VOL.5

Reuniões em Matemática Aplicada e Computação Científica

X Congresso Nacional de Matemática Aplicada e Computacional

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MINICURSO

Gramado RS - 21 a 25 de Setembro de 1987

REUNIÕES EM MATEMÁTICA APLICADA E COMPUTAÇÃO CIENTÍFICA

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THE TWO-PHASE STEFAN PROBLEM AND SOME RELATED CONDUCTION PROBLEMS (*)

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ABSTRACT

The goal of this minicourse is to study the two-phase steady-state Stefan problem and some related conduction problems with or without phase change through the elliptic variational inequalities (EVI) theory. In chapter I we give some generalities on EVI with a symetric bilinear form and the variational formulation of several elliptic boundary valve problems in Sololev spaces. In chapters II and III we apply the EVI theory (in particular, the properties given before for elliptic variational equalities) to the two steady-state two-phase Stefan problems and some related conduction problems with or without phase change in a bounded domain $\Omega \subset R^n$ (n=1,2,3 in practice). In chapter IV we give some conditions for several heat fluxes on the fixed face to obtain an instantaneous evolution two-phase Stefan problem for a semi-infinite material with constant initial temperature. In appendix 1 we give three steady-state examples with explicit solution related to problems presented in chapters II and III. Moreover, for all these examples, the sufficient condition (given in chapters II and III) is also necessary. In appendix 2 we present a short review on the approximate, numerical and theoretical methods to solve free boundary problems for the heat-diffusion equation of Stefan type. In appendix 3 we present a short review on the Stefan problem through elliptic and parabolic variational inequalities and their numerical approximations.

NOTE: We shall note with (J-N-i) the formula (i) of paragraph N in chapter J (we shall omit N in the same paragraph and J in the same chapter). We shall also omit the space variable $x \in \Omega$ for every function defined in the domain $\Omega \subset R^n$.

CONTENTS

CHAPTER 1

SOME GENERALITIES ON ELLIPTIC VARIATIONAL INEQUALITIES

1.1 - INTRODUCTION

A very useful class of nonlinear problems ariging from mechanics, physics, etc. consists in solving the following elliptic variational inequality (EVI): To find $u \in K$ such that

$$\begin{cases} a(u, v-u) \geq L(v-u), \ \forall v \in K, \\ u \in K, \end{cases} \tag{1}$$

where

i) V: real Hilbert space with scalar product (.,.) and associated norm $\|\cdot\|$,

ii) V': the dual space of V,

iii) $a: V \times V \longrightarrow \mathbb{R}$ is a bilinear (i.e. linear in both variables), symmetric, continuous and coercive (or V-elliptic) form, that is, respectively

$$\begin{cases} \text{a) } a(u,v) = a(v,u), \ \forall u, v \in V, \\ \text{b) } M>0 \ / \ |a(u,v)| \leq M \|u\| \ \|v\|, \ \forall u, v \in V, \\ \text{c) } \alpha>0 \ / \ a(v,v) \geq \alpha \|v\|^2, \ \forall v \in V, \end{cases} \tag{2}$$

iv) $K \subset V$ is a closed, non-empty convex set,

v) $L \in V'$, i.e. $L: V \longrightarrow \mathbb{R}$ is a linear and continuous functional on V ($L(v) = \langle L, v \rangle$), which verifies

$$|L(v)| \leq \|L\| \ \|v\|, \ \forall v \in V. \tag{3}$$

1.2 - EXISTENCE AND UNIQUENESS THEOREM

We have the following

THEOREM 1.2.1 - Under the above hypotheses, there exists a unique solution to problem (1-1). In addition, the mapping $L\in V' \longrightarrow u\in K$ is Lipschitz, that is, if u_1, u_2 are solutions to problem (1-1) corresponding to L_1, $L_2 \in V'$, then

$$\|u_1-u_2\|_V \leq \frac{1}{\alpha} \|L_1-L_2\|_{V'}. \tag{1}$$

PROOF - The application $((.,.)): V\times V \longrightarrow \mathbf{R}$, defined by

$$((u,v)) = a(u,v), \ \forall u, v\in V, \tag{2}$$

is a scalar product on V. The associated norm $|||.|||$ is equivalent to the norm $\|.\|$ on V because

$$\alpha\|v\|^2 \leq a(v,v) \equiv |||v|||^2 \leq M \|v\|^2, \ \forall v\in V. \tag{3}$$

Hence by (3) we have

$$|L(v)| \leq \|L\|_{V'} \|v\| \leq \frac{\|L\|_{V'}}{\sqrt{\alpha}} |||v|||, \ \forall v\in V, \tag{4}$$

and therefore L is also continuous on V with respect to the new norm $|||.|||$.

By using the Riesz's representation theorem there exists a unique element $x_L \in V$ such that

$$L(v) = ((x_L, v)), \ \forall v\in V. \tag{5}$$

Moreover, the EVI (1-1) is equivalent to

$$\begin{cases} ((u, v-u)) \geq (x_L, v-u)), \forall v \in K, \\ u \in K, \end{cases} \tag{6}$$

which has a unique solution $u \in K$, given by (Exercise 1.2.1)

$$u = P_K(x_L) \tag{7}$$

where $P_K : V \longrightarrow K$ is the projection operator on the closed and convex subset K of V with respect to the norm $|||\cdot|||$. In addition, the element $u \in K$ satisfies the following minimization problem

$$\begin{cases} |||u - x_L||| \leq |||v - x_L|||, \forall v \in K, \\ u \in K. \end{cases} \tag{8}$$

To prove (1), we suppose that there exists $u_1, u_2 \in K$ solutions of the EVI

$$\begin{cases} a(u_i, v-u_i) \geq L_i(v-u_i), \forall v \in K, \\ u_i \in K \quad (i=1,2). \end{cases} \tag{9}$$

Setting $v=u_2$ in the EVI for u_1 and $v=u_1$ in that for u_2 we obtain, upon adding,

$$\begin{aligned} \alpha \|u_1-u_2\|^2 &\leq a(u_1-u_2, u_1-u_2) \leq (L_1-L_2)(u_1-u_2) \leq \\ &\leq \|L_1-L_2\| \, \|u_1-u_2\|, \end{aligned} \tag{10}$$

and therefore (1) holds.

COROLLARY 1.2.2 - (Exercise 1.2.2)

i) If $K \subset V$ is a closed, non-empty convex cone with vertex 0, then the problem (1-1) is equivalent to

$$\begin{cases} a(u,v) \geq L(v), \ \forall v \in K, \\ a(u,u) = L(u), \ u \in K. \end{cases} \tag{11}$$

ii) If $K \equiv V$, then the problem (1-1) is equivalent to the following elliptic variational equality (EVE)

$$\begin{cases} a(u,v) = L(v), \ \forall v \in V, \\ u \in V. \end{cases} \tag{12}$$

Morever, the problem (12) is equivalent to

$$\begin{cases} Au = L \in V' \\ u \in V, \end{cases} \tag{13}$$

where $A: V \longrightarrow V'$ is a linear and continuous operator, defined by

$$\langle Au, v \rangle = a(u,v), \ \forall u, v \in V. \tag{14}$$

REMARK 1.2.1 - In the case $K \equiv V$, the Theorem 2.1 is the well-know Lax-Milgram Theorem.

1.3 - RELATIONSHIP BETWEEN ELLIPTIC VARIATIONAL INEQUALITIES AND MINIMIZATION PROBLEMS

Now we shall connect the EVI (1-1) with the following minimization problem

$$\begin{cases} J(u) \leq J(v), \ \forall v \in K, \\ u \in K, \end{cases} \tag{1}$$

where the functional $J: V \longrightarrow \mathbb{R}$ is defined by

$$J(v) = \frac{1}{2} a(v,v) - L(v), \ \forall v \in V. \tag{2}$$

THEOREM 1.3.1 - If $L \in V'$ and a is a bilinear, symmetric, coercive and continuous form on V, then the problem (1-1) is equivalent to problem (1).

PROOF - (1-1) $\Rightarrow$ (1): For any $v \in K$, we have that

$$J(v) = J(u-(v-u)) = \frac{1}{2} a(u+(v-u), u+(v-u)) - L(u+(v-u)) =$$

$$= [\frac{1}{2} a(u,u) - L(u)] + \frac{1}{2} a(v-u, v-u) + [a(u, v-u) - L(v-u)] \geq \tag{3}$$

$$\geq J(u)$$

and therefore (1) holds.

