# A new elliptic mixed boundary value problem with $(p, q)$-Laplacian and Clarke subdifferential: Existence, comparison and convergence results 

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The goal of this paper is to investigate a new class of elliptic mixed boundary value problems involving a nonlinear and nonhomogeneous partial differential operator $(p, q)$ Laplacian, and a multivalued term represented by Clarke's generalized gradient. First, we apply a surjectivity result for multivalued pseudomonotone operators to examine the existence of weak solutions under mild hypotheses. Then, a comparison theorem is delivered, and a convergence result, which reveals the asymptotic behavior of solution when the parameter (heat transfer coefficient) tends to infinity, is obtained. Finally, we

[^0]establish a continuous dependence result of solution to the boundary value problem on the data.

Keywords: Mixed boundary value problem; $(p, q)$-Laplacian; Clarke's generalized gradient; comparison; asymptotic behavior.

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## 1. Introduction

Let $\Omega$ be a bounded domain in $\mathbb{R}^{N}$ with a Lipschitz boundary $\Gamma:=\partial \Omega$ which is divided into three measurable and mutually disjoint parts $\Gamma_{1}, \Gamma_{2}$, and $\Gamma_{3}$ such that $\Gamma_{1}$ has a positive measure. Let $1<q<p<+\infty, \alpha, \beta, \mu>0, b \in \mathbb{R}$ and $\theta<p^{*}$, where $p^{*}$ is the critical exponent to $p$ (see (2.1) in Sec. 2). Given functions $g: \Omega \rightarrow \mathbb{R}, r: \Gamma_{2} \rightarrow \mathbb{R}$ and $j: \Gamma_{3} \times \mathbb{R} \rightarrow \mathbb{R}$, in the paper we consider the following nonlinear mixed boundary value problems:

Problem 1. Find $u: \Omega \rightarrow \mathbb{R}$ such that

$$
\begin{aligned}
& -\Delta_{p} u(x)-\mu \Delta_{q} u(x)+\beta|u(x)|^{\theta-2} u(x)=g(x) \quad \text { in } \Omega, \\
& u=0 \quad \text { on } \Gamma_{1}, \\
& -\frac{\partial_{(p, q)} u}{\partial \nu}=r(x) \text { on } \Gamma_{2}, \\
& u=b \quad \text { on } \Gamma_{3} .
\end{aligned}
$$

Problem 2. Find $u: \Omega \rightarrow \mathbb{R}$ such that

$$
\begin{aligned}
& -\Delta_{p} u(x)-\mu \Delta_{q} u(x)+\beta|u(x)|^{\theta-2} u(x)=g(x) \quad \text { in } \Omega, \\
& u=0 \quad \text { on } \Gamma_{1}, \\
& -\frac{\partial_{(p, q)} u}{\partial \nu}=r(x) \quad \text { on } \Gamma_{2}, \\
& -\frac{\partial_{(p, q)} u}{\partial \nu} \in \alpha \partial j(u) \quad \text { on } \Gamma_{3},
\end{aligned}
$$

where $\Delta_{p}$ denotes the $p$-Laplace differential operator of the form

$$
\Delta_{p} u=\operatorname{div}\left(|\nabla u|^{p-2} \nabla u\right) \quad \text { for all } u \in W^{1, p}(\Omega)
$$

$\nu$ is the outward unit normal at the boundary $\Gamma$

$$
\frac{\partial_{(p, q)} u}{\partial \nu}:=\left(|\nabla u|^{p-2} \nabla u+\mu|\nabla u|^{q-2} \nabla u\right) \cdot \nu
$$

and $\partial j$ stands for the Clarke subgradient of $j$ with respect to its last variable (see Definition (3).

Our motivation to study this type of problem comes from two sources. First, the $(p, q)$-Laplacian operator has been used to model steady-state solutions to reactiondiffusion problems arising in numerous applications in biophysics, plasma physics
and in the study of chemical reactions, where the unknown generally denotes a concentration of a substance, see, e.g., [1, 6] and the references therein. Second, we have the mathematical interest in these type of problems mainly regarding the existence and the convergence of solutions. Our present work is a continuation of a very recent paper [8] in which a particular variational form of Problems 1 and 2 was investigated for $\beta=0, \mu=0$ and $p=2$. There, $u$ represents a temperature, $g$ is the internal energy, $r$ represents the heat flux on $\Gamma_{2}, b$ is the prescribed temperature on $\Gamma_{3}$, and $\alpha>0$ is the heat transfer coefficient on $\Gamma_{3}$.

The purposes of this paper are threefold. The first is to establish the existence theorem for the weak solutions of nonsmooth elliptic mixed boundary value problem, Problem 2. Such existence result is based on a surjectivity theorem for a class of multivalued pseudomonotone operators. The second purpose is to explore the comparison principle and to study the asymptotic behavior of solution to Problem 2 as the heat transfer coefficient $\alpha \rightarrow+\infty$. The last aim is to prove a result on the continuous dependence of solution with respect to parameters $\alpha, \beta>0$ and functions $(g, r) \in L^{p^{\prime}}(\Omega) \times L^{p^{\prime}}\left(\Gamma_{2}\right)$.

In Problem 2 the function $j: \Gamma_{3} \times \mathbb{R} \rightarrow \mathbb{R}$ is such that $j(x, \cdot)$ is locally Lipschitz for a.e. $x \in \Gamma_{3}$ and not necessarily differentiable. In general $j(x, \cdot)$ is nonconvex, so the multivalued condition on $\Gamma_{3}$ in Problem 2 is described by a nonmonotone relation expressed by the generalized gradient of Clarke. Such multivalued relation in Problem 2 is met in modeling of certain types of steady-state heat conduction problems (the behavior of a semipermeable membrane of finite thickness, a temperature control problems, etc., see for example [2, 3, 9, 10, 13, 14, 19). Also, Problem 2 can be considered as a prototype of several nonmonotone boundary semipermeability models, see [12, 15, 17, 18, 24, 28, which appear in hydraulics, fluid flow problems through porous media, and electrostatics, where a solution represents the pressure and the electric potentials. The analogous problems with maximal monotone multivalued boundary relations when $j(x, \cdot)$ is a convex function have been first studied in [5], see also references therein. Further, Problems 1 and 2 have been treated in steady-state two phase Stefan models, see, e.g., [7, 21, 22. It turns out that the weak formulation of Problem 2 is a hemivariational inequality. More information on this kind of inequalities can be found in $[16,415,17,20,24]$.

The outline of the paper is as follows. Section 2 provides the necessary notation and collects preliminary results. In Sec. 3, we prove the existence of weak solutions to Problem 2, and discuss a comparison principle. Section 4 is concerned with the asymptotic behavior of solution to Problem 2 and of a continuous dependence result of Problem 2 with respect to the data $(\alpha, \beta, g, r)$.

## 2. Mathematical Prerequisites

In this section, we recall basic notation, definitions and necessary preliminary material, which will be used in this paper. More details can be found, for instance, in 4, 16, 23.

Everywhere below, the symbols $\xrightarrow{w}$ and $\rightarrow$ stand for the weak convergence and the strong convergence, respectively. Given a Banach space $Y$, we also adopt the notation $\|\cdot\|_{Y}$ and $Y^{*}$ for a norm and the dual space of $Y$, respectively. The duality brackets for the pair $\left(Y^{*}, Y\right)$ is denoted by $\langle\cdot, \cdot\rangle$. We say that a mapping $F: Y \rightarrow Y^{*}$ is of the type $(S)_{+}$(or $F$ satisfies the $\left(S_{+}\right)$-property), if for any sequence $\left\{u_{n}\right\} \subset Y$ with $u_{n} \xrightarrow{w} u$ in $Y$ as $n \rightarrow \infty$ for some $u$ and $\lim \sup \left\langle F u_{n}, u_{n}-u\right\rangle \leq 0$, the sequence $\left\{u_{n}\right\}$ converges strongly to $u$ in $Y$.

