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Explicit solutions related to the Rubinstein binary-alloy solidification problem with a heat flux or a convective condition at the fixed face

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Similarity solutions for the two-phase Rubinstein binary-alloy solidification problem in a semi-infinite material are developed. These new explicit solutions are obtained by considering two cases: A heat flux or a convective boundary conditions at the fixed face, and the necessary and sufficient conditions on data are also given in order to have an instantaneous solidification process. We also show that all solutions for the binary-alloy solidification problem are equivalent under some restrictions for data. Moreover, this implies that the coefficient that characterizes the solidification front for the Rubinstein solution must verify an inequality as a function of all thermal and boundary conditions.

KEYWORDS

binary alloy, explicit solution, free boundary problem, liquidus and solidus curves, phase change process, similarity, solidification, Stefan problem

MSC CLASSIFICATION

35R35, 80A22, 35C05

1 | INTRODUCTION

Heat transfer during the solidification of alloys has been of particular interest in many engineering applications, especially in the field of casting, welding, thermal energy storage systems, and crystal growth in semiconductors.

A semi-infinite material of a binary alloy consisting of two components A and B is considered. Let C be the concentration of component B, and T the temperature. We assume that the solidification of the alloy is governed by a phase equilibrium diagram consisting of a "liquidus" curve $C = f_l(T)$ and a "solidus" curve $C = f_s(T)$. We suppose that f_s and f_l are increasing functions in the variable T assuming that they verify the following inequalities:

$$f_l(T_A) = f_s(T_A) < f_l(T) < f_s(T) < f_l(T_B) = f_s(T_B),$$

where T_A and T_B are the melting temperatures of A and B, respectively. The material is in the solid phase if $C > f_s(T)$ and in the liquid phase if $C < f_l(T)$. When C is between $f_s(T)$ and $f_l(T)$, the state of the material is not well defined and it is known by mushy region according to the description of the model proposed in [1–4], which can be appreciated in Figure 1.

The alloy is considered to be initially in a liquid state at constant temperature T_0 and constant concentration C_0. Then, a heat flow characterized by the constant q_0 is imposed on the fixed face $x = 0$ and a front of solidification $x = s(t)$ begins instantly separating the alloy in solid state $(0 < x < s(t))$ and liquid state $(x > s(t))$. The mathematical formulation of this crystallization process consists of finding the temperature $T = T(x, t)$ and the concentration $C = C(x, t)$, both defined for

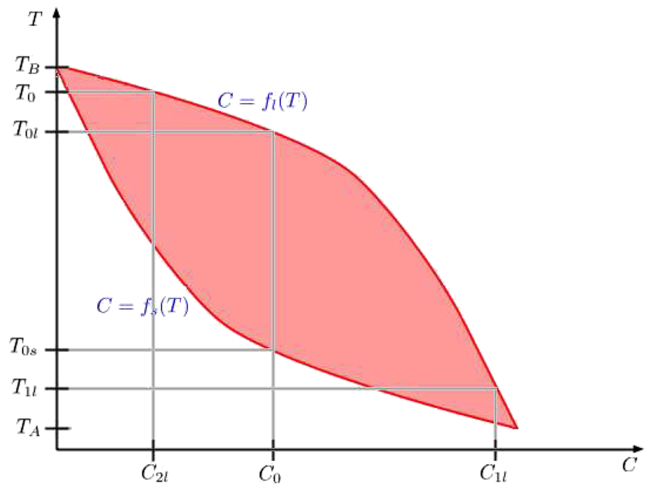

FIGURE 1 Concentration versus temperature (phase equilibrium diagram with liquidus and solidus curves). [Colour figure can be viewed at wileyonlinelibrary.com]

$x > 0$ and $t > 0$, the free boundary $x = s(t)$, defined for $t > 0$, and the critical solidification temperature T_k such that the following conditions are satisfied (problem (P_1)):

$$
\begin{array}{rlll}
i.\ \alpha_s T_{s_{xx}} = T_{s_t} & 0 < x < s(t), & t > 0, & \\
ii.\ \alpha_l T_{l_{xx}} = T_{l_t} & s(t) < x, & t > 0, & \\
iii.\ d_s C_{s_{xx}} = C_{s_t} & 0 < x < s(t), & t > 0, & \\
iv.\ d_l C_{l_{xx}} = C_{l_t} & s(t) < x, & t > 0, & \\
v.\ k_s T_{s_x}(0,t) = \dfrac{q_0}{\sqrt{t}} & (q_0 > 0), & t > 0, & \\
vi.\ T_s(s(t),t) = T_l(s(t),t) = T_k & & t > 0, & \\
vii.\ T_l(x,0) = T_0 & T_A < T_0 < T_B, & x > 0, & (1) \\
viii.\ T_l(\infty,t) = T_0 & T_A < T_0 < T_B, & t > 0, & \\
ix.\ C_{s_x}(0,t) = 0 & & t > 0, & \\
x.\ C_s(s(t),t) = f_s(T_k) & & t > 0, & \\
xi.\ C_l(x,0) = C_0 & x > 0, & & \\
xii.\ C_l(s(t),t) = f_l(T_k) & & t > 0, & \\
xiii.\ k_s T_{s_x}(s(t),t) - k_l T_{l_x}(s(t),t) = \gamma \rho s'(t) & & t > 0, & \\
xiv.\ d_l C_{l_x}(s(t),t) - d_s C_{s_x}(s(t),t) = [f_s(T_k) - f_l(T_k)]\, s'(t) & & t > 0, &
\end{array}
$$

where $\rho, k, \alpha, d, \gamma$ represent the mass density, the thermal conductivity, the thermal diffusivity, the mass diffusion, and the latent heat of fusion, with s and l denoting the solid and the liquid phase, respectively. When the condition on the fixed boundary $x = 0$ is given by a constant temperature $T_A < T_s(0,t) = T_1 < T_0$ (instead of condition (1.v)), the corresponding solidification problem was solved in [1, 2, 4].

A study of binary-alloy problems can be seen in [5–12]. Recent works on the solidification of a binary alloy are [6, 8, 13–24]. The heat flux condition (1.v) imposed at the fixed face was first considered in [25] for a two-phase Stefan problem for a semi-infinite homogeneous material. In [26], the same heat flux condition was considered when a density jump is supposed for the two-phase Stefan problem.

We can also consider the problem (P_2), which consists in finding the temperature $T = T(x,t)$ and the concentration $C = C(x,t)$, both defined for $x > 0$ and $t > 0$, the free boundary $x = s(t)$, defined for $t > 0$, and the critical solidification temperature T_k, such that the following conditions (1)(i)-(iv),(vi)-(xiv), and (v') are verified, where

$$v'.\ k_s T_{s_x}(0,t) = \frac{h_0}{\sqrt{t}}(T_s(0,t) - T_1),\ (h_0 > 0),\ t > 0.$$

The convective condition ($1.v'$) imposed at the fixed face was considered in [27] for a two-phase Stefan problem for a semi-infinite homogeneous material.

The goal of this paper is to find the necessary and/or sufficient conditions for data (initial and boundary conditions, and thermal coefficients of the binary alloy) in order to obtain an instantaneous phase-change process with the corresponding explicit solution of the similarity type. A review of explicit solution for Stefan-like problems is [28].

In Section 2, we obtain the necessary and sufficient condition (2) for problem (P_1) in order to obtain the explicit solution (3)–(8). In Section 3, we deduce the inequality (39) for the coefficient μ that characterizes the Rubinstein free boundary $x = s(t)$ for problem (P_1) defined in (3). In Section 4, we obtain the necessary and sufficient condition (41) for problem (P_2) in order to obtain the explicit solution (42)–(47). Finally, in Section 5, we deduce the inequality (52) for the coefficient μ that characterizes the Rubinstein free boundary $x = s(t)$ for problem (P_2) defined in (42).

2 | EXPLICIT SOLUTION FOR THE SOLIDIFICATION OF A BINARY ALLOY WITH A HEAT FLUX BOUNDARY CONDITION

In this section, we consider the problem (P_1) defined by differential equations and conditions (1) (i)–(xiv).