(1) $\Rightarrow$ (1-1): Applying (1), and keeping in mind that K is convex, we have that

$$\frac{J(u+t(v-u)) - J(u)}{t} \geq 0, \ \forall v \in K, \ \forall t \in (0,1]. \tag{4}$$

Using the following limit (Exercise 1.3.1)

$$\lim_{t \to 0^+} \frac{J(u+t(v-u) - J(u)}{t} = a(u, v-u) - L(v-u), \ \forall v \in K, \tag{5}$$

we obtain (1-1).

REMARK 1.3.1 - The theorem 1.2.1 is valid even when the bilinear form a is not symmetric [K_iSt, L_iSt, Stam1], but in this case the Theorem 1.3.1 is not valid (for the equivalence between the EVI (1-1) and the minimization problem (1) the symmetry of a is essential).

REMARK 1.3.2 - For generalities on EVI or EVE from a theoretical point of view, see [BaCa, Brez, Diaz, Duva3, EkTe, Frie 2, K_iSt, Lion 2, L_iSt, Oden, OdKi, Rodr, Ruas, Stam 1, Tarz 3], from a numerical point of view, see [Ciar, Glow, GLT, Noch, RaTh]. Some additional references are given within the books cited above.

1.4 - VARIATIONAL FORMULATION OF SEVERAL ELLIPTIC BOUNDARY VALUE PROBLEMS IN SOBOLEV SPACES

We shall introduce some useful functional spaces and properties [Adam, BaCa, BrCi, Brez, Dali, Lion 1, LiMa, Morr, Neca, RaTh, Stam 2].

Let Ω be an open set of $\mathbf{R}^n$ (n=1,2,3 in practice) with a regular boundary $\Gamma = \partial\Omega$. Let $H^1(\Omega)$ be the Sobolev space of order 1 on Ω, defined by

$$H^1(\Omega)= \left\{ v\in L^2(\Omega) / \frac{\partial v}{\partial x_i} \in L^2(\Omega),\ 1 \leq i \leq n \right\} \tag{1}$$

which is a Hilbert space with the following scalar product

$$(u,v)_1 = (u,v)_0 + \sum_{i=1}^{n} \left(\frac{\partial u}{\partial x_i}, \frac{\partial v}{\partial x_i}\right)_0 \tag{2}$$

and associated norm $||v||_1 = \sqrt{(v,v)_1}$. We denote with $(u,v)_0$ the usual scalar product on $L^2(\Omega)$, defined by

$$(u,v)_0 = (u,v) = (u,v)_{L^2(\Omega)} = \int_\Omega uv\ dx, \tag{3}$$

and the associated norm $||v||_0 = \sqrt{(v,v)_0}$.

Let $H^1_0(\Omega)$ be functional space, given by

$$H^1_0(\Omega) = \{v \in H^1(\Omega) / \gamma_0(v) \equiv v/_\Gamma = 0\}, \tag{4}$$

where $\gamma_0 : H^1(\Omega) \longrightarrow L^2(\Gamma)$ is the trace operator which is a linear and continuos application with $Im(\gamma_0) \equiv H^{1/2}(\Gamma) \subset L^2(\Gamma)$. We can also interpret $H^1_0(\Omega)$ as the closure of $D(\Omega) = C_0^\infty(\Omega)$ in $H^1(\Omega)$. We also define $H^{-1}(\Omega) = (H^1_0(\Omega))'$ as the dual of the space $H^1_0(\Omega)$ when we identify $L^2(\Omega)$ with its dual.

We can define the spaces $H^2(\Omega)$, $H^{3/2}(\Gamma)$, etc. analogously.

THEOREM 1.4.1 - (Poincaré-Friedrichs inequality) If Ω is a bounded domain in $\mathbb{R}^n$ with a regular boundary Γ, then there exists a constant $C_0 = C_0(\Omega) > 0$ such that

$$\|v\|_0 \leq C_0(\Omega) \left(\sum_{i=1}^{n} \left\|\frac{\partial v}{\partial x_i}\right\|_0^2 \right)^{1/2}, \quad \forall v \in H^1_0(\Omega), \tag{5}$$

that is, the seminorm on $H^1(\Omega)$

$$v \longrightarrow \left(\int_\Omega |\nabla v|^2 \, dx\right)^{1/2} \tag{6}$$

is a norm on $H^1_0(\Omega)$ and it is equivalent to the norm on $H^1_0(\Omega)$ induced from $H^1(\Omega)$.

REMARK 1.4.1 - The above Theorem 1.4.1 is also valid when:

i) Ω is a domain in $\mathbb{R}^n$ which is bounded in at least one direction.

ii) We replace $H^1_0(\Omega)$ by the space

$$\begin{cases} V_0 = \{v \in H^1(\Omega) / v/\Gamma_1 = 0\}, \\ \Gamma_1 \subset \Gamma \text{ with meas } (\Gamma_1) \equiv |\Gamma_1| > 0 \text{ (measure of surface in } \mathbb{R}^{n-1}) \end{cases} \tag{7}$$

which verifies $H_0^1(\Omega) \subset V_0 \subset H^1(\Omega)$.

We denote with v^+ and v^- the positive and the negative parts of v, that is,

$$v^+ = Max(v,0), \quad v^- = Max(-v,0). \tag{8}$$

<u>THEOREM 1.4.2</u> - Let V be the space $H^1(\Omega)$ or $H_0^1(\Omega)$. If $v \in V$, then we have that v^+, $v^- \in V$. Morever, the applications $v \longrightarrow v^+$ and $v \longrightarrow v^-$ are continuous from $V \longrightarrow V$.

Now we shall give the variational formulation (i.e. the weak formulation) of some elliptic boundary value problems for the Laplace equation in a domain $\Omega \subset \mathbb{R}^n$ with a regular boundary Γ.

i) <u>Dirichlet problem</u>: Find a function u, defined in Ω, such that

$$\begin{cases} \text{i)} \quad -\Delta u = f \text{ in } \Omega, \\ \text{ii)} \quad u/_{\Gamma} = 0 \end{cases} \tag{9}$$

where $f \in L^2(\Omega)$ is a given function.

If we multiply (9i) by a "test function" $v \in H_0^1(\Omega)$, we integrate on Ω and we apply the following Green formula (valid for $u \in H^2(\Omega)$ and $v \in H^1(\Omega)$):

$$-\int_\Omega \Delta u \, v \, dx = \int_\Omega \nabla u . \nabla v \, dx - \int_\Gamma \frac{\partial u}{\partial n} v \, d\gamma \tag{10}$$

then we obtain for u the following EVE

$$\begin{cases} a_1(u,v)= (f,v), \ \forall v\varepsilon H_0^1(\Omega), \\ u\varepsilon H_0^1(\Omega), \end{cases} \qquad (11)$$

where

$$a_1(u,v)= \int_\Omega \nabla u.\nabla v \ dx. \qquad (12)$$

LEMMA 1.4.3 - There exists a unique solution to problem (11). Morever, if $f\geq 0$ in Ω then $u\geq 0$ in Ω.

PROOF - Problem (11) has a unique solution by application of Lax-Milgram Theorem and Poincaré-Friedrichs inequality. To prove $u\geq 0$ in Ω we choose $v= u^-\varepsilon \ H_0^1(\Omega)$ in (11) and we obtain that

$$0 \geq -a_1(u^-,u^-)= (f, u_0) \geq 0$$

because $u= u^+-u^-$ and $a_1(u^+, u^-)=0$. Therefore $u^-=0$ in Ω and then $u\geq 0$ in Ω.

ii) Neumann problem: Find a function u, defined in Ω, such that

$$\begin{cases} i) \quad -\Delta u + u= f \text{ in } \Omega, \\ ii) \ \frac{\partial u}{\partial n}\Big| \Gamma= 0, \end{cases} \qquad (13)$$

where $f\varepsilon L^2(\Omega)$ is a given function.

The variational formulation of (13) is given by

$$\begin{cases} a_2(u,v)= (f,v), \ \forall v\varepsilon H^1(\Omega), \\ u\varepsilon H^1(\Omega), \end{cases} \qquad (14)$$

where

$$a_2(u,v)= \int_\Omega (\nabla u.\nabla v+uv)dx= (u,v)_1. \tag{15}$$

LEMMA 1.4.4 - There exists a unique solution to problem (14). Morever, if $f\geq 0$ in Ω then $u\geq 0$ in Ω.