Let $\left(Z,\|\cdot\|_{Z}\right)$ be a reflexive Banach space. We say that a function $J: Z \rightarrow \mathbb{R}$ is locally Lipschitz at $u \in Z$ if there exist a neighborhood $N(u)$ of $u$ in $Z$ and a constant $L_{u}>0$ such that

$$
|J(w)-J(z)| \leq L_{u}\|w-z\|_{Z} \quad \text { for all } w, z \in N(u)
$$

Definition 3. Given a locally Lipschitz function $J: Z \rightarrow \mathbb{R}$, we denote by $J^{0}(u ; v)$ the directional derivative in the sense of Clarke (or generalized directional derivative) of $J$ at $u \in Z$ in the direction $v \in Z$ defined by

$$
J^{0}(u ; v)=\limsup _{\lambda \rightarrow 0^{+}, w \rightarrow u} \frac{J(w+\lambda v)-J(w)}{\lambda}
$$

The generalized gradient of $J: Z \rightarrow \mathbb{R}$ at $u \in Z$ is given by

$$
\partial J(u)=\left\{\xi \in Z^{*} \mid J^{0}(u ; v) \geq\langle\xi, v\rangle \text { for all } v \in Z\right\}
$$

The properties of the generalized directional derivative and the generalized gradient of a locally Lipschitz function are collected in what follows, see, e.g., [16, Proposition 3.23].

Proposition 4. Let $J: Z \rightarrow \mathbb{R}$ be a locally Lipschitz function on a Banach space Z. Then, we have
(i) for every $u \in Z$, the function $Z \ni v \mapsto J^{0}(u ; v) \in \mathbb{R}$ is positively homogeneous and subadditive, i.e.

$$
J^{0}(u ; \lambda v)=\lambda J^{0}(u ; v) \quad \text { for all } \lambda \geq 0 \quad \text { and } \quad u, v \in Z
$$

and

$$
J^{0}\left(u ; v_{1}+v_{2}\right) \leq J^{0}\left(u ; v_{1}\right)+J^{0}\left(u ; v_{2}\right) \quad \text { for all } u, v_{1}, v_{2} \in Z
$$

(ii) Let $u \in Z$ be fixed, for each $v \in Z$, there exists an element $\xi(v) \in \partial J(u)$ such that

$$
J^{0}(u ; v)=\langle\xi(v), v\rangle, \quad \text { i.e. } J^{0}(u ; v)=\max \{\langle\xi, v\rangle \mid \xi \in \partial J(u)\} .
$$

(iii) The function $Z \times Z \ni(u, v) \mapsto J^{0}(u ; v) \in \mathbb{R}$ is upper semicontinuous.
(iv) The multivalued mapping $u \mapsto \partial J(u)$ is upper semicontinuous from $Z$ into $w^{*}-Z^{*}$.

Let $\Omega \subset \mathbb{R}^{N}$ be a bounded domain such that its Lipschitz boundary $\Gamma=\partial \Omega$ is divided into three measurable and mutually disjoint parts $\Gamma_{1}, \Gamma_{2}$, and $\Gamma_{3}$ with $\Gamma_{1}$ being of positive measure. Let $1<p<+\infty$ and $p^{\prime}>1$ be the conjugate exponent of $p$, i.e. $\frac{1}{p}+\frac{1}{p^{\prime}}=1$. In the sequel, we denote by $p^{*}$ the critical exponent to $p$ given by

$$
p^{*}= \begin{cases}\frac{N p}{N-p} & \text { if } p<N  \tag{2.1}\\ +\infty & \text { if } p \geq N\end{cases}
$$

Throughout the paper, the norms of the Lebesgue space $L^{p}(\Omega)$ and Sobolev space $W^{1, p}(\Omega)$ are defined by

$$
\|u\|_{L^{p}(\Omega)}:=\left(\int_{\Omega}|u(x)|^{p} d x\right)^{\frac{1}{p}} \quad \text { for all } u \in L^{p}(\Omega)
$$

and

$$
\|u\|_{W^{1, p}(\Omega)}:=\|u\|_{L^{p}(\Omega)}+\|\nabla u\|_{L^{p}\left(\Omega ; \mathbb{R}^{N}\right)} \quad \text { for all } u \in W^{1, p}(\Omega)
$$

respectively. We introduce a subspace $V$ of $W^{1, p}(\Omega)$ given by

$$
V:=\left\{u \in W^{1, p}(\Omega) \mid u=0 \text { on } \Gamma_{1}\right\} .
$$

It follows from the fact that $\Gamma_{1}$ has a positive measure and the Poincaré inequality that $V$ endowed with the norm

$$
\|u\|_{V}:=\left(\int_{\Omega}|\nabla u|^{p} d x\right)^{\frac{1}{p}} \quad \text { for all } u \in V
$$

is a reflexive Banach space. Further, we consider the subsets $K$ and $K_{0}$ of $V$ defined by

$$
\begin{align*}
K & :=\left\{u \in V \mid u=b \text { on } \Gamma_{3}\right\},  \tag{2.2}\\
K_{0} & :=\left\{u \in V \mid u=0 \text { on } \Gamma_{3}\right\}, \tag{2.3}
\end{align*}
$$

respectively, where $b \in \mathbb{R}$ is given in Problem 1 .

## 3. Existence and Comparison Results

In the section, we shall study the existence of a weak solution to Problem 2, and discuss a comparison principle which reveals the relations between the solutions of Problems 11 and 2 and the constant $b$. In what follows, we assume that $(g, r) \in$ $L^{p^{\prime}}(\Omega) \times L^{p^{\prime}}\left(\Gamma_{2}\right)$.

The weak solutions of Problems 1 and 2 are understood as follows.

Definition 5. We say that
(i) a function $u: \Omega \rightarrow \mathbb{R}$ is a weak solution of Problem if $u \in W^{1, p}(\Omega)$ is such that $u=0$ on $\Gamma_{1}, u=b$ on $\Gamma_{3}$ and

$$
\begin{aligned}
& \int_{\Omega}\left(|\nabla u(x)|^{p-2} \nabla u(x)+\mu|\nabla u(x)|^{q-2} \nabla u(x), \nabla v(x)\right)_{\mathbb{R}^{N}} d x \\
& \quad+\int_{\Omega} \beta|u(x)|^{\theta-2} u(x) v(x) d x=\int_{\Omega} g(x) v(x) d x-\int_{\Gamma_{2}} r(x) v(x) d \Gamma
\end{aligned}
$$

for all $v \in W^{1, p}(\Omega)$ with $v=0$ on $\Gamma_{1}$ and $v=0$ on $\Gamma_{3}$.
(ii) a function $u: \Omega \rightarrow \mathbb{R}$ is a weak solution of Problem 2, if $u \in W^{1, p}(\Omega)$ satisfies $u=0$ on $\Gamma_{1}$ and

$$
\begin{aligned}
& \int_{\Omega}\left(|\nabla u(x)|^{p-2} \nabla u(x)+\mu|\nabla u(x)|^{q-2} \nabla u(x), \nabla v(x)\right)_{\mathbb{R}^{N}} d x \\
& \quad+\int_{\Omega} \beta|u(x)|^{\theta-2} u(x) v(x) d x+\alpha \int_{\Gamma_{3}} j^{0}(x, u(x) ; v(x)) d \Gamma \\
& \geq \int_{\Omega} g(x) v(x) d x-\int_{\Gamma_{2}} r(x) v(x) d \Gamma
\end{aligned}
$$

for all $v \in W^{1, p}(\Omega)$ with $v=0$ on $\Gamma_{1}$.
We introduce nonlinear mappings $A: V \rightarrow V^{*}$ and $B: V \rightarrow V^{*}$ defined by

$$
\begin{equation*}
\langle A u, v\rangle:=\int_{\Omega}\left(|\nabla u(x)|^{p-2} \nabla u(x)+\mu|\nabla u(x)|^{q-2} \nabla u(x), \nabla v(x)\right)_{\mathbb{R}^{N}} d x \tag{3.1}
\end{equation*}
$$

and

$$
\begin{equation*}
\langle B u, v\rangle:=\int_{\Omega} \beta|u(x)|^{\theta-2} u(x) v(x) d x \tag{3.2}
\end{equation*}
$$

for all $u, v \in V$, and the element $f \in V^{*}$ given by

$$
\begin{equation*}
\langle f, v\rangle=\int_{\Omega} g(x) v(x) d x-\int_{\Gamma_{2}} r(x) v(x) d \Gamma \quad \text { for } v \in V \tag{3.3}
\end{equation*}
$$

Under these notations for $A, B$ and $f$, an alternative form of Definition 5 reads as follows:
(i) ${ }^{\prime}$ a function $u_{\alpha} \in V$ is a weak solution to Problem 2 corresponding to $\alpha>0$, if the following inequality holds:

$$
\begin{equation*}
\left\langle A u_{\alpha}+B u_{\alpha}, v\right\rangle+\alpha \int_{\Gamma_{3}} j^{0}\left(x, u_{\alpha}(x) ; v(x)\right) d \Gamma \geq\langle f, v\rangle \quad \text { for all } v \in V \text {. } \tag{3.4}
\end{equation*}
$$

(ii) a function $u_{\infty} \in K$ is a weak solution to Problem if the following equality is true

$$
\begin{equation*}
\left\langle A u_{\infty}+B u_{\infty}, v\right\rangle=\langle f, v\rangle \quad \text { for all } v \in K_{0} \tag{3.5}
\end{equation*}
$$

where $K$ and $K_{0}$ are given by (2.2) and (2.3), respectively.