Theorem 1. *If q_0 verifies the following inequality:*

$$\frac{(T_0 - T_{0l})k_l}{\sqrt{\pi\alpha_l}} < q_0 < \frac{(T_0 - T_{0s})k_l}{\sqrt{\pi\alpha_l}}, \tag{2}$$

where $T_{0l} = f_l^{-1}(C_0)$ and $T_{0s} = f_s^{-1}(C_0)$, with $f_l^{-1}(C) = T$ is the inverse function of f_l and $f_s^{-1}(C) = T$ is the inverse function of f_s respectively, then there exists a unique solution of the similarity type for the free boundary problem (P_1), which is given by

$$s(t) = 2\lambda\sqrt{\alpha_s t},\ t > 0, \tag{3}$$

$$T_s(x,t) = \left(T_k - \frac{q_0}{k_s}\sqrt{\pi\alpha_s}erf(\lambda)\right) + \frac{q_0}{k_s}\sqrt{\pi\alpha_s}erf\left(\frac{x}{2\sqrt{\alpha_s t}}\right), 0 < x < s(t), t > 0, \tag{4}$$

$$T_l(x,t) = T_0 + \frac{T_k - T_0}{erfc\left(\frac{\sqrt{\alpha_s}}{\sqrt{\alpha_l}}\lambda\right)} + \frac{T_0 - T_k}{erfc\left(\frac{\sqrt{\alpha_s}}{\sqrt{\alpha_l}}\lambda\right)}erf\left(\frac{x}{2\sqrt{\alpha_l t}}\right),\ s(t) < x,\ t > 0, \tag{5}$$

$$C_s(x,t) = f_s(T_k),\ 0 < x < s(t),\ t > 0, \tag{6}$$

$$C_l(x,t) = C_0 + \frac{f_l(T_k) - C_0}{erfc\left(\frac{\sqrt{\alpha_s}}{\sqrt{d_l}}\lambda\right)}erfc\left(\frac{x}{2\sqrt{d_l t}}\right),\ s(t) < x,\ t > 0, \tag{7}$$

where the unknowns T_k and λ (coefficient that characterizes the free boundary $x = s(t)$) must satisfy the following equations:

$$T_k = F(\lambda),\ M(\lambda) = \phi(T_k),\ \lambda > 0,\ T_A < T_k < T_B, \tag{8}$$

where the real functions F, M, and ϕ are defined as follows:

$$\begin{aligned} erf(x) &= \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} \mathrm{d}t; \; erfc(x) = 1 - erf(x), \\ Q(x) &= \sqrt{\pi} x e^{x^2} erfc(x), \; F_1(x) = erfc(x) e^{x^2}, \; x > 0 \\ F(x) &= T_0 + \frac{\gamma \rho \alpha_l}{k_l} Q\left(\frac{\sqrt{\alpha_s}}{\sqrt{\alpha_l}} x\right) - \frac{q_0}{k_l} \sqrt{\alpha_l \pi} e^{-x^2} F_1\left(\frac{\sqrt{\alpha_s}}{\sqrt{\alpha_l}} x\right), \; x > 0, \\ M(x) &= \left[Q\left(\frac{\sqrt{\alpha_s}}{\sqrt{d_l}} x\right)\right]^{-1}, \; x > 0, \;\; \phi(x) = \frac{f_s(x) - f_l(x)}{C_0 - f_l(x)}, \; x \in (T_A, T_{0l}) \cup (T_{0l}, T_B). \end{aligned} \tag{9}$$

The proof of the theorem is based on the next proposition, which will be done below.

Proposition 1. *The following properties are valid,*

(a) *$erf(x)$ is an strictly increasing function, with $erf(0^+) = 0$ and $erf(+\infty) = 1$.*
(b) *Q is an strictly increasing function, with $Q(0^+) = 0$ and $Q(+\infty) = 1$.*
(c) *F_1 is an strictly decreasing function, with $F_1(0^+) = 1$ and $F_1(+\infty) = 0$.*
(d) *F is an strictly increasing function, with $F(0^+) = T_0 - \frac{\sqrt{\pi\alpha_l} q_0}{k_l}$ and $F(+\infty) = T_0 + \frac{\gamma\rho\alpha_l}{k_l}$.*
(e) *ϕ is an strictly increasing function on $[T_{0s}, T_{0l})$, with $\phi(T_{0s}) = 1$ and $\phi(T_{0l}^-) = \lim_{x \to T_{0l}^-} \phi(x) = +\infty$.*

Proof. (a), (b), and (c) follow from the definition of erf function, [25] and [3].

(d) Since Q is an increasing function and F_1 is a decreasing function, F is an increasing function. Further,

$$F(0) = T_0 + \frac{\gamma\rho\alpha_l}{k_l} Q(0) - \frac{q_0}{k_l} \sqrt{\alpha_l \pi} e^{-0^2} F_1(0) = T_0 - \frac{q_0}{k_l} \sqrt{\alpha_l \pi},$$

and

$$\begin{aligned} F(+\infty) &= \lim_{x \to +\infty} T_0 + \frac{\gamma\rho\alpha_l}{k_l} Q\left(\frac{\sqrt{\alpha_s}}{\sqrt{\alpha_l}} x\right) - \frac{q_0}{k_l} \sqrt{\alpha_l \pi} e^{-x^2} F_1\left(\frac{\sqrt{\alpha_s}}{\sqrt{\alpha_l}} x\right) \\ &= T_0 + \frac{\gamma\rho\alpha_l}{k_l}. \end{aligned}$$

(e) We have,

$$\phi(x) = \frac{f_s(x) - C_0}{C_0 - f_l(x)} + 1,$$

and $f_s(x) \geq C_0$ for all $x \geq T_{0s}$, $f_l(x) < C_0$ for all $x < T_{0l}$. Consequently, since f_s and f_l are increasing functions, ϕ is increasing on $[T_{0s}, T_{0l})$. Moreover,

$$\phi(T_{0s}) = \frac{f_s(T_{0s}) - f_l(T_{0s})}{C_0 - f_l(T_{0s})} = 1,$$

and since $f_s(x) > f_l(x)$ for all $x \in (T_A, T_B)$, in particular $f_s(T_{0l}) > f_l(T_{0l})$, from where

$$\phi(T_{0l}^-) = \lim_{x \to T_{0l}^-} \frac{f_s(x) - f_l(x)}{C_0 - f_l(x)} = +\infty.$$

□

We now prove Theorem 1.

Proof. The similarity solutions to the heat equation $\alpha u_{xx} = u_t$ has the form $u(x,t) = A + Berf\left(\frac{x}{2\sqrt{\alpha t}}\right)$ with A and B real numbers to be determined. Then, we can suppose that

$$\begin{aligned} T_s(x,t) &= A_s^T + B_s^T erf\left(\frac{x}{2\sqrt{\alpha_s t}}\right), \ T_l(x,t) = A_l^T + B_l^T erf\left(\frac{x}{2\sqrt{\alpha_l t}}\right), \\ C_s(x,t) &= A_s^C + B_s^C erf\left(\frac{x}{2\sqrt{d_s t}}\right), \ C_l(x,t) = A_l^C + B_l^C erf\left(\frac{x}{2\sqrt{d_l t}}\right). \end{aligned} \tag{10}$$

By (1) (vi), it results

$$T_s(s(t),t) = A_s^T + B_s^T erf\left(\frac{s(t)}{2\sqrt{\alpha_s t}}\right) = T_k,$$

where T_k is a constant to be determined. Then it follows that $s(t) = 2\lambda\sqrt{\alpha_s t}$ for some $\lambda > 0$, and again by (1) (vi), we have

$$T_s(s(t),t) = A_s^T + B_s^T erf(\lambda) = T_k, \tag{11}$$

$$T_l(s(t),t) = A_l^T + B_l^T erf\left(\frac{\sqrt{\alpha_s}}{\sqrt{\alpha_l}}\lambda\right) = T_k. \tag{12}$$

On the other hand, from (1) (v), we obtain that

$$k_s T_{s_x}(0,t) = \frac{k_s B_s^T}{\sqrt{\pi\alpha_s t}} = \frac{q_0}{\sqrt{t}},$$

from where

$$B_s^T = \frac{q_0}{k_s}\sqrt{\pi\alpha_s}. \tag{13}$$

Then, replacing (13) in (11), we have

$$A_s^T = T_k - \frac{q_0}{k_s}\sqrt{\pi\alpha_s} erf(\lambda).$$

From (1) (vii), it follows that

$$T_l(x,0) = A_l^T + B_l^T = T_0. \tag{14}$$

Subtracting (12) and (14), we obtain that

$$B_l^T erfc\left(\frac{\sqrt{\alpha_s}}{\sqrt{\alpha_l}}\lambda\right) = T_0 - T_k,$$

or equivalently,

$$B_l^T = \frac{T_0 - T_k}{erfc\left(\frac{\sqrt{\alpha_s}}{\sqrt{\alpha_l}}\lambda\right)}. \tag{15}$$

Replacing (15) in (14), we conclude that

$$A_l^T = T_0 - \frac{T_0 - T_k}{erfc\left(\frac{\sqrt{\alpha_s}}{\sqrt{\alpha_l}}\lambda\right)}.$$

Therefore, we obtain (4) and (5). Similarly, considering the equalities ix–xii from (1), we deduce that

$$C_{s_x}(0,t) = \frac{B_s^C}{\sqrt{\pi\alpha_s t}} = 0,$$

from where $B_s^C = 0$, and

$$C_s(s(t),t) = A_s^C + B_s^C erf\left(\frac{\sqrt{\alpha_s}}{\sqrt{d_s}}\lambda\right) = A_s^C = f_s(T_k).$$

Hence, $C_s \equiv f_s(T_k)$. On the other hand,

$$C_l(x,0) = A_l^C + B_l^C = C_0,$$

and

$$C_l(s(t),t) = A_l^C + B_l^C erf\left(\frac{\sqrt{\alpha_s}}{\sqrt{d_l}}\lambda\right) = f_l(T_k),$$

from where

$$A_l^C = C_0 - \frac{C_0 - f_l(T_k)}{erfc\left(\frac{\sqrt{\alpha_s}}{\sqrt{d_l}}\lambda\right)}, \quad B_l^C = \frac{C_0 - f_l(T_k)}{erfc\left(\frac{\sqrt{\alpha_s}}{\sqrt{d_l}}\lambda\right)}.$$

Therefore,

$$\begin{aligned} C_l(x,t) &= C_0 - \frac{C_0 - f_l(T_k)}{erfc\left(\frac{\sqrt{\alpha_s}}{\sqrt{d_l}}\lambda\right)} + \frac{C_0 - f_l(T_k)}{erfc\left(\frac{\sqrt{\alpha_s}}{\sqrt{d_l}}\lambda\right)} erf\left(\frac{x}{2\sqrt{d_l t}}\right) \\ &= C_0 + \frac{f_l(T_k) - C_0}{erfc\left(\frac{\sqrt{\alpha_s}}{\sqrt{d_l}}\lambda\right)} erfc\left(\frac{x}{2\sqrt{d_l t}}\right), \end{aligned}$$

that is, (7).