PROOF-(Exercise 1.4.1)

EXERCISE 1.4.2 - Let $f\varepsilon L^2(\Omega)$, $V=H^1(\Omega)$ and $L(v)= (f,v)$. Then, we have that

i) $L\varepsilon V$,

ii) $\exists! \; u_L\varepsilon V/L(v)= (u_L,v)_1$ (Apply the Lax-Milgram or Reisz Representation Theorem),

iii) The element $u_L\varepsilon V$ can be interpreted as the solution of the elliptic boundary value problem (13) (u_L satisfies (13i) in the distributional sense, and then in $L^2(\Omega)$).

iii) Mixed Dirichlet-Neumann problem - Find a function u, defined in Ω, such that

$$\begin{cases} -\Delta u= f \quad \text{in } \Omega, \\ u/_{\Gamma_1} = 0, \quad \frac{\partial u}{\partial n}/_{\Gamma_2} = 0, \end{cases} \tag{16}$$

where $f\varepsilon L^2(\Omega)$, $\Gamma_1 c \Gamma$ with $|\Gamma_1|>0$ and $\Gamma_2= \Gamma-\Gamma_1$.

EXERCISE 1.4.3 - i) The variational formulation of problem (16) is given by

$$\begin{cases} a_1(u,v) = (f,v), \ \forall v \in V_0, \\ u \in V_0, \end{cases} \tag{17}$$

where V_0 and a_1 are given by (7) and (12) respectively.

ii) There exists a unique solution to problem (17).

ii) If $f \geq 0$ in Ω, then $u \geq 0$ in Ω.

<u>EXERCISE 1.4.4</u> - Let $\overline{\Omega} = \overline{\Omega}_1 \cup \overline{\Omega}_2$ be a partition of $\overline{\Omega}$ such that

a) $\Omega_i \subset \Omega$ is an open set of $\mathbb{R}^n$ with a regular boundary $\partial(\Omega_i)$ for $i=1,2$.

b) $\Omega_1 \cap \Omega_2 = \emptyset$.

Let $v \in C^0(\overline{\Omega})$ be a continuous function such that its restriction to Ω_i verifies $v/_{\Omega_i} \in H^1(\Omega_i)$ $(i=1,2)$. Then $v \in H^1(\Omega)$.

<u>HINT</u> - Use the distributional sense and the following Green formula

$$\int_{\Omega_j} \frac{\partial u}{\partial x_i} v \, dx = -\int_{\Omega_j} u \frac{\partial v}{\partial x_i} dx + \int_{\partial(\Omega_j)} u v n_i \, d\gamma \ , \ (1 \leq i \leq n) \tag{18}$$

which is valid for all $u, v \in H^1(\Omega_j)$ $(j=1,2)$ where n_i is the i-component of the normal versor to $\partial(\Omega_j)$.

iv) <u>A transmission problem</u> - Let Ω_1 and Ω_2 be two bodies in thermal contact through a surface S. We consider the set $\Omega = \Omega_1 \cup \Omega_2 \cup S$ (see Exercise 1.4.4 and Figure 1.4.1) with a boundary $\Gamma = \Gamma_1 \cup \Gamma_2$ where $\Gamma_i = \Gamma \cap \partial(\Omega_i)$ $(i=1,2)$).

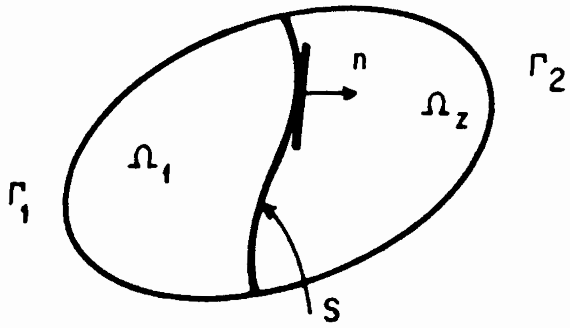


FIGURE I.4.1.

The problem consists in finding the functions u_1 and u_2, defined in Ω_1 and Ω_2 respectively, such that they satisfy the following conditions

$$\begin{cases} \text{i)} & -\Delta u_1 = f_1 \quad \text{in } \Omega_1, \\ \text{ii)} & -\Delta u_2 = f_2 \quad \text{in } \Omega_2, \\ \text{iii)} & u_1 = u_2, \ \dfrac{\partial u_1}{\partial n} = \dfrac{\partial u_2}{\partial n} \ \text{on } S, \\ \text{iv)} & u_1/_{\Gamma_1} = 0, \quad u_2/_{\Gamma_2} = 0, \end{cases} \tag{19}$$

where $f_i \varepsilon L^2(\Omega_i)$ $(i=1,2)$ and n is a normal versor to S.

From a physical point of view the first condition of (19iii) reflects a natural requirement, namely the absence of dicontinuities in the medium, while the second reflects the requirement that the forces acting on the interface S must be in equilibrium.

<u>EXERCISE 1.4.5</u> - The variational formulation of (19) is given by

$$\begin{cases} a(u,v) = L(v), \ \forall v = \{v_1, v_2\} \ \varepsilon \ V, \\ u = \{u_1, u_2\} \ \varepsilon \ V, \end{cases} \tag{20}$$

where

$$\begin{cases} V = \{v = \{v_1, v_2\} \in H^1(\Omega_1) \times H^1(\Omega_2) / \ v_1 = v_2 \text{ on } S\}, \\ a(u,v) = \sum_{i=1}^{2} \int_{\Omega_i} \nabla u_i . \nabla v_i \, dx, \\ L(v) = \sum_{i=1}^{2} \int_{\Omega_i} f_i v_i \, dx. \end{cases} \tag{21}$$

Morever, there exists a unique solution to the EVE (20).

The boundary conditions (19iv) can be modified by other Dirichlet or mixed conditions on Γ_1 and Γ_2.

CHAPTER 2

ON A FIRST STEADY-STATE TWO-PHASE STEFAN PROBLEM

2.1 - INTRODUCTION

We consider a bounded domain Ω of $\mathbb{R}^n$ (n=1,2,3 in practice), with a regular boundary $\Gamma=\Gamma_1 U \Gamma_2 U \Gamma_3$ (wich $|\Gamma_1|>0$ and $|\Gamma_2|>0$). We assume, without loss of generality, that the phase-change temperature is 0^{o}C. We impose a temperature $b=b(x)$ on Γ_1 and a heat flux $q=q(x)$ on Γ_2; we also suppose that the portion of the boundary Γ_3 (when it exists) is a wall impermeable to heat, i.e. the heat flux on Γ_3 is null.

If we consider in Ω a steady-state heat conduction problem, then we are interested in finding out necessary and/or suffcicient conditions for the heat flux q on Γ_2 (e.g. in the case $q>0$ on Γ_2 and $b>0$ on Γ_1) to obtain a change of phase in Ω, that is, a steady-state two-phase Stefan problem in Ω. We shall consider several problems related to it.

Following [Tarz1] we study the temperature $\theta=\theta(x)$ defined for $x\epsilon\Omega$. The set Ω can be expressed in the form

$$\Omega = \Omega_1 \text{ U } \Omega_2 \text{ U} \pounds \tag{1}$$

where

$$\begin{cases} \Omega_1=\{x\epsilon\Omega/\theta(x)<0\}, \quad \Omega_2=\{x\epsilon\Omega/\theta(x)>0\}, \\ \pounds=\{x\epsilon\Omega/\theta(x)=0\}, \end{cases} \tag{2}$$

are the solid phase, the liquid phase and the free boundary (e.g. a surface in R^3) that separates them, respectively (see Figure 2.1.1.).

The temperature Ω can be represent in Ω in the following way:

$$\theta(x) = \begin{cases} \theta_1(x) < 0 & \text{if } x \in \Omega_1, \\ 0 & \text{if } x \in \pounds, \\ \theta_2(x) > 0 & \text{if } x \in \Omega_2, \end{cases} \tag{3}$$

and satisfies the conditions below:

$$\begin{cases} \text{i) } \Delta\theta_1 = 0 \text{ in } \Omega_1, \\ \text{ii) } \Delta\theta_2 = 0 \text{ in } \Omega_2, \\ \text{iii) } \theta_1 = \theta_2 = 0, \quad k_1 \dfrac{\partial\theta_1}{\partial n} = k_2 \dfrac{\partial\theta_2}{\partial n} \text{ on } \pounds, \\ \text{iv) } \theta/\Gamma_1 = b, \\ \text{v) } \begin{cases} -k_2 \dfrac{\partial\theta_2}{\partial n} /\Gamma_2 = q & \text{if } \theta/\Gamma_2 > 0, \\ -k_1 \dfrac{\partial\theta_1}{\partial n} /\Gamma_2 = q & \text{if } \theta/\Gamma_2 < 0, \end{cases} \\ \text{vi) } \dfrac{\partial\theta}{\partial n}/\Gamma_3 = 0, \end{cases} \tag{4}$$

where $k_i > 0$ is the thermal conductivity of phase i (i=1: solid phase, i=2: liquid phase), b is the temperature given on Γ_1, q is the heat flux given on Γ_2, and n represents the exterior normal versor to Γ and $\pounds$ (in the direction of Ω_1 to Ω_2).

