$

for all u, $v \in V$. This together with [24, Theorem 3.69] implies that L is a pseudomonotone operator. Therefore, by using Theorem 4(ii), we infer that $A + B + \alpha L: V \to V^*$ is pseudomonotone.

Next, let $\varepsilon > 0$ be arbitrary. From hypothesis $H(l)$, we get the estimate

$$\langle Lu, u\rangle = \int_{\Gamma_3} l(x,u(x))u(x) \,\mathrm{d}\Gamma = \int_{\Gamma_3} l(x,u(x))(u(x) - b) \,\mathrm{d}\Gamma + \int_{\Gamma_3} l(x,u(x))b \,\mathrm{d}\Gamma$$

$$\geq \int_{\Gamma_3} l(x,b)(u(x)-b)\,\mathrm{d}\Gamma + \int_{\Gamma_3} l(x,u(x))b\,\mathrm{d}\Gamma$$
$$\geq -\int_{\Gamma_3} (a_l(x)+b_l(1+|u(x)|^{p-1}))b\,\mathrm{d}\Gamma$$
$$\geq -b\,|\Gamma_3|^{\frac{1}{p}}\,\|a_l\|_{L^{p'}(\Gamma_3)} - b_l\,b\,|\Gamma_3| - b_l\,b\int_{\Gamma_3}|u(x)|^{p-1}\,\mathrm{d}\Gamma$$
$$\geq -b\,|\Gamma_3|^{\frac{1}{p}}\,\|a_l\|_{L^{p'}(\Gamma_3)} - b_l\,b\,|\Gamma_3| - \varepsilon\,\|u\|^p_{L^p(\Gamma_3)} - c(\varepsilon)$$

with some $c(\varepsilon) > 0$, where the last inequality is obtained by using the Young inequality. From the estimates above and definitions of A and B, we obtain

$$\begin{aligned}
&\langle Au + Bu + \alpha Lu, u\rangle \\
&\quad\geq \|u\|_V^p + \mu\|\nabla u\|^q_{L^q(\Omega;\mathbb{R}^N)} + \beta\|u\|^\theta_{L^\theta(\Omega)} - \alpha\,b\,|\Gamma_3|^{\frac{1}{p}}\,\|a_l\|_{L^{p'}(\Gamma_3)} - \alpha\,b_l\,b\,|\Gamma_3| \\
&\quad\quad -\varepsilon\,\alpha\,\|u\|^p_{L^p(\Gamma_3)} - \alpha\,c(\varepsilon) \\
&\quad\geq \|u\|_V^p + \mu\|\nabla u\|^q_{L^q(\Omega;\mathbb{R}^N)} + \beta\|u\|^\theta_{L^\theta(\Omega)} - \alpha\,b\,|\Gamma_3|^{\frac{1}{p}}\,\|a_l\|_{L^{p'}(\Gamma_3)} - \alpha\,b_l\,b\,|\Gamma_3| \\
&\quad\quad -\varepsilon\,\alpha\,c_V^p\,\|u\|_V^p - \alpha\,c(\varepsilon)
\end{aligned} \tag{3.9}$$

for all $u \in V$. We set $\varepsilon = \frac{1}{2\alpha c_V^p}$, where $c_V > 0$ is the constant for the embedding of V into $L^p(\Gamma_3)$, that is, $\|v\|_{L^p(\Gamma_3)} \leq c_V\|v\|_V$ for $v \in V$. Then, because of $p > 1$, we conclude that $A + B + \alpha L$ is coercive. Therefore, all conditions of Theorem 5 are verified. Using this theorem, we deduce that Problem 2 has at least one solution. Furthermore, the strict monotonicity of A allows us to apply a standard method to show that Problem 2 has a unique solution $u_\alpha \in V$.

(iii) Let $u_\infty \in K$ be the unique solution of Problem 1. We set $w = (u_\infty - b)^+$. Then, one has $u_\infty = b$ on Γ_3 and $w = 0$ on Γ_3. Thus, $w \in K_0$. We take $v = w$ in (3.5) to get

$$\int_\Omega \left(|\nabla u_\infty|^{p-2}\nabla u_\infty + \mu|\nabla u_\infty|^{q-2}\nabla u_\infty, \nabla(u_\infty - b)^+\right)_{\mathbb{R}^N}\,\mathrm{d}x$$
$$+\int_\Omega \beta|u_\infty(x)|^{\theta-2}u_\infty(x)(u_\infty(x)-b)^+\,\mathrm{d}x = \langle Au_\infty + Bu_\infty, w\rangle = \langle f, w\rangle$$
$$= \int_\Omega g(x)(u_\infty(x)-b)^+\,\mathrm{d}x - \int_{\Gamma_2} r(x)(u_\infty(x)-b)^+\,\mathrm{d}\Gamma.$$

From condition $H(0)$, we deduce

$$\int_\Omega g(x)(u_\infty(x)-b)^+\,\mathrm{d}x - \int_{\Gamma_2} r(x)(u_\infty(x)-b)^+\,\mathrm{d}\Gamma \leq 0,$$

while the monotonicity of B and nonnegativity of b guarantee that

$$\int_\Omega \beta|u_\infty(x)|^{\theta-2}u_\infty(x)(u_\infty(x)-b)^+\,\mathrm{d}x \geq 0.$$

Taking into account the last two inequalities and the fact $\nabla b = 0$, we have

$$\langle Au_\infty - Ab, w\rangle \leq 0.$$

The latter combined with the strict monotonicity of A implies that $w = 0$. This means that $u_\infty \leq b$ in Ω.