Lemma 6. Under the above notation, for any $b \in \mathbb{R}$, Problem 1 has a unique weak solution $u_{\infty} \in K$.

Proof. Observe that $u_{\infty} \in K$ solves (3.5) if and only if $z_{\infty} \in K_{0}$ defined by $z_{\infty}:=u_{\infty}-b$ is a solution to

$$
\begin{equation*}
\left\langle A z_{\infty}+B z_{\infty}, v\right\rangle=\langle f-B b, v\rangle \quad \text { for all } v \in K_{0} \tag{3.6}
\end{equation*}
$$

We use [11, Chap. 3, Example 1.7, p. 303] to infer that $A$ is bounded, continuous, strictly monotone (hence maximal monotone), of the type ( $S_{+}$), and

$$
\begin{equation*}
\|A u\|_{V^{*}} \leq\|u\|_{V}^{p-1}+\mu\|\nabla u\|_{L^{(q-1) p^{\prime}}\left(\Omega ; \mathbb{R}^{N}\right)}^{q-1} \quad \text { for all } u \in V \text {. } \tag{3.7}
\end{equation*}
$$

From [16, Theorem 3.69], we can see that $A$ is a pseudomonotone operator. The operator $B$ is monotone and continuous, and such that

$$
\begin{equation*}
\|B u\|_{V^{*}} \leq c_{2}\|u\|_{V}^{\theta-1} \quad \text { for all } u \in V \tag{3.8}
\end{equation*}
$$

with some $c_{2}>0$. Recall that the embedding of $V$ into $L^{\theta}(\Omega)$ is compact (thanks to $\left.\theta<p^{*}\right)$, so, $B$ is completely continuous. This implies that $B$ is pseudomonotone as well. By the easily verifiable inequality

$$
\begin{equation*}
\langle A u+B u, u\rangle \geq\|u\|_{V}^{p}+\mu\|\nabla u\|_{L^{q}\left(\Omega ; \mathbb{R}^{N}\right)}^{q}+\beta\|u\|_{L^{\theta}(\Omega)}^{\theta} \quad \text { for all } u \in V, \tag{3.9}
\end{equation*}
$$

we deduce that $A+B$ is pseudomonotone and coercive. Therefore, by 4, Theorem 1.3.70], Problem 1 is solvable. Moreover, $A+B$ is strictly monotone, so, we can use a standard way to prove that Problem has a unique solution in $K$. This completes the proof.

Next, we give the existence result to Problem 2 To this end, the following assumption on the potential $j$ is needed.
$\underline{H(j)}: j: \Gamma_{3} \times \mathbb{R} \rightarrow \mathbb{R}$ is such that
(i) $x \mapsto j(x, r)$ is measurable on $\Gamma_{3}$ for all $r \in \mathbb{R}$,
(ii) $r \mapsto j(x, r)$ is locally Lipschitz continuous for a.e. $x \in \Gamma_{3}$,
(iii) there exist constants $c_{0}, c_{1} \geq 0$ such that

$$
|\partial j(x, r)| \leq c_{0}+c_{1}|r|^{p-1} \quad \text { for all } r \in \mathbb{R} \quad \text { and for a.e. } x \in \Gamma_{3},
$$

(iv) $j^{0}(x, r ; b-r) \leq 0$ for all $r \in \mathbb{R}$ and for a.e. $x \in \Gamma_{3}, b \geq 0$ is given in Problem

Theorem 7. Assume that $H(j)$ holds. Then, Problem 2 has at least one weak solution in $V$.

Proof. First, analogously as in the proof of Lemma 6, we deduce that operator $A+$ $B$ is pseudomonotone and coercive. Next, we consider the functional $J: L^{p}\left(\Gamma_{3}\right) \rightarrow \mathbb{R}$ defined by

$$
\begin{equation*}
J(w):=\int_{\Gamma_{3}} j(x, w(x)) d \Gamma \quad \text { for } w \in L^{p}\left(\Gamma_{3}\right) \tag{3.10}
\end{equation*}
$$

Since $j$ satisfies hypothesis $H(j)$, we could apply the same arguments as in the proof of [16, Theorem 3.47] to obtain
(a) $J$ given by (3.10) is well-defined and locally Lipschitz continuous in $L^{p}\left(\Gamma_{3}\right)$,
(b) it holds

$$
\left\{\begin{array}{l}
J^{0}(w ; z) \leq \int_{\Gamma_{3}} j^{0}(x, w(x) ; z(x)) d \Gamma  \tag{3.11}\\
\partial J(w) \subset \int_{\Gamma_{3}} \partial j(x, w(x)) d \Gamma
\end{array}\right.
$$

for all $w, z \in L^{p}\left(\Gamma_{3}\right)$,
(c) there are constants $c_{J}, d_{J} \geq 0$ such that

$$
\begin{equation*}
\|\partial J(w)\|_{L^{p^{\prime}}\left(\Gamma_{3}\right)} \leq c_{J}+d_{J}\|w\|_{L^{p}\left(\Gamma_{3}\right)}^{p} \tag{3.12}
\end{equation*}
$$

for all $w \in L^{p}\left(\Gamma_{3}\right)$.
Moreover, we claim that the set-valued operator $V \ni u \mapsto \gamma^{*} \partial J(\gamma u) \subset V^{*}$ is pseudomonotone, where $\gamma$ is the trace operator from $V$ into $L^{p}\left(\Gamma_{3}\right)$. From Proposition 4 (iv) and (3.12), we can see that for each $u \in V$, the set $\gamma^{*} \partial J(\gamma u)$ is nonempty, bounded, closed and convex in $V^{*}$. Having in mind [16, Proposition 3.58], it is sufficient to show that $V \ni u \mapsto \gamma^{*} \partial J(\gamma u) \subset V^{*}$ is a generalized pseudomonotone operator. Let $\left\{u_{n}\right\} \subset V$ and $\left\{\xi_{n}\right\} \subset V^{*}$ be sequences such that $\xi_{n} \in \gamma^{*} \partial J\left(\gamma u_{n}\right)$ and

$$
u_{n} \xrightarrow{w} u \text { in } V, \quad \xi_{n} \xrightarrow{w} \xi \text { in } V^{*}, \quad \text { and } \lim \sup _{n \rightarrow \infty}\left\langle\xi_{n}, u_{n}-u\right\rangle \leq 0 .
$$

Then, we are able to find a sequence $\left\{\eta_{n}\right\} \subset L^{p^{\prime}}\left(\Gamma_{3}\right)$ such that $\xi_{n}=\gamma^{*} \eta_{n}$ for all $n \in \mathbb{N}$. The inequality (3.12) implies that $\left\{\eta_{n}\right\}$ is bounded in $L^{p^{\prime}}\left(\Gamma_{3}\right)$. Passing to a subsequence if necessary, we may find a function $\eta \in L^{p^{\prime}}\left(\Gamma_{3}\right)$ such that $\eta_{n} \xrightarrow{w} \eta$ in $L^{p^{\prime}}\left(\Gamma_{3}\right)$, as $n \rightarrow \infty$. The continuity and linearity of $\gamma^{*}$ implies that $\xi_{n}=\gamma^{*} \eta_{n} \xrightarrow{w}$ $\xi=\gamma^{*} \eta$ in $V^{*}$. Recall that the graph of the map $w \mapsto \partial J(w)$ is strongly-weakly closed. This combined with the compactness of $\gamma$ shows that $\eta \in \partial J(\gamma u)$, that is, $\xi \in \gamma^{*} \partial J(\gamma u)$. In addition, from the convergence

$$
\left\langle\xi_{n}, u_{n}\right\rangle=\left\langle\eta_{n}, \gamma u_{n}\right\rangle_{L^{p^{\prime}}\left(\Gamma_{3}\right) \times L^{p}\left(\Gamma_{3}\right)} \rightarrow\langle\eta, \gamma u\rangle_{L^{p^{\prime}}\left(\Gamma_{3}\right) \times L^{p}\left(\Gamma_{3}\right)}=\left\langle\gamma^{*} \eta, u\right\rangle=\langle\xi, u\rangle
$$

we conclude that $V \ni u \mapsto \gamma^{*} \partial J(\gamma u) \subset V^{*}$ is generalized pseudomonotone. Hence we deduce that the operator $V \ni u \mapsto \gamma^{*} \partial J(\gamma u) \subset V^{*}$ is pseudomonotone.