From (1) (*xiii*), we have

$$q_0 e^{-\lambda^2} + k_l \frac{T_k - T_0}{erfc\left(\frac{\sqrt{\alpha_s}}{\sqrt{\alpha_l}}\lambda\right)} \frac{e^{-\left(\frac{\sqrt{\alpha_s}}{\sqrt{\alpha_l}}\lambda\right)^2}}{\sqrt{\alpha_l \pi}} = \gamma\rho\lambda\sqrt{\alpha_s}.$$

Thus,

$$\begin{aligned} T_k &= T_0 + \left(\gamma\rho\lambda\sqrt{\alpha_s} - q_0 e^{-\lambda^2}\right) \frac{\sqrt{\alpha_l \pi}\, erfc\left(\frac{\sqrt{\alpha_s}}{\sqrt{\alpha_l}}\lambda\right)}{k_l e^{-\left(\frac{\sqrt{\alpha_s}}{\sqrt{\alpha_l}}\lambda\right)^2}} \\ &= T_0 + \frac{\gamma\rho\alpha_l}{k_l} Q\left(\frac{\sqrt{\alpha_s}}{\sqrt{\alpha_l}}\lambda\right) - q_0 e^{-\lambda^2} \frac{\sqrt{\alpha_l \pi}}{k_l} F_1\left(\frac{\sqrt{\alpha_s}}{\sqrt{\alpha_l}}\lambda\right) \\ &= F(\lambda), \end{aligned} \tag{16}$$

where Q, F_1 y F are given by (9).

From (1) (*xiv*), we obtain

$$\frac{1}{\sqrt{\pi}\frac{\sqrt{\alpha_s}}{\sqrt{d_l}}\lambda e^{\left(\frac{\sqrt{\alpha_s}}{\sqrt{d_l}}\lambda\right)^2} erfc\left(\frac{\sqrt{\alpha_s}}{\sqrt{d_l}}\lambda\right)} = \frac{f_s(T_k) - f_l(T_k)}{C_0 - f_l(T_k)}, \tag{17}$$

that is,

$$M(\lambda) = \phi(T_k). \tag{18}$$

By Proposition 1, Q and F are increasing functions. Then, M is a decreasing function, and

$$M(0^+) = \lim_{x\to 0^+} \frac{1}{Q(x)} = +\infty, \quad M(+\infty) = \frac{1}{Q(+\infty)} = 1, \tag{19}$$

$$F(0^+) = T_0 - \frac{\sqrt{\pi\alpha_l} q_0}{k_l}, \quad F(+\infty) = T_0 + \frac{\gamma\rho\alpha_l}{k_l}. \tag{20}$$

By (16) and (17), we have

$$M(\lambda) = \phi(F(\lambda)). \tag{21}$$

Then, taking into account (19), (20), and Proposition 1(e), we can assure the existence and uniqueness of λ verifying (21), if $T_{0s} \leq F(0) < F(+\infty) < T_{0l}$, or equivalently (2). □

Remark 1. Define $T_1 = T_s(0,t)$. Then,

$$T_1 = T_k - \frac{q_0}{k_s}\sqrt{\alpha_s \pi}erf(\lambda),$$

which is a constant independent of time t. Moreover,

$$T_1 \leq T_s(x,t) < T_k,\ 0 < x < s(t),\ t > 0.$$

In fact, since $erf(x)$ is an increasing function, and for $x \in (0, s(t))$, $0 < erf\left(\frac{x}{2\sqrt{\alpha_s t}}\right) < erf(\lambda)$, we have

$$T_1 = T_s(0,t) < T_s(x,t) < T_s(s(t),t) = T_k,\ 0 < x < s(t),\ t > 0.$$

Corollary 1. *Define $C_{1l} = f_l(T_1)$ and $C_{2l} = f_l(T_0)$. Then, $C_0 \in [C_{1l}, C_{2l}]$.*

Proof. By definition, $C_0 = f_l(T_{0l})$. By hypothesis, $f_l(T_l) \geq C_l$. Hence

$$C_{2l} = f_l(T_0) = f_l(T_l(x,0)) \geq C_l(x,0) = C_0. \tag{22}$$

On the other hand, there are values x_0 and t_0, with $0 < x_0 < s(t_0)$, such that

$$T_{0s} = f_s^{-1}(C_0) = T_s(x_0, t_0),$$

and since $f_l(T) < f_s(T)$, it results $T_{0s} = f_s^{-1}(C_0) < f_l^{-1}(C_0) = T_{0l}$.

Thus, by Remark 1, it results $T_1 \leq T_s(x_0,t_0) = T_{0s} \leq T_{0l}$, from where, taking into account that f_l is an increasing function, we obtain that

$$C_{1l} = f_l(T_1) \leq f_l(T_{0l}) = C_0. \tag{23}$$

By (22) and (23) we conclude that $C_0 \in [C_{1l}, C_{2l}]$. □

Remark 2. Following the steps on [4, Section 4] we can obtain that a mushy region will appear for the free boundary problem (P_1) if the condition

$$\frac{T_0 - T_k}{C_0 - f_l(T_k)} < (f_l^{-1})'(f_l(T_k))\frac{\alpha_l}{d_l}\frac{Q\left(\frac{\sqrt{\alpha_s}}{\sqrt{\alpha_l}}\lambda\right)}{Q\left(\frac{\sqrt{\alpha_s}}{\sqrt{d_l}}\lambda\right)}. \tag{24}$$

From Proposition (1)-(b), it follows that

$$\frac{\alpha_l}{d_l}\frac{Q\left(\frac{\sqrt{\alpha_s}}{\sqrt{\alpha_l}}\lambda\right)}{Q\left(\frac{\sqrt{\alpha_s}}{\sqrt{d_l}}\lambda\right)} > \frac{\alpha_l}{d_l}Q\left(\frac{\sqrt{\alpha_s}}{\sqrt{\alpha_l}}\lambda\right). \tag{25}$$

Then the condition (24) will hold if d_l is sufficiently small.

3 | AN INEQUALITY FOR THE COEFFICIENT μ OF THE RUBINSTEIN FREE BOUNDARY

For the solution given in Theorem 1, the temperature on the fixed face $x = 0$ is given by

$$\tilde{T}_1 = T_s(0,t) = T_k - \frac{q_0}{k_s}\sqrt{\alpha_s \pi}erf(\lambda). \tag{26}$$

Since $\tilde{T}_1 < T_k$, we can consider the problem (P_R) given by (1) (i)–(iv),$(\tilde{v})$,(vi)–(xiv), where

$$\tilde{v}.\ \tilde{T}_s(0,t) = T_1,\ \ t > 0. \tag{27}$$

Problem (P_R) has a unique solution, known as Rubinstein solution, given by [1], that is

$$\tilde{T}_s(x,t) = \mathscr{A}_s^T + \mathscr{B}_s^T erf\left(\frac{x}{2\sqrt{\alpha_s t}}\right),\ \ \tilde{T}_l(x,t) = \mathscr{A}_l^T + \mathscr{B}_l^T erf\left(\frac{x}{2\sqrt{\alpha_l t}}\right), \tag{28}$$

$$\tilde{C}_s(x,t) = \mathscr{A}_s^C + \mathscr{B}_s^C erf\left(\frac{x}{2\sqrt{d_s t}}\right),\ \ \tilde{C}_l(x,t) = \mathscr{A}_l^C + \mathscr{B}_l^C erf\left(\frac{x}{2\sqrt{d_l t}}\right), \tag{29}$$

$$\tilde{s}(t) = 2\mu\sqrt{\alpha_s t}, \tag{30}$$

where

$$\mathscr{A}_s^T(\mu) = T_1,\ \ \mathscr{B}_s^T(\mu) = \frac{\tilde{T}_k - T_1}{erf(\mu)}, \tag{31}$$

$$\mathscr{A}_l^T(\mu) = T_0 + \frac{\tilde{T}_k - T_0}{erfc\left(\frac{\sqrt{\alpha_s}}{\sqrt{\alpha_l}}\mu\right)},\ \ \mathscr{B}_l^T(\mu) = \frac{T_0 - \tilde{T}_k}{erfc\left(\frac{\sqrt{\alpha_s}}{\sqrt{\alpha_l}}\mu\right)}, \tag{32}$$

$$\mathscr{A}_s^C(\mu) = f_s(\tilde{T}_k),\ \ \mathscr{B}_s^C(\mu) = 0, \tag{33}$$

$$\mathscr{A}_l^C(\mu) = C_0 + \frac{f_l(\tilde{T}_k) - C_0}{erfc\left(\frac{\sqrt{\alpha_s}}{\sqrt{d_l}}\mu\right)},\ \ \mathscr{B}_l^C(\mu) = \frac{C_0 - f_l(\tilde{T}_k)}{erfc\left(\frac{\sqrt{\alpha_s}}{\sqrt{d_l}}\mu\right)}, \tag{34}$$

and the unknowns $\tilde{T}_k$ and μ (coefficient that characterizes the free boundary $x = s(t)$) satisfy the following equations:

$$\tilde{T}_k = G(\mu),\ \ M(\mu) = \phi(\tilde{T}_k),\ \ \mu > 0,\ \ T_A < \tilde{T}_k < T_B, \tag{35}$$

where G is defined as follows:

$$G(x) = \frac{\gamma\rho\alpha_s\alpha_l Q_1(x) Q\left(\frac{\sqrt{\alpha_s}}{\sqrt{\alpha_l}}x\right) + T_1 k_s\alpha_l Q\left(\frac{\sqrt{\alpha_s}}{\sqrt{\alpha_l}}x\right) + T_0 k_l\alpha_s Q_1(x)}{k_s\alpha_l Q\left(\frac{\sqrt{\alpha_s}}{\sqrt{\alpha_l}}x\right) + k_l\alpha_s Q_1(x)},\ \ x > 0,$$

$$Q_1(x) = \sqrt{\pi}xe^{x^2}erf(x). \tag{36}$$

Theorem 2. *If in the Rubinstein solution (28)–(35) we take $T_1 = \tilde{T}_1$, under condition (2) and (26), we obtain that the two free boundaries problems (P_1) and (P_R) are equivalents, that is,*

$$\mu = \lambda,\ \ \tilde{T}_k = T_k, \tag{37}$$

$$T_s(x,t) = \tilde{T}_s(x,t),\ \ T_l(x,t) = \tilde{T}_l(x,t),\ \ C_s(x,t) = \tilde{C}_s(x,t),\ \ C_l(x,t) = \tilde{C}_l(x,t). \tag{38}$$

Corollary 2. *Under the hypothesis of Theorem 2, the coefficient μ of the free boundary $\tilde{s}(t)$ and the initial temperature $\tilde{T}_k$ of solidification corresponding to the Rubinstein solution to the free boundary problem (P_R) satisfies the inequalities*

$$\frac{\sqrt{\alpha_l}}{\sqrt{\alpha_s}}\frac{\tilde{T}_k - T_1}{T_0 - T_{0s}}\frac{k_s}{k_l} < erf(\mu) < \frac{\sqrt{\alpha_l}}{\sqrt{\alpha_s}}\frac{\tilde{T}_k - T_1}{T_0 - T_{0l}}\frac{k_s}{k_l}. \tag{39}$$

Proof. From the solution to problem P_R, equivalent to problem P_1, by using condition (1) (v), we have (26), from where

$$q_0 = \frac{k_s(\tilde{T}_k - T_1)}{\sqrt{\alpha_s \pi} erf(\mu)},$$

where T_1 is the temperature boundary condition for the Rubinstein solution.

By the equivalence between problems (P_1) and (P_R), q_0 must satisfy condition (2). Then, the next inequalities hold:

$$\frac{(T_0 - T_{0l})k_l}{\sqrt{\pi \alpha_l}} < \frac{k_s(T_k - T_1)}{\sqrt{\alpha_s \pi} erf(\mu)} < \frac{(T_0 - T_{0s})k_l}{\sqrt{\pi \alpha_l}},$$

or equivalently (39). □

Remark 3. The inequalities (39) has a complete physical meaning for the solution to problem P_R when the corresponding parameters verifies the inequality

$$\frac{\sqrt{\alpha_l}}{\sqrt{\alpha_s}} \frac{\tilde{T}_k - T_1}{T_0 - T_{0l}} \frac{k_s}{k_l} < 1.$$

Remark 4. The results in this section generalizes [25]. In fact, if $C(x, t)$ is constant and if we consider only the thermal problem, then condition (2) is the inequality given in [25, Lemma 1] obtained for the two-phase Stefan problem with a heat flux condition at $x = 0$.

4 | EXPLICIT SOLUTION FOR THE BINARY ALLOY SOLIDIFICATION PROBLEM WITH A CONVECTIVE BOUNDARY CONDITION

In this section, we consider the next two-phase Rubinstein type binary-alloy solidification problem defined by (Problem (P_2)):

$$\begin{array}{llll}
i. & \alpha_s T_{s_{xx}} = T_{s_t} & 0 < x < s(t), & t > 0, \\
ii. & \alpha_l T_{l_{xx}} = T_{l_t} & s(t) < x, & t > 0, \\
iii. & d_s C_{s_{xx}} = C_{s_t} & 0 < x < s(t), & t > 0, \\
iv. & d_l C_{l_{xx}} = C_{l_t} & s(t) < x, & t > 0, \\
v. & k_s T_{s_x}(0,t) = \frac{h_0}{\sqrt{t}}(T_s(0,t) - T_\infty) & (h_0 > 0), & t > 0, \\
vi. & T_s(s(t),t) = T_l(s(t),t) = T_k & & t > 0, \\
vii. & T_l(x,0) = T_0 & T_A < T_0 < T_B, & x > 0, \\
viii. & T_l(\infty,t) = T_0 & T_A < T_0 < T_B, & t > 0, \\
ix. & C_{s_x}(0,t) = 0 & & t > 0, \\
x. & C_s(s(t),t) = f_s(T_k) & & t > 0, \\
xi. & C_l(x,0) = C_0 & x > 0, & \\
xii. & C_l(s(t),t) = f_l(T_k) & & t > 0, \\
xiii. & k_s T_{s_x}(s(t),t) - k_l T_{l_x}(s(t),t) = \gamma \rho s'(t) & & t > 0, \\
xiv. & d_l C_{l_x}(s(t),t) - d_s C_{s_x}(s(t),t) = [f_s(T_k) - f_l(T_k)]\, s'(t) & & t > 0,
\end{array} \tag{40}$$

where T_∞ is the bulk temperature at a large distance from the fixed face $x = 0$.

Following the study done in Section 2, we can obtain the next result.

Theorem 3. *If the coefficient h_0, which characterized the convective boundary condition (40) (v), verifies the following inequalities:*

$$\frac{(T_0 - T_{0l})k_l}{(T_{0l} - T_\infty)\sqrt{\pi \alpha_l}} < h_0 < \frac{(T_0 - T_{0s})k_l}{(T_{0s} - T_\infty)\sqrt{\pi \alpha_l}}, \tag{41}$$

where $T_{0l} = f_l^{-1}(C_0)$ *and* $T_{0s} = f_s^{-1}(C_0)$, *where* $f_l^{-1}(C) = T$ *is the inverse function of* f_l *and* $f_s^{-1}(C) = T$ *is the inverse function of* f_s *respectively, the free boundary problem (40) has a unique similarity type solution given by*

$$\hat{s}(t) = 2\delta\sqrt{\alpha_s t},\ t > 0, \tag{42}$$

$$\hat{T}_s(x,t) = \hat{T}_k + \frac{h_0\sqrt{\pi\alpha_s}(\hat{T}_k - T_\infty)}{k_s + h_0\sqrt{\pi\alpha_s}erf(\delta)}\left(erf\left(\frac{x}{2\sqrt{\alpha_s t}}\right) - erf(\delta)\right),\ 0 < x < \hat{s}(t),\ t > 0, \tag{43}$$

$$\hat{T}_l(x,t) = T_0 + \frac{\hat{T}_k - T_0}{erfc\left(\frac{\sqrt{\alpha_s}}{\sqrt{\alpha_l}}\delta\right)}erfc\left(\frac{x}{2\sqrt{\alpha_l t}}\right),\ \hat{s}(t) < x,\ t > 0, \tag{44}$$

$$\hat{C}_s(x,t) = f_s(\hat{T}_k),\ 0 < x < \hat{s}(t),\ t > 0, \tag{45}$$

$$\hat{C}_l(x,t) = C_0 + \frac{f_l(\hat{T}_k) - C_0}{erfc\left(\frac{\sqrt{\alpha_s}}{\sqrt{d_l}}\delta\right)}erfc\left(\frac{x}{2\sqrt{d_l t}}\right),\ \hat{s}(t) < x,\ t > 0, \tag{46}$$

where the unknowns $\hat{T}_k$ *and* δ *(coefficient that characterizes the free boundary* $x = s(t)$*) satisfy the following equations:*

$$\hat{T}_k = W(\delta),\ M(\delta) = \phi(\hat{T}_k),\ \delta > 0,\ T_A < \hat{T}_k < T_B, \tag{47}$$

where the real function W is defined as follows:

$$W(x) = T_\infty + \frac{T_0 - T_\infty}{F_2(x) + 1} + \frac{\gamma\rho\sqrt{\pi\alpha_s\alpha_l}}{H(x)},$$

$$F_2(x) = \frac{h_0 k_s\sqrt{\pi\alpha_l}e^{-x^2}F_1\left(\frac{\sqrt{\alpha_s}}{\sqrt{\alpha_l}}x\right)}{k_l\left(k_s + h_0\sqrt{\pi\alpha_s}erf(x)\right)},$$

$$H(x) = \frac{h_0 k_s\sqrt{\pi\alpha_l}}{xe^{x^2}\left(k_s + h_0\sqrt{\pi\alpha_s}erf(x)\right)} + \frac{k_l\sqrt{\pi\alpha_s}}{\sqrt{\alpha_l}Q\left(\frac{\sqrt{\alpha_s}}{\sqrt{\alpha_l}}x\right)}.$$

The proof of the theorem is based on the next proposition.

