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# Analytical study of steady-state heat transfer in two solids with continuous flow and thermal jump at the interface

**Domingo A. Tarzia**[1,2] · **Guillermo F. Umbricht**[3] · **Mara Rossani**[1,2]

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**Abstract**
In this work, we analytically address the general steady-state heat conduction problem between two solids in thermal contact, accounting for heat flux continuity and a temperature jump at the interface. The novelty of the proposed approach lies in its generality, as it is applicable regardless of the geometric configuration of the solids, and in the use of variational methods combined with the fixed-point theorem, which allow us to rigorously establish sufficient conditions for the existence and uniqueness of the solution. Moreover, we demonstrate the continuous dependence of the solution on internal heat sources in both solids, providing a robust framework for the analysis of similar systems. The practical relevance and applicability of the method are illustrated through three explicit solutions given for Cartesian, polar and spherical coordinates and four explicit examples with different characteristics, highlighting the added value of this approach for understanding and solving interface heat transfer problems. Unlike previous studies that assume specific geometries or idealized contacts, our method provides a general and rigorous framework applicable to a wide range of interface heat transfer problems.

**Keywords** Elliptic variational problems · Fixed-point theory · Thermal contact resistance · Interface heat transfer problems

Guillermo F. Umbricht and Mara Rossani contributed equally to this work.

✉ Domingo A. Tarzia
DTarzia@austral.edu.ar

Guillermo F. Umbricht
GUmbricht@austral.edu.ar

Mara Rossani
MRossani@austral.edu.ar

[1] Departamento de Matemática, Facultad de Ciencias Empresariales, Universidad Austral, Paraguay 1950, S2000FZF Rosario, Santa Fe, Argentina

[2] Consejo Nacional de Investigaciones Científicas y Técnicas (CONICET), Godoy Cruz 2290, C1425FQB Autonomous City of Buenos Aires, Buenos Aires, Argentina

[3] Laboratorio de Investigación, Desarrollo y Transferencia de la Universidad Austral (LIDTUA), Facultad de Ingeniería, Universidad Austral, Mariano Acosta 1611, B1629WWA Pilar, Buenos Aires, Argentina

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## 1 Introduction

Steady-state and transient heat and mass transfer problems in adjacent solids [1, 2] have been extensively studied in recent years. This growing interest is largely due to their direct applications in various fields of science, engineering, and industry. The breadth of these applications is evident from the wealth of published literature. For example, studies cover topics such as brain tumor growth [3], microelectronic analysis and thermal conduction in composite materials [4], drug release in stents [5], skin permeability studies [6], moisture analysis in composite tissues [7], contamination determination in porous media [8], greenhouse gas emission analysis [9], lithium-ion cell analysis [10], innovations in wool cleaning techniques [11], and heat conduction through skin analysis [12], among others.

Steady-state and transient heat and mass transfer problems in adjacent solids have been tackled analytically using a variety of methods, including recursive image methods [13], separation of variables [14], and solutions involving integral transforms such as Laplace and Fourier transforms [15]. Numerical techniques have also been employed, such as the method of fundamental solutions [16], finite difference methods, and finite element methods [14]. A comprehensive and up-to-date review of heat and mass transfer in multilayer materials and the mathematical techniques used can be found in [17].

Although the literature on conduction problems in adjacent solids is extensive, it lacks generality. There is no record of a general approach to the problem, as all studies focus on specific cases defined by a particular geometry. For example, the steady-state one-dimensional problem in Cartesian coordinates was addressed in [18–20], while the transient one-dimensional problem in Cartesian coordinates was treated in [14, 17, 21, 22]. The two-dimensional problem in Cartesian coordinates was studied in [23, 24], and the two-dimensional problem in cylindrical coordinates was solved in [25], to name a few.

Variational methods [26–28], on the other hand, constitute a fundamental tool in mathematical analysis and originate from the calculus of variations. This approach seeks to find functions that minimize (or maximize) a functional, which typically expresses the energy of a physical system or the action of a mechanical theory. Variational equations have become one of the most powerful techniques for solving problems across a wide range of disciplines. Some of the most significant applications include solid and fluid mechanics [29, 30], boundary problems in physics [31], image processing [32, 33], optimal control and game theory [34, 35], and contact mechanics [36, 37].

The main contribution of this manuscript lies in providing a general analytical framework for steady-state heat conduction between two solids in thermal contact, independent of their geometric configuration. The novelty of the proposed approach is twofold: first, it combines variational methods with the fixed-point theorem to rigorously establish existence, uniqueness, and continuous dependence results for the temperature fields; second, it allows explicit calculation of temperatures and heat fluxes at the interface, even in the presence of a temperature jump due to interfacial resistance. This framework not only generalizes previous models restricted to specific geometries but also provides a robust tool for analyzing a wide class of interface heat transfer problems.

The objective of this work is threefold. First, we aim to provide a geometry-independent mathematical model for the steady-state heat conduction in a body composed of two solids in thermal contact through a standard interface, considering the continuity of heat flux and a temperature jump due to contact resistance. Then, a variational formulation is presented using the theory of elliptic variational equalities [26, 27, 38]. Second, we seek to determine the temperatures in both solids and the common heat fluxes by applying a fixed-point theorem

[39]. Third, we obtain a continuous dependence of the solution to the problem with respect to the sources. Additionally, some examples with explicit solutions are given to illustrate the interfacial thermal resistance problem.

The remainder of the manuscript is structured as follows: In Sect. 2, we present the modeling of the problem. In Sect. 3, we establish the corresponding existence and uniqueness of the solution for the variational equalities for each domain corresponding to the two solids as a function of the common heat fluxes at the interface. In Sect. 4, we derive the temperatures and heat fluxes at the interface of the two solids in thermal contact by using the Banach fixed-point theorem. In Sect. 5, we obtain the continuous dependence of the solution with respect to the source in each solid. Finally, in Sect. 6, we show three explicit solutions given for Cartesian, polar and spherical coordinates and four explicit examples.

## 2 Mathematical modeling

The problem of interest is to study the steady-state heat conduction in a body $S$, composed of two solids $S_1$ and $S_2$ in thermal contact. To define the geometry of the body, we introduce the following domain in $\mathbb{R}^n$:

$$\begin{cases} \Omega = \Omega_1 \cup \Omega_2, \quad \mathring{\Omega}_1 \cap \mathring{\Omega}_2 = \emptyset, \quad \overline{\Omega}_1 \cap \overline{\Omega}_2 = \Gamma_0, \\ \partial\Omega = \Gamma_1 \cup \Gamma_2, \quad \partial\Omega_1 = \Gamma_1 \cup \Gamma_0, \quad \partial\Omega_2 = \Gamma_2 \cup \Gamma_0, \end{cases} \tag{1}$$

where $\Omega$, $\Omega_i$, $\partial\Omega$, and $\partial\Omega_i$ denote the domains and the boundary of the body $S$ and the solid $S_i$, respectively. In particular, $\Gamma_0$ denotes the interface shared by both solids (see Fig. 1).

The phenomenon of steady heat transfer between solids $S_1$ and $S_2$ is governed by the following elliptic equations:

$$-\kappa_1 \Delta u_1 = g_1 \ \text{ in } \ \Omega_1, \tag{2}$$

$$-\kappa_2 \Delta u_2 = g_2 \ \text{ in } \ \Omega_2. \tag{3}$$

In Eqs. (2)–(3), the subscripts 1 and 2 refer to the solids defined by $\Omega_1$ and $\Omega_2$, respectively. The functions $u_i$ denote the temperature fields, the functions $g_i$ are source terms (which may model an electric current flowing through the body, chemical or nuclear reactions occurring within the body, for instance), and $\kappa_i \in \mathbb{R}^+$ represents the thermal conductivity for the solid $S_i$ $(i = 1, 2)$.

At the exterior boundary of each solid, zero Dirichlet boundary conditions are imposed, that is

$$u_1 = 0 \ \text{ on } \ \Gamma_1, \tag{4}$$

$$u_2 = 0 \ \text{ on } \ \Gamma_2. \tag{5}$$

Finally, at the interface between the two solids, the continuity of thermal flux (energy conservation) is imposed and a temperature jump caused by the thermal contact resistance between the material surfaces, namely:

$$R\ (u_2 - u_1) = \kappa_1 \frac{\partial u_1}{\partial n} \ \text{ on } \ \Gamma_0, \tag{6}$$

$$\kappa_1 \frac{\partial u_1}{\partial n} = \kappa_2 \frac{\partial u_2}{\partial n} \ \text{ on } \ \Gamma_0, \tag{7}$$

where condition (6) describes the temperature jump at the contact interface, which is proportional to the fluxes, and $R > 0$ corresponds to resistance at the interface, and condition

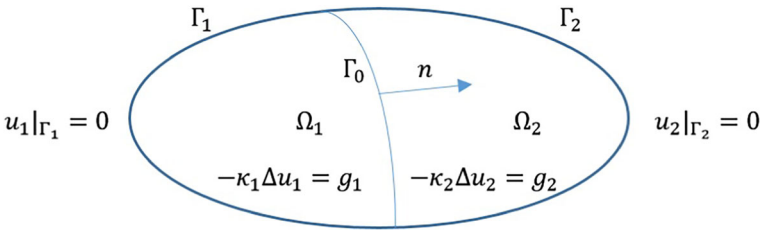



**Fig. 1** Schematic representation of the two-solid in thermal contact problem

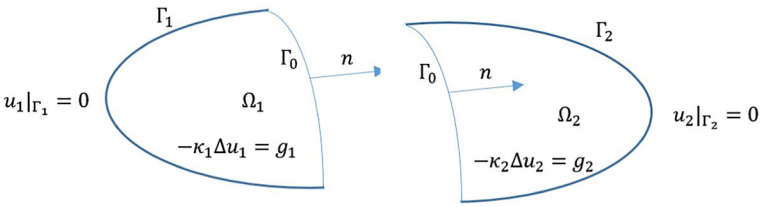



**Fig. 2** Schematic representation of the two subproblems

(7) represents the equality of heat fluxes across the interface. Here, $n$ denotes the unit normal vector to $\Omega_1$ oriented toward $\Omega_2$. Figure 1 illustrates the problem under consideration.