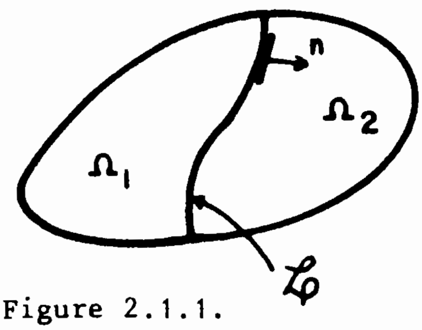


Figure 2.1.1.

Problem (4) represents a free boundary elliptic problem (when $\pounds \neq \phi$) where the free boundary $\pounds$ (unknown a priori) is characterized by the three conditions (4iii).

The goal of this chapter is to determine necessary **and/or** sufficient conditions for the data b=const.> 0 on Γ_1 and q=const. > 0 on Γ_2 so that problem (4) is a free boundary problem, that is $\mathcal{L} \neq \phi$.

2.2 - VARIATIONAL FORMULATION AND SOME PROPERTIES

Following the idea of [Baio, BaCa, Duva1, Duva2, Tarz1] we shall transform the free boundary elliptic problem (1-4) into a new elliptic problem, but now without a free boundary.

Let T_2, T_1 and u be the functions, defined in Ω, as follows

$$T_2 = \theta^+ = \begin{cases} \theta_2 > 0 & \text{in } \Omega_2, \\ 0 & \text{in } \Omega - \Omega_2, \end{cases} \tag{1}$$

$$T_1 = -\theta^- = \begin{cases} \theta_1 < 0 & \text{in } \Omega_1, \\ 0 & \text{in } \Omega - \Omega_1, \end{cases} \tag{2}$$

$$u = k_2 \theta^+ - k_1 \theta^- \quad \text{in } \Omega, \tag{3}$$

where θ^+ and θ^- represent the positive and the negative parts of function θ respectively, and u is the new unknown function.

We can find the temperature θ, when the new function u is known, through the inverse transformation of (3) that is

$$\theta = \frac{1}{k_2} u^+ - \frac{1}{k_1} u^- \quad \text{in } \Omega. \tag{4}$$

Let $D(\Omega)$ be the space

$$D(\Omega) = C_0^\infty(\Omega) = \{\gamma \in C^\infty(\Omega) / \text{supp}(\gamma) \text{ is compact in } \Omega\} \tag{5}$$

and let $\mathcal{D}'(\Omega)$ be the distributional space on Ω, that is the space of linear and continuous functionals of $D(\Omega)$ into $\mathbb{R}$ (i.e. the dual space of $D(\Omega)$).

We obtain the following properties

Theorem 2.2.1 i) We have that

$$<-k_2\ \Delta T_2, \gamma> = -k_2 \int_{\mathcal{L}} \frac{\partial \theta_2}{\partial n} \gamma\ d\gamma\ , \qquad \forall \gamma \in D(\Omega), \tag{6}$$

$$<-k_1\ \Delta T_1, \gamma> = k_1 \int_{\mathcal{L}} \frac{\partial \theta_1}{\partial n} \gamma\ d\gamma, \qquad \forall \gamma \in D(\Omega), \tag{7}$$

where $<.,.>$ represents the duality pairing between $D'(\Omega)$ and $D(\Omega)$.

ii) The new unknown function u satisfies the following elliptic problem

$$\begin{cases} \text{i)} & \Delta u = 0 \quad \text{in } D'(\Omega), \\ \text{ii)} & u/\Gamma_1 = B, \\ \text{iii)} & -\dfrac{\partial u}{\partial n} /\Gamma_2 = q, \\ \text{iv)} & \dfrac{\partial u}{\partial n} /\Gamma_3 = 0, \end{cases} \tag{8}$$

where $B=B(x)$ is defined by

$$B = k_2\ b^+ - k_1 b^- \text{ in } \Gamma_1. \tag{9}$$

Proof. To prove (6) for all $\gamma \in D(\Omega)$, we use a Green formula in Ω_2 as follows

$$<-k_2\ \Delta T_2, \gamma> = -k_2 \int_{\Omega} T_2\ \Delta\gamma\ dx = -k_2 \int_{\Omega_2} \theta_2\ \Delta\gamma\ dx =$$

$$= -k_2 \left[\int_{\Omega_2} \Delta\theta_2\ \gamma\ dx + \int_{\partial(\Omega_2)} \left(\theta_2 \frac{\partial \gamma}{\partial \nu} - \frac{\partial \theta_2}{\partial \nu} \right) d\gamma \right] =$$

$$= - k_2 \int_{\pounds} \gamma \frac{\partial \theta_2}{\partial n} \, d\gamma \ ,$$

because $\partial(\Omega_2) = (\partial(\Omega) \cap \Omega_2) \cup (-\pounds)$.

Exercise 2.2.1. Prove (7) and (8).

Theorem 2.2.2. i) The variational formulation of problem (8) is given by the following EVE

$$\begin{cases} a(u, v-u) = L(v-u), & \forall v \varepsilon K, \\ u \varepsilon K, \end{cases} \tag{10}$$

where

$$\begin{cases} V = H^1(\Omega), \quad V_0 = \{v \varepsilon V \ / \ v/\Gamma_1 = 0\}, \\ K = \{v \varepsilon V \ / \ v/\Gamma_1 = B \quad (=K_B)\ , \\ a(u,v) = \int_\Omega \nabla u . \nabla v \ \ dx, \quad L(v) = -\int_{\Gamma_2} q v d\gamma (=L_q(v)). \end{cases} \tag{11}$$

ii) Under the hypothesis $L \varepsilon V_0'$ (e.g. $q \varepsilon L^2(\Gamma_2)$) and $b \varepsilon H^{1/2}(\Gamma_1)$, there exists a unique solution of (10).

iii) The solution u to (10) is characterized by the following minimization problem

$$\begin{cases} J_q(u) \leq J_q(v), & \forall v \varepsilon K, \\ u \varepsilon K, \end{cases} \tag{12}$$

where

$$J_q(v) = \frac{1}{2} \ a(v,v) - L(v) = \frac{1}{2} \ a(v,v) + \int_{\Gamma_2} q v \ d\gamma. \tag{13}$$

Proof. Exercise 2.2.2. (Apply the Theorems 1.2.1, 1.3.1. and the following equivalence

$$\begin{cases} a(U,v) = F(v), \ \forall v \in V_o, \\ U \in V_o \end{cases} \qquad (10) \iff (14)$$

where

$$\begin{cases} U = u - B_o \in V_o, \\ B_o \in V \ / \ B_o/\Gamma_1 = B, \\ F(v) = L(v) - a(B_o, v), \ \forall v \in V_o). \end{cases} \qquad (15)$$

Next we give a property of monotonicity of the solution to problem (10) as a function of the data b (or B) and q.

Lemma 2.2.3. If $u=u_{qB}$ is the unique solution of problem (10) for data B and q, then we have that

$$\left. \begin{array}{l} B_1 \leq B_2 \ (\text{or } b_1 \leq b_2) \text{ on } \Gamma_1 \\ q_2 \leq q_1 \text{ on } \Gamma_2 \end{array} \right\} \Rightarrow u_{q_1 B_1} \leq u_{q_2 B_2} \text{ in } \bar{\Omega}. \qquad (16)$$

Proof. To prove (16) we shall take into account the following equivalence (we note $u_i = u_{q_i B_i}$ for $i=1,2$):

$$u_1 \leq u_2 \text{ in } \bar{\Omega} \iff w=0 \text{ in } \bar{\Omega}, \qquad (17)$$

where $w=(u_2-u_1)^-$.

Since $w \in V_0$, then, if we use $v=u_1+w \in K_{B_1}$ in the EVE corresponding to u_1, and $v=u_2+w \in K_{B_2}$ in the one corresponding to u_2 and we later substract them, we have

$$0 \leq a(w,w) = -\int_{\Gamma_2} (q_1 - q_2) w \, d\gamma \leq 0, \qquad (18)$$

that is, $w=0$ in $\bar{\Omega}$.