(iv) Let $u_\alpha \in V$ be the unique solution of Problem 2 corresponding to $\alpha > 0$. We put $w = (u_\alpha - b)^+$. Inserting $v = w$ into (3.4), it yields

$$\langle Au_\alpha + Bu_\alpha, w\rangle + \alpha\langle Lu_\alpha, w\rangle = \langle f, w\rangle. \tag{3.10}$$

Hence, we have

$$\begin{aligned}\langle Au_\alpha, w\rangle &\le -\alpha\langle Lu_\alpha, w\rangle = -\alpha\int_{\Gamma_3} l(x,u_\alpha(x))(u_\alpha(x)-b)^+\,\mathrm{d}\Gamma\\ &= -\alpha\int_{\{u_\alpha>b\}\cap\Gamma_3} l(x,u_\alpha(x))(u_\alpha(x)-b)\,\mathrm{d}\Gamma \le -\alpha\int_{\{u_\alpha>b\}\cap\Gamma_3} l(x,b)(u_\alpha(x)-b)\,\mathrm{d}\Gamma\\ &= 0,\end{aligned}$$

where we have used the monotonicity of the function $s \mapsto l(x,s)$ and hypothesis $H(l)$(iii), and the set $\{u_\alpha > b\}$ is defined by $\{u_\alpha > b\} := \{x \in \Gamma_3 \mid u_\alpha(x) > b\}$. Therefore, one has

$$\langle Au_\alpha - Ab, w\rangle \le 0.$$

Then, it is true that $w = 0$, i.e., $u_\alpha \le b$ in Ω.

From Eq. (3.10) and the fact $u_\alpha \le b$ in Ω, we have

$$\begin{aligned}0 \ge \alpha\,\langle Lu_\alpha, w\rangle &= \alpha\int_{\Gamma_3} l(x,u_\alpha(x))(u_\alpha(x)-b)^+\,\mathrm{d}\Gamma\\ &= \alpha\int_{\{u_\alpha>b\}\cap\Gamma_3} l(x,u_\alpha(x))(u_\alpha(x)-b)\,\mathrm{d}\Gamma.\end{aligned} \tag{3.11}$$

Using the monotonicity of $s \mapsto l(x,s)$ and hypothesis $H(l)$(iii) again, we obtain $l(x,u_\alpha(x))\,(u_\alpha(x)-b) \ge l(x,b)(u_\alpha(x)-b) = 0$ for a.e. $x \in \{u_\alpha > b\}\cap\Gamma_3$. This together with inequality (3.11) implies $l(x,u_\alpha(x)) = 0$ due to $u_\alpha(x) > b$. On the other hand, condition $H(l)$(iii) turns out $u_\alpha(x) = b$. This leads to a contradiction. Therefore, we conclude that $u_\alpha \le b$ on Γ_3 as claimed.

(v) For any $\alpha > 0$ fixed, let $u_\alpha \in V$ and $u_\infty \in K$ be the unique solutions of Problems 2 and 1, respectively. We set $w = (u_\alpha - u_\infty)^+$. From assertion (iv) it follows that $u_\alpha \le b$ on Γ_3. We use the definition of K (i.e., $u_\infty = b$ on Γ_3) to get $w = (u_\alpha - u_\infty)^+ = 0$ on Γ_3, so, $w \in K_0$. Taking $v = w$ into (3.4) and (3.5), respectively, we have

$$\langle Au_\infty + Bu_\infty, w\rangle = \langle f, w\rangle \quad\text{and}\quad \langle Au_\alpha + Bu_\alpha, w\rangle + \alpha\langle\, Lu_\alpha, w\rangle = \langle f, w\rangle.$$

Summing up the equalities above, it gives

$$\begin{aligned}&\langle Au_\alpha - Au_\infty + Bu_\alpha - Bu_\infty, w\rangle = -\alpha\,\langle Lu_\alpha, w\rangle\\ &\quad= -\alpha\int_{\Gamma_3} l(x,u_\alpha(x))(u_\alpha(x)-u_\infty(x))^+\,\mathrm{d}\Gamma = -\alpha\int_{\Gamma_3} l(x,u_\alpha(x))(u_\alpha(x)-b)^+\,\mathrm{d}\Gamma\\ &\quad= 0.\end{aligned}$$

Employing the monotonicity of A and B, we have $w = 0$, which implies $u_\alpha \le u_\infty$ in Ω.

(vi) Let $0 < \alpha_1 \le \alpha_2$ and $u_i := u_{\alpha_i}$ be the unique solution of Problem 2 associated with $\alpha = \alpha_i$ for $i = 1, 2$. Hence,

$$\langle Au_i + Bu_i, v\rangle + \alpha_i\langle Lu_i, v\rangle = \langle f, v\rangle \quad\text{for all}\quad v \in V.$$

Let $w := (u_1 - u_2)^+$. Putting $v = w$ into the equality above and summing up the resulting equations, one has

$$\langle Au_1 - Au_2 + Bu_1 - Bu_2, w\rangle = \langle \alpha_2 Lu_2 - \alpha_1 Lu_1, w\rangle = \alpha_2\left\langle Lu_2 - \frac{\alpha_1}{\alpha_2}Lu_1, w\right\rangle$$

$$= \alpha_2 \int\limits_{\Gamma_3} \left(l(x, u_2(x)) - \frac{\alpha_1}{\alpha_2} l(x, u_1(x)) \right) (u_1(x) - u_2(x))^+ \, \mathrm{d}\Gamma$$

$$= \alpha_2 \int\limits_{\{u_1 > u_2\} \cap \Gamma_3} \left(l(x, u_2(x)) - \frac{\alpha_1}{\alpha_2} l(x, u_1(x)) \right) (u_1(x) - u_2(x)) \, \mathrm{d}\Gamma. \tag{3.12}$$