***Remark 1*** The resistance parameter $R$ at the interface represents the barrier to transfer between two media. It controls how strongly the interface affects the overall behavior of the system and can significantly influence source reconstruction or material property estimation in inverse problems.

## 3 Variational formulation

In order to decompose problem (2)–(7), into two simpler subproblems (see Fig. 2), we define

$$f : \Gamma_0 \longrightarrow \mathbb{R},\ f = -\kappa_1 \frac{\partial u_1}{\partial n} = -\kappa_2 \frac{\partial u_2}{\partial n} \ \text{ on } \ \Gamma_0, \tag{8}$$

and our goal is to find functions $u_i$ in $\Omega_i$, $(i = 1, 2)$ such that $u_1$ satisfies conditions (2), (4), (8) while $u_2$ satisfies conditions (3), (5), (8), respectively, for given $f$ on $\Gamma_0$. The problem is solvable if and only if the integral equation for $f$, using condition (6), admits a unique solution.

We define the following functional spaces, bilinear and linear functionals:

$$Q = L^2(\Gamma_0), \quad H_i = L^2(\Omega_i), \quad W_i = H^1(\Omega_i), \tag{9}$$

$$V_i = \left\{ v \in W_i / v\,|_{\Gamma_i} = 0 \right\}, \quad i = 1, 2, \tag{10}$$

$$a_i : W_i \times W_i \longrightarrow \mathbb{R}, \quad \text{such that}$$

$$a_i(u_i, v_i) = \kappa_i \int_{\Omega_i} \nabla u_i \cdot \nabla v_i \, dx, \quad \forall\, u_i, v_i \in W_i, \tag{11}$$

and

$$\begin{cases} L_{1_f}(v) = \displaystyle\int_{\Omega_1} g_1 v \, dx - \int_{\Gamma_0} f \, v \, dS, & \forall \, v \in W_1, \\ L_{2_f}(v) = \displaystyle\int_{\Omega_2} g_2 v \, dx + \int_{\Gamma_0} f \, v \, dS, & \forall \, v \in W_2. \end{cases} \tag{12}$$

By applying classical techniques such as the Lax–Milgram Theorem [26, 27, 38], can be established the following results for function $f \in Q$ satisfying (8):

**Lemma 1**

1. *The elliptic variational equality for the problem* (2), (4), *and* (8) *is given by*

$$u_1 \in V_1 : a_1(u_1, v) = L_{1_f}(v), \quad \forall v \in V_1. \tag{13}$$

   *We denote by* $u_{1_f}$ *the unique solution of the elliptic variational equality* (13) *for each* $f \in Q$.
2. *The elliptic variational equality for the problem* (3), (5), *and* (8) *is given by*

$$u_2 \in V_2 : a_2(u_2, v) = L_{2_f}(v), \quad \forall v \in V_2. \tag{14}$$

   *We denote by* $u_{2_f}$ *the unique solution of the elliptic variational equality* (14) *for each* $f \in Q$.

***Proof*** We have that $a_i$ with $i = 1, 2$ is a bilinear, continuous, coercive, and symmetric form on $W_i \times W_i$ because there exists $\lambda_i > 0 \, (i = 1, 2)$ such that

$$\int_{\Omega_i} \nabla v \cdot \nabla v \, dx \geq \lambda_i \, \|v\|^2_{W_i} \quad \forall v \in V_i, \tag{15}$$

and then

$$a_i(v, v) \geq \kappa_i \, \lambda_i \, \|v\|^2_{W_i}, \quad \forall v \in V_i. \tag{16}$$

Furthermore, $L_{i_f}$ is a linear and continuous functional. Then, for each $f \in Q$, there exists a unique $u_{i_f} \in V_i \subset W_i$ for $i = 1, 2$ that solves problems (13) and (14), respectively [26, 27, 38]. □

Furthermore, we present the following result that reciprocally complements Lemma 1 which is important to solve problem (2)–(7) by using a fixed point theorem.

**Lemma 2** *Let* $u_{1_f} \in V_1$ *and* $u_{2_f} \in V_2$ *be the solutions of the elliptic variational equations* (13) *and* (14), *respectively. Then, there exists a unique function* $f \in Q$ *that satisfies the heat flux condition* (8).

***Proof*** Let $u_{1_f} \in V_1$ be the unique solution to the elliptic variational equation (13), and we consider a function $\varphi \in \mathscr{D}(\Omega_1)$, set of distributions in $\Omega_1$, then $\varphi \in V_1$.

Substituting $v = \varphi$ into the elliptic variational equation (13) we have:

$$\left\langle -\kappa_1 \Delta u_{1_f}, \varphi \right\rangle = \int_{\Omega_1} g_1 \, \varphi \, dx - \int_{\Gamma_0} f \, \varphi \, dS = \int_{\Omega_1} g_1 \, \varphi \, dx, \qquad \forall \varphi \in \mathscr{D}(\Omega_1). \tag{17}$$

Equality (17) implies that:

$$-\kappa_1 \Delta u_{1_f} = g_1 \ \text{ in } \ \mathscr{D}(\Omega_1), \tag{18}$$

which in turn yields $\Delta u_{1_f} \in H_1$ since $g_1 \in H_1$. Then, there exists $\left.\frac{\partial u_{1_f}}{\partial n}\right|_{\Gamma_0} \in H^{-1/2}(\Gamma_0)$ such that, using the duality $H^{-1/2}(\Omega_1) \times H^{1/2}(\Omega_2)$ [27, 28, 38], which implies:

$$-\int_{\Omega_1} \Delta u_{1_f}\, v\, dx = \int_{\Omega_1} \nabla u_{1_f}\, \nabla v\, dx - \left\langle \frac{\partial u_{1_f}}{\partial n}, \gamma_0\, v \right\rangle, \quad \forall\, v \in W_1. \tag{19}$$

Multiplying expression (18) by $v \in V_1 \subset W_1$ and integrate over $\Omega_1$ we get

$$\begin{aligned} \int_{\Omega_1} g\, v\, dx &= -\int_{\Omega_1} \kappa_1 \Delta u_{1_f}\, v\, dx \\ &= \kappa_1 \int_{\Omega_1} \nabla u_{1_f} \cdot \nabla v\, dx - \kappa_1 \left\langle \frac{\partial u_{1_f}}{\partial n}, \gamma_0\, v \right\rangle_{\Gamma_0} \\ &= a_1(u_{1_f}, v) - \kappa_1 \left\langle \frac{\partial u_{1_f}}{\partial n}, \gamma_0\, v \right\rangle_{H^{-1/2}(\Gamma_0)\times H^{1/2}(\Gamma_0)}. \end{aligned} \tag{20}$$

Comparing expressions (13) and (20) yields:

$$\kappa_1 \left\langle \frac{\partial u_{1_f}}{\partial n}, \gamma_0\, v \right\rangle_{H^{-1/2}(\Gamma_0)\times H^{1/2}(\Gamma_0)} = -\int_{\Gamma_0} f\, v\, dS, \quad \forall\, v \in V_1. \tag{21}$$

Since $f \in Q$ we deduce:

$$f = -\kappa_1 \left.\frac{\partial u_{1_f}}{\partial n}\right|_{\Gamma_0}. \tag{22}$$

Similarly, if $u_{2_f} \in V_2$ is the unique solution to the elliptic variational equation (14), and we consider a function $\varphi \in \mathscr{D}(\Omega_2)$, set of distributions in $\Omega_2$, then $\varphi \in V_2$.