Exercise 2.2.3. If $q=q(x)$ 0>on Γ_2, then we have that

$$u_{qB} \geq \underset{\Gamma_1}{\mathrm{Max}}\, B \text{ in } \bar{\Omega}. \tag{19}$$

Remark 2.2.1. In this Chapter 2 we consider that the domain Ω and the data b (or B) on Γ_1 and q on Γ_2 are sufficiently regular to have the regularity property $u \in H^2(\Omega) \cap C^0(\bar{\Omega})$ (for $n \leq 3$, $H^2(\Omega) \subset C^0(\bar{\Omega})$) [Frie2, Gris, Mu St, Neca]. Moreover, in the three examples (See Appendix 1), the solution u satisfies this condition.

2.3 - AN INEQUALITY FOR THE CONSTANT HEAT FLUX TO OBTAIN A TWO-PHASE STEFAN PROBLEM

We assume a constant temperature $b>0$ on Γ_1 and a constant outcoming heat flux $q>0$ on Γ_2. If we consider in Ω a steady-state heat conduction problem, from the physical point of view, we arrive at the following conclusions:

i) If q is small, then the temperature in Ω will be positive, and so a change of phase in the material will not occur. In this case, the resulting problem will be one of conduction, only for the liquid phase.

ii) If q is large, then the temperature in Ω will take positive and negative values, and so a change of phase in the material will occur.

In this paragraphy, we shall find for q a sufficient condition for the occurence of a change of phase in Ω, i.e., we shall prove that for all $B>0$ ($B=k_2 b>0$) there exists $q_0 = q_0(B)>0$ so that for all $q>q_0$ we have a steady-state two-phase Stefan problem in Ω [Tarz5].

Lemma 2.3.1. Let $u=u_q$ be the unique solution to the EVE (2-10) for $q>0$ and $B>0$. We have the following properties:

$$a(u_q^-, u_q^-) = \int_{\Gamma_2} q\, u_q^-\, d\gamma, \tag{1}$$

$$u_q^- \neq 0 \text{ in } \Omega \iff u_q^- \neq 0 \text{ on } \Gamma_2. \tag{2}$$

Proof. It is enough to choose $v=u_q^+ \in K$ in (2-10) to obtain (1). From (1) and $u_q^- \in V_0$, we deduce (2).

Remark 2.3.1. For a given value of $q>0$ there will be a change of phase in Ω (u_q or θ_q takes positive and negative values in Ω) iff the function u_q takes negative values on the boundary Γ_2. In other words, the function u_q will begin to take negative values on Γ_2 (We can also obtain this fact by using the maximum principle [PrWe]).

Moreover, this fact will be taken into account when we carry out, the numerical simulation for the computation of the coefficient q_0 [Tarz5] and also for several optimization problems with temperature constraints [GoTa1] (See next paragraph).

Exercise 2.3.1. The Lemma 2.3.1 is also valid for the general case $b=b(x)>0$ on Γ_1 and $q=q(x)>0$ on Γ_2 (This property will be used in § 2.5.).

Lemma 2.3.2. If $u_i=u_{q_i}$ is the solution of (2-10) for q_i $(i=1,2)$, then we have the following properties:

i)

$$\begin{cases} \text{i)} \quad a(u_2-u_1, u_2-u_1) = (q_1-q_2) \displaystyle\int_{\Gamma_2} (u_2-u_1)\, d\gamma\,, \\ \text{ii)} \quad a(u_2,u_2) - a(u_1,u_1) = a(u_2+u_1,\, u_2-u_1) = \\ \qquad\qquad = (q_1+q_2) \displaystyle\int_{\Gamma_2} (u_1-u_2)\, d\gamma\,. \end{cases} \tag{3}$$

ii) If $q_2 \geq q_1$ then

$$\begin{cases} \text{i) } u_1 \leqq u_2 \text{ in } \Omega, \\ \text{ii) } \int_{\Gamma_2} u_1 \, d\gamma \leqq \int_{\Gamma_2} u_2 \, d\gamma. \end{cases} \tag{4}$$

iii) The applications $q \to u_q$ and $q \to \int_{\Gamma_2} u_q \, d\gamma$ are strictly decreasing functions, i.e.,

$$q_2 < q_1 \Rightarrow (5) \begin{cases} \text{a) } u_1 \leqq u_2, \quad u_1 \neq u_2 \text{ in } \Omega, \\ \text{b) } \int_{\Gamma_2} u_1 \, d\gamma < \int_{\Gamma_2} u_2 \, d\gamma. \end{cases} \tag{5}$$

Proof. i) If we take $v = u_2 \in K$ in the EVE corresponding to u_1, and $v = u_1 \in K$ in the one corresponding to u_2 and we add up and substract both equalities, then we obtain (3i) and (3ii) respectively.

ii) Condition (4ii) follows directly from (3i). To prove (4i) we shall take into account the following equivalence:

$$\begin{cases} u_1 \leqq u_2 \quad \text{in} \quad \Omega \Longleftrightarrow \quad w = 0 \quad \text{in} \quad \Omega, \\ \text{where} \quad w = (u_2 - u_1)^-. \end{cases} \tag{6}$$

Since $w \in V_0$, then, if we use $v = u_2 + w \in K$ in the EVE corresponding to u_1, and $v = u_1 + w \in K$ in the one corresponding to u_2 and we later add them up, we have

$$0 \leqq (q_1 - q_2) \int_{\Gamma_2} w \, d\gamma = a\,(u_2 - u_1, w) = -\,a\,(w, w) \leqq 0, \tag{7}$$

that is, $w = 0$ in Ω.

iii) To prove (5a,b) we use the following results:

(A) $u_1 = u_2$ in $\Omega \Rightarrow q_1 = q_2$ or $\int_{\Gamma_2} (u_2 - u_1) \, d\gamma = 0$,

$$(B) \int_{\Gamma_2} (u_2-u_1)\, d\gamma=0 \Rightarrow \begin{cases} (Bi) & u_2=u_1 \quad \text{in} \quad \Omega, \\ (Bii) & q_1=q_2. \end{cases}$$

Condition (A) results directly from (3i) and condition (Bi) is deduced from (3i) and from the fact that $u_2-u_1 \varepsilon V_o$. Taking into account (B)'s hypothesis, the result (Bi) and the variational equalities corresponding to u_2 and u_1, we obtain

$$- q_1 \int_{\Gamma_2} (v-u_1)\, d\gamma = a(u_1,v-u_1) = a(u_2,v-u_2) =$$

$$= - q_2 \int_{\Gamma_2} (v-u_2)\, d\gamma = -q_2 \int_{\Gamma_2} (v-u_1)\, d\gamma, \quad \forall v \varepsilon K,$$

i.e.,

$$(q_1-q_2) \int_{\Gamma_2} (v-u_1)\, d\gamma = 0, \ \forall v \varepsilon K. \qquad (8)$$

Taking one element $v_o \varepsilon V_o$ so that $\int_{\Gamma_2} v_o d\gamma \neq 0$ and choosing $v=u_1+v_o \varepsilon K$, from (8) we deduced (Bii).

Let $f: \mathbb{R}^+ \to \mathbb{R}$ be the real function, defined in the following way:

$$f(q) = J_q(u_q) = \frac{1}{2} a(u_q,u_q) + q \int_{\Gamma_2} u_q\, d\gamma. \qquad (9)$$

<u>Remark 2.3.3.</u> To find the element q_o, and taking into account (2), (5b) and function f, it will be enough to find a valve $q>0$ for which we have $f(q)<0$. We shall further that this technique can still be improved.

<u>Lemma 2.3.3.</u> For all $q>0$ and h such that $q+h>0$, we have the following estimates:

i)

$$\left\| \frac{1}{h}(u_q-u_{q+h}) \right\|_V \leq \frac{\|\gamma_o\| \sqrt{|\Gamma_2|}}{\alpha_o} \tag{10}$$

$$\left\| \frac{1}{h}(u_q-u_{q+h}) \right\|_{L^2(\Gamma_2)} \leq \frac{\|\gamma_o\|^2 \sqrt{|\Gamma_2|}}{\alpha_o} \ (=C_o), \tag{11}$$

where γ_o is the trace operator (linear and continuous, defined on V), and $\alpha_o > 0$ is the coercivity constant on V_o of the bilinear form a, i.e.:

$$a(v,v) \geq \alpha_o \|v\|_V^2, \ \forall v \in V_o. \tag{12}$$

ii) For all q>0 and h>0 we have

$$0 < \int_{\Gamma_2} (u_q-u_{q+h})\, d\gamma < C_o\, h, \tag{13}$$

and therefore the function $q \to \int_{\Gamma_2} u_q\, d\gamma$ is continuous.