Recalling that $u_i \leq b$ on Γ_3 (see assertion (iv)), we use condition $H(l)$(ii) to find that $0 = l(x, b) \geq l(x, u_i(x))$ for a.e. $x \in \Gamma_3$ and $i = 1, 2$. From hypothesis $H(l)$(ii) again and the fact $\frac{\alpha_1}{\alpha_2} \leq 1$, we get

$$\begin{aligned} -\frac{\alpha_1}{\alpha_2} \, l(x, u_1(x))(u_1(x) - u_2(x)) &\leq -\frac{\alpha_1}{\alpha_2} \, l(x, u_2(x))(u_1(x) - u_2(x)) \\ &\leq -l(x, u_2(x))(u_1(x) - u_2(x)) \text{ for a.e. } x \in \{u_1 > u_2\}. \end{aligned}$$

Inserting the inequality above into (3.12) and using hypothesis $H(l)$(ii), it yields

$$\begin{aligned} &\langle Au_1 - Au_2 + Bu_1 - Bu_2, w \rangle \\ &\leq \alpha_2 \int\limits_{\{u_1 > u_2\} \cap \Gamma_3} (l(x, u_2(x)) - l(x, u_1(x))) \, (u_1(x) - u_2(x)) \, \mathrm{d}\Gamma \leq 0. \end{aligned}$$

Therefore, we conclude that $w = 0$, and finally $u_1 \leq u_2$ in Ω.

(vii) Let $\{\alpha_n\}$ be a sequence such that $\alpha_n > 0$ for all $n \in \mathbb{N}$ and $\alpha_n \to \infty$ as $n \to \infty$. For each $n \in \mathbb{N}$, let $u_n := u_{\alpha_n}$ be the unique solution of Problem 2 corresponding to $\alpha = \alpha_n$. We claim that sequence $\{u_n\}$ is bounded in V. For each $n \in \mathbb{N}$, we have

$$\|f\|_{V^*}(\|u_n\|_V + \|u_\infty\|_V) \geq \langle f, u_n - u_\infty \rangle = \langle Au_n + Bu_n + \alpha_n L u_n, u_n - u_\infty \rangle.$$

Applying conditions $H(l)$(ii) and (iii), one finds

$$\begin{aligned} \alpha_n \langle L u_n, u_n - u_\infty \rangle &= \alpha_n \int\limits_{\Gamma_3} l(x, u_n(x))(u_n(x) - u_\infty(x)) \, \mathrm{d}\Gamma \\ &= \alpha_n \int\limits_{\Gamma_3} l(x, u_n(x))(u_n(x) - b) \, \mathrm{d}\Gamma \geq \alpha_n \int\limits_{\Gamma_3} l(x, b)(u_n(x) - b) \, \mathrm{d}\Gamma = 0. \end{aligned}$$

Then, we have

$$\|f\|_{V^*}(\|u_n\|_V + \|u_\infty\|_V) + \langle Au_n + Bu_n, u_\infty \rangle \geq \langle Au_n + Bu_n, u_n \rangle.$$

We use the Hölder inequality and the monotonicity of B to get

$$\begin{aligned} &\|u_n\|_V^p + \mu \|\nabla u_n\|_{L^q(\Omega;\mathbb{R}^N)}^q \leq \|u_n\|_V^{p-1} \|u_\infty\|_V + \beta \|u_\infty\|_{L^\theta(\Omega)}^\theta \\ &\quad + \mu \, \|\nabla u_n\|_{L^{p'(q-1)}(\Omega;\mathbb{R}^N)}^{q-1} \|u_\infty\|_V + \|f\|_{V^*} \left(\|u_n\|_V + \|u_\infty\|_V\right) + c_2 \|u_\infty\|_{L^{(\theta-1)p'}(\Omega)}^{\theta-1} \|u_n\|_V \end{aligned}$$

for some $c_2 > 0$. This reveals that sequence $\{u_n\}$ is bounded in V. Passing to the subsequence if necessary, we may assume that $u_n \xrightarrow{w} u$ in V as $n \to \infty$ with some $u \in V$. We are going to show that $u \in K$, i.e., $u = b$ on Γ_3. The boundedness of operators A, B and of sequence $\{u_n\}$ guarantee that there exists a constant $c_3 > 0$ independent of n such that

$$\langle Au_n + Bu_n - f, u_\infty - u_n \rangle \leq c_3$$

for all $n \in \mathbb{N}$. Then, for each $n \in \mathbb{N}$, we have

$$\langle L u_n, u_n - u_\infty \rangle = \frac{1}{\alpha_n} \langle Au_n + Bu_n - f, u_\infty - u_n \rangle \leq \frac{c_3}{\alpha_n}.$$

Keeping in mind that the embedding of V into $L^p(\Gamma_3)$ is compact, we have $u_n \to u$ in $L^p(\Gamma_3)$. By the Lebesgue dominated convergence theorem, we get

$$\begin{aligned} 0 &= \lim_{n\to\infty} \frac{c_3}{\alpha_n} \geq \lim_{n\to\infty} \langle Lu_n, u_n - u_\infty \rangle = \lim_{n\to\infty} \int_{\Gamma_3} l(x, u_n(x))(u_n(x) - u_\infty(x))\, \mathrm{d}\Gamma \\ &= \int_{\Gamma_3} l(x, u(x))(u(x) - u_\infty(x))\, \mathrm{d}\Gamma = \int_{\Gamma_3} l(x, u(x))(u(x) - b)\, \mathrm{d}\Gamma \\ &\geq \int_{\Gamma_3} l(x, b)(u(x) - b)\, \mathrm{d}\Gamma = 0. \end{aligned}$$

So, it holds $l(x, u(x))(u(x) - b) = 0$ for a.e. $x \in \Gamma_3$. Condition $H(l)$(iii) points out that $u(x) = b$ for a.e. $x \in \Omega$. This means that $u \in K$.