Substituting $v = \varphi$ into the elliptic variational equation (14) yields:

$$\langle -\kappa_2 \Delta u_{2_f}, \varphi \rangle = \int_{\Omega_2} g_2\, \varphi\, dx + \int_{\Gamma_0} f\, \varphi\, dS = \int_{\Omega_2} g_2\, \varphi\, dx, \qquad \forall \varphi \in \mathscr{D}(\Omega_2). \tag{23}$$

Equality (23) implies that:

$$-\kappa_2 \Delta u_{2_f} = g_2 \ \text{ in } \ \mathscr{D}(\Omega_2), \tag{24}$$

since $g_2 \in H_2$ implies that $\Delta u_{2_f} \in H_2$. Then, there exists $\left.\frac{\partial u_{2_f}}{\partial n}\right|_{\Gamma_0} \in H^{-1/2}(\Gamma_0)$ such that, using the duality $H^{-1/2}(\Omega_1) \times H^{1/2}(\Omega_2)$ [27, 28, 38], hence:

$$-\int_{\Omega_2} \Delta u_{2_f}\, v\, dx = \int_{\Omega_2} \nabla u_{2_f}\, \nabla v\, dx + \left\langle \frac{\partial u_{2_f}}{\partial n}, \gamma_0\, v \right\rangle, \quad \forall\, v \in W_2. \tag{25}$$

Multiplying expression (24) by $v \in V_2 \subset W_2$ and integrating over $\Omega_2$, we obtain

$$-\int_{\Omega_2} \kappa_2 \Delta u_{2_f}\, v\, dx = \int_{\Omega_2} g\, v\, dx, \tag{26}$$

equivalently

$$\begin{aligned} &\kappa_2 \int_{\Omega_2} \nabla u_{2_f} \cdot \nabla v\, dx + \kappa_2 \left\langle \frac{\partial u_{2_f}}{\partial n}, \gamma_0\, v \right\rangle_{\Gamma_0} \\ &= a_2(u_{2_f}, v) + \kappa_2 \left\langle \frac{\partial u_{2_f}}{\partial n}, \gamma_0\, v \right\rangle_{H^{-1/2}(\Gamma_0)\times H^{1/2}(\Gamma_0)}. \end{aligned} \tag{27}$$

Comparing expressions (14) and (27) yields:

$$\kappa_2 \left\langle \frac{\partial u_{2_f}}{\partial n}, \gamma_0 v \right\rangle_{H^{-1/2}(\Gamma_0)\times H^{1/2}(\Gamma_0)} = -\int_{\Gamma_0} f\, v\, dS, \quad \forall\, v \in V_2. \tag{28}$$

Since $f \in Q$ hence:

$$f = -\kappa_2 \left. \frac{\partial u_{1_f}}{\partial n} \right|_{\Gamma_0}. \tag{29}$$

From expressions (22) and (29), we conclude relation (8), as desired.

□

Lemmas 1 and 2 can be unified into the following result.

**Theorem 3** *There exist unique functions $u_{1_f} \in V_1$ and $u_{2_f} \in V_2$ that solve the elliptic variational equations (13) and (14) respectively, if and only if, there exists a unique function $f \in Q$ that satisfies the heat flux condition (8).*

It remains to determine that function $f \in Q$ satisfies the thermal jump condition (6). This problem will be addressed in the following section.

## 4 Solution of the resistance problem by using fixed-point

In order to solve problem (2)–(7), Theorem 3 indicates that we must determine a function $f \in Q$ such that

$$f = R\left(u_{1_f} - u_{2_f}\right) \text{ on } \Gamma_0, \tag{30}$$

where $u_{1_f} \in V_1$ and $u_{2_f} \in V_2$ are the unique solutions of the elliptic variational equalities (13) and (14) respectively, for each $f \in Q$. To achieve this, we define the operator

$$T : Q \to Q \text{ such that } T(f) = R\left(\gamma_{0_{\Omega_1}}(u_{1_f}) - \gamma_{0_{\Omega_2}}(u_{2_f})\right). \tag{31}$$

The problem reduces to solving the functional equation $T(f) = f$ in $Q$ where $\gamma_{0_{\Omega_i}}(u_{i_f}) = u_{i_f}\big|_{\Gamma_0}$ denotes the trace from $\Omega_i$ on $\Gamma_0$. Therefore, it remains to determine under what conditions $T$ is a contractive operator. By classical results [26, 27], it is known that

$$\|v_i\|_Q \leq \left\|\gamma_{0_{\Omega_i}}\right\| \|v_i\|_{W_i}, \quad \forall v_i \in V_i,\ i = 1, 2. \tag{32}$$

**Theorem 4** *The expression*

$$R < R_e = \frac{1}{\left(\dfrac{\left\|\gamma_{0_{\Omega_1}}\right\|^2}{\kappa_1\lambda_1} + \dfrac{\left\|\gamma_{0_{\Omega_2}}\right\|^2}{\kappa_2\lambda_2}\right)}, \tag{33}$$

*provides a sufficient condition for the conduction problem (2)–(7) to admit a unique solution.*

***Proof*** We consider that $f, g \in Q$. Let $u_{1_f}, u_{2_f}$ be the solution of the variational equalities (13) and (14) respectively, corresponding to data $f \in Q$, and $u_{1_g}, u_{2_g}$ the solution of the variational equalities (13) and (14) respectively for data $g \in Q$. We prove the theorem in the following steps.

Step 1: If we use $v = u_{1_f} - u_{1_g} \in V_1$ in the elliptic variational equality corresponding to $u_{1_g}$, and $v = u_{1_g} - u_{1_f} \in V_1$ in the elliptic variational equality corresponding to $u_{1_f}$, we get:

$$\begin{aligned} a_1(u_{1_g}, u_{1_f} - u_{1_g}) &= L_{1_g}(u_{1_f} - u_{1_g}) \\ &= \int_{\Omega_1} g_1(u_{1_f} - u_{1_g})\, dx - \int_{\Gamma_0} g(u_{1_f} - u_{1_g})\, dS, \end{aligned} \tag{34}$$

$$\begin{aligned} a_1(u_{1_f}, u_{1_g} - u_{1_f}) &= L_{1_f}(u_{1_g} - u_{1_f}) \\ &= \int_{\Omega_1} g_1(u_{1_g} - u_{1_f})\, dx - \int_{\Gamma_0} f(u_{1_g} - u_{1_f})\, dS. \end{aligned} \tag{35}$$

By adding expressions from (34) and (35), we have

$$-a_1(u_{1_g} - u_{1_f}, u_{1_g} - u_{1_f}) = \int_{\Gamma_0} (u_{1_g} - u_{1_f})(g - f)\, dS. \tag{36}$$

Moreover, by substituting (32) and (16) into (36), we derive:

$$\begin{aligned} \kappa_1 \lambda_1 \left\| u_{1_g} - u_{1_f} \right\|_{W_1}^2 &\leq \left\| u_{1_g} - u_{1_f} \right\|_Q \|g - f\|_Q \\ &\leq \left\| \gamma_{0_{\Omega_1}} \right\| \left\| u_{1_g} - u_{1_f} \right\|_{W_1} \|g - f\|_Q , \end{aligned} \tag{37}$$

which implies that

$$\left\| u_{1_g} - u_{1_f} \right\|_{W_1} \leq \frac{\left\| \gamma_{0_{\Omega_1}} \right\|}{\kappa_1 \lambda_1} \|g - f\|_Q . \tag{38}$$

Step 2: Analogously, by taking $v = u_{2_f} - u_{2_g} \in V_2$ in the elliptic variational equality corresponding to $u_{2_g}$, and $v = u_{2_g} - u_{2_f} \in V_2$ in the elliptic variational equality corresponding to $u_{2_f}$, we deduce:

$$\begin{aligned} a_2(u_{2_f}, u_{2_g} - u_{2_f}) &= L_{2_f}(u_{2_g} - u_{2_f}) \\ &= \int_{\Omega_2} g_2(u_{2_g} - u_{2_f})\, dx + \int_{\Gamma_0} f(u_{2_g} - u_{2_f})\, dS, \end{aligned} \tag{39}$$

$$\begin{aligned} a_2(u_{2_g}, u_{2_f} - u_{2_g}) &= L_{2_g}(u_{2_f} - u_{2_g}) \\ &= \int_{\Omega_2} g_2(u_{2_f} - u_{2_g})\, dx + \int_{\Gamma_0} g(u_{2_f} - u_{2_g})\, dS. \end{aligned} \tag{40}$$

By adding expressions from (39) and (40), we get

$$-a_2(u_{2_g} - u_{2_f}, u_{2_g} - u_{2_f}) = \int_{\Gamma_0} (u_{2_f} - u_{2_g})(g - f)\, dS. \tag{41}$$

Moreover, by substituting (32) and (16) into (41), we derive:

$$\begin{aligned} \kappa_2 \lambda_2 \left\| u_{2_g} - u_{2_f} \right\|_{W_2}^2 &\leq \left\| u_{2_g} - u_{2_f} \right\|_Q \|g - f\|_Q \\ &\leq \left\| \gamma_{0_{\Omega_2}} \right\| \left\| u_{2_g} - u_{2_f} \right\|_{W_2} \|g - f\|_Q , \end{aligned} \tag{42}$$

which implies that

$$\left\| u_{2_g} - u_{2_f} \right\|_{W_2} \leq \frac{\left\| \gamma_{0_{\Omega_2}} \right\|}{\kappa_2 \lambda_2} \|g - f\|_Q . \tag{43}$$

Step 3: Therefore, by combining (38) and (43), it follows that

$$\begin{aligned} \left\| u_{i_g} - u_{i_f} \right\|_Q &= \left\| \gamma_0 (u_{i_g} - u_{i_f}) \right\|_Q \leq \left\| \gamma_{0_{\Omega_i}} \right\| \left\| u_{i_g} - u_{i_f} \right\|_{W_i} \\ &\leq \frac{\left\| \gamma_{0_{\Omega_i}} \right\|^2}{\kappa_i \lambda_i} \left\| g - f \right\|_Q , \quad i = 1, 2. \end{aligned} \tag{44}$$

Step 4: Finally, using (44) in the definition of $T$, we can obtain that

$$\begin{aligned} \left\| T(g) - T(f) \right\|_Q &\leq R \left[ \left\| u_{1_g} - u_{1_f} \right\|_Q + \left\| u_{2_g} - u_{2_f} \right\|_Q \right] \\ &\leq R \left[ \frac{\left\| \gamma_{0_{\Omega_1}} \right\|^2}{\kappa_1 \lambda_1} \left\| g - f \right\|_Q + \frac{\left\| \gamma_{0_{\Omega_2}} \right\|^2}{\kappa_2 \lambda_2} \left\| g - f \right\|_Q \right] \\ &= R \left( \frac{\left\| \gamma_{0_{\Omega_1}} \right\|^2}{\kappa_1 \lambda_1} + \frac{\left\| \gamma_{0_{\Omega_2}} \right\|^2}{\kappa_2 \lambda_2} \right) \left\| g - f \right\|_Q \\ &= \frac{R}{R_e} \left\| g - f \right\|_Q . \end{aligned}$$

Therefore, $T$ is a Lipschitz operator. Moreover, $T$ becomes a contraction operator to apply the Banach theorem if and only if the inequality (33) holds, as we wanted to prove.