Furthermore, we are going to verify that the operator $A+B+\alpha \gamma^{*} \partial J(\gamma \cdot)$ is coercive. Using hypotheses $H(j)($ iii $), H(j)(i v)$ and inequality (3.11), one has

$$
\begin{aligned}
\alpha\langle\xi, u\rangle & =-\alpha\langle\xi,-u\rangle \geq-\alpha J^{0}(u ;-u)=-\alpha J^{0}(u ; b-u-b) \\
& \geq-\alpha J^{0}(u ; b-u)-\alpha J^{0}(u ;-b) \\
& \geq-\alpha \int_{\Gamma_{3}} j^{0}(x, u(x) ; b-u(x)) d \Gamma-\alpha \int_{\Gamma_{3}} j^{0}(x, u(x) ;-b) d \Gamma
\end{aligned}
$$

$$
\begin{aligned}
& \geq-\alpha \int_{\Gamma_{3}} j^{0}(x, u(x) ;-b) d \Gamma \geq-\int_{\Gamma_{3}} \alpha\left(c_{0}+c_{1}|u(x)|^{p-1}\right) b d \Gamma \\
& =-\alpha c_{0} b\left|\Gamma_{3}\right|-\alpha c_{1} b\|u\|_{L^{p-1}\left(\Gamma_{3}\right)}^{p-1} .
\end{aligned}
$$

From the last inequality and (3.9), we obtain

$$
\begin{aligned}
& \left\langle A u+B u+\alpha \gamma^{*} \partial J(\gamma u), u\right\rangle \\
& \quad \geq\|u\|_{V}^{p}+\mu\|\nabla u\|_{L^{q}\left(\Omega ; \mathbb{R}^{N}\right)}^{q}+\beta\|u\|_{L^{\theta}(\Omega)}^{\theta}-\alpha c_{0} b\left|\Gamma_{3}\right|-\alpha c_{1} b\|u\|_{L^{p-1}\left(\Gamma_{3}\right)}^{p-1},
\end{aligned}
$$

which easily implies that $V \ni u \mapsto A u+B u+\alpha \gamma^{*} \partial J(\gamma u) \subset V^{*}$ is coercive.
Therefore, all conditions of [4, Theorem 1.3.70] are verified. Using this theorem, we are able to find a function $u \in V$ such that

$$
\begin{equation*}
A u+B u+\alpha \gamma^{*} \partial J(\gamma u) \ni f \tag{3.13}
\end{equation*}
$$

where $f \in V^{*}$ is given by (3.3). Multiplying (3.13) by $v \in V$ and using the definition of $f$ and the Clarke subgradient, it yields

$$
\langle A u+B u, v\rangle+\alpha J^{0}(\gamma u ; \gamma v) \geq \int_{\Omega} g(x) v(x) d x-\int_{\Gamma_{2}} r(x) v(x) d \Gamma \quad \text { for all } v \in V
$$

This together with (3.11) and the definition of $A$ and $B$ entails

$$
\begin{aligned}
& \int_{\Omega}\left(|\nabla u|^{p-2} \nabla u+\mu|\nabla u|^{q-2} \nabla u, \nabla v\right)_{\mathbb{R}^{N}} d x+\int_{\Omega} \beta|u(x)|^{\theta-2} u(x) v(x) d x \\
& \quad+\alpha \int_{\Gamma_{3}} j^{0}(x, \gamma u(x) ; \gamma v(x)) d \Gamma \geq \int_{\Omega} g(x) v(x) d x-\int_{\Gamma_{2}} r(x) v(x) d \Gamma
\end{aligned}
$$

for all $v \in V$. This shows that $u$ is also a weak solution of Problem 2, This completes the proof of the theorem.

Further, we need the following condition.
$\underline{H(0)}: g \in L^{p^{\prime}}(\Omega)$ with $g \leq 0$ in $\Omega$ and $r \in L^{p^{\prime}}\left(\Gamma_{2}\right)$ with $r \geq 0$ on $\Gamma_{2}$.
The corresponding comparison result for Problem 2 reads as follows.
Theorem 8. Suppose that $H(j)$ and $H(0)$ hold. Let $u_{\alpha}$ be a solution of Problem 2 corresponding to $\alpha>0$, and $u_{\infty}$ be the unique solution of Problem 11. Then, for each $\alpha>0$, the following statements hold:
(i) $u_{\alpha} \leq b$ in $\Omega$,
(ii) $u_{\alpha} \leq u_{\infty}$ in $\Omega$.

Proof. (i) Set $w=u_{\alpha}-b$. It suffices to show that $w^{+}=0$ in $\Omega$. Since $u_{\alpha} \in V$, then it holds $w=-b$ on $\Gamma_{1}$. This shows that $w^{+}=0$ on $\Gamma_{1}$, thus, $-w^{+} \in V$. It
allows one to take $v=-w^{+}$as a test function in (3.4) to get

$$
\left\langle A u_{\alpha}+B u_{\alpha},-w^{+}\right\rangle+\alpha \int_{\Gamma_{3}} j^{0}\left(x, u_{\alpha}(x) ;-w^{+}(x)\right) d \Gamma \geq\left\langle f,-w^{+}\right\rangle
$$

Because $A b=0$, we have

$$
\begin{align*}
-\left\langle A u_{\alpha}-A b, w^{+}\right\rangle-\left\langle B u_{\alpha}-B b, w^{+}\right\rangle & -\left\langle B b, w^{+}\right\rangle \\
+\alpha \int_{\Gamma_{3}} j^{0}\left(x, u_{\alpha}(x) ;-w^{+}(x)\right) d \Gamma & \geq\left\langle f,-w^{+}\right\rangle \tag{3.14}
\end{align*}
$$

By the monotonicity of $B$, we have

$$
\left\langle B u_{\alpha}-B b, w^{+}\right\rangle \geq 0 \quad \text { and } \quad\left\langle B b, w^{+}\right\rangle \geq 0
$$

So, (3.14) gives

$$
\left\langle f,-w^{+}\right\rangle+\left\langle A u_{\alpha}-A b, w^{+}\right\rangle \leq \alpha \int_{\Gamma_{3}} j^{0}\left(x, u_{\alpha}(x) ;-w^{+}(x)\right) d \Gamma
$$

The latter combined with the definition of $f$ and hypothesis $H(0)$ implies

$$
\begin{equation*}
\left\langle A u_{\alpha}-A b, w^{+}\right\rangle \leq \alpha \int_{\Gamma_{3}} j^{0}\left(x, u_{\alpha}(x) ;-w^{+}(x)\right) d \Gamma \tag{3.15}
\end{equation*}
$$

From condition $H(j)($ iv $)$ and the fact $j^{0}(x, t, 0)=0$ for all $t \in \mathbb{R}$ and $x \in \Gamma_{3}$, it follows that

$$
\begin{aligned}
& \alpha \int_{\Gamma_{3}} j^{0}\left(x, u_{\alpha}(x) ;-w^{+}(x)\right) d \Gamma \\
& \quad=\alpha \int_{\left\{u_{\alpha}>b\right\}} j^{0}\left(x, u_{\alpha}(x) ; b-u_{\alpha}(x)\right) d \Gamma+\alpha \int_{\left\{u_{\alpha} \leq b\right\}} j^{0}\left(x, u_{\alpha}(x) ; 0\right) d \Gamma \leq 0 .
\end{aligned}
$$