Proposition 2. *The following properties are valid,*

(a) F_2 *is an strictly decreasing function, with* $F_2(0^+) = \frac{h_0}{k_l}\sqrt{\alpha_l\pi}$ *and* $F_2(+\infty) = 0$.
(b) H *is an strictly decreasing function, with* $H(0^+) = +\infty$ *and* $H(+\infty) = \frac{\sqrt{\pi\alpha_s}}{\sqrt{\alpha_l}}k_l$.
(c) W *is an strictly increasing function, with* $W(0^+) = T_\infty + \frac{T_0 - T_\infty}{1 + \frac{h_0\sqrt{\pi\alpha_l}}{k_l}}$ *and* $W(+\infty) = T_0 + \frac{\gamma\rho\alpha_l}{k_l}$.

Proof. We obtain (a) and (b) by Proposition 1, and (c) by using properties (a) and (b). □

Remark 5. Similarly to Remark 2, we can obtain analogous inequalities to (24) and (25) for the appearance of a mushy region of the free boundary problem (P_2). Then, a mushy region appears if d_l is sufficiently small.

5 | AN INEQUALITY FOR THE COEFFICIENT μ OF THE RUBINSTEIN FREE BOUNDARY

For the solution given in Theorem 3, the temperature in the fixed face $x = 0$ is given by

$$\hat{T}_1 = \hat{T}_s(0,t) = \hat{T}_k - \frac{h_0\sqrt{\pi\alpha_s}erf(\delta)(\hat{T}_k - T_\infty)}{k_s + h_0\sqrt{\pi\alpha_s}erf(\delta)}. \tag{48}$$

Since $\hat{T}_1 < \hat{T}_k$, we can consider the problem (P_R) defined by (40) (i)–(iv), $(\tilde{v})$, (vi)–(xiv), where

$$\tilde{v}.\ \hat{T}_s(0,t) = T_1,\ t > 0, \tag{49}$$

which has a unique solution, known as Rubinstein solution, and was given by (28)–(35).

Theorem 4. *If $T_1 = \hat{T}_1$, under condition (41) and (48), we obtain that the two free boundaries problems (P_2) and (P_R) are equivalents, that is,*

$$\mu = \delta,\ \hat{T}_k = \tilde{T}_k, \tag{50}$$

$$\hat{T}_s(x,t) = \tilde{T}_s(x,t),\ \hat{T}_l(x,t) = \tilde{T}_l(x,t),\ \hat{C}_s(x,t) = \tilde{C}_s(x,t),\ \hat{C}_l(x,t) = \tilde{C}_l(x,t). \tag{51}$$

Corollary 3. *Under the hypothesis of Theorem 4, the coefficient μ of the free boundary $\tilde{s}(t)$ and the initial temperature $\tilde{T}_k$ of solidification, corresponding to the Rubinstein solution to the free boundary problem (P_R) satisfies the inequalities*

$$\frac{\sqrt{\alpha_l}}{\sqrt{\alpha_s}}\frac{k_s}{k_l}\frac{\tilde{T}_k - T_1}{T_0 - T_{0s}}\frac{T_{0s} - T_\infty}{T_1 - T_\infty} < erf(\mu) < \frac{\sqrt{\alpha_l}}{\sqrt{\alpha_s}}\frac{k_s}{k_l}\frac{\tilde{T}_k - T_1}{T_0 - T_{0l}}\frac{T_{0l} - T_\infty}{T_1 - T_\infty}. \tag{52}$$

Proof. From the solution to problem P_R equivalent to problem P_2, by using condition (40) (v), we obtain

$$h_0 = \frac{k_s}{\sqrt{\alpha_s\pi}erf(\mu)}\frac{\tilde{T}_k - T_1}{T_1 - T_\infty}.$$

By the equivalence between problems (P_2) and (P_R), h_0 must satisfy inequalities (41). Then, the next inequality holds:

$$\frac{(T_0 - T_{0l})k_l}{(T_{0l} - T_\infty)\sqrt{\pi\alpha_l}} < \frac{k_s}{\sqrt{\alpha_s\pi}erf(\mu)}\frac{\tilde{T}_k - T_1}{T_1 - T_\infty} < \frac{(T_0 - T_{0s})k_l}{(T_{0s} - T_\infty)\sqrt{\pi\alpha_l}}, \tag{53}$$

or equivalently (52). □

If we wish to obtain an inequality for the coefficient μ of the Rubinstein solution, we must eliminate the dependence on T_∞. Taking the maximum of $\frac{T_{0s}-T_\infty}{T_1-T_\infty}$ respect to T_∞ on the left hand side, and the minimum of $\frac{T_{0l}-T_\infty}{T_1-T_\infty}$ respect to T_∞ on the right hand side in (52), we get

Remark 6.

$$\frac{\sqrt{\alpha_l}}{\sqrt{\alpha_s}}\frac{k_s}{k_l}\frac{\tilde{T}_k - T_1}{T_0 - T_{0s}} < erf(\mu) < \frac{\sqrt{\alpha_l}}{\sqrt{\alpha_s}}\frac{k_s}{k_l}\frac{\tilde{T}_k - T_1}{T_0 - T_{0l}}, \tag{54}$$

which is the same inequalities given by (39). Therefore, Remark 3 holds.

Remark 7. The results in this section generalizes [27]. In fact, if $C(x,t)$ is constant and if we consider only the thermal problem, then condition (41) is the inequality given in [27, Theorem 1] obtained for the two-phase Stefan problem with a heat flux condition at $x=0$.

6 | CONCLUSION

Two explicit solutions of a similarity type for the temperature and the concentration in the two-phase binary-alloy solidification problem in a semi-infinite material were obtained for two different boundary conditions at the fixed face $x=0$. When the boundary condition is the heat flux condition given by (1) (v), then there exists an instantaneous solidification process if and only if the coefficient q_0 satisfies the inequalities

$$\frac{(T_0-T_{0l})k_l}{\sqrt{\pi\alpha_l}} < q_0 < \frac{(T_0-T_{0s})k_l}{\sqrt{\pi\alpha_l}}.$$

When the boundary condition is a convective condition given by (40) (v), then there exists an instantaneous solidification process if and only if the coefficient h_0 satisfies the inequalities

$$\frac{(T_0-T_{0l})k_l}{(T_{0l}-T_1)\sqrt{\pi\alpha_l}} < h_0 < \frac{(T_0-T_{0s})k_l}{(T_{0s}-T_1)\sqrt{\pi\alpha_l}}.$$

Moreover, if μ is the coefficient of the free boundary corresponding to the Rubinstein solution (28)–(35) to the free boundary problem (P_R), then it satisfies the inequalities

$$\frac{\sqrt{\alpha_l}}{\sqrt{\alpha_s}}\frac{\tilde{T}_k-T_1}{T_0-T_{0s}}\frac{k_s}{k_l} < erf(\mu) < \frac{\sqrt{\alpha_l}}{\sqrt{\alpha_s}}\frac{\tilde{T}_k-T_1}{T_0-T_{0l}}\frac{k_s}{k_l}. \tag{55}$$

NOMENCLATURE

C_0	initial concentration
d	mass diffusion
f_l	liquidus curve
f_s	solidus curve
h_0	coefficient that characterizes the heat transfer condition at x=0
k	thermal conductivity
q_0	coefficient that characterizes the heat flux at x=0
$s(t)$	position of the solidification front at time t
t	time
T_0	initial temperature
T_k	critical temperature of solidification
α	thermal diffusivity
γ	latent heat of fusion by density of mass
λ	constant which characterizes the moving boundary for heat flux boundary condition
δ	constant which characterizes the moving boundary for convective boundary condition
ρ	mass density

SUBSCRIPT

s	solid phase
l	liquid phase

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CONFLICT OF INTEREST STATEMENT

This work does not have any conflicts of interest.

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REFERENCES

1. L. I. Rubinstein, *The Stefan problem*, Translations of Mathematical Monographs 27, American Mathematical Society, 1971.
2. A. D. Solomon, D. G. Wilson, and V. Alexiades, *Explicit solutions to phase change problems*, Quart. Appl. Math. **41** (1983), 237–243.
3. D. G. Wilson, *Existence and uniqueness for similarity solutions of one dimensional multi-phase Stefan problems*, SIAM J. Appl. Math. **35** (1978), 135–147.
4. D. G. Wilson, A. D. Solomon, and V. Alexiades, *A shortcoming of the explicit solution for the binary alloy solidification problem*, Lett. Heat Mass Transf. **9** (1982), 421–428.
5. D. V. Alexandrov, A. P. Malygin, and A.D. Solomon, *Macroscopic global modeling of binary alloy solidification processes*, Quart. Appl. Math. **43** (1985), no. 2, 143–158.
6. S. Carpy and H. Mathis, *Modeling binary alloy solidification by a random projection method*, Numer. Methods Partial Differential Eq. **35** (2019), 733–760.
7. A. Jakhar, P. Rath, and S.K. Mahapatra, *A similarity solution for phase change of binary alloy with shrinkage or expansion*, Eng. Sci. Technol - Int. J. **19** (2016), 1390–1399.
8. F. B. Planella, C. P. Please, and R. A. Van Gorder, *Extended Stefan problem for the solidification of binary alloys in a sphere*, Eur. J. Appl. Math. **32** (2021), no. 2, 242–279.
9. S. Sundarraj and V. R. Voller, *The binary alloy problem in an expanding domain: The microsegregation problem*, Int. J. Heat Mass Transfer **36** (1993), no. 3, 713–723.
10. L. N. Tao, *On solidification of a binary alloy*, Q. J. Mech. Appl. Math. **33** (1980), no. 2, 211–225.
11. R. E. White, *The binary alloy problem: Existence, uniqueness, and numerical approximations*, SIAM J. Numer. Anal. **22** (1985), no. 2, 205–244.
12. D. G. Wilson, A. D. Solomon, and V. Alexiades, *A model of binary alloy solidification*, Int. J. Numer. Methods Eng. **20** (1984), no. 6, 1067–1084.
13. D. V. Alexandrov, D. Lee, and H.N. Huang, *Numerical modeling of one-dimensional binary solidification with a mushy layer evolution*, Numer. Math., Theory Methods Appl. **5** (2012), no. 2, 157–185.
14. D. V. Alexandrov and A. P. Malygin, *Self-similar solidification of an alloy from a cooled boundary*, Int. J. Heat Mass Transfer **49** (2006), 763–769.
15. M. Assunção, M. Vynnycky, and S.L. Mitchell, *On small-time similarity-solution behaviour in the solidification shrinkage of binary alloys*, Eur. J. Appl. Math. **32** (2021), no. 2, 199–225.
16. J. D. Chung, J. S. Lee, S.T. Ro, and H. Yoo, *An analytical approach to the conduction-dominated solidification of binary mixtures*, Int. J. Heat Mass Transfer **42** (1999), 373–377.
17. U. K. Mohanty and H. Sarangi, Solidification of metals and alloys, *Casting Processes and Modelling of Metallic Materials, chapter 2*, Z. Abdallah and N. Aldoumani, (eds.), IntechOpen, Rijeka, 2020, pp. 19–40.
18. J. R. G. Parkinson, D. F. Martin, A.J. Wells, and R. F. Katz, *Modelling binary alloy solidification with adaptive mesh refinement*, J. Comput. Phys.: X **5** (2020), 100043.
19. A. G. Petrova, *On the problem of control the composition of material in the binary alloy solidification process*, J. Inverse Ill-Posed Probl. **18** (2010), no. 3, 307–320.
20. S. L. Sobolev, *Driving force for binary alloy solidification under far from local equilibrium conditions*, Acta Mater. **93** (2015), 256–263.
21. V. R. Voller, *A numerical method for the Rubinstein binary-alloy problem in the presence of an under-cooled liquid*, Int. J. Heat Mass Transfer **51** (2008), 696–706.
22. V. R. Voller, *A similarity solution for solidification of an under-cooled binary alloy*, Int. J. Heat Mass Transfer **49** (2006), 1981–1985.
23. V. R. Voller, *An enthalpy method for modeling dendritic growth in a binary alloy*, Int. J. Heat Mass Transfer **51** (2008), 823–834.
24. A. Zielonka, E. Hetmaniok, and D. Słota, *Inverse alloy solidification problem including the material shrinkage phenomenon solved by using the bee algorithm*, Int. Commun. Heat Mass Transf. **87** (2017), 295–301.
25. D. A. Tarzia, *An inequality for the coefficient σ of the free boundary $s(t) = 2\sigma\sqrt{t}$ of the Neumann solution for the two-phase Stefan problem*, Quart. Appl Math. **39** (1981), 491–497.

26. A. B. Bancora and D. A. Tarzia, *On the Neumann solution for the two-phase Stefan problem including the density jump at the free boundary*, Latin Am. Heat Mass Transfer **9** (1985), 215–222.
27. D. A. Tarzia, *Relationship between Neumann solutions for two-phase lame-Clapeyron-Stefan problems with convective and temperature boundary conditions*, Therm. Sci. **21** (2017), 187–197.
28. D. A. Tarzia, *Explicit and approximated solutions for heat and mass transfer problems with a moving interface Edited by* M. El-Amin, chapter 20, Advanced Topics in Mass Transfer, Intech, Rijeka, 2011.

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APPENDIX A

A.1 | Proof of Theorem 2

Proof. By (8), we have that $M(\lambda) = \phi(T_k)$. Taking into account that $T_1 = \tilde{T}_1$ is given by (26), we have

$$\begin{aligned}
G(\lambda) &= \frac{\gamma\rho\alpha_s\alpha_l Q_1(\lambda)Q\left(\frac{\sqrt{\alpha_s}}{\sqrt{\alpha_l}}\lambda\right) + \tilde{T}_1 k_s\alpha_l Q\left(\frac{\sqrt{\alpha_s}}{\sqrt{\alpha_l}}\lambda\right) + T_0 k_l\alpha_s Q_1(\lambda)}{k_s\alpha_l Q\left(\frac{\sqrt{\alpha_s}}{\sqrt{\alpha_l}}\lambda\right) + k_l\alpha_s Q_1(\lambda)} \\
&= \frac{\gamma\rho\alpha_s\alpha_l Q_1(\lambda)Q\left(\frac{\sqrt{\alpha_s}}{\sqrt{\alpha_l}}\lambda\right) + T_k k_s\alpha_l Q\left(\frac{\sqrt{\alpha_s}}{\sqrt{\alpha_l}}\lambda\right) - q_0\sqrt{\alpha_s\pi}erf(\lambda)\alpha_l Q\left(\frac{\sqrt{\alpha_s}}{\sqrt{\alpha_l}}\lambda\right) + T_0 k_l\alpha_s Q_1(\lambda)}{k_s\alpha_l Q\left(\frac{\sqrt{\alpha_s}}{\sqrt{\alpha_l}}\lambda\right) + k_l\alpha_s Q_1(\lambda)} \\
&= \frac{\alpha_s k_l Q_1(\lambda)\left(T_0 + \frac{\gamma\rho\alpha_l}{k_l}Q\left(\frac{\sqrt{\alpha_s}}{\sqrt{\alpha_l}}\lambda\right) - \frac{q_0\alpha_l}{\sqrt{\alpha_s}k_l}\frac{e^{-\lambda^2}}{\lambda}Q\left(\frac{\sqrt{\alpha_s}}{\sqrt{\alpha_l}}\lambda\right)\right) + T_k k_s\alpha_l Q\left(\frac{\sqrt{\alpha_s}}{\sqrt{\alpha_l}}\lambda\right)}{k_s\alpha_l Q\left(\frac{\sqrt{\alpha_s}}{\sqrt{\alpha_l}}\lambda\right) + k_l\alpha_s Q_1(\lambda)} \\
&= \frac{\alpha_s k_l Q_1(\lambda)\left(T_0 + \frac{\gamma\rho\alpha_l}{k_l}Q\left(\frac{\sqrt{\alpha_s}}{\sqrt{\alpha_l}}\lambda\right) - \frac{q_0}{k_l}\sqrt{\alpha_l\pi}e^{-\lambda^2}F_1\left(\frac{\sqrt{\alpha_s}}{\sqrt{\alpha_l}}\lambda\right)\right) + T_k k_s\alpha_l Q\left(\frac{\sqrt{\alpha_s}}{\sqrt{\alpha_l}}\lambda\right)}{k_s\alpha_l Q\left(\frac{\sqrt{\alpha_s}}{\sqrt{\alpha_l}}\lambda\right) + k_l\alpha_s Q_1(\lambda)} \\
&= \frac{\alpha_s k_l Q_1(\lambda)F(\lambda) + T_k k_s\alpha_l Q\left(\frac{\sqrt{\alpha_s}}{\sqrt{\alpha_l}}\lambda\right)}{k_s\alpha_l Q\left(\frac{\sqrt{\alpha_s}}{\sqrt{\alpha_l}}\lambda\right) + k_l\alpha_s Q_1(\lambda)} = \frac{\alpha_s k_l Q_1(\lambda)T_k + T_k k_s\alpha_l Q\left(\frac{\sqrt{\alpha_s}}{\sqrt{\alpha_l}}\lambda\right)}{k_s\alpha_l Q\left(\frac{\sqrt{\alpha_s}}{\sqrt{\alpha_l}}\lambda\right) + k_l\alpha_s Q_1(\lambda)} \\
&= T_k.
\end{aligned} \tag{A1}$$

Then,

$$\begin{cases} M(\lambda) = \phi(T_k), \\ G(\lambda) = T_k, \end{cases}$$

and by the uniqueness of the pair $\tilde{T}_k, \mu$ in (35), we have $\lambda = \mu$, and $T_k = \tilde{T}_k$.

On the other hand, taking into account that $\lambda = \mu$, we have

$$\begin{aligned} A_s^T(x) &= \mathscr{A}_s^T(x),\ \forall x > 0, \quad A_l^T(x) = \mathscr{A}_l^T(x),\ \forall x > 0, \\ A_s^C(x) &= \mathscr{A}_s^C(x),\ \forall x > 0, \quad A_l^C(x) = \mathscr{A}_l^C(x),\ \forall x > 0, \\ B_s^T(x) &= \mathscr{B}_s^T(x),\ \forall x > 0, \quad B_l^T(x) = \mathscr{B}_l^T(x),\ \forall x > 0, \\ B_s^C(x) &= \mathscr{B}_s^C(x),\ \forall x > 0, \quad B_l^C(x) = \mathscr{B}_l^C(x),\ \forall x > 0, \end{aligned}$$

that is, (38). □

A.2 | Proof of Theorem 3

Proof. The similarity solutions to the heat equation $\alpha u_{xx} = u_t$ has now the form $u(x,t) = D + E erf\left(\frac{x}{2\sqrt{\alpha t}}\right)$, where the real coefficients D and E must be determined. Then, we can write

$$\begin{aligned} \hat{T}_s(x,t) &= D_s^T + E_s^T erf\left(\frac{x}{2\sqrt{\alpha_s t}}\right), \quad \hat{T}_l(x,t) = D_l^T + E_l^T erf\left(\frac{x}{2\sqrt{\alpha_l t}}\right), \\ \hat{C}_s(x,t) &= D_s^C + E_s^C erf\left(\frac{x}{2\sqrt{d_s t}}\right), \quad \hat{C}_l(x,t) = D_l^C + E_l^C erf\left(\frac{x}{2\sqrt{d_l t}}\right). \end{aligned} \tag{A2}$$

By (40) (vi), it results

$$\hat{T}_s(s(t),t) = D_s^T + E_s^T erf\left(\frac{s(t)}{2\sqrt{\alpha_s t}}\right) = \hat{T}_k,$$

where $\hat{T}_k$ is a constant to be determined. Then it follows that $s(t) = 2\delta\sqrt{\alpha_s t}$ for some $\delta > 0$, and again by (1) (vi), we get

$$\hat{T}_s(s(t),t) = D_s^T + E_s^T erf(\delta) = \hat{T}_k, \tag{A3}$$

$$\hat{T}_l(s(t),t) = D_l^T + E_l^T erf\left(\frac{\sqrt{\alpha_s}}{\sqrt{\alpha_l}}\delta\right) = \hat{T}_k. \tag{A4}$$

On the other hand,

$$\hat{T}_s(0,t) = D_s^T,$$

and then, from (40) (v), we obtain that

$$k_s T_{s_x}(0,t) = k_s E_s^T \frac{2}{\sqrt{\pi}} \frac{1}{2\sqrt{\alpha_s t}} = \frac{k_s E_s^T}{\sqrt{\pi \alpha_s t}} = \frac{h_0}{\sqrt{t}}\left(D_s^T - T_\infty\right),$$

that is,

$$D_s^T = T_\infty + \frac{k_s E_s^T}{h_0\sqrt{\pi\alpha_s}}. \tag{A5}$$

Then, replacing (A5) in (A3), we have

$$E_s^T = \frac{h_0\sqrt{\pi\alpha_s}(\hat{T}_k - T_\infty)}{k_s + h_0\sqrt{\pi\alpha_s} erf(\delta)},$$

and then we have

$$D_s^T = T_\infty + \frac{k_s(\hat{T}_k - T_\infty)}{k_s + h_0\sqrt{\pi\alpha_s}erf(\delta)}.$$

From (40) (*vii*), it follows that

$$T_0 = \hat{T}_l(x, 0) = D_l^T + E_l^T. \tag{A6}$$

Subtracting (A4) and (A6), we obtain that

$$E_l^T = \frac{T_0 - \hat{T}_k}{erfc\left(\frac{\sqrt{\alpha_s}}{\sqrt{\alpha_l}}\delta\right)}. \tag{A7}$$

Replacing (A7) in (A6), we conclude that

$$D_l^T = T_0 + \frac{\hat{T}_k - T_0}{erfc\left(\frac{\sqrt{\alpha_s}}{\sqrt{\alpha_l}}\delta\right)}.$$

Therefore,

$$\hat{T}_s(x,t) = \hat{T}_k + \frac{h_0\sqrt{\pi\alpha_s}(\hat{T}_k - T_\infty)}{k_s + h_0\sqrt{\pi\alpha_s}erf(\delta)}\left(erf\left(\frac{x}{2\sqrt{\alpha_s t}}\right) - erf(\delta)\right),$$

and

$$\hat{T}_l(x,t) = T_0 + \frac{\hat{T}_k - T_0}{erfc\left(\frac{\sqrt{\alpha_s}}{\sqrt{\alpha_l}}\delta\right)}erfc\left(\frac{x}{2\sqrt{\alpha_l t}}\right),$$

that is, (43) and (44).

Similarly, considering the equalities *ix–xii* from (1), we deduce that

$$D_s^C = f_s(\hat{T}_k), \quad E_s^C = 0,$$

$$D_l^C = C_0 + \frac{f_l(\hat{T}_k) - C_0}{erfc\left(\frac{\sqrt{\alpha_s}}{\sqrt{d_l}}\delta\right)}, \quad E_l^C = \frac{C_0 - f_l(\hat{T}_k)}{erfc\left(\frac{\sqrt{\alpha_s}}{\sqrt{d_l}}\delta\right)},$$

that is, (45) and (46).

From (40) (*xiii*), we have

$$k_s\frac{h_0(\hat{T}_k - T_\infty)}{k_s + h_0\sqrt{\pi\alpha_s}erf(\delta)}e^{-\delta^2} + k_l\frac{\hat{T}_k - T_0}{erfc\left(\frac{\sqrt{\alpha_s}}{\sqrt{\alpha_l}}\delta\right)}\frac{1}{\sqrt{\pi\alpha_l}}e^{-\left(\frac{\sqrt{\alpha_s}}{\alpha_l}\delta\right)^2} = \gamma\rho\delta\sqrt{\alpha_s}, \tag{A8}$$

from where

$$
\begin{aligned}
\hat{T}_k &= \frac{\gamma\rho\sqrt{\pi\alpha_s\alpha_l}\left(k_s+h_0\sqrt{\pi\alpha_s}erf(\delta)\right)\delta erfc\left(\frac{\sqrt{\alpha_s}}{\sqrt{\alpha_l}}\delta\right)}{h_0k_s\sqrt{\pi\alpha_l}e^{-\delta^2}erfc\left(\frac{\sqrt{\alpha_s}}{\sqrt{\alpha_l}}\delta\right)+k_le^{-\left(\frac{\sqrt{\alpha_s}}{\sqrt{\alpha_l}}\delta\right)^2}\left(k_s+h_0\sqrt{\pi\alpha_s}erf(\delta)\right)} \\
&\quad+\frac{h_0k_sT_\infty\sqrt{\pi\alpha_l}e^{-\delta^2}erfc\left(\frac{\sqrt{\alpha_s}}{\sqrt{\alpha_l}}\delta\right)+k_lT_0\left(k_s+h_0\sqrt{\pi\alpha_s}erf(\delta)\right)e^{-\left(\frac{\sqrt{\alpha_s}}{\alpha_l}\delta\right)^2}}{h_0k_s\sqrt{\pi\alpha_l}e^{-\delta^2}erfc\left(\frac{\sqrt{\alpha_s}}{\sqrt{\alpha_l}}\delta\right)+k_l\left(k_s+h_0\sqrt{\pi\alpha_s}erf(\delta)\right)e^{-\left(\frac{\sqrt{\alpha_s}}{\alpha_l}\delta\right)^2}} \\
&= \frac{\gamma\rho\sqrt{\pi\alpha_s\alpha_l}\left(k_s+h_0\sqrt{\pi\alpha_s}erf(\delta)\right)\delta}{h_0k_s\sqrt{\pi\alpha_l}e^{-\delta^2}+k_l\frac{e^{-\left(\frac{\sqrt{\alpha_s}}{\sqrt{\alpha_l}}\delta\right)^2}}{erfc\left(\frac{\sqrt{\alpha_s}}{\sqrt{\alpha_l}}\delta\right)}\left(k_s+h_0\sqrt{\pi\alpha_s}erf(\delta)\right)} \\
&\quad+\frac{\frac{h_0k_s\sqrt{\pi\alpha_l}e^{-\delta^2}e^{\left(\frac{\sqrt{\alpha_s}}{\alpha_l}\delta\right)^2}erfc\left(\frac{\sqrt{\alpha_s}}{\sqrt{\alpha_l}}\delta\right)}{k_l\left(k_s+h_0\sqrt{\pi\alpha_s}erf(\delta)\right)}T_\infty+T_0}{\frac{h_0k_s\sqrt{\pi\alpha_l}e^{-\delta^2}e^{\left(\frac{\sqrt{\alpha_s}}{\alpha_l}\delta\right)^2}erfc\left(\frac{\sqrt{\alpha_s}}{\sqrt{\alpha_l}}\delta\right)}{k_l\left(k_s+h_0\sqrt{\pi\alpha_s}erf(\delta)\right)}+1} \\
&= \frac{\gamma\rho\sqrt{\pi\alpha_s\alpha_l}}{H(\delta)}+T_\infty+\frac{T_0-T_\infty}{F_2(\delta)+1} \\
&= W(\delta).
\end{aligned}
\tag{A9}
$$

From (40) (*xiv*), we obtain

$$
\frac{C_0-f_l(\hat{T}_k)}{erfc\left(\frac{\sqrt{\alpha_s}}{\sqrt{d_l}}\delta\right)}\frac{\sqrt{d_l}}{\sqrt{\pi}}e^{-\left(\frac{\sqrt{\alpha_s}}{\sqrt{d_l}\delta}\right)^2}=\left[f_s(\hat{T}_k)-f_l(\hat{T}_k)\right]\delta\sqrt{\alpha_s}
$$

that is,

$$
\frac{1}{Q\left(\frac{\sqrt{\alpha_s}}{\sqrt{d_l}}\delta\right)}=\phi(\hat{T}_k), \tag{A10}
$$

and then (47) holds.

By (A9) and (A10), we have

$$
M(\delta)=\phi(W(\delta)).
$$

We know that M is a strictly decreasing function with $M(0^+)=+\infty, M(+\infty)=1$. On the other hand, by Proposition 2(c), W is an strictly increasing function. Then, taking into account Proposition 1(e), we can assure the existence and uniqueness of δ verifying (47), if $T_{0s}\leq W(0)<W(+\infty)<T_{0l}$, or equivalently (41), and the thesis holds. □

A.3 | Proof of Theorem 4

Proof. By (47), we obtain that $M(\delta)=\phi(\hat{T}_k)$. We prove now that $G(\delta)=\hat{T}_k$.

Observe that

$$
\hat{T}_k=\frac{k_s\alpha_lQ\left(\frac{\sqrt{\alpha_s}}{\sqrt{\alpha_l}}\delta\right)\hat{T}_k+k_l\alpha_sQ_1(\delta)\hat{T}_k}{k_s\alpha_lQ\left(\frac{\sqrt{\alpha_s}}{\sqrt{\alpha_l}}\delta\right)+k_l\alpha_sQ_1(\delta)}=\frac{A+B}{C}. \tag{A11}
$$

Now, we work with B.

$$B = k_l\alpha_s Q_1(\delta)\hat{T}_k = k_l\alpha_s Q_1(\delta)\hat{T}_k \pm \frac{h_0 k_s \alpha_l \sqrt{\pi\alpha_s}erf(\delta)}{k_s + h_0\sqrt{\pi\alpha_s}erf(\delta)} Q\left(\frac{\sqrt{\alpha_s}}{\sqrt{\alpha_l}}\delta\right)\hat{T}_k$$
$$= \frac{k_l\alpha_s Q_1(\delta)\left[k_s + h_0\sqrt{\pi\alpha_s}erf(\delta)\right] + h_0 k_s\alpha_l\sqrt{\pi\alpha_s}erf(\delta)Q\left(\frac{\sqrt{\alpha_s}}{\sqrt{\alpha_l}}\delta\right)}{k_s + h_0\sqrt{\pi\alpha_s}erf(\delta)}\hat{T}_k - \frac{h_0 k_s \alpha_l \sqrt{\pi\alpha_s}erf(\delta)}{k_s + h_0\sqrt{\pi\alpha_s}erf(\delta)} Q\left(\frac{\sqrt{\alpha_s}}{\sqrt{\alpha_l}}\delta\right)\hat{T}_k$$
$$= \frac{k_l\alpha_s Q_1(\delta)\left[k_s + h_0\sqrt{\pi\alpha_s}erf(\delta)\right] + h_0 k_s\alpha_l\sqrt{\pi\alpha_s}erf(\delta)Q\left(\frac{\sqrt{\alpha_s}}{\sqrt{\alpha_l}}\delta\right)}{k_s + h_0\sqrt{\pi\alpha_s}erf(\delta)} \cdot$$
$$\left[\frac{k_l\sqrt{\alpha_s}\delta\left[k_s + h_0\sqrt{\pi\alpha_s}erf(\delta)\right]}{h_0 k_s\alpha_l e^{-\delta^2}Q\left(\frac{\sqrt{\alpha_s}}{\sqrt{\alpha_l}}\delta\right) + k_l\sqrt{\alpha_s}\delta\left[k_s + h_0\sqrt{\pi\alpha_s}erf(\delta)\right]} \cdot \left(\frac{h_0 k_s\alpha_l e^{-\delta^2}Q\left(\frac{\sqrt{\alpha_s}}{\sqrt{\alpha_l}}\delta\right)}{k_l\sqrt{\alpha_s}\delta\left[k_s + h_0\sqrt{\pi\alpha_s}erf(\delta)\right]}T_\infty + T_0\right)\right.$$
$$\left. + \frac{\delta\rho\alpha_l\sqrt{\alpha_s}\delta\left[k_s + h_0\sqrt{\pi\alpha_s}erf(\delta)\right]Q\left(\frac{\sqrt{\alpha_s}}{\sqrt{\alpha_l}}\delta\right)}{h_0 k_s\alpha_l e^{-\delta^2}Q\left(\frac{\sqrt{\alpha_s}}{\sqrt{\alpha_l}}\delta\right) + k_l\sqrt{\alpha_s}\delta\left[k_s + h_0\sqrt{\pi\alpha_s}erf(\delta)\right]}\right] - \frac{h_0 k_s \alpha_l \sqrt{\pi\alpha_s}erf(\delta)}{k_s + h_0\sqrt{\pi\alpha_s}erf(\delta)} Q\left(\frac{\sqrt{\alpha_s}}{\sqrt{\alpha_l}}\delta\right)\hat{T}_k \tag{A12}$$
$$= k_l\alpha_s Q_1(\delta)\left(\frac{h_0 k_s\alpha_l e^{-\delta^2}Q\left(\frac{\sqrt{\alpha_s}}{\sqrt{\alpha_l}}\delta\right)}{k_l\sqrt{\alpha_s}\delta\left[k_s + h_0\sqrt{\pi\alpha_s}erf(\delta)\right]}T_\infty + T_0\right) + \gamma\rho\alpha_l\alpha_s Q_1(\delta)Q\left(\frac{\sqrt{\alpha_s}}{\sqrt{\alpha_l}}\delta\right)$$
$$- \frac{h_0 k_s \alpha_l \sqrt{\pi\alpha_s}erf(\delta)}{k_s + h_0\sqrt{\pi\alpha_s}erf(\delta)} Q\left(\frac{\sqrt{\alpha_s}}{\sqrt{\alpha_l}}\delta\right)\hat{T}_k = -\frac{h_0 k_s \alpha_l \sqrt{\pi\alpha_s}erf(\delta)}{k_s + h_0\sqrt{\pi\alpha_s}erf(\delta)} Q\left(\frac{\sqrt{\alpha_s}}{\sqrt{\alpha_l}}\delta\right)(\hat{T}_k - T_\infty) + k_l\alpha_s Q_1(\delta)T_0$$
$$+ \gamma\rho\alpha_l\alpha_s Q_1(\delta)Q\left(\frac{\sqrt{\alpha_s}}{\sqrt{\alpha_l}}\delta\right).$$

Then, by (A11) and (A12) and taking into account that $T_1 = \hat{T}_1$ is given by (48), we have

$$\hat{T}_k = \frac{k_s\alpha_l Q\left(\frac{\sqrt{\alpha_s}}{\sqrt{\alpha_l}}\delta\right)\hat{T}_k + k_l\alpha_s Q_1(\delta)\hat{T}_k}{k_s\alpha_l Q\left(\frac{\sqrt{\alpha_s}}{\sqrt{\alpha_l}}\delta\right) + k_l\alpha_s Q_1(\delta)}$$
$$= \frac{k_s\alpha_l Q\left(\frac{\sqrt{\alpha_s}}{\sqrt{\alpha_l}}\delta\right)\hat{T}_k - \frac{h_0 k_s\alpha_l\sqrt{\pi\alpha_s}erf(\delta)}{k_s + h_0\sqrt{\pi\alpha_s}erf(\delta)}Q\left(\frac{\sqrt{\alpha_s}}{\sqrt{\alpha_l}}\delta\right)(\hat{T}_k - T_\infty)}{k_s\alpha_l Q\left(\frac{\sqrt{\alpha_s}}{\sqrt{\alpha_l}}\delta\right) + k_l\alpha_s Q_1(\delta)}$$
$$+ \frac{k_l\alpha_s Q_1(\delta)T_0 + \gamma\rho\alpha_l\alpha_s Q_1(\delta)Q\left(\frac{\sqrt{\alpha_s}}{\sqrt{\alpha_l}}\delta\right)}{k_s\alpha_l Q\left(\frac{\sqrt{\alpha_s}}{\sqrt{\alpha_l}}\delta\right) + k_l\alpha_s Q_1(\delta)} \tag{A13}$$
$$= \frac{k_s\alpha_l Q\left(\frac{\sqrt{\alpha_s}}{\sqrt{\alpha_l}}\delta\right)\hat{T}_1 + k_l\alpha_s Q_1(\delta)T_0 + \gamma\rho\alpha_l\alpha_s Q_1(\delta)Q\left(\frac{\sqrt{\alpha_s}}{\sqrt{\alpha_l}}\delta\right)}{k_s\alpha_l Q\left(\frac{\sqrt{\alpha_s}}{\sqrt{\alpha_l}}\delta\right) + k_l\alpha_s Q_1(\delta)}$$
$$= G(\delta).$$

Thus, by the uniqueness of the pair $\tilde{T}_k, \mu$ in (35), $\mu = \delta$, and $\tilde{T}_k = \hat{T}_k$.
On the other hand, taking into account that $\delta = \mu$, we have

$$\begin{aligned}
D_s^T(x) = \mathscr{A}_s^T(x),\ \forall x > 0, \quad D_l^T(x) = \mathscr{A}_l^T(x),\ \forall x > 0,\\
D_s^C(x) = \mathscr{A}_s^C(x),\ \forall x > 0, \quad D_l^C(x) = \mathscr{A}_l^C(x),\ \forall x > 0,\\
E_s^T(x) = \mathscr{B}_s^T(x),\ \forall x > 0, \quad E_l^T(x) = \mathscr{B}_l^T(x),\ \forall x > 0,\\
E_s^C(x) = \mathscr{B}_s^C(x),\ \forall x > 0, \quad E_l^C(x) = \mathscr{B}_l^C(x),\ \forall x > 0,
\end{aligned}$$

that is, (51). □