□

***Remark 2*** From Theorems 3 and 4, we deduce, the steady-state heat transfer problem defined by equations and conditions (2)–(7) admits a unique solution whenever condition (33) is fulfilled.

## 5 The analytical solution varies continuously with the source

The fact that the solution depends continuously on the source is of great importance for practical applications. In general, in real-world problems, the source terms are not known exactly and are obtained through experimental measurements. These data are subject to calibration errors and measurement uncertainties of the instruments used. It is desirable that a small modification in the data leads to a small modification in the solutions. In other words, it is expected that the solution varies continuously with respect to the source.

In order to study the continuous dependence of the problem (2)–(7) with respect to the energy data, we define the following spaces:

$$H = H_1 \times H_2, \quad W = W_1 \times W_2 \tag{45}$$

Let $f_g$, $f_h$ denote the unique solutions to the Eq. (6) (or the fixed point (16)) corresponding to data $g = (g_1, g_2) \in H$ and $h = (h_1, h_2) \in H$ respectively.

Let $u_{1_{f_g}} \in W_1$ and $u_{1_{f_h}} \in W_1$ be the solutions to the variational equalities (13) for data $g_1 \in H_1$ and $f_g \in Q$, and $h_1 \in H_1$ and $f_h \in Q$ respectively. Similarly, let $u_{2_{f_g}} \in W_2$, and $u_{2_{f_h}} \in W_2$ be the solutions to the variational equalities (14) for data $g_2 \in H_2$ and $f_g \in Q$, and $h_2 \in H_2$ and $f_h \in Q$, respectively.

We consider in the spaces $H$ and $W$ the classical norms defined by:

$$\|(g_1, g_2)\|_H = \|g_1\|_{H_1} + \|g_2\|_{H_2}, \quad \forall\, g = (g_1, g_2) \in H, \tag{46}$$

$$\|(\omega_1, \omega_2)\|_W = \|\omega_1\|_{W_1} + \|\omega_2\|_{W_2}, \quad \forall\, \omega = (\omega_1, \omega_2) \in W. \tag{47}$$

We now define the operator

$$\tau : H \longrightarrow W \text{ such that } \tau(g) = u_g = (u_{1_{f_g}}, u_{2_{f_g}}) \in W \tag{48}$$

where $u_{1_{f_g}}$ and $u_{2_{f_g}}$ are the unique solutions of the variational equalities (13) and (14) respectively for data $g = (g_1, g_2) \in H$.

We aim to prove that $\tau$ is a Lipschitzian operator.

**Theorem 5** *Under the previously established notations, the following results hold:*

i) *We have*

$$\|u_{1_{f_g}} - u_{1_{f_h}}\|_{W_1} \leq \frac{1}{\kappa_1\lambda_1}\left(\|g_1 - h_1\|_{H_1} + \|\gamma_{0_{\Omega_1}}\|\|f_g - f_h\|_Q\right). \tag{49}$$

ii) *We have*

$$\|u_{2_{f_g}} - u_{2_{f_h}}\|_{W_2} \leq \frac{1}{\kappa_2\lambda_2}\left(\|g_2 - h_2\|_{H_2} + \|\gamma_{0_{\Omega_2}}\|\|f_g - f_h\|_Q\right). \tag{50}$$

iii) *We have*

$$\left\|f_g - f_h\right\|_Q \leq RC_1\|u_g - u_h\|_W \tag{51}$$

*with*

$$C_1 = \max\left(\|\gamma_{0_{\Omega_1}}\|; \|\gamma_{0_{\Omega_2}}\|\right). \tag{52}$$

iv) *We have*

$$\begin{aligned} ||\tau(g) - \tau(h)||_W &= \|u_g - u_h\|_W \\ &\leq C_2\|g - h\|_H + RC_1\left(\frac{\|\gamma_{0_{\Omega_1}}\|}{\kappa_1\lambda_1} + \frac{\|\gamma_{0_{\Omega_2}}\|}{\kappa_2\lambda_2}\right)\|u_g - u_h\|_W \end{aligned} \tag{53}$$

*with*

$$C_2 = max\left(\frac{1}{\kappa_1\lambda_1}; \frac{1}{\kappa_2\lambda_2}\right). \tag{54}$$

v) *Let $R_d$ be the positive constants defined by:*

$$R_d = \frac{1}{C_1\left(\frac{\|\gamma_{0_{\Omega_1}}\|}{k_1\lambda_1} + \frac{\|\gamma_{0_{\Omega_2}}\|}{k_2\lambda_2}\right)}. \tag{55}$$

*Then, if*

$$R < R_d, \tag{56}$$

*there exists a positive constant $C_3$ such that*

$$||\tau(g) - \tau(h)||_W = \|u_g - u_h\|_W \leq \frac{C_2}{C_3}\|g - h\|_H \tag{57}$$

*where*

$$C_3 = 1 - RC_1\left(\frac{\|\gamma_{0_{\Omega_1}}\|}{k_1\lambda_1} + \frac{\|\gamma_{0_{\Omega_2}}\|}{k_2\lambda_2}\right) > 0. \tag{58}$$

***Proof*** i) If we use $v = u_{1_{f_h}} - u_{1_{f_g}} \in V_1$ in the elliptic variational equality for $u_{1_{f_g}}$, and $v = u_{1_{f_g}} - u_{1_{f_h}} \in V_1$ in the elliptic variational equality for $u_{1_{f_h}}$, then we obtain

$$\begin{aligned} a_1(u_{1_{f_g}}, u_{1_{f_h}} - u_{1_{f_g}}) &= L_{1_{f_g}}(u_{1_{f_h}} - u_{1_{f_g}}) \\ &= \int_{\Omega_1} g_1(u_{1_{f_h}} - u_{1_{f_g}})\, dx - \int_{\Gamma_0} f_g(u_{1_{f_h}} - u_{1_{f_g}})\, dS, \\ a_1(u_{1_{f_h}}, u_{1_{f_g}} - u_{1_{f_h}}) &= L_{1_{f_h}}(u_{1_{f_g}} - u_{1_{f_h}}) \\ &= \int_{\Omega_1} h_1(u_{1_{f_g}} - u_{1_{f_h}})\, dx - \int_{\Gamma_0} f_h(u_{1_{f_g}} - u_{1_{f_h}})\, dS. \end{aligned}$$

If we add the last two equalities and use the coerciveness of the bilinear form $a_1$, hence:

$$\begin{aligned} \kappa_1\lambda_1\|u_{1_{f_g}} - u_{1_{f_h}}\|^2_{W_1} &\le a_1(u_{1_{f_g}} - u_{1_{f_h}}, u_{1_{f_g}} - u_{1_{f_h}}) \\ &= \int_{\Omega_1} (g_1 - h_1)(u_{1_{f_g}} - u_{1_{f_h}})\, dx \\ &\quad + \int_{\Gamma_0} (f_h - f_g)(u_{1_{f_g}} - u_{1_{f_h}})\, dS \\ &\le \|g_1 - h_1\|_{H_1}\|u_{1_g} - u_{1_h}\|_{H_1} + \|f_h - f_g\|_Q\|u_{1_{f_g}} - u_{1_{f_h}}\|_Q \\ &\le \left(\|g_1 - h_1\|_{H_1} + \|\gamma_{0_{\Omega_1}}\|\|f_h - f_g\|_Q\right)\|u_{1_{f_g}} - u_{1_{f_h}}\|_{W_1} \end{aligned}$$

that is (49) holds.

ii) Similarly, if we use $v = u_{2_{f_h}} - u_{2_{f_g}} \in V_2$ in the elliptic variational equality for $u_{2_{f_g}}$, and $v = u_{2_{f_g}} - u_{2_{f_h}} \in V_2$ in the elliptic variational equality for $u_{2_{f_h}}$, then we obtain

$$\begin{aligned} a_2(u_{2_{f_g}}, u_{2_{f_h}} - u_{2_{f_g}}) &= L_{2_{f_g}}(u_{2_{f_h}} - u_{2_{f_g}}) \\ &= \int_{\Omega_2} g_2(u_{2_{f_h}} - u_{2_{f_g}})\, dx + \int_{\Gamma_0} f_g(u_{2_{f_h}} - u_{2_{f_g}})\, dS, \\ a_2(u_{2_{f_h}}, u_{2_{f_g}} - u_{2_{f_h}}) &= L_{2_{f_h}}(u_{2_{f_g}} - u_{2_{f_h}}) \\ &= \int_{\Omega_2} h_2(u_{2_{f_g}} - u_{2_{f_h}})\, dx + \int_{\Gamma_0} f_h(u_{2_{f_g}} - u_{2_{f_h}})\, dS. \end{aligned}$$