Subsequently, we shall show that $u = u_\infty$. For any $w \in K$, we have

$$\langle Au_n + Bu_n, u_n - w\rangle + \alpha_n \langle Lu_n, u_n - w\rangle = \langle f, u_n - w\rangle.$$

Because of $w = b$ on Γ_3, the following inequality holds

$$\begin{aligned} \langle Lu_n, u_n - w\rangle &= \int_{\Gamma_3} l(x, u_n(x))(u_n(x) - w(x))\, \mathrm{d}\Gamma \\ &= \int_{\Gamma_3} l(x, u_n(x))(u_n(x) - b)\, \mathrm{d}\Gamma \geq \int_{\Gamma_3} l(x, b)(u_n(x) - b)\, \mathrm{d}\Gamma = 0. \end{aligned}$$

This implies

$$\langle Au_n + Bu_n, w - u_n\rangle \geq \langle f, w - u_n\rangle. \tag{3.13}$$

From the monotonicity of A and B, we infer that

$$\langle Aw + Bw, w - u_n\rangle \geq \langle f, w - u_n\rangle.$$

Passing to the limit as $n \to \infty$ in the inequality above, one gets

$$\langle Aw + Bw, w - u\rangle \geq \langle f, w - u\rangle \quad \text{for all} \quad w \in K.$$

Due to $u \in K$, for any $t \in (0, 1)$ and $v \in K$, we have $w_t := tv + (1 - t)u \in K$. Inserting $w = w_t$ into the inequality above, it gives

$$\langle Au + Bu, v - u\rangle = \lim_{t\to 0} \langle Aw_t + Bw_t, v - u\rangle \geq \langle f, v - u\rangle$$

for all $v \in K$, namely,

$$\langle Au + Bu, v\rangle = \langle f, v\rangle$$

for all $v \in K_0$. From assertion (i), we know that u_∞ is the unique solution of Problem 1. Therefore, we deduce that $u = u_\infty$. Since, every weakly convergent subsequence of $\{u_n\}$ converges weakly to the same limit u_∞, it follows that the whole sequence $\{u_n\}$ converges weakly to u_∞.

Finally, it is easy to prove that u_n converges strongly in V to u_∞. Indeed, putting $w = u_\infty$ into (3.13), passing to the lower limit as $n \to \infty$ in the resulting inequality, and taking into account the monotonicity of B, we obtain

$$\limsup_{n\to\infty} \langle Au_n, u_n - u_\infty\rangle \leq \limsup_{n\to\infty} \langle f, u_n - u_\infty\rangle + \limsup_{n\to\infty} \langle Bu_\infty, u_\infty - u_n\rangle = 0.$$

This inequality combined with the (S_+)-property of operator A implies that $u_n \to u_\infty$ in V as $n \to \infty$. □

4. Optimal control and asymptotic analysis

In this section, we investigate two optimal control problems driven by mixed boundary value problems, Problems 1 and 2, respectively. We prove existence of optimal controls and establish a result on the asymptotic convergence of optimal control-state pairs, when the parameter α tends to infinity.

Let $H = L^{p'}(\Omega)$. Given a measured datum $z_d \in L^p(\Omega)$ and two regularization parameters λ, $\rho > 0$, we consider the following distributed optimal control problems governed by Problems 1 and 2, respectively.

Problem 9. *Find $g^* \in H$ such that*

$$J(g^*) = \min_{g\in H} J(g), \tag{4.1}$$

where the cost functional J is defined by

$$J(g) = \frac{\lambda}{p}\|u_g - z_d\|^p_{L^p(\Omega)} + \frac{\rho}{p'}\|g\|^{p'}_{L^{p'}(\Omega)}, \tag{4.2}$$

and u_g is the unique solution to Problem 1 corresponding to $g \in L^{p'}(\Omega)$.

and

Problem 10. *Given $\alpha > 0$, find $g^* \in H$ such that*

$$J_\alpha(g^*) = \min_{g\in H} J_\alpha(g), \tag{4.3}$$

where the cost functional J_α is defined by

$$J_\alpha(g) = \frac{\lambda}{p}\|u_{\alpha g} - z_d\|^p_{L^p(\Omega)} + \frac{\rho}{p'}\|g\|^{p'}_{L^{p'}(\Omega)}, \tag{4.4}$$

and $u_{\alpha g}$ is the unique solution to Problem 2 corresponding to $g \in L^{p'}(\Omega)$ and $\alpha > 0$.

A control-state pair (g^*, u_{g^*}) on which the infimum of (4.1) is attained is called an optimal solution to Problem 9. An analogous notion is applied to Problem 10.

The first result of this section is on existence of solutions to Problems 9 and 10.