Proof. i) Taking into account (12), (3i) with $q_1=q+h$ and $q_2=q$, the Cauchy-Schwarz inequality and the continuity of γ_o, we obtain (10). Taking into account (10) and the continuity of γ_o, we deduce (11).

ii) From (5b) and (11) we deduce (13).

Lemma 2.3.4. The function f is differentiable. Moreover, f' is a continuous and strictly decreasing function, and it is given by the following expression

$$f'(q)= \int_{\Gamma_2} u_q\, d\gamma. \tag{14}$$

Proof. We deduce (14) by using the fact that

$$\frac{f(q+h) - f(q)}{h} = \frac{1}{2} \int_{\Gamma_2} u_q \, d\gamma + \frac{1}{2} \int_{\Gamma_2} u_{q+h} \, d\gamma, \qquad (15)$$

which is obtained from (2-10) after elementary manipulations.

Lemma 2.3.5. From all $q>0$ we have the following expressions

$$a(u_q,u_q) = B \int_{\Gamma_1} \frac{\partial u_q}{\partial n} \, d\gamma - q \int_{\Gamma_2} u_q \, d\gamma, \qquad (16)$$

$$\int_{\Gamma_1} \frac{\partial u_q}{\partial n} \, d\gamma = q \, |\Gamma_2| , \qquad (17)$$

$$f(q) = B|\Gamma_2| \, q - \frac{1}{2} \, a(u_q,u_q) , \qquad (18)$$

$$\frac{d}{dq} \left[a(u_q,u_q) \right] = 2 \left[B|\Gamma_2| - \int_{\Gamma_2} u_q \, d\gamma \right] = \frac{2}{q} \, a(u_q,u_q), \qquad (19)$$

$$f'(q) = B \, |\Gamma_2| - \frac{1}{q} \, a(u_q,u_q), \qquad (20)$$

$$f''(q) = -\frac{1}{q^2} \, a(u_q,u_q) < 0. \qquad (21)$$

Proof. Expressions (16) and (17) are obtained by multiplying the differential equation (2-8i) by u_q and 1 respectively, by integrating on Ω and by using Grren's formula. Expression (18) is deduced from (9), (16) and (17). Expression (19) is obtained by taking the derivative of (18) with respect to q and by using (14).

Expression (20) is deduced from (14), (16) and (17), and expression (21) is obtained by using (19) and by differentiation of (20) with respect to q.

Theorem 2.3.6. i) There exists a constant C>0 such that

$$a(u_q,u_q) = C\,q^2,\ \forall q>0. \tag{22}$$

ii) Function f, defined by (9), is given by:

$$f(q) = -\frac{C}{2}\,q^2 + B\,|\Gamma_2|\,q,\ \forall q>0. \tag{23}$$

Proof. Let Y be the real function defined by

$$Y(q) = \frac{1}{q}\,a(u_q,u_q) \tag{24}$$

for q>0. It satisfies the following Cauchy problem:

$$\begin{cases} Y'(q) = \frac{1}{q}\,Y(q), \quad q>0, \\ Y(0^+) = \lim_{q\to 0^+} 2\left[B\,|\Gamma_2| - \int_{\Gamma_2} u_q\,d\gamma\right] = 0. \end{cases} \tag{25}$$

The solution of (25) is given by

$$Y(q) = C\,q, \quad \text{with } C = \text{const.}>0. \tag{26}$$

and therefore we obtain (22) as well as (23).

Remark 2.3.3. Constant C has the following physical dimension:

$$[C] = (\text{cm})^n \tag{27}$$

where n is the dimension of space $\mathbf{R}^n$ and it can also be calculated by the expression:

$$C = \frac{1}{q} \int_{\Gamma_2} (B-u_q)\, d\gamma \ , \text{ for some } q>0. \tag{28}$$

Exercise 2.3.2. Constant C is given by

$$C = a(u_3,\, u_3) = \int_{\Gamma_2} u_3 \, d\gamma, \tag{29}$$

where u_3 os the solution of the following EVE

$$\begin{cases} a(u_3,v) = \int_{\Gamma_2} v \, d\gamma, \ \forall v \varepsilon V_0, \\ u_3 \varepsilon V_0. \end{cases} \tag{30}$$

Then, we have that $C = C(\Omega,\Gamma)$.

Theorem 2.3.7. If $q>q_0$, where

$$q_0 = q_0(B) = \frac{B}{C} \, |\Gamma_2|, \tag{31}$$

then we obtain a two-phase Stefan problem in Ω.

Proof. It follows from Lemma 2.3.1 and the fact that

$$f'(q_0) = 0 = \int_{\Gamma_2} u_{q_0} \, d\gamma \ . \tag{32}$$

Corollary 2.3.8. In the case where, by reason of symmetry, we find that the function u_q is constant on Γ_2, the sufficient condition, given by Theorem 2.3.7., is also necessary for problem (2-10) to be a two-phase Stefan problem.

Proof. Since $u_q/\Gamma_2 =$ Const., the property follows from the following equivalence:

$$\int_{\Gamma_2} u_q \, d\gamma = 0 \iff u_q/\Gamma_2 = 0. \tag{33}$$

Lemma 2.3.9. If we consider the general case $b=b(x)>0$ on Γ_1 and $q=q(x)>0$ on Γ_2, we obtain: If q satisfies

$$\operatorname*{Inf}_{x\in\Gamma_2} q(x) > \frac{k_2}{C} |\Gamma_2| \operatorname*{Sup}_{x\in\Gamma_1} b(x) , \tag{34}$$

then we have a two-phase Stefan problem in Ω.

Proof. Exercise 2.3.3. (Use Lemma 2.2.3.).

Remark 2.3.4. In [Tarz5], numerical results were obtained by using the software MODULEF [Bern] (finite element code) in two cases for which Corollary 2.3.8. is valid and the solution is explicitly known [Tarz2] (See also Appendix 1).

Remark 2.3.5. In the Appendix 1 we give explicit solutions and the expressions of C and $q_0(B)$ for three special domains.

2.4 - SOME ESTIMATES FOR THE CRITICAL HEAT FLUX WHICH CHARACTERIZES A TWO-PHASE STEFAN PROBLEM

We continue to consider the case $b=\text{Const.}>0$ on Γ_1 and $q=\text{Const.}>0$ on Γ_2.

Let q_c be the critical heat outgoing flux which characterizes a two-phase Stefan problem in Ω, that is

$$\begin{cases} q>q_c \iff \exists \text{ 2-phase}, \\ q\le q_c \iff \exists \text{ 1-phase (in our case, the liquid phase)}, \end{cases} \tag{1}$$

because the monotony property (2.3.4).

We shall give now some estimates for q_c [BST].

<u>Theorem 2.4.1.</u> i) Let ω denote the solution to

$$\begin{cases} \Delta\omega = 0 \text{ in } \Omega, \\ \omega/\Gamma_1 = B, \quad \omega/\Gamma_2 = 0, \quad \frac{\partial\omega}{\partial n}/\Gamma_3 = 0. \end{cases} \tag{2}$$

If we define

$$q_i = \underset{\Gamma_2}{\mathrm{Min}} \left(-\frac{\partial\omega}{\partial n}/\Gamma_2\right) \tag{3}$$

then $u_q \geq \omega \geq 0$ in $\overline{\Omega}$, $\forall q \leq q_i$.

Moreover, we have

$$q_i \leq q_c \tag{4}$$

ii) Let $d = \underset{x\varepsilon\Gamma_2}{\mathrm{Sup}} \ \mathrm{dist}(x,\overline{\Gamma}_1)>0$ and let $P_1 \varepsilon \ \overline{\Gamma}_1$ and $P_2 \varepsilon \Gamma_2$ be such that $d = \mathrm{dist}\ (P_1,P_2)$. Now let π be an affine function such that

$$\begin{cases} \pi(P_1)=B, \quad \pi/\Gamma_1 \geq B, \\ \pi(P_2)=0, \quad \pi/\Gamma_2 \ \ 0, \\ \frac{\partial\pi}{\partial n}/\Gamma_3 \geq 0. \end{cases} \tag{5}$$

If we define

$$q_s = \underset{\Gamma_2}{\mathrm{Max}} \left(-\frac{\partial\pi}{\partial n}/\Gamma_2\right) \tag{6}$$

then $u_q \leq \pi$ in $\bar{\Omega}$, $\forall q \geq q_s$.