By adding the last two equalities, and use the coerciveness of the bilinear form $a_2$, we get:

$$\begin{aligned} k_2\lambda_2\|u_{2_{f_g}} - u_{2_{f_h}}\|^2_{W_2} &\le a_2(u_{2_{f_g}} - u_{2_{f_h}}, u_{2_{f_g}} - u_{2_{f_h}}) \\ &= \int_{\Omega_2} (g_2 - h_2)(u_{2_{f_g}} - u_{2_{f_h}})\, dx \\ &\quad + \int_{\Gamma_0} (f_g - f_h)(u_{2_{f_g}} - u_{2_{f_h}})\, dS \\ &\le \|g_2 - h_2\|_{H_2}\|u_{2_{f_g}} - u_{2_{f_h}}\|_{H_2} + \|f_g - f_h\|_Q\|u_{2_{f_g}} - u_{2_{f_h}}\|_Q \\ &\le \left(\|g_2 - h_2\|_{H_2} + \|\gamma_{0_{\Omega_2}}\|\|f_h - f_g\|_Q\right)\|u_{2_{f_g}} - u_{2_{f_h}}\|_{W_2} \end{aligned}$$

and thus, (50) holds.

iii) If we assume that $f_g = T(f_g) = R\left(\gamma_{0\Omega_1}(u1_{f_g}) - \gamma_{0\Omega_2}(u2_{f_g})\right)$ on $\Gamma_0$, and $f_h = T(f_h) = R\left(\gamma_{0\Omega_1}(u1_{f_h}) - \gamma_{0\Omega_2}(u2_{f_h})\right)$ on $\Gamma_0$, then we have

$$\begin{aligned} f_g - f_h &= R\left[\left(\gamma_{0\Omega_1}(u1_{f_g}) - \gamma_{0\Omega_1}(u1_{f_h})\right) - \left(\gamma_{0\Omega_2}(u2_{f_g} - \gamma_{0\Omega_2}(u2_{f_h})\right)\right] \\ &= R\left[\gamma_{0\Omega_1}(u1_{f_g} - u1_{f_h}) - \gamma_{0\Omega_2}(u2_{f_g} - u2_{f_h})\right]. \end{aligned}$$

Moreover, we obtain:

$$\begin{aligned} \|f_g - f_h\|_Q &\le R\left[\left\|\gamma_{0\Omega_1}(u1_{f_g} - u1_{f_h})\right\|_Q + \left\|\gamma_{0\Omega_2}(u2_{f_g} - u2_{f_h})\right\|_Q\right] \\ &\le R\left[\|\gamma_{0\Omega_1}\|\|u1_{f_g} - u1_{f_h}\|_{W_1} + \|\gamma_{0\Omega_2}\|\|u2_{f_g} - u2_{f_h}\|_{W_2}\right] \\ &\le RC_1\left(\|u1_{f_g} - u1_{f_h}\|_{W_1} + \|u2_{f_g} - u2_{f_h}\|_{W_2}\right) \\ &= RC_1\|u_{f_g} - u_{f_h}\|_W \end{aligned}$$

and therefore (51) holds.

iv) We have

$$\begin{aligned} ||\tau(g) - \tau(h)||_W &= \|u_g - u_h\|_W = \|u1_{f_g} - u1_{f_h}\|_{W_1} + \|u2_{f_g} - u2_{f_h}\|_{W_2} \\ &\le \frac{1}{\kappa_1\lambda_1}\left(\|g_1 - h_1\|_{H_1} + \|\gamma_{0\Omega_1}\|\|f_g - f_h\|_Q\right) \\ &\quad + \frac{1}{\kappa_2\lambda_2}(\|g_2 - h_2\|_{H_2} + \|\gamma_{0\Omega_2}\|\|f_g - f_h\|_Q) \\ &= \frac{1}{\kappa_1\lambda_1}\|g_1 - h_1\|_{H_1} + \frac{1}{\kappa_2\lambda_2}\|g_2 - h_2\|_{H_2} \\ &\quad + \left(\frac{\|\gamma_{0\Omega_1}\|}{\kappa_1\lambda_1} + \frac{\|\gamma_{0\Omega_2}\|}{\kappa_2\lambda_2}\right)\|f_g - f_h\|_Q \\ &\le C_2\|g - h\|_H + RC_1\left(\frac{\|\gamma_{0\Omega_1}\|}{\kappa_1\lambda_1} + \frac{\|\gamma_{0\Omega_2}\|}{\kappa_2\lambda_2}\right)\|u_g - u_h\|_W \end{aligned}$$

which shows that inequality (53) holds.

v) If $R$ satisfies the inequality (56), then parameter $C_3 > 0$. Therefore, by using equation (53) we get

$$C_3||\tau(g) - \tau(h)||_W \le C_2\|g - h\|_H$$

and (57) holds.

□

***Remark 3*** By Theorem (5), $\tau$ is a Lipschitz operator with constant $\frac{C_2}{C_3} > 0$, and we have the continuous dependence of the problem (2)–(7) with respect to data $g = (g_1, g_2) \in H$.

**Lemma 6** *Let $R_e$ and $R_d$ be the positive constant defined by (33) and (55) respectively. Then, the following inequality holds:*

$$R_d \le R_e. \tag{59}$$

***Proof*** In order to prove the inequality (59), we divide the analysis into two cases.

i) If $C_1 = max\left(\|\gamma_{0\Omega_1}\|; \|\gamma_{0\Omega_2}\|\right) = \|\gamma_{0\Omega_1}\|$, we have:

$$\begin{aligned}\frac{1}{R_d} - \frac{1}{R_e} &= \|\gamma_{0\Omega_1}\|\left(\frac{\|\gamma_{0\Omega_1}\|}{\kappa_1\lambda_1} + \frac{\|\gamma_{0\Omega_2}\|}{\kappa_2\lambda_2}\right) - \left(\frac{\|\gamma_{0\Omega_1}\|^2}{\kappa_1\lambda_1} + \frac{\|\gamma_{0\Omega_2}\|^2}{\kappa_2\lambda_2}\right)\\ &= \frac{\|\gamma_{0\Omega_1}\|^2}{\kappa_1\lambda_1} + \frac{\|\gamma_{0\Omega_1}\|\|\gamma_{0\Omega_2}\|}{\kappa_2\lambda_2} - \frac{\|\gamma_{0\Omega_1}\|^2}{\kappa_1\lambda_1} - \frac{\|\gamma_{0\Omega_2}\|^2}{\kappa_2\lambda_2}\\ &= \frac{\|\gamma_{0\Omega_2}\|}{\kappa_2\lambda_2}\left(\|\gamma_{0\Omega_1}\| - \|\gamma_{0\Omega_2}\|\right) \geq 0.\end{aligned}$$

ii) If $C_1 = max\left(\|\gamma_{0\Omega_1}\|; \|\gamma_{0\Omega_2}\|\right) = \|\gamma_{0\Omega_2}\|$, we have:

$$\begin{aligned}\frac{1}{R_d} - \frac{1}{R_e} &= \|\gamma_{0\Omega_2}\|\left(\frac{\|\gamma_{0\Omega_1}\|}{\kappa_1\lambda_1} + \frac{\|\gamma_{0\Omega_2}\|}{\kappa_2\lambda_2}\right) - \left(\frac{\|\gamma_{0\Omega_1}\|^2}{\kappa_1\lambda_1} + \frac{\|\gamma_{0\Omega_2}\|^2}{\kappa_2\lambda_2}\right)\\ &= \frac{\|\gamma_{0\Omega_2}\|^2}{\kappa_2\lambda_2} + \frac{\|\gamma_{0\Omega_1}\|\|\gamma_{0\Omega_2}\|}{\kappa_1\lambda_1} - \frac{\|\gamma_{0\Omega_2}\|^2}{\kappa_2\lambda_2} - \frac{\|\gamma_{0\Omega_1}\|^2}{\kappa_1\lambda_1}\\ &= \frac{\|\gamma_{0\Omega_1}\|}{\kappa_1\lambda_1}\left(\|\gamma_{0\Omega_2}\| - \|\gamma_{0\Omega_1}\|\right) \geq 0.\end{aligned}$$

therefore, we obtain $\frac{1}{R_d} - \frac{1}{R_e} \geq 0$ which proves (59). □

***Remark 4*** We summarized the obtained results:

a) If $0 < R < R_d$, then there exists a unique solution to problem (2)–(7), that depends continuously to data $g = (g_1, g_2) \in H = H_1 \times H_2$.
b) If $R_d \leq R < R_e$ then there still exists a unique solution to problem (2)–(7).

***Remark 5*** We can improve the part $v$) of Theorem 5 by the following fact.
Let $R_d^\varepsilon$ be the positive constant defined by the expression

$$R_d^\varepsilon = \frac{1-\varepsilon}{C_1\left(\frac{\|\gamma_{0\Omega_1}\|}{k_1\lambda_1} + \frac{\|\gamma_{0\Omega_2}\|}{k_2\lambda_2}\right)}, \text{ for any } \varepsilon \in (0,1). \tag{60}$$

If $R \leq R_d^\varepsilon$, then the inequality (57) can be replaced by

$$||\tau(g) - \tau(h)||_W = \|u_g - u_h\|_W \leq \frac{C_2}{\varepsilon}\|g - h\|_H, \quad \forall\, \varepsilon \in (0,1). \tag{61}$$

## 6 Study of particular cases

In what follows, we present three illustrative cases with explicit solutions, each corresponding to Cartesian, cylindrical, and spherical coordinate systems, respectively.