Theorem 11. *Assume that $H(l)$ and $r \in L^{p'}_+(\Gamma_2)$ hold. Then, we have*

(i) *Problem 9 has at least one optimal solution $(g^*, u^*_{g^*}) \in H \times K$,*

(ii) *for each $\alpha > 0$, Problem 10 has at least one optimal solution $(g^*_\alpha, u^*_{\alpha g^*_\alpha}) \in H \times V$.*

Proof. We prove statement (ii), while assertion (i) can be obtained in a similar way. For any $\alpha > 0$ fixed, it follows from definition (4.4) that J_α is bounded from below. This permits us to find a minimizing sequence $\{g_n\} \subset H$ of Problem 10 such that

$$\lim_{n\to\infty} J_\alpha(g_n) = \inf_{g\in H} J_\alpha(g) := m_\alpha \geq 0. \tag{4.5}$$

By the coercivity of J_α, we can see that sequence $\{g_n\}$ is bounded in $L^{p'}(\Omega)$. By the reflexivity of $L^{p'}(\Omega)$, we may assume, passing to a subsequence if necessary, that

$$g_n \xrightarrow{w} g \text{ in } L^{p'}(\Omega) \tag{4.6}$$

for some $g \in H$. Let us denote by $u_n \in V$ the unique solution to Problem 2 corresponding to $g = g_n$ and $\alpha > 0$. We claim that $\{u_n\}$ is bounded in V. Let $\varepsilon := \frac{1}{2\alpha c_V^p}$. For every $n \in \mathbb{N}$, a simple computation gives (see (3.9), for example)

$$\begin{aligned}
&\|g_n\|_{L^{p'}(\Omega)}\|u_n\|_{L^p(\Omega)} + \|r\|_{L^{p'}(\Gamma_2)}\|u_n\|_{L^p(\Gamma_2)}\\
&\quad\geq \langle f_n, u_n\rangle = \langle Au_n + Bu_n + \alpha Lu_n, u_n\rangle\\
&\quad\geq \|u_n\|^p_V + \mu\|\nabla u_n\|^q_{L^q(\Omega;\mathbb{R}^N)} + \beta\|u_n\|^\theta_{L^\theta(\Omega)} - \alpha b|\Gamma_3|^{\frac{1}{p}}\|a_l\|_{L^{p'}(\Gamma_3)} - \alpha b_l b|\Gamma_3|
\end{aligned}$$

$$-\varepsilon\, \alpha\, c_V^p\, \|u_n\|_V^p - \alpha\, c(\varepsilon), \tag{4.7}$$

where $f_n \in V^*$ is defined by

$$\langle f_n, v\rangle = \int_\Omega g_n(x)v(x)\,\mathrm{d}x - \int_{\Gamma_2} r(x)v(x)\,\mathrm{d}\Gamma \quad \text{for all} \quad v \in V.$$

The latter combined with the continuity of the embeddings of V to $L^p(\Omega)$ and of V to $L^p(\Gamma_2)$ implies that sequence $\{u_n\}$ is bounded in V. Without any loss of generality, we may suppose that

$$u_n \xrightarrow{\,w\,} u \ \text{ in } \ V \text{ as } n \to \infty \tag{4.8}$$

with some $u \in V$.

Next, we verify that u is the unique solution of Problem 2 corresponding to $g \in L^{p'}(\Omega)$ and $\alpha > 0$. In fact, for each $n \in \mathbb{N}$, one has

$$\langle Au_n + Bu_n + \alpha Lu_n, w\rangle = \langle f_n, w\rangle \tag{4.9}$$

for all $w \in V$. We insert $w = u - u_n$ into (4.9) to get

$$\langle Au_n, u_n - u\rangle = \langle Bu_n + \alpha Lu_n - f_n, u - u_n\rangle.$$

Passing to the upper limit as $n \to \infty$ in this equality and using the compactness of embeddings of V to $L^p(\Omega)$ and of V to $L^p(\Gamma_2)$, and the monotonicity of B and L, we obtain

$$\limsup_{n\to\infty} \langle Au_n, u_n - u\rangle \le \limsup_{n\to\infty} \langle Bu + \alpha Lu - f_n, u - u_n\rangle = 0.$$

Taking into account the above result and the fact that A satisfies (S_+)-property, we find that $u_n \to u$ in V as $n \to \infty$. Letting $n \to \infty$ in Eq. (4.9), one has

$$\langle Au + Bu + \alpha Lu, w\rangle = \langle f, w\rangle$$

for all $w \in V$. Now, it is obvious that u is the unique solution of Problem 2 corresponding to $g \in L^{p'}(\Omega)$ and $\alpha > 0$.

Finally, from the weak lower semicontinuity of the norm function $g \mapsto \|g\|_{L^{p'}(\Omega)}$, we infer that

$$\begin{aligned}
\liminf_{n\to\infty} J_\alpha(g_n) &= \liminf_{n\to\infty} \left(\frac{\lambda}{p}\,\|u_n - z_d\|_{L^p(\Omega)}^p + \frac{\rho}{p'}\,\|g_n\|_{L^{p'}(\Omega)}^{p'}\right)\\
&= \lim_{n\to\infty} \frac{\lambda}{p}\,\|u_n - z_d\|_{L^p(\Omega)}^p + \liminf_{n\to\infty} \frac{\rho}{p'}\,\|g_n\|_{L^{p'}(\Omega)}^{p'}\\
&\ge \frac{\lambda}{p}\,\|u - z_d\|_{L^p(\Omega)}^p + \frac{\rho}{p'}\,\|g\|_{L^{p'}(\Omega)}^{p'} = J_\alpha(g).
\end{aligned}$$

This together with (4.5) entails that $(g, u) \in H \times V$ is an optimal solution to Problem 10. This completes the proof. □

The second result of this section is on the asymptotic behavior of the optimal solutions to Problem 10.

Theorem 12. *Assume that $H(l)$ and $r \in L_+^{p'}(\Gamma_2)$ hold. Let $\{\alpha_n\}$ be a sequence such that $\alpha_n > 0$ and $\alpha_n \to +\infty$ as $n \to \infty$, and let $(g_{\alpha_n}, u_{\alpha_n g_{\alpha_n}})$ be an optimal solution for Problem 10. Then, there exist an optimal solution $(g_\infty^*, u_{\infty g_\infty^*}^*)$ for Problem 9 and a subsequence of $\{(g_{\alpha_n}, u_{\alpha_n g_{\alpha_n}})\}$, still denoted by the same way, such that*

$$g_{\alpha_n} \to g_\infty^* \ \text{ in } \ L^{p'}(\Omega) \ \text{ and } \ u_{\alpha_n g_{\alpha_n}} \to u_{\infty g_\infty^*}^* \ \text{ in } \ V \text{ as } n \to \infty.$$

Moreover, the sequence $\{J_{\alpha_n}(g_{\alpha_n})\}$ of optimal values for Problem 10 converges to the optimal value $J(g_\infty^*)$ of Problem 9.

If Problem 9 has a unique optimal solution, then the whole sequence $\{(g_{\alpha_n}, u_{\alpha_n g_{\alpha_n}})\}$ converges in $L^{p'}(\Omega) \times V$ to $(g_\infty^*, u_{\infty g_\infty^*}^*)$ as $n \to \infty$.

Proof. Let $\{\alpha_n\}$ be a sequence such that $\alpha_n > 0$ and $\alpha_n \to +\infty$ as $n \to \infty$, and let $(g_n, u_n) := (g_{\alpha_n}, u_{\alpha_n g_{\alpha_n}})$ be an optimal solution to Problem 10 corresponding to $\alpha_n > 0$. It is obvious that

$$\frac{\lambda}{p}\|u_n - z_d\|^p_{L^p(\Omega)} + \frac{\rho}{p'}\|g_n\|^{p'}_{L^{p'}(\Omega)} = J_{\alpha_n}(g_n) \le J_{\alpha_n}(g) = \frac{\lambda}{p}\|\widetilde{u}_n - z_d\|^p_{L^p(\Omega)} + \frac{\rho}{p'}\|g\|^{p'}_{L^{p'}(\Omega)},$$

for any $g \in H$, where $\widetilde{u}_n \in V$ is the unique solution of Problem 2 corresponding to $\alpha_n > 0$ and $g \in L^{p'}(\Omega)$. Employing Theorem 8(vii), we have $\widetilde{u}_n \to u_\infty$ in V as $n \to \infty$, where u_∞ is the solution of Problem 1 associated with $g \in L^{p'}(\Omega)$. We can observe that sequence $\{g_n\}$ is bounded in $L^{p'}(\Omega)$. Without any loss of generality, we may assume that

$$g_n \xrightarrow{w} g \text{ in } L^{p'}(\Omega) \text{ as } n \to \infty$$

for some $g \in H$. Next, for each $n \in \mathbb{N}$, we have

$$\langle Au_n + Bu_n, u_\infty - u_n\rangle + \alpha_n\langle Lu_n, u_\infty - u_n\rangle = \langle f_n, u_\infty - u_n\rangle.$$

Using the same arguments as in the proof of (4.7), we obtain that sequence $\{u_n\}$ is bounded in V. Passing to a subsequence if necessary, we may suppose that

$$u_n \xrightarrow{w} u \text{ in } V \text{ as } n \to \infty$$

for some $u \in V$. Since $\{u_n\}$ is bounded in V, by the former argument, one has

$$\langle Lu_n, u_n - u_\infty\rangle \le \frac{1}{\alpha_n}\left[\langle Au_n + Bu_n - f_n, u_\infty - u_n\rangle\right] \le \frac{c_4}{\alpha_n} \tag{4.10}$$

for some $c_4 > 0$ which is independent of n. We use the compactness of embedding of V to $L^p(\Gamma_3)$, apply (4.10) and the Lebesgue dominated convergence theorem to get

$$0 = \lim_{n\to\infty}\frac{c_4}{\alpha_n} \ge \lim_{n\to\infty}\langle Lu_n, u_n - u_\infty\rangle = \langle Lu, u - u_\infty\rangle \ge \langle Lu_\infty, u - u_\infty\rangle = 0.$$

Hence, it holds $u(x) = b$ for a.e. $x \in \Gamma_3$, i.e., $u \in K$. We take the limit as $n \to \infty$ in the following inequality

$$\langle Au_n + Bu_n, w - u_n\rangle \ge \langle f_n, w - u_n\rangle \tag{4.11}$$

for all $w \in K$ to get $\langle Au + Bu, v\rangle = \langle f, v\rangle$ for all $v \in K_0$. This means that $u \in K$ is the unique solution of Problem 1 corresponding to g. Choosing $w = u$ into (4.11) and passing to the upper limit as $n \to \infty$, we use the (S_+)-property of A to get that $u_n \to u$ in V as $n \to \infty$. Note that from

$$J_{\alpha_n}(g_n) \le J_{\alpha_n}(h) \text{ for all } h \in H,$$

we have

$$\begin{aligned} J(g) &= \frac{\lambda}{p}\|u - z_d\|^p_{L^p(\Omega)} + \frac{\rho}{p'}\|g\|^{p'}_{L^{p'}(\Omega)} \\ &\le \liminf_{n\to\infty}\left(\frac{\lambda}{p}\|u_n - z_d\|^p_{L^p(\Omega)} + \frac{\rho}{p'}\|g_n\|^{p'}_{L^{p'}(\Omega)}\right) = \liminf_{n\to\infty} J_{\alpha_n}(g_n) \le \liminf_{n\to\infty} J_{\alpha_n}(h) \\ &= \liminf_{n\to\infty}\left(\frac{\lambda}{p}\|\widehat{u}_n - z_d\|^p_{L^p(\Omega)} + \frac{\rho}{p'}\|h\|^{p'}_{L^{p'}(\Omega)}\right) = \frac{\lambda}{p}\|\widehat{u} - z_d\|^p_{L^p(\Omega)} + \frac{\rho}{p'}\|h\|^{p'}_{L^{p'}(\Omega)} = J(h) \end{aligned}$$

for all $h \in H$, where we have applied Theorem 8(vii) and $\widehat{u}_n$ is the unique solution of Problem 2 corresponding to $h \in L^{p'}(\Omega)$ and $\alpha_n > 0$. Therefore, we can see that g is also a solution of Problem 9. In the meanwhile, we have $g = g^*_\infty$ and $u = u^*_{\infty g^*_\infty}$.

Finally, we show that g_n converges strongly to g_∞^* in $L^{p'}(\Omega)$. Keeping in mind that

$$\|g_\infty^*\|_{L^{p'}(\Omega)} \le \liminf_{n\to\infty} \|g_n\|_{L^{p'}(\Omega)},$$

and $J_{\alpha_n}(g_n) \le J_{\alpha_n}(g_\infty^*)$, we have

$$J(g_\infty^*) \le \liminf_{n\to\infty} J_{\alpha_n}(g_n) \le \limsup_{n\to\infty} J_{\alpha_n}(g_n) \le \limsup_{n\to\infty} J_{\alpha_n}(g_\infty^*) = J(g_\infty^*). \tag{4.12}$$

Hence

$$J(g_\infty^*) = \lim_{n\to\infty} J_{\alpha_n}(g_n) \quad \text{and} \quad \|g_\infty^*\|_{L^{p'}(\Omega)} = \lim_{n\to\infty} \|g_n\|_{L^{p'}(\Omega)}. \tag{4.13}$$

Thus, we conclude that $g_n \to g_\infty^*$ in $L^{p'}(\Omega)$ by using the triangle inequality and the fact that $g_n \xrightarrow{w} g$ in $L^{p'}(\Omega)$. The convergence of the sequence $\{J_{\alpha_n}(g_{\alpha_n})\}$ of optimal values for Problem 10 to the optimal value $J(g_\infty^*)$ of Problem 9 is a consequence of (4.13). This completes the proof. □

Remark 13. Let $y_d \in L^p(\Omega;\mathbb{R}^N)$ be a desired element. Theorems 11 and 12 established in this section are valid when the cost functionals (4.2) and (4.4) are replaced by the ones

$$J(g) = \frac{\lambda}{p}\|\nabla u_g - y_d\|^p_{L^p(\Omega;\mathbb{R}^N)} + \frac{\rho}{p'}\|g\|^{p'}_{L^{p'}(\Omega)},$$
$$J_\alpha(g) = \frac{\lambda}{p}\|\nabla u_{\alpha g} - y_d\|^p_{L^p(\Omega;\mathbb{R}^N)} + \frac{\rho}{p'}\|g\|^{p'}_{L^{p'}(\Omega)},$$

respectively.

Acknowledgements

The authors wish to thank the two knowledgeable referees for their useful remarks in order to improve the paper. This project has received funding from the NNSF of China Grant Nos. 12001478, 12026255, 12026256 and 12001463, the European Union's Horizon 2020 Research and Innovation Programme under the Marie Skłodowska-Curie grant agreement No. 823731 CONMECH, National Science Center of Poland under Project No. 2021/41/B/ST1/01636, and the Startup Project of Doctor Scientific Research of Yulin Normal University No. G2020ZK07. It is also supported by Natural Science Foundation of Guangxi Grant Nos. 2021GXNSFFA196004, 2020GXNSFBA297137 and 2018GXNSFAA281353, and the Ministry of Science and Higher Education of Republic of Poland under Grants Nos. 4004/GGPJII/H2020/2018/0 and 440328/PnH2/2019.

Data availability Data sharing not applicable to this article as no datasets were generated or analyzed during the current study.

References

[1] Alves, M.J., Assunçao, R.B., Miyagaki, O.: Existence result for a class of quasilinear elliptic equations with (p-q)-Laplacian and vanishing potentials. Ill. J. Math. **59**, 545–575 (2015)
[2] Axelsson, A., Keith, S., McIntosh, A.: The Kato square root problem for mixed boundary value problems. J. Lond. Math. Soc. **74**, 113–130 (2006)
[3] Bai, Y.R., Papageorgiou, N.S., Zeng, S.D.: A singular eigenvalue problem for the Dirichlet (p,q)-Laplacian. Math. Z. **300**, 325–345 (2022)
[4] Barboteu, M., Bartosz, K., Han, W., Janiczko, T.: Numerical analysis of a hyperbolic hemivariational inequality arising in dynamic contact. SIAM J. Numer. Anal. **53**, 527–550 (2015)

[5] Denkowski, Z., Migórski, S.: Control problems for parabolic and hyperbolic equations via the theory of G- and Γ-convergence. Ann. Mat. Pura Appl. **149**, 23–39 (1987)
[6] Denkowski, Z., Migórski, S., Papageorgiou, N.S.: An Introduction to Nonlinear Analysis: Applications. Kluwer Academic/Plenum Publishers, Boston, Dordrecht, London, New York (2003)
[7] Denkowski, Z., Mortola, S.: Asymptotic behavior of optimal solutions to control problems for systems described by differential inclusions corresponding to partial differential equations. J. Optim. Theory Appl. **78**, 365–391 (1993)
[8] Duvaut, G., Lions, J.-L.: Inequalities in Mechanics and Physics. Springer, Berlin (1976)
[9] Figueiredo, G.M.: Existence of positive solutions for a class of $p\&q$ elliptic problems with critical growth on $\mathbb{R}^N$. J. Math. Anal. Appl. **378**, 507–518 (2011)
[10] Gariboldi, C., Tarzia, D.: Convergence of distributed optimal controls on the internal energy in mixed elliptic problems when the heat transfer coefficient goes to infinity. Appl. Math. Optim. **47**, 213–230 (2003)
[11] Gasiński, L., Papageorgiou, N.S.: Asymmetric $(p, 2)$-equations with double resonance. Calc. Var. PDEs **56**(88), 23p (2017)
[12] Gasiński, L., Winkert, P.: Existence and uniqueness results for double phase problems with convection term. J. Differ. Equ. **268**, 4183–4193 (2020)
[13] Han, W.: Singular perturbations of variational-hemivariational inequalities. SIAM J. Math. Anal. **52**, 1549–1566 (2020)
[14] Hu, S., Papageorgiou, N.S.: Handbook of Multivalued Analysis, Vol. I: Theory. Kluwer Academic Publishers, Dordrecht (1997)
[15] Liu, Y.J., Migórski, S., Nguyen, V.T., Zeng, S.D.: Existence and convergence results for an elastic frictional contact problem with nonmonotone subdifferential boundary conditions. Acta Math. Sci. **41**, 1151–1168 (2021)
[16] Liu, Z.H., Motreanu, D., Zeng, S.D.: Positive solutions for nonlinear singular elliptic equations of p-Laplacian type with dependence on the gradient. Calc. Var. PDEs **58**(28), 22p (2019)
[17] Liu, Z.H., Motreanu, D., Zeng, S.D.: Generalized penalty and regularization method for differential variational-hemivariational inequalities. SIAM J. Optim. **31**, 1158–1183 (2021)
[18] Liu, Z., Papageorgiou, N.S.: Positive Solutions for Resonant (p, q)-equations with convection. Adv. Nonlinear Anal. **10**, 217–232 (2021)
[19] Maz'ya, V., Rossmann, J.: Mixed boundary value problems for the stationary Navier–Stokes system in polyhedral domains. Arch. Ration. Mech. An. **194**, 669–712 (2009)
[20] Migórski, S.: Sensitivity analysis of distributed-parameter optimal control problems for nonlinear parabolic equations. J. Optim. Theory Appl. **87**, 595–613 (1995)
[21] Migórski, S.: Convergence of optimal solutions in control problems for hyperbolic equations. Ann. Polon. Math. **62**, 111–121 (1995)
[22] Migórski, S., Khan, A.A., Zeng, S.D.: Inverse problems for nonlinear quasi-variational inequalities with an application to implicit obstacle problems of p-Laplacian type. Inverse Probl. **35**, 035004 (2019)
[23] Migórski, S., Khan, A.A., Zeng, S.D.: Inverse problems for nonlinear quasi-hemivariational inequalities with application to mixed boundary value problems. Inverse Probl. **36**, 024006 (2020)
[24] Migórski, S., Ochal, A., Sofonea, M.: Nonlinear Inclusions and Hemivariational Inequalities. Models and Analysis of Contact Problems. Advances in Mechanics and Mathematics, vol. 26. Springer, New York (2013)
[25] Mihăilescu, M., Rădulescu, V.: Sublinear eigenvalue problems associated to the Laplace operator revisited. Israel J. Math. **181**, 317–326 (2011)
[26] Mitrea, M.: Mixed boundary-value problems for Maxwell's equations. Trans. Am. Math. Soc. **362**, 117–143 (2010)
[27] Papageorgiou, N.S., Rădulescu, V.D., Repovš, D.D.: Positive solutions for nonlinear Neumann problems with singular terms and convection. J. Math. Pures Appl. **136**, 1–21 (2020)
[28] Papageorgiou, N.S., Rădulescu, V.D., Repovš, D.D.: Anisotropic (p, q)-equations with gradient dependent reaction. Nonlinearity **34**, 5319–5343 (2021)
[29] Papageorgiou, N.S., Qin, D., Rădulescu, V.D.: Nonlinear eigenvalue problems for the (p, q)-Laplacian. Bull. Sci. Math. **172**, 103039 (2021)
[30] Papageorgiou, N.S., Rădulescu, V.D., Repovš, D.D.: Nonlinear Analysis Theory and Methods. Springer, Switzerland (2019)
[31] Sofonea, M., Migórski, S.: Variational-Hemivariational Inequalities with Applications. Monographs and Research Notes in Mathematics, Chapman & Hall/CRC, Boca Raton (2017)
[32] Yu, S., Feng, Z.S.: Lions-type theorem of the p-Laplacian and applications. Adv. Nonlinear Anal. **10**, 1178–1200 (2021)
[33] Zeng, S.D., Migórski, S., Tarzia, D.: A new elliptic mixed boundary value problem with (p, q)-Laplacian and Clarke subdifferential: existence, comparison and convergence results. Anal. Appl. (2021). https://doi.org/10.1142/S0219530521500287
[34] Zeng, S.D., Rădulescu, V.D., Winkert, P.: Douvle phase implicit obstacle problems with convection and multivalued mixed boundary value conditions. SIAM J. Math. Anal. **54**, 1898–1926 (2022)
[35] Zeng, S.D., Migórski, S., Khan, A.A.: Nonlinear quasi-hemivariational inequalities: existence and optimal control. SIAM J. Control Optim. **59**(2), 1246–1274 (2021)

[36] Zeng, S.D., Bai, Y.R., Gasiński, L., Winkert, P.: Existence results for double phase implicit obstacle problems involving multivalued operators. Calc. Var. Partial Differ. Equ. **59**, 18 (2020)
[37] Zeng, S.D., Migórski, S., Liu, Z.H.: Well-posedness, optimal control and sensitivity analysis for a class of differential variational-hemivariational inequalities. SIAM J. Optim. **31**, 2829–2862 (2021)

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(Received: February 2, 2022; revised: May 9, 2022; accepted: May 30, 2022)