Taking into account the above inequality and (3.15), we obtain

$$
\left\langle A u_{\alpha}-A b, w^{+}\right\rangle \leq 0
$$

Therefore, by the strict monotonicity of $A$, one finds $\nabla w^{+}=0$, i.e. $w^{+}=0$. This proves that $u_{\alpha} \leq b$ in $\Omega$.
(ii) Denote $w=u_{\alpha}-u_{\infty}$. We are going to show that $w^{+}=0$ in $\Omega$. Recalling that $u_{\alpha}=u_{\infty}=0$ on $\Gamma_{1}$, we have $-w^{+} \in V$ due to $w=0$ on $\Gamma_{1}$. Inserting $v=-w^{+}$ into (3.4) we deduce

$$
\begin{align*}
& -\left\langle A u_{\alpha}-A u_{\infty}, w^{+}\right\rangle-\left\langle A u_{\infty}, w^{+}\right\rangle-\left\langle B u_{\alpha}, w^{+}\right\rangle \\
& \quad+\alpha \int_{\Gamma_{3}} j^{0}\left(x, u_{\alpha}(x) ;-w^{+}(x)\right) d \Gamma \geq\left\langle f,-w^{+}\right\rangle \tag{3.16}
\end{align*}
$$

Choosing $v=w^{+} \in K_{0}$ (because of $u_{\infty} \in K$, i.e. $u_{\infty}=b$ on $\Gamma_{3}$, and assertion (i), $u_{\alpha} \leq b$ ) in (3.5), we get

$$
-\left\langle A u_{\infty}, w^{+}\right\rangle=\left\langle-f+B u_{\infty}, w^{+}\right\rangle
$$

Putting the above equality into (3.16) implies

$$
-\left\langle A u_{\alpha}-A u_{\infty}, w^{+}\right\rangle-\left\langle B u_{\alpha}-B u_{\infty}, w^{+}\right\rangle+\alpha \int_{\Gamma_{3}} j^{0}\left(x, u_{\alpha}(x) ;-w^{+}(x)\right) d \Gamma \geq 0
$$

Taking into account the monotonicity of $B$, we have

$$
\left\langle A u_{\alpha}-A u_{\infty}, w^{+}\right\rangle \leq \alpha \int_{\Gamma_{3}} j^{0}\left(x, u_{\alpha}(x) ;-w^{+}(x)\right) d \Gamma
$$

Note that $u_{\infty} \in K$, so $u_{\infty}=b$ on $\Gamma_{3}$. Then

$$
\begin{aligned}
\int_{\Gamma_{3}} j^{0}\left(x, u_{\alpha}(x) ;-w^{+}(x)\right) d \Gamma= & \int_{\Gamma_{3}} j^{0}\left(x, u_{\alpha}(x) ;-\left(u_{\alpha}-b\right)^{+}\right) d \Gamma \\
= & \int_{\left\{u_{\alpha}>b\right\}} j^{0}\left(x, u_{\alpha}(x) ; b-u_{\alpha}(x)\right) d \Gamma \\
& +\int_{\left\{u_{\alpha} \leq b\right\}} j^{0}\left(x, u_{\alpha}(x) ; 0\right) d \Gamma \leq 0,
\end{aligned}
$$

where we have used hypothesis $H(j)(i v)$. Therefore, the last two inequalities give

$$
\left\langle A u_{\alpha}-A u_{\infty}, w^{+}\right\rangle \leq 0
$$

By virtue of the strict monotonicity of $A$, we conclude that $w^{+}=0$. This means that $u_{\alpha} \leq u_{\infty}$ on $\Omega$ which completes the proof of the theorem.

We complete this section with the monotonicity property of solutions to Problem 2 with respect to the parameter $\alpha>0$.

Proposition 9. Suppose that $H(j)$ and $H(0)$ are fulfilled. If, in addition, the following inequality holds

$$
\begin{equation*}
j^{0}\left(x, t ;-(t-s)^{+}\right)+c j^{0}\left(x, s ;(t-s)^{+}\right) \leq 0 \tag{3.17}
\end{equation*}
$$

for all $c \geq 1$, all $s, t \in \mathbb{R}, s \leq b, t \leq b$ and a.e. $x \in \Gamma_{3}$, then
(i) for each $\alpha>0$, Problem 2 has a unique solution $u_{\alpha} \in V$,
(ii) if $u_{i}:=u_{\alpha_{i}}$ is the unique solution to Problem 2 associated with $\alpha_{i}>0$ for $i=1,2$, and $\alpha_{1} \leq \alpha_{2}$, then one has

$$
u_{1} \leq u_{2} \quad \text { in } \Omega
$$

Proof. (i) For any $\alpha>0$ fixed, let $u_{1}$ and $u_{2}$ be two solutions to Problem 2, Then, we have

$$
\left\langle A u_{i}+B u_{i}, v\right\rangle+\alpha \int_{\Gamma_{3}} j^{0}\left(x, u_{i}(x) ; v(x)\right) d \Gamma \geq\langle f, v\rangle
$$

for all $v \in V$. We take $v=u_{2}-u_{1}$ into the above inequality with $i=1$ and $v=u_{1}-u_{2}$ for the inequality with $i=2$, and sum up the resulting inequalities to
obtain

$$
\begin{aligned}
& \left\langle A u_{1}-A u_{2}, u_{2}-u_{1}\right\rangle+\left\langle B u_{1}-B u_{2}, u_{2}-u_{1}\right\rangle \\
& \quad+\alpha \int_{\Gamma_{3}} j^{0}\left(x, u_{1}(x) ; u_{2}(x)-u_{1}(x)\right) d \Gamma \\
& \quad+\alpha \int_{\Gamma_{3}} j^{0}\left(x, u_{2}(x) ; u_{1}(x)-u_{2}(x)\right) d \Gamma \geq 0 .
\end{aligned}
$$

Next, we use the monotonicity of $B$ to get

$$
\begin{aligned}
\left\langle A u_{1}-A u_{2}, u_{1}-u_{2}\right\rangle \leq & \alpha \int_{\Gamma_{3}} j^{0}\left(x, u_{1}(x) ; u_{2}(x)-u_{1}(x)\right) d \Gamma \\
& +\alpha \int_{\Gamma_{3}} j^{0}\left(x, u_{2}(x) ; u_{1}(x)-u_{2}(x)\right) d \Gamma
\end{aligned}
$$

which together with inequality (3.17) entails

$$
\left\langle A u_{1}-A u_{2}, u_{1}-u_{2}\right\rangle \leq 0 .
$$

Therefore, from the strict monotonicity of $A$, we conclude that $u_{1}=u_{2}$. This means that for each $\alpha>0$, Problem 2 admits a unique solution.
(ii) Let $\alpha_{1}, \alpha_{2}>0$ be such that $\alpha_{1} \leq \alpha_{2}$. Let $w=u_{1}-u_{2}$. By Theorem 8 (i), we have $u_{i} \leq b$ in $\Omega$ for $i=1,2$. We shall verify that $w^{+}=0$ in $\Omega$. Since $u_{1}, u_{2} \in V$, one finds that $w=0$ on $\Gamma_{1}$, and $-w^{+} \in V$. Hence, we have

$$
\left\langle A u_{1}+B u_{1},-w^{+}\right\rangle+\alpha_{1} \int_{\Gamma_{3}} j^{0}\left(x, u_{1}(x) ;-w^{+}(x)\right) d \Gamma \geq\left\langle f,-w^{+}\right\rangle
$$

and

$$
\left\langle A u_{2}+B u_{2}, w^{+}\right\rangle+\alpha_{2} \int_{\Gamma_{3}} j^{0}\left(x, u_{2}(x) ; w^{+}(x)\right) d \Gamma \geq\left\langle f, w^{+}\right\rangle
$$

Summing up the inequalities above, it gives

$$
\begin{aligned}
\left\langle A u_{1}-\right. & \left.A u_{2}, w^{+}\right\rangle \\
\leq & \left\langle B u_{1}-B u_{2},-w^{+}\right\rangle \\
& +\alpha_{1}\left(\int_{\Gamma_{3}} j^{0}\left(x, u_{1}(x) ;-w^{+}(x)\right) d \Gamma+\frac{\alpha_{2}}{\alpha_{1}} \int_{\Gamma_{3}} j^{0}\left(x, u_{2}(x) ; w^{+}(x)\right) d \Gamma\right) \\
\leq & \alpha_{1}\left(\int_{\Gamma_{3}} j^{0}\left(x, u_{1}(x) ;-w^{+}(x)\right) d \Gamma+\frac{\alpha_{2}}{\alpha_{1}} \int_{\Gamma_{3}} j^{0}\left(x, u_{2}(x) ; w^{+}(x)\right) d \Gamma\right),
\end{aligned}
$$

where the last inequality is obtained by the monotonicity of $B$. Finally, from (3.17), we have

$$
\left\langle A u_{1}-A u_{2}, w^{+}\right\rangle \leq 0
$$

which shows that $w^{+}=0$, that is, $u_{1} \leq u_{2}$. The proof is complete.