The thermal conductivities employed in Examples 1, 2, 3, and 4 were obtained from [40] and are summarized in the Table 1 below.

***Remark 6*** Note that in Examples 1, 3, and 4, a deliberately large spatial domain was considered in order to clearly illustrate the temperature gradient between each boundary and the interface location. Furthermore, internal heat sources were set to zero, ensuring that the resulting temperature differences can be attributed exclusively to the thermal properties of the materials involved and the thermal resistance at the interface.

**Table 1** Thermal conductivities of different materials

| Material | Symbol | $\kappa$ [W/(m · °C)] |
|---|---|---|
| Lead | Pb | 35 |
| Iron | Fe | 73 |
| Nickel | Ni | 90 |
| Aluminum | Al | 204 |
| Copper | Cu | 386 |
| Silver | Ag | 419 |

### 6.1 One-dimensional case in cartesian coordinates

We consider the following domains and boundaries (see Fig. 3):

$$\Omega_1 = (-l_1, 0),\ \Omega_2 = (0, l_2),\ \Gamma_0 = \{0\},\ \Gamma_1 = \{-l_1\}\ and\ \Gamma_2 = \{l_2\}. \tag{62}$$

The corresponding steady-state heat transfer problem, with continuity of heat flow and a thermal jump at the interface is modeled by:

$$\begin{aligned} &-k_1 u_1''(x) = g_1, \quad -l_1 < x < 0 \\ &-k_2 u_2''(x) = g_2, \quad 0 < x < l_2 \\ &u_1(-l_1) = \theta_1,\ u_2(l_2) = \theta_2 \\ &k_1 u_1'(0) = k_2 u_2'(0) \\ &R(u_2(0) - u_1(0)) = k_1 u_1'(0) = -f \end{aligned} \tag{63}$$

The explicit solution to problem (63) is given by:

$$\begin{cases} u_1(x) = \theta_1 + \frac{g_1}{2k_1}(l_1^2 - x^2) + \frac{R}{k_1}\dfrac{\left((\theta_2-\theta_1)+\left(\frac{g_2 l_2^2}{2k_2} - \frac{g_1 l_1^2}{2k_1}\right)\right)}{\mu}(x + l_1)\ \ in\ -l_1 < x < 0 \\ u_2(x) = \theta_2 + \frac{g_2}{2k_2}(l_2^2 - x^2) - \frac{R}{k_2}\dfrac{\left((\theta_2-\theta_1)+\left(\frac{g_2 l_2^2}{2k_2} - \frac{g_1 l_1^2}{2k_1}\right)\right)}{\mu}(l_2 - x)\ \ in\ 0 < x < l_2 \end{cases} \tag{64}$$

where the parameter $\mu$ is defined by:

$$\mu = \mu(R) = 1 + R\left(\frac{l_1}{k_1} + \frac{l_2}{k_2}\right). \tag{65}$$

Moreover, the temperature jump at the interface $x = 0$ is given by:

$$\frac{-f}{R} = u_2(0) - u_1(0) = \frac{\left((\theta_2 - \theta_1) + \left(\frac{g_2 l_2^2}{2k_2} - \frac{g_1 l_1^2}{2k_1}\right)\right)}{1 + R\left(\frac{l_1}{k_1} + \frac{l_2}{k_2}\right)} \tag{66}$$

which is nonzero in general.

***Example 1*** As an example, we consider a particular case of the one detailed in Sect. 6.1. The physical parameters used are: $-l_1 = -100\,m$; $l_2 = 100\,m$; $\theta_1 = 100\,°C$; $\theta_2 = 80\,°C$; $R = 0.05\ W/(m^2 \cdot °C)$; $g_1(x) = 0$ and $g_2(x) = 0$.

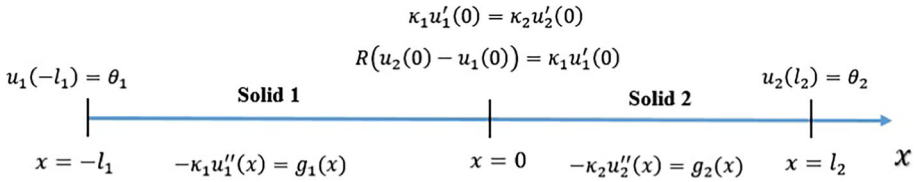



**Fig. 3** General scheme of the one-dimensional thermal contact problem in Cartesian coordinates

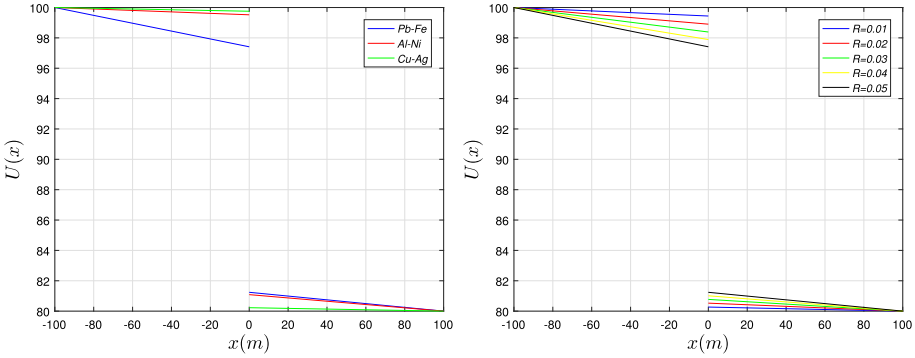



**Fig. 4** Steady-state temperature profiles. For different materials with $R = 0.05$ (left), and for different values of $R$ in a $Pb - Fe$ body (right)

In addition, the temperature field is defined:

$$U(x) = \begin{cases} u_1(x), & -l_1 < x < 0, \\ u_2(x), & 0 < x < l_2. \end{cases}$$

Figure 4 illustrates the temperature profile for Example 1, showing two dashed straight lines separated by a temperature jump at the interface. This discontinuity highlights the effect of the contact resistance, a phenomenon that cannot be captured by models assuming perfect thermal contact. This behavior is consistent with previous studies [20].

The left graph demonstrates that more conductive materials yield an interface temperature closer to that of the adjacent boundary, reflecting their greater capacity to transfer heat. In contrast, the right graph shows that as the thermal resistance increases, the temperature gradient at the interface decreases. This is expected, as a larger value of $R$ requires a smaller temperature difference across the interface to sustain the same heat flux.

Overall, the figure confirms that the proposed formulation reproduces the expected physical behavior of conduction in bicontact systems and provides a clear illustration of the role of material properties and interface resistance in shaping the temperature distribution.

***Example 2*** For comparative purposes and to understand the specific influence of thermal sources on the studied phenomenon, the problem described in Example 1 is reconsidered, but with non-zero and non-constant sources. The thermal sources used are:

$$g(x) = \begin{cases} g_1(x) = -\dfrac{1}{5}\, x(x + l_1), & -l_1 < x < 0, \\ g_2(x) = -\dfrac{1}{5}\, x(x - l_2), & 0 < x < l_2. \end{cases} \tag{67}$$

where $-l_1 = -10$ m and $l_2 = 10$ m.

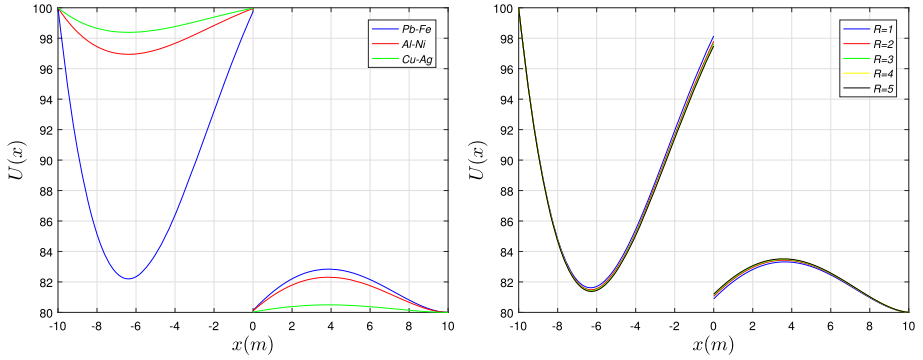



**Fig. 5** Steady-state temperature profiles. For different materials with $R = 0.05$ (left), and for different values of $R$ in a $Pb - Fe$ body (right). The thermal sources are given by (67)

Figure 5 illustrates the temperature profile for the new example with nonzero thermal sources defined by (67). As in Example 1, a temperature jump at the interface is observed, highlighting the effect of the contact resistance. However, unlike the previous case, the temperature profiles are no longer straight lines due to the spatial variation introduced by the nonzero sources.

The left graph shows that less conductive materials reach lower temperature values, which in this case is due to the functional form of the source, strongly conditioning the qualitative shape of the temperature profile. The right graph demonstrates that increasing the interface resistance reduces the temperature gradient at the interface, consistent with physical expectations as previously explained.

Overall, the figure confirms that the proposed formulation accurately captures the influence of material properties and nonuniform thermal sources on the temperature distribution, while preserving the characteristic discontinuity at the interface.