Moreover, we have $u_q(P_2) < 0$, $\forall q \geq q_s$ and then

$$q_c \leq q_s \tag{7}$$

iii) We have $\omega \leq \pi$ in $\bar{\Omega}$. Moreover,

$$\omega \neq \pi \quad \text{in } \bar{\Omega} \Rightarrow q_i < q_s. \tag{8}$$

Proof. i) We have $\omega > 0$ in Ω and $\omega \geq 0$ in $\bar{\Omega}$ due to the maximum principle [PrWe]; then $\frac{\partial \omega}{\partial n}/\Gamma_2 < 0$.

To prove $u_q \geq \omega$ in $\bar{\Omega}$, we define $z = u_q - \omega$ that satisfies $\Delta z = 0$ in Ω, $z/\Gamma_1 = 0$ and $\frac{\partial z}{\partial n}/\Gamma_3 = 0$. If $z/\Gamma_2 \geq 0$, then $z \geq 0$ in $\bar{\Omega}$ as desired. Then, if we assume that z has a negative minimum at some point $P \in \Gamma_2$, we obtain

$$0 > \frac{\partial z}{\partial n}(P) = \frac{\partial u_q}{\partial n}(P) - \frac{\partial \omega}{\partial n}(P) \geq -q + q = 0, \quad \forall q \leq q_i \ ,$$

which is a contradiction.

ii), iii): Exercise 2.4.1.

Remark 2.4.1. There is not necessarily uniqueness for the points $P_1 \in \bar{\Gamma}_1$ and $P_2 \in \Gamma_2$ in the above Theorem. In the three examples, given in the Appendix 1, there exist infinite pairing points P_1 and P_2 which satisfy the required condition; e.g. in Example 1 we have

$$P_1 = (0, y), \quad P_2(x_0, y), \text{ with } y \in [0, y_0]. \tag{9}$$

Remark 2.4.2. A sufficient condition for π to exist is the convexity of Ω; then one takes a supporting hyperphane to Ω at the point P_2 and we construct an affine function π vanishing on it and taking the value B at P_1. This construction fails if Γ_2 is a flat portion of Γ, e.g. the side of a triangle $\Omega\subset\mathbb{R}^2$, Γ_1 being the other two sides and $\Gamma_3=\phi$.

Remark 2.4.3. $u_q(P_2)<0$ give us that the second phase (in our case, the solid phase) appears at $P_2\in\Gamma_2$, the farthest point from Γ_1 (See also (2.3.1) and Remark 2.3.1.).

We shall consider $q_c=q_c(\Omega)$ as a function of the domain Ω. Let Ω_1 and Ω_2 be two bounded domains, with regular boundaries, such that (See Figure 2.4.1.) [BST]:

$$\begin{cases} \Omega_1\subset \Omega_2, \\ \partial(\Omega_1)= \Gamma_1^{(1)} \cup \Gamma_2 \cup \Gamma_3, \\ \partial(\Omega_2)= \Gamma_1^{(2)} \cup \Gamma_2 \cup \Gamma_3, \end{cases} \tag{10}$$

where the boundary conditions on $\Gamma_1^{(i)}$ $(i=1,2)$, Γ_2 and Γ_3 are of the same type as the ones defined before.

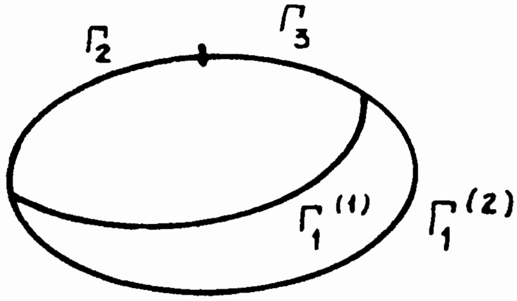


Figure 2.4.1.

Let u_i $(i=1,2)$ be the solution to problem (2-10) for the domain Ω_i with data B on $\Gamma_1^{(i)}$ and $q_i=q_i(x)$ on Γ_2 $(i=1,2)$, that is

$$\begin{cases} a_i(u_i, v-u_i) = -\int_{\Gamma_2} q_i(v-u_i)\, d\gamma, & \forall v\varepsilon K_i, \\ u_i \varepsilon K_i \quad (i=1,2) \end{cases} \tag{11}$$

where

$$\begin{cases} K_i = \{v\varepsilon H^1(\Omega_i) \;/\; v/\Gamma_1^{(i)}=B\}, \\ a_i(u,v) = \int_{\Omega_i} \nabla u.\nabla v\, dx \quad (i=1.2). \end{cases} \tag{12}$$

Lemma 2.4.2. Under the above hypotheses, we obtain the following property:

$$q_1 \leq q_2 \text{ on } \Gamma_2 \implies u_2 \leq u_1 \text{ in } \overline{\Omega}_1. \tag{13}$$

Proof. To prove (13) we shall take into account the following equivalence

$$u_2 \leq u_1 \text{ in } \overline{\Omega}_1 \iff z=0 \text{ in } \overline{\Omega}_1, \tag{14}$$

where $z=(u_2-u_1)^+\varepsilon H^1(\Omega_1)$. Moreover, $z/\Gamma_1^{(1)}=0$ because $u_2 \leq B$ in Ω_2, i.e. $u_2/\Gamma_1^{(1)} \leq B$. By using (11) with $v=u_1+z\ \varepsilon K_1$, we have

$$a_1(u_1,z) = -\int_{\Gamma_2} q_1 z\, d\gamma. \tag{15}$$

If we extend by o the function z to the whole set Ω_2 and we put $v=u_2+z\varepsilon K_2$ in (11), we obtain

$$\begin{aligned} -\int_{\Gamma_2} q_2 z\, d\gamma = a_2(u_2,z) &= \int_{\Omega_2} \nabla u_2.\nabla z\, dx = \\ &= \int_{\Omega_1} \nabla u_2.\nabla z\, dx = a_1\,(u_2,z) \end{aligned} \tag{16}$$

From (15) and (16) we obtain

$$0\leq a_1(z,z)= a_1(u_2-u_1,z)=\int_{\Gamma_2}(q_1-q_2)z\,d\gamma\leq 0,$$

that is $z=0$ in $\overline{\Omega}_1$.

Corollary 2.4.3. Let Ω_1 and Ω_2 be as above, i.e. they satisfy the conditions in (10). Then, we have that

$$q_c(\Omega_2)\leq q_c(\Omega_1), \tag{17}$$

that is, $q_c=q_c(\Omega)$ is a non-increasing function of the domain Ω where the order is represented by conditions (10).

Proof. It is enough to put $q_1\equiv q_2(=q)$ in Lemma 2.4.2.

Exercise 2.4.1. Let $\Omega\subset\mathbb{R}^2$ be the set defined, in polar coordinates, by

$$\begin{cases} \Omega =\{(r,\gamma) \ / \ r_o(\ )<r<R,\ 0\leq\gamma<2\pi\}, \\ 0<R_2\leq r_o(\ )\leq R_1<R,\ \forall\gamma\in[0,2\pi)\ (R_2<R_1), \end{cases} \tag{18}$$

then we have that (Use Example 2 in Appendix 1)

$$\frac{B}{R\log\frac{R}{R_2}} \leq q_c(\Omega) \leq \frac{B}{R\log\frac{R}{R_1}} \tag{19}$$

Remark 2.4.4. Other useful estimates can be found in [BST].

2.5 - SOME HEAT FLUX OPTIMIZATION PROBLEMS WITH TEMPERATURE CONSTRAINTS

A) For the general case $b=b(x)>0$ on Γ_1 and $q=q(x)$ on Γ_2, we consider the following optimization problem: Find $q\in Q^+$ that produces the total maximum heat flux on Γ_2, without change of phase within Ω, i.e. [GoTa1]

$$\operatorname*{Max}_{q\varepsilon Q^+} F(q) \tag{1}$$

where the functional $F:Q \to \mathbf{R}$ is defined by

$$F(q) = \int_{\Gamma_2} q \, d\gamma \ , \tag{2}$$

and

$$\begin{cases} S = \{v\varepsilon K \ / \ \Delta v=0 \text{ in } \Omega, \ \frac{\partial v}{\partial n}/\Gamma_3=0\}, \\ S^+ = \{v\varepsilon S \ / \ v\geq 0 \text{ in } \Omega\}, \quad Q=H^{1/2}(\Gamma_2), \\ Q^+=T^{-1}(S^+) = \{q\varepsilon Q \ / \ u_q\geq 0 \text{ in } \Omega\}. \end{cases} \tag{3}$$

The application $T:Q \to S$ is defined by

$$T(q)=u_q \tag{4}$$

where $u=u_q$ is the unique solution of (2-10), with the hypothesis $b\varepsilon H^{3/2}(\Gamma_1)$ (See Remark 2.2.1.); then we have that in Ω there will not exist a phase change for any heat flux $q\varepsilon Q^+$.