Example 10. Note that hypothesis (3.17) is satisfied in the classical case, see [21, for the function $j(r)=\frac{1}{2}(r-b)^{2}$. Indeed, in this case, we have $\partial j(r)=\{r-b\}$ and $j^{0}(r ; s)=(r-b) s$ for all $r, s \in \mathbb{R}$. Let $s, t \in \mathbb{R}$ with $s \leq b, t \leq b$. Hence, for all $c \geq 1$, we get

$$
I:=j^{0}\left(t ;-(t-s)^{+}\right)+c j^{0}\left(s ;(t-s)^{+}\right)=(t-s)^{+}((1-c) b+c s-t) .
$$

For $t \leq s$, we have $(t-s)^{+}=0$ and $I=0$. For $s \leq t$, we obtain

$$
I:=(t-s)((1-c) b+c s-t) \leq(t-s)(c-1)(t-b) \leq 0 .
$$

Consequently, (3.17) holds.
Remark 11. It should be mentioned that if $p=2, \beta=0$, and $\mu=0$, then Theorems 7 and 8, and Proposition 9 have been recently obtained in [8.

## 4. Asymptotic Analysis and Dependence Result

In this section, we focus our attention on the investigation of the asymptotic behavior of solution to Problem 2, and we prove a continuous dependence theorem for solution to Problem 2 with respect to the data $(\alpha, \beta, g, r) \in \mathbb{R}^{2} \times L^{p^{\prime}}(\Omega) \times L^{p^{\prime}}\left(\Gamma_{2}\right)$.

We need one more hypothesis on the potential $j$.
$\underline{H(1)}:$ If $j^{0}(x, t ; b-t)=0$ for a.e. $x \in \Gamma_{3}$ and for some $t \in \mathbb{R}$, then $t=b$.
We start the section by providing the following convergence result which reveals that any sequence of solutions to Problem 2 converges to the unique solution of Problem 1 as the heat transfer coefficient $\alpha \rightarrow+\infty$.

Theorem 12. Suppose that $H(j), H(0)$ and $H(1)$ are satisfied. Let $\left\{\alpha_{n}\right\} \subset \mathbb{R}$, $\alpha_{n}>0$ be such that $\alpha_{n} \rightarrow+\infty$ as $n \rightarrow \infty$, and $u_{n}:=u_{\alpha_{n}}$ be a solution to Problem 2 with $\alpha=\alpha_{n}$, and $u_{\infty} \in K$ be the unique solution to Problem 11. Then

$$
\begin{equation*}
u_{n} \rightarrow u_{\infty} \quad \text { in } V, \quad \text { as } n \rightarrow \infty . \tag{4.1}
\end{equation*}
$$

Proof. Let $\left\{\alpha_{n}\right\}$ be a sequence of real positive numbers such that $\alpha_{n} \rightarrow+\infty$ as $n \rightarrow \infty$, and $\left\{u_{n}\right\}$ be a sequence of solutions to Problem 2 such that $u_{n}$ is a solution of Problem 2 corresponding to $\alpha=\alpha_{n}$. First, we verify that sequence $\left\{u_{n}\right\}$ is bounded in $V$. To this end, we take $v=u_{\infty}-u_{n} \in V, u_{\alpha}=u_{n}$ and $\alpha=\alpha_{n}$ in (3.4) to get

$$
\left\langle A u_{n}+B u_{n}, u_{n}-u_{\infty}\right\rangle \leq \alpha_{n} \int_{\Gamma_{3}} j^{0}\left(x, u_{n}(x) ; u_{\infty}(x)-u_{n}(x)\right) d \Gamma-\left\langle f, u_{\infty}-u_{n}\right\rangle .
$$

From $u_{\infty} \in K$, it is clear that $u_{\infty}=b$ on $\Gamma_{3}$. From hypothesis $H(j)(i v)$, one has

$$
\alpha_{n} \int_{\Gamma_{3}} j^{0}\left(x, u_{n}(x) ; u_{\infty}(x)-u_{n}(x)\right) d \Gamma=\alpha_{n} \int_{\Gamma_{3}} j^{0}\left(x, u_{n}(x) ; b-u_{n}(x)\right) d \Gamma \leq 0 .
$$

By the definition of $A$ and the monotonicity of $B$, we obtain

$$
\begin{aligned}
& \left\|u_{n}\right\|_{V}^{p}+\mu\left\|\nabla u_{n}\right\|_{L^{q}\left(\Omega ; \mathbb{R}^{N}\right)}^{q} \leq\left\langle A u_{n}, u_{n}\right\rangle+\left\langle B u_{\infty}, u_{\infty}-u_{n}\right\rangle \\
& \quad+\|f\|_{V^{*}}\left(\left\|u_{n}\right\|_{V}+\left\|u_{\infty}\right\|_{V}\right),
\end{aligned}
$$

and by using the Hölder inequality, we derive

$$
\begin{aligned}
\left\|u_{n}\right\|_{V}^{p} & +\mu\left\|\nabla u_{n}\right\|_{L^{q}\left(\Omega ; \mathbb{R}^{N}\right)}^{q} \\
\leq & \left\|u_{n}\right\|_{V}^{p-1}\left\|u_{\infty}\right\|_{V}+\mu\left\|\nabla u_{n}\right\|_{L^{p^{\prime}(q-1)}\left(\Omega ; \mathbb{R}^{N}\right)}^{q-1}\left\|u_{\infty}\right\|_{V}+\|f\|_{V^{*}}\left(\left\|u_{n}\right\|_{V}+\left\|u_{\infty}\right\|_{V}\right) \\
& +\beta\left\|u_{\infty}\right\|_{L^{\theta}(\Omega)}^{\theta}+M_{3}\left\|u_{\infty}\right\|_{L^{(\theta-1) p^{\prime}}(\Omega)}^{\theta-1}\left\|u_{n}\right\|_{V}
\end{aligned}
$$

with some $M_{3}>0$ which is independent of $n$, where we have used the fact that the embedding from $V$ into $L^{p}(\Omega)$ is continuous. Hence, it is easy to find that the sequence $\left\{u_{n}\right\}$ is bounded in $V$. Passing to a subsequence if necessary, we are able to find an element $u \in V$ such that

$$
\begin{equation*}
u_{n} \xrightarrow{w} u \text { in } V, \quad \text { as } n \rightarrow \infty . \tag{4.2}
\end{equation*}
$$

Next, we are going to prove that $u=u_{\infty}$. For any $w \in K$, we have $w-u_{n} \in V$ for each $n \in \mathbb{N}$. Inserting $v=w-u_{n}$ into (3.4), we have

$$
\begin{equation*}
\left\langle A u_{n}+B u_{n}, w-u_{n}\right\rangle+\alpha_{n} \int_{\Gamma_{3}} j^{0}\left(x, u_{n}(x) ; w(x)-u_{n}(x)\right) d \Gamma \geq\left\langle f, w-u_{n}\right\rangle \tag{4.3}
\end{equation*}
$$

for all $w \in K$. Since $w \in K$, it holds $w=b$ on $\Gamma_{3}$. The hypothesis $H(j)(i v)$ implies

$$
j^{0}\left(x, u_{n}(x) ; w(x)-u_{n}\right)=j^{0}\left(x, u_{n}(x) ; b-u_{n}(x)\right) \leq 0 \quad \text { for a.e. } x \in \Gamma_{3} .
$$