### 6.2 Two-dimensional case in polar coordinate system

We now consider the following domains and boundaries in two variables ($0 < r_0 < r_1 < r_2$) (see Fig. 6):

$$\begin{aligned} \Omega_1 &= \{(x, y) \in \mathbb{R}^2 \,/\, r_0 < r = \sqrt{x^2 + y^2} < r_1\} \\ \Omega_2 &= \{(x, y) \in \mathbb{R}^2 \,/\, r_1 < r = \sqrt{x^2 + y^2} < r_2\} \\ \Gamma_0 &= \{r = r_1\},\ \Gamma_1 = \{r = r_0\},\ \Gamma_2 = \{r = r_2\} \end{aligned} \tag{68}$$

where $r = \sqrt{x^2 + y^2}$ is the radial coordinate in $\mathbb{R}^2$.

We consider radial symmetry, and in this case, the problem (2)–(7) is now formulated as follows:

$$\begin{aligned} -k_1 \left( \frac{1}{r} u_1'(r) + u_1''(r) \right) &= g_1, \quad r_0 < r < r_1 \\ -k_2 \left( \frac{1}{r} u_2'(r) + u_2''(r) \right) &= g_2, \quad r_1 < r < r_2 \\ u_1(\Gamma_0) = \theta_1,\ u_2(\Gamma_2) &= \theta_2 \end{aligned}$$

$$
\begin{aligned}
& k_1 u_1^{'}(r_1) = k_2 u_2^{'}(r_1) \\
& R(u_2(r_1) - u_1(r_1)) = k_1 u^{'}(r_1)
\end{aligned} \tag{69}
$$

The explicit solution to problem (69) is given by:

$$
\begin{cases}
u_1(r) = \theta_1 - \frac{g_1}{4k_1}(r^2 - r_0^2) + \beta \log(\frac{r}{r_0}) \;\; in\ \Omega_1 \\
u_2(r) = \theta_2 + \frac{g_2}{4k_2}(r_2^2 - r^2) - \left((g_2 - g_1)\frac{r_1^2}{2k_2} + \frac{k_1}{k_2}\beta\right)\log(\frac{r_2}{r}) \;\; in\ \Omega_2
\end{cases} \tag{70}
$$

where $\beta$ is a constant defined as:

$$
\beta = \frac{(\theta_2 - \theta_1) + \frac{g_1 r_1}{2R} + \frac{g_2}{4k_2}(r_2^2 - r_1^2) + \frac{g_1}{4k_1}(r_1^2 - r_0^2) - (g_2 - g_1)\frac{r_1^2}{2k_2}\log(\frac{r_2}{r_1})}{\frac{k_1}{Rr_1} + \frac{k_1}{k_2}\log(\frac{r_2}{r_1}) + \log(\frac{r_1}{r_0})}. \tag{71}
$$

Moreover, the temperature jump at the interface $r = r_1$ is given by:

$$
\begin{aligned}
\frac{-f}{R} &= u_2(r_1) - u_1(r_1) \\
&= (\theta_2 - \theta_1) + \frac{1}{4}\left(\frac{g_2}{k_2}(r_2^2 - r_1^2) + \frac{g_1}{k_1}(r_1^2 - r_0^2)\right) \\
&\quad - (g_2 - g_1)\frac{r_1^2}{2k_2}\log\left(\frac{r_2}{r}\right) - \beta\left(\frac{k_1}{k_2}\log\left(\frac{r_2}{r_1}\right) + \log\left(\frac{r_1}{r_0}\right)\right)
\end{aligned} \tag{72}
$$

which is nonzero in general.

***Example 3*** As an example, we consider a particular case of the one detailed in Sect. 6.2. The physical parameters used are: $r_0 = 0.05\,m$; $r_1 = 100\,m$; $r_2 = 200\,m$; $\theta_1 = 100\,°C$; $\theta_2 = 80\,°C$; $R = 0.05\,W/(m^2 \cdot °C)$; $g_1(x) = 0$ and $g_2(x) = 0$.

In addition, the temperature field is defined:

$$
U(r) = \begin{cases} u_1(r), & r_0 < r < r_1, \\ u_2(r), & r_1 < r < r_2. \end{cases}
$$

Figure 7 illustrates the temperature profile corresponding to Example 3. Since this problem is formulated in polar coordinates with radial symmetry, the temperature distributions in each solid are no longer straight lines, in contrast with Example 1. This highlights the role of geometry in shaping the steady-state temperature field.

The graph on the left shows that more conductive materials again yield an interface temperature closer to that of the adjacent boundary, confirming the trend already observed in Example 1. The graph on the right reveals that as the thermal resistance increases, the temperature gradient at the interface decreases. As in Example 1, this is expected because a larger value of $R$ requires a smaller temperature difference across the interface to sustain a constant heat flux.

Overall, this figure illustrates how the proposed framework captures the combined effects of material properties, interfacial resistance, and geometry. The ability to account for radial symmetry is particularly relevant for practical applications involving cylindrical or spherical domains, such as pipelines, coated wires, or biological tissues.

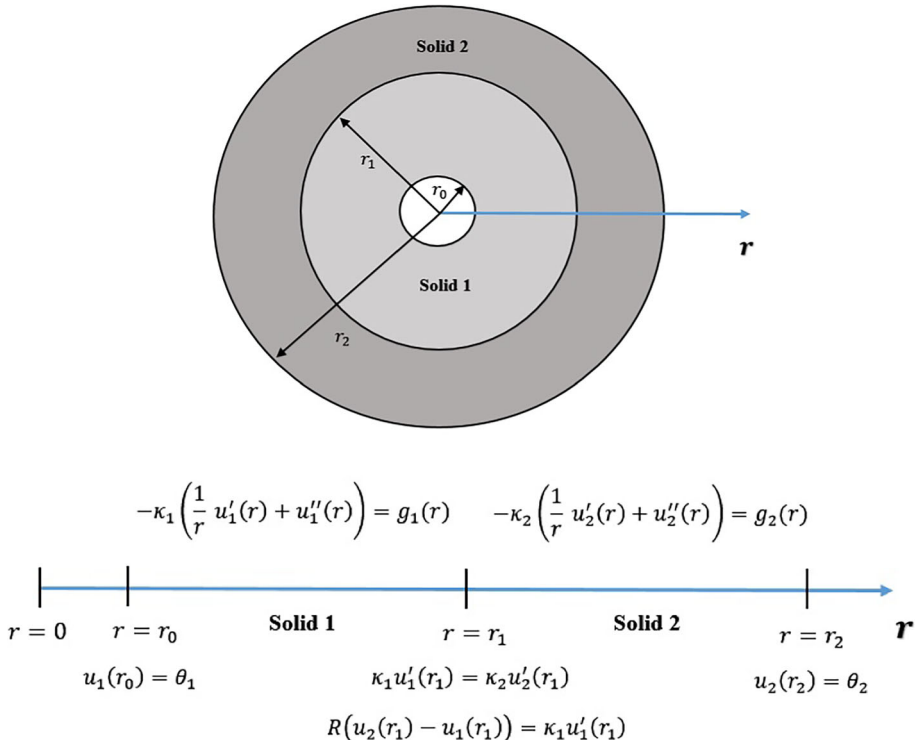



**Fig. 6** General scheme of the two-dimensional thermal contact problem in polar coordinates

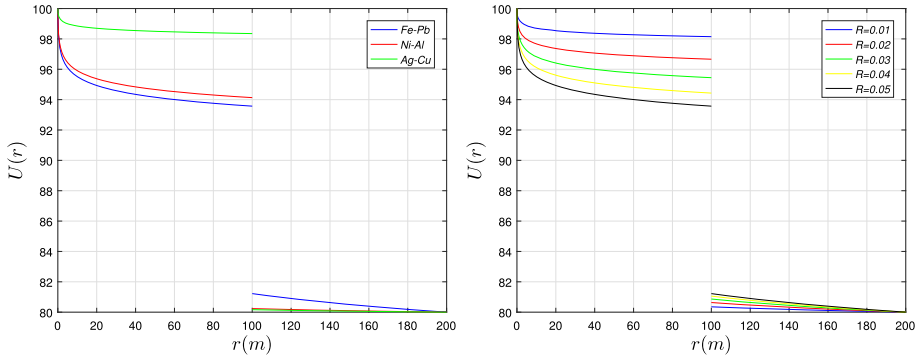



**Fig. 7** Steady-state temperature profiles. For different materials with $R = 0.05$ (left), and for different values of $R$ in a $Fe - Pb$ body (right)

### 6.3 Three-dimensional case in spherical coordinate system

Now we consider the domains and boundary in three variables ($0 < r_0 < r_1 < r_2$)(see Fig. 8):

$$\begin{aligned}
\Omega_1 &= \left\{(x, y, z) \in \mathbb{R}^3 \ / \ r_0 < r = \sqrt{x^2 + y^2 + z^2} < r_1\right\} \\
\Omega_2 &= \left\{(x, y, z) \in \mathbb{R}^3 \ / \ r_1 < r = \sqrt{x^2 + y^2 + z^2} < r_2\right\} \\
\Gamma_0 &= \{r = r_1\},\ \Gamma_1 = \{r = r_0\},\ r_2 = \{r = r_2\}
\end{aligned} \tag{73}$$

where $r = \sqrt{x^2 + y^2 + z^2}$ is the radial in spherical coordinates in $\mathbb{R}^3$.

We consider radial symmetry, and in this case, the problem (2)–(7) is formulated as follows:

$$\begin{aligned}
&-k_1\left(\frac{2}{r}u_1^{'}(r) + u_1^{''}(r)\right) = g_1, \quad r_0 < r < r_1 \\
&-k_2\left(\frac{2}{r}u_2^{'}(r) + u_2^{''}(r)\right) = g_2, \quad r_1 < r < r_2 \\
&u_1(\Gamma_0) = \theta_1,\ u_2(\Gamma_2) = \theta_2 \\
&k_1 u_1^{'}(r_1) = k_2 u_2^{'}(r_1) \\
&R(u_2(r_1) - u_1(r_1)) = k_1 u^{'}(r_1)
\end{aligned} \tag{74}$$

The explicit solution to problem (74) is given by:

$$\begin{cases}
u_1(r) = \theta_1 - \frac{g_1}{6k_1}(r^2 - r_0^2) + \alpha\left(\frac{1}{r_0} - \frac{1}{r}\right) \ \ in\ \Omega_1 \\
u_2(r) = \theta_2 + \frac{g_2}{6k_2}(r_2^2 - r^2) + \left(\frac{1}{r_2} - \frac{1}{r}\right)\left((g_2 - g_1)\frac{r_1^3}{3k_2} + \frac{k_1}{k_2}\alpha\right) \ \ in\ \Omega_2
\end{cases} \tag{75}$$

where the constant $\alpha$ is defined by:

$$\alpha = \frac{r_1(\theta_2 - \theta_1) + \frac{r_1^3}{6}(\frac{g_1}{k_1} - \frac{g_2}{k_2}) + \frac{g_2}{6k_2}r_1 r_2^2 - \frac{g_1}{6k_1}r_1 r_0^2 + (\frac{r_1}{r_2} - 1)(g_2 - g_1)\frac{r_1^3}{3k_2}}{\frac{k_1}{Rr_1} - \frac{k_1}{k_2}\left(\frac{r_1}{r_2} - 1\right) + \frac{r_1}{r_0} - 1}. \tag{76}$$

Moreover, the temperature jump at the interface $r = r_1$ is given by:

$$\begin{aligned}
\frac{-f}{R} &= u_2(r_1) - u_1(r_1) \\
&= (\theta_2 - \theta_1) + \frac{1}{6}\left(\frac{g_2}{k_2}(r_2^2 - r_1^2) + \frac{g_1}{k_1}(r_1^2 - r_0^2)\right) \\
&\quad + (g_2 - g_1)\frac{r_1^3}{3k_2}\left(\frac{1}{r_2} - \frac{1}{r_1}\right) + \alpha\left(\frac{k_1}{k_2}\left(\frac{1}{r_2} - \frac{1}{r_1}\right) - \left(\frac{1}{r_0} - \frac{1}{r_1}\right)\right)
\end{aligned} \tag{77}$$

which is nonzero in general.

***Example 4*** As an example, we consider a particular case of the one detailed in Sect. 6.3. The physical parameters used are: $r_0 = 0.05\,m$; $r_1 = 10\,m$; $r_2 = 20\,m$; $\theta_1 = 100\,^\circ C$; $\theta_2 = 80\,^\circ C$; $R = 0.05\,W/(m^2 \cdot\,^\circ C)$; $g_1(x) = 0$ and $g_2(x) = 0$.

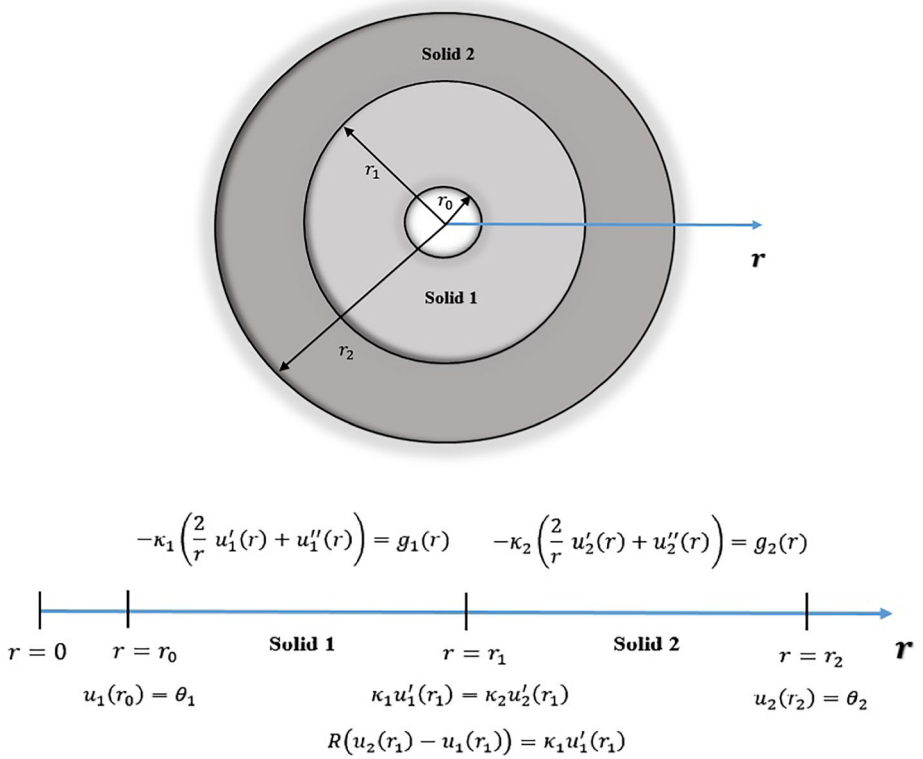



**Fig. 8** General scheme of the three-dimensional thermal contact in spherical coordinates

In addition, the temperature field is defined:

$$U(r)=\begin{cases}u_1(r), & r_0<r<r_1,\\ u_2(r), & r_1<r<r_2.\end{cases}$$

Figure 9 illustrates the temperature profile corresponding to Example 4. Unlike Example 1 (Cartesian) and Example 3 (polar with radial symmetry), this case addresses a spherically symmetric configuration. The temperature distribution is therefore influenced not only by the material conductivities and interfacial resistance but also by the curvature of the geometry.

As in the previous examples, the left plot shows that materials with higher conductivity yield an interface temperature closer to that of the adjacent boundary. Similarly, the right plot confirms that increasing the thermal resistance reduces the temperature gradient across the interface, since a larger $R$ requires a smaller temperature difference to sustain a constant heat flux.

This example demonstrates that the proposed analytical framework is flexible enough to handle spherical geometries, which are especially relevant in contexts such as heat conduction in coated particles, spherical capsules, or inclusions in composite media. The results remain qualitatively consistent with physical expectations, thereby reinforcing the robustness of the theoretical approach.

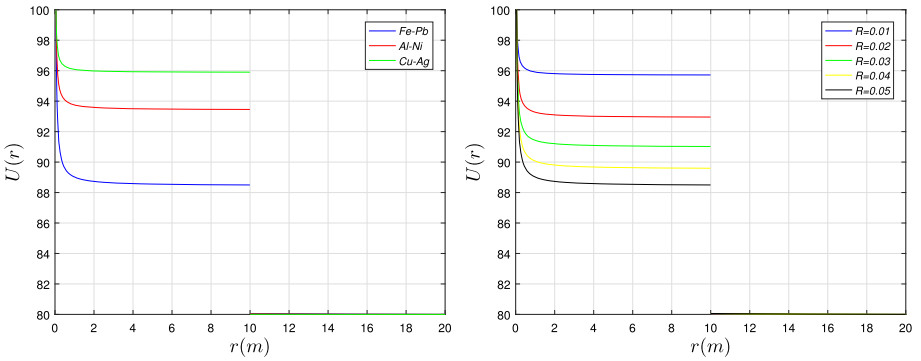



**Fig. 9** Steady-state temperature profiles. For different materials with $R = 0.05$ (left), and for different values of $R$ in a $Fe - Pb$ body (right)

## 7 Conclusion

This paper presents a mathematical analysis of heat conduction between two adjacent solids in thermal contact, regardless of their geometric configuration. The model assumes continuity of the heat flux and allows for a temperature discontinuity at the contact interface. The problem is formulated and solved using elliptic variational equalities and a fixed-point theorem, with the resistance parameter at the interface playing a central role. The continuous dependence of the solution on the internal heat sources in each solid is also established. Finally, three explicit solutions are given for Cartesian, polar and spherical coordinates and four illustrative examples are provided, whose results are consistent with the expected physical behavior of the system.

It is important to note that the present framework is limited to steady-state heat conduction and considers only linear interfacial effects. While this allows for a rigorous analytical treatment, extensions to transient problems, nonlinear interfacial phenomena, or more complex boundary conditions are beyond the scope of this study and represent promising directions for future research.

Moreover, the proposed formulation provides a foundation for analyzing practical engineering problems involving composite materials, microscale heat transfer, or other complex systems. Although the current study focuses on linear, steady-state conditions, the approach can be adapted to investigate nonlinear effects and transient behaviors, which are of significant interest in real-world applications.

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**Data availability** Not applicable.

**Code availability** Not applicable.

## Declarations

**Conflict of interest** The authors declare no conflict of interest.

**Ethics approval and consent to participate** Not applicable.

**Consent for publication** Not applicable.

**Materials availability** Not applicable.

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