<u>Lemma 2.5.1.</u> i) The operator T, defined by (4), is an affine and monotone increasing operator, that is, there exist $u\,\varepsilon\, S$ and two new operators T_1 and T_2 so that $T=T_1+T_2$, where

$$\begin{cases} T_1:Q \to S \ / \ T_1(q)=u_1\varepsilon S, \ \forall q\varepsilon Q, \\ T_2:Q \to V_0 \ / \ T_2 \text{ is linear and continuous.} \end{cases} \tag{5}$$

ii) Q^+ is a convex set and F is a linear (then, convex) functional.

<u>Proof.</u> i) Let $u_1 \varepsilon K$ and $u_2=u_2(q)\varepsilon V_o$ be the unique solutions to the following problems:

$$\begin{cases} a(u_1, v-u_1)=0, \ \forall v\varepsilon K, \\ u_1 \varepsilon K, \end{cases} \tag{6}$$

$$\begin{cases} a(u_2,v)= - \int_{\Gamma_2} q \ v \ d\gamma, \ \forall v\varepsilon V_o, \\ u_2 \varepsilon V_o . \end{cases} \tag{7}$$

We have that $u=u_1+u_2$ from the uniqueness of problem (2-10); then T_1 and T_2 can be defined as follows:

$$T_1(q)=u_1, \quad T_2(q)=u_2, \quad \forall q\varepsilon Q. \tag{8}$$

Operator T_2 is linear and continuous because

$$\| T_2(q) \|_{V_o} \leq \frac{\|\gamma_o\|}{\alpha_o} \| q \|_{H^{1/2}(\Gamma_2)}, \quad \forall q\varepsilon Q, \tag{9}$$

where $\gamma_o : V \to V_o$ is the trace operator and $\alpha_o>0$ is the coercivity constant on V_o of the bilinear form a (See (3-12)). The monotony property of T is valid from Lemma 2.2.3.

ii) Q^+ is a convex set due to the fact that T is an affine operator and S^+ is a convex set.

We have the following theorem of existence and uniqueness of solution for the problem (1)

<u>Theorem 2.5.2.</u> There exists a unique $\bar{q}\varepsilon Q^+$ such that

$$F(\overline{q}) = \underset{q\varepsilon Q^+}{\text{Max}} \; F(q). \tag{10}$$

Moreover, the element $\overline{q}$ is defined by

$$\overline{q} = -\frac{\partial\omega}{\partial n}/\Gamma_2 \tag{11}$$

where ω is given by (4-2).

Proof. The element $\overline{q}\varepsilon Q^+$ verifies $F(q) \leq F(\overline{q})$, $\forall q\varepsilon Q^+$, by using the maximum principle (Exercise 2.5.1.).

Let $I:S \to \mathbb{R}$ be the linear functional, defined by:

$$I(v) = -\int_{\Gamma_2} \frac{\partial v}{\partial n}\, d\gamma, \; \forall v\varepsilon S. \tag{12}$$

We can consider a new formulation of the optimization problem (1), as follows:

$$\underset{v\varepsilon S^+}{\text{Max}} \; I(v) \tag{13}$$

Let Ψ, P and G_o be

$$\begin{cases} \Psi = C^o(\Gamma_2), \quad P = \{p\varepsilon\Psi \;/\; p \geq 0 \text{ on } \Gamma_2\}, \\ G_o : S \to \Psi \;/\; G_o(v) = -v/\Gamma_2, \end{cases} \tag{14}$$

then the problem (13) is equivalent to

$$\underset{v\varepsilon S,\; G_o(v) \leq 0}{\text{Max}} \; I(v) \;, \tag{15}$$

by using Exercise 2.3.1.

If u is a solution of (15), there exists a Lagrange multiplier $\mu\varepsilon\Psi'$ (dual of Ψ) with $\mu \geq 0$ (i.e. $\langle\mu, p\rangle = \int_{\Gamma_2} \mu p \, d\gamma \geq 0$,

$\forall p \in P$) that satisfies the following conditions [Bens, EkTe]:

$$\begin{cases} I(u) + < \mu, G_o(v)> \geq I(v), \quad \forall v \in S, \\ < \mu, G_o(u)> = 0. \end{cases} \tag{16}$$

We can deduce that $u=\omega$ and $\mu = \frac{\partial z_o}{\partial n}/\Gamma_2$, where the element z_o is given by (Exercise 2.5.2.)

$$\begin{cases} \Delta z_o = 0 \quad \text{in} \quad \Omega, \\ z_o/\Gamma_1 = 0, \quad z_o/\Gamma_2 = 1, \quad \frac{\partial z_o}{\partial n}/\Gamma_3 = 0, \end{cases} \tag{17}$$

and therefore we obtain the uniqueness of problem (1) or (10).

B) For the general case $b=b(x)>0$ on Γ_1 and $q=q(x)>0$ on Γ_2, we consider the following optimization problem: Find the maximum upper bound for q such that there is no change of phase within Ω, i.e. [GoTa1]

$$\text{Find } q_M^o>0 \; / \; u_q \geq 0 \text{ in } \Omega, \; \forall q=q(x) \leq q_M^o \text{ on } \Gamma_2. \tag{18}$$

<u>Theorem 2.5.3.</u> i) For the case $q=\text{Const.}>0$, we obtain that

$$q_M^o = \inf_{x\in\Gamma_2} \frac{u_1(x)}{u_3(x)}, \tag{19}$$

where u_1 and u_3 are given respectively by

$$\begin{cases} \Delta u_1 = 0 \quad \text{in} \quad \Omega, \\ u_1/\Gamma_1 = B, \quad \frac{\partial u_1}{\partial n}/\Gamma_2 U \Gamma_3 = 0 \qquad \text{(See (2-5-6)}, \end{cases} \tag{20}$$

$$\begin{cases} \Delta u_3 = 0 \quad \text{in } \Omega, \\ u_3/\Gamma_1 = 0, \quad \dfrac{\partial u_3}{\partial n}/\Gamma_2 = 1, \quad \dfrac{\partial u_3}{\partial n}/\Gamma_3 = 0 \quad (\text{See}(2.3.30)). \end{cases} \tag{21}$$

ii) If $q=q(x)>0$ on Γ_2 satisfies the condition

$$\sup_{x\in\Gamma_2} q(x) \leqslant q_M^o , \tag{22}$$

where q_M^o is defined by (19), then $u_q \geq 0$ in Ω.

iii) We have that

$$q_M^o = q_c , \tag{23}$$

where q_c is the critical heat outgoing flux (4-1).

Proof. Exercise 2.5.3.

Remark 2.5.1. The element q_M^o coincides with q_c in the three examples (See Appendix 1), i.e., it is the critical heat outgoing flux which characterizes a two-phase Stefan problem in Ω.

We shall consider now a new problem which is analogous to the above problem B.

C) For the general case $b=b(x)$ on Γ_1 and

$$q(x) = Q \quad q_1(x) \quad \text{on} \quad \Gamma_2, \tag{24}$$

where Q=Const.>0 and $q_1=q_1(x)>0$ on Γ_2 is a given function, find the maximum upper bound for Q such that there is no change of phase within Ω, i.e. [GoTa1]

Find $Q_M^o > 0$ / $u_q \geq 0$ in Ω, $\forall Q \leq Q_M$. (25)

Lemma 2.5.4. The element Q_M^o is given by

$$Q_M^o = \operatorname*{Inf}_{x\epsilon\Gamma_2} \frac{u_1(x)}{u_4(x)} \tag{26}$$

where u_1 and u_4 are given by (20) as well as by the following:

$$\begin{cases} \Delta u_4 = 0 \text{ in } \Omega, \\ u_4/\Gamma_1 = 0, \quad \dfrac{\partial u_4}{\partial n}/\Gamma_2 = q_1, \quad \dfrac{\partial u_4}{\partial n}/\Gamma_3 = 0. \end{cases} \tag{27}$$

Proof. Exercise 2.5.4.

Remark 2.5.2. In [MeGo], numerical results on $\overline{q}$, q_M^o and Q_M^o were obtained by using the software MODULEF [Bern] (finite element code).

CHAPTER 3

ON A SECOND STEADY-STATE TWO-PHASE STEFAN PROBLEM

3.1 - INTRODUCTION

We consider the same problem posed in