Combining the last two inequalities, it yields

$$
\left\langle A u_{n}+B u_{n}, w-u_{n}\right\rangle \geq\left\langle f, w-u_{n}\right\rangle \quad \text { for all } w \in K
$$

Next, by the monotonicity of $A$ and $B$, we get

$$
\left\langle A w+B w, w-u_{n}\right\rangle \geq\left\langle f, w-u_{n}\right\rangle \quad \text { for all } w \in K
$$

Letting $n \rightarrow \infty$ for the inequality above, one has

$$
\langle A w+B w, w-u\rangle \geq\langle f, w-u\rangle \quad \text { for all } w \in K
$$

Because $A$ and $B$ are both continuous and $K$ is nonempty closed and convex, we employ the Minty trick to deduce

$$
\langle A u+B u, w-u\rangle \geq\langle f, w-u\rangle \quad \text { for all } w \in K
$$

which implies

$$
\langle A u+B u, w\rangle=\langle f, w\rangle \quad \text { for all } w \in K_{0} .
$$

Keeping in mind that $u_{\infty} \in K$ is the unique solution of Problem so, when we verify that $u \in K$, then by the uniqueness of $u_{\infty}$, we get $u=u_{\infty}$. Obviously,
it suffices to demonstrate that $u=b$ on $\Gamma_{3}$. From the compactness of the trace operator of $V$ into $L^{p}\left(\Gamma_{3}\right)$ and the upper semicontinuity of the function $(u, v) \mapsto$ $j^{0}(x, u ; v)$, we use hypothesis $H(j)(i i i)$ and the Fatou lemma to find

$$
\begin{align*}
& \limsup _{n \rightarrow \infty} \int_{\Gamma_{3}} j^{0}\left(x, u_{n}(x) ; u_{\infty}(x)-u_{n}(x)\right) d \Gamma \\
& \quad \leq \int_{\Gamma_{3}} \limsup _{n \rightarrow \infty} j^{0}\left(x, u_{n}(x) ; u_{\infty}(x)-u_{n}(x)\right) d \Gamma \\
& \quad \leq \int_{\Gamma_{3}} j^{0}\left(x, u(x) ; u_{\infty}(x)-u(x)\right) d \Gamma=\int_{\Gamma_{3}} j^{0}(x, u(x) ; b-u(x)) d \Gamma \leq 0, \tag{4.4}
\end{align*}
$$

where we have used hypothesis $H(j)$ (iv). Combining the boundedness of $\left\{u_{n}\right\}$, the monotonicity of $A$ and $B$, and (3.4), we obtain

$$
\begin{aligned}
& -\alpha_{n} \int_{\Gamma_{3}} j^{0}\left(x, u_{n}(x) ; u_{\infty}(x)-u_{n}(x)\right) d \Gamma \\
& \quad \leq\left\langle A u_{\infty}+B u_{\infty}, u_{\infty}-u_{n}\right\rangle-\left\langle f, u_{\infty}-u_{n}\right\rangle \leq M_{0}
\end{aligned}
$$

for some $M_{0}>0$, which is independent of $n$. Hence

$$
\begin{equation*}
-\int_{\Gamma_{3}} j^{0}\left(x, u_{n}(x) ; u_{\infty}(x)-u_{n}(x)\right) d \Gamma \leq \frac{M_{0}}{\alpha_{n}} \tag{4.5}
\end{equation*}
$$

From (4.4) and (4.5), we have

$$
0 \leq-\int_{\Gamma_{3}} j^{0}(x, u(x) ; b-u(x)) d \Gamma \leq \limsup _{n \rightarrow \infty} \frac{M_{0}}{\alpha_{n}}=0
$$

The latter combined with hypothesis $H(j)(i v)$ gives $j^{0}(x, u(x) ; b-u(x))=0$ for a.e. $x \in \Gamma_{3}$. It follows from condition $H(1)$ that $u(x)=b$ for a.e. $x \in \Gamma_{3}$. This means that $u \in K$. Therefore, we conclude that $u=u_{\infty}$. Note that every weakly convergent subsequence of $\left\{u_{n}\right\}$ converges to the same limit $u_{\infty}$, so we conclude that the whole sequence of $\left\{u_{n}\right\}$ converges weakly to $u_{\infty}$.

Finally, we shall prove that $\left\{u_{n}\right\}$ converges strongly in $V$ to $u_{\infty}$. We insert $w=u \in K$ into (4.3), use $H(j)$ (iv) and the compactness of the embedding of $V$ to $L^{\theta}(\Omega)$ (owing to $\theta<p^{*}$ ) and derive

$$
\begin{aligned}
& \limsup _{n \rightarrow \infty}\left\langle A u_{n}, u_{n}-u\right\rangle \leq \limsup _{n \rightarrow \infty}\left\langle B u_{n}, u-u_{n}\right\rangle \\
& \quad+\limsup _{n \rightarrow \infty} \alpha_{n} \int_{\Gamma_{3}} j^{0}\left(x, u_{n}(x) ; b-u_{n}(x)\right) d \Gamma+\limsup _{n \rightarrow \infty}\left\langle f, u_{n}-u\right\rangle \leq 0 .
\end{aligned}
$$

The $\left(S_{+}\right)$-property of the operator $A$ demonstrates that $u_{n} \rightarrow u$ in $V$, as $n \rightarrow \infty$. This completes the proof of the theorem.

In what follows, we explore the continuous dependence result for solution to Problem 2 with respect to $(\alpha, \beta, g, r) \in \mathbb{R}^{2} \times L^{p^{\prime}}(\Omega) \times L^{p^{\prime}}\left(\Gamma_{2}\right)$.

Theorem 13. Let $\left\{\alpha_{n}\right\},\left\{\beta_{n}\right\}$ be sequences of real positive numbers such that $\alpha_{n} \rightarrow$ $\alpha, \beta_{n} \rightarrow \beta$ as $n \rightarrow \infty$ with $\alpha, \beta>0$, and $\left\{\left(g_{n}, r_{n}\right)\right\} \subset L^{p^{\prime}}(\Omega) \times L^{p^{\prime}}\left(\Gamma_{2}\right),(g, r) \in$ $L^{p^{\prime}}(\Omega) \times L^{p^{\prime}}\left(\Gamma_{2}\right)$ be such that

$$
\begin{equation*}
\left(g_{n}, r_{n}\right) \xrightarrow{w}(g, r) \quad \text { in } L^{p^{\prime}}(\Omega) \times L^{p^{\prime}}\left(\Gamma_{2}\right) . \tag{4.6}
\end{equation*}
$$

Let $\left\{u_{n}\right\}$ be a sequence such that $u_{n}$ is a solution of Problem 2 associated with $\alpha=$ $\alpha_{n}, \beta=\beta_{n}$ and $(g, r)=\left(g_{n}, r_{n}\right)$ for each $n \in \mathbb{N}$. Then, there exists a subsequence $\left\{u_{n_{k}}\right\}$ of $\left\{u_{n}\right\}$ such that $u_{n_{k}} \rightarrow u$ in $V$, where $u \in V$ is a solution of Problem 2 corresponding to $(\alpha, \beta, g, r) \in \mathbb{R}^{2} \times L^{p^{\prime}}(\Omega) \times L^{p^{\prime}}\left(\Gamma_{2}\right)$.

Proof. Let $u_{n}$ be a solution of Problem 2 corresponding to $\alpha=\alpha_{n}, \beta=\beta_{n}$ and $(g, r)=\left(g_{n}, r_{n}\right)$ for each $n \in \mathbb{N}$. Also, let $u_{\infty} \in K$ be the unique solution of Problem [1. Then, $u_{\infty}-u_{n} \in V$ and we have

$$
\begin{align*}
& \left\langle A u_{n}, u_{\infty}-u_{n}\right\rangle+\beta_{n} \int_{\Omega}\left|u_{n}\right|^{\theta-2} u_{n}\left(u_{\infty}-u_{n}\right) d x \\
& \quad+\alpha_{n} \int_{\Gamma_{3}} j^{0}\left(x, u_{n}(x) ; u_{\infty}(x)-u_{n}(x)\right) d \Gamma \\
& \geq \tag{4.7}
\end{align*}
$$

We claim that the sequence $\left\{u_{n}\right\}$ is bounded in $V$. By virtue of $H(j)$ (iv), we get

$$
\begin{aligned}
\left\langle A u_{n},\right. & \left.u_{n}\right\rangle \\
\leq & \left\langle A u_{n}, u_{\infty}\right\rangle+\beta_{n} \int_{\Omega}\left|u_{\infty}\right|^{\theta-2} u_{\infty}\left(u_{\infty}-u_{n}\right) d x \\
& +\alpha_{n} \int_{\Gamma_{3}} j^{0}\left(x, u_{n}(x) ; u_{\infty}(x)-u_{n}(x)\right) d \Gamma \\
& -\int_{\Omega} g_{n}(x)\left(u_{\infty}(x)-u_{n}(x)\right) d x+\int_{\Gamma_{2}} r_{n}(x)\left(u_{\infty}(x)-u_{n}(x)\right) d \Gamma \\
\leq & \left\langle A u_{n}, u_{\infty}\right\rangle-\int_{\Omega} g_{n}(x)\left(u_{\infty}(x)-u_{n}(x)\right) d x+\int_{\Gamma_{2}} r_{n}(x)\left(u_{\infty}(x)-u_{n}(x)\right) d \Gamma \\
& +\beta_{n} \int_{\Omega}\left|u_{\infty}\right|^{\theta-2} u_{\infty}\left(u_{\infty}-u_{n}\right) d x
\end{aligned}
$$

Employing the Hölder inequality, the Sobolev embedding theorem and the trace theorem, one finds

$$
\begin{aligned}
\left\|u_{n}\right\|_{V}^{p} & +\mu\left\|\nabla u_{n}\right\|_{L^{q}\left(\Omega ; \mathbb{R}^{N}\right)}^{q} \leq\left\langle A u_{n}, u_{n}\right\rangle d x \\
\leq & \left(\left\|u_{n}\right\|_{V}^{p-1}+\left\|u_{n}\right\|_{L^{p^{\prime}(q-1)}\left(\Omega ; \mathbb{R}^{N}\right)}^{q-1}\right)\left\|u_{\infty}\right\|_{V}+\beta_{n} M_{1}\left(\left\|u_{n}\right\|_{V}\left\|u_{\infty}\right\|_{V}^{\theta-1}+\left\|u_{\infty}\right\|_{V}^{\theta}\right) \\
& +M_{2}\left(\left\|g_{n}\right\|_{L^{p^{\prime}}(\Omega)}+\left\|r_{n}\right\|_{L^{p^{\prime}}\left(\Gamma_{2}\right)}\right)\left(\left\|u_{\infty}\right\|_{V}+\left\|u_{n}\right\|_{V}\right)
\end{aligned}
$$

for some $M_{1}, M_{2}>0$. This implies that $\left\{u_{n}\right\}$ is bounded in $V$. Passing to a subsequence if necessary, we may assume that $u_{n} \xrightarrow{w} u$ in $V$, as $n \rightarrow \infty$ with some $u \in V$. Next, we apply the monotonicity of $A$ and $B$ to obtain

$$
\begin{aligned}
& \left\langle A v, v-u_{n}\right\rangle+\beta_{n} \int_{\Omega}|v|^{\theta-2} v\left(v-u_{n}\right) d x+\alpha_{n} \int_{\Gamma_{3}} j^{0}\left(x, u_{n}(x) ; v(x)-u_{n}(x)\right) d \Gamma \\
& \quad \geq \int_{\Omega} g_{n}(x)\left(v(x)-u_{n}(x)\right) d x-\int_{\Gamma_{2}} r_{n}(x)\left(v(x)-u_{n}(x)\right) d \Gamma
\end{aligned}
$$

for all $v \in V$. Passing to the upper limit as $n \rightarrow \infty$, using the compactness of the embedding $V$ to $L^{\theta}(\Omega)$, and of traces $V$ to $L^{p}\left(\Gamma_{2}\right)$, and $V$ to $L^{p}\left(\Gamma_{3}\right)$, and the Lebesgue dominated convergence theorem, we have

$$
\begin{aligned}
\langle A v+ & B v, v-u\rangle+\alpha \int_{\Gamma_{3}} j^{0}(x, u(x) ; v(x)-u(x)) d \Gamma \\
\geq & \limsup _{n \rightarrow \infty}\left\langle A v+B v, v-u_{n}\right\rangle+\limsup _{n \rightarrow \infty}\left(\alpha_{n}-\alpha\right) \\
& \times \int_{\Gamma_{3}} j^{0}\left(x, u_{n}(x) ; v(x)-u_{n}(x)\right) d \Gamma \\
& +\limsup _{n \rightarrow \infty} \alpha \int_{\Gamma_{3}} j^{0}\left(x, u_{n}(x) ; v(x)-u_{n}(x)\right) d \Gamma+\limsup _{n \rightarrow \infty}\left(\beta_{n}-\beta\right) \\
& \times \int_{\Omega}|v|^{\theta-2} v\left(v-u_{n}\right) d x \\
\geq & \limsup _{n \rightarrow \infty} \int_{\Omega} g_{n}(x)\left(v(x)-u_{n}(x)\right) d x-\liminf _{n \rightarrow \infty} \int_{\Gamma_{2}} r_{n}(x)\left(v(x)-u_{n}(x)\right) d \Gamma \\
= & \int_{\Omega} g(x)(v(x)-u(x)) d x-\int_{\Gamma_{2}} r(x)(v(x)-u(x)) d \Gamma
\end{aligned}
$$

for all $v \in V$. Invoking the Minty argument, we obtain

$$
\begin{aligned}
\langle A u & +B u, v-u\rangle+\alpha \int_{\Gamma_{3}} j^{0}(x, u(x) ; v(x)-u(x)) d \Gamma \\
& \geq \int_{\Omega} g(x)(v(x)-u(x)) d x-\int_{\Gamma_{2}} r(x)(v(x)-u(x)) d \Gamma
\end{aligned}
$$

for all $v \in V$. This points out that $u \in V$ is a solution of Problem 2 corresponding to $(\alpha, \beta, g, r) \in \mathbb{R}^{2} \times L^{p^{\prime}}(\Omega) \times L^{p^{\prime}}\left(\Gamma_{2}\right)$.

Furthermore, we shall prove that $\left\{u_{n}\right\}$ converges strongly in $V$ to $u$. A simple calculation gives

$$
\begin{aligned}
& \left\langle A u_{n}, u_{n}-u\right\rangle \\
& \qquad \beta_{n} \int_{\Omega}\left|u_{n}\right|^{\theta-2} u_{n}\left(u-u_{n}\right) d x+\alpha_{n} \int_{\Gamma_{3}} j^{0}\left(x, u_{n}(x) ; u(x)-u_{n}(x)\right) d \Gamma \\
& \quad-\int_{\Omega} g_{n}(x)\left(u(x)-u_{n}(x)\right) d x+\int_{\Gamma_{2}} r_{n}(x)\left(u(x)-u_{n}(x)\right) d \Gamma .
\end{aligned}
$$

We pass to the upper limit as $n \rightarrow \infty$ to get $\lim _{\sup _{n \rightarrow \infty}}\left\langle A u_{n}, u_{n}-u\right\rangle \leq 0$. The latter combined with the $\left(S_{+}\right)$-property shows that $\left\{u_{n}\right\}$ converges strongly in $V$ to $u$. This completes the proof of the theorem.

Remark 14. If $p=2, \beta=0$, and $\mu=0$, then Theorem 12 coincides with the one obtained in [8. Particularly, when $p=2, \beta=0$, and $\mu=0$, Theorem 13 extends the result established in [8] from two perspectives:
(1) in [8], the authors derived the continuous dependence result with respect to $(g, r)$ only, while this paper deals with a more complicated situation that Problem 2 is perturbated by parameters $(\alpha, g, r)$;
(2) we prove that any sequence of perturbed solutions $\left\{u_{n}\right\}$ has a subsequence converging to a solution of Problem 2 corresponding to $(\alpha, g, r)$, whereas [8, Theorem 9] delivered only a convergence result in the weak topology.

Besides, it should be mentioned that [8, Sec. 6] provides several examples of functional $j$ which satisfy hypotheses $H(j)$ and $H(1)$. Finally, it is an open problem to study a more general case when $b$ is a suitable nonconstant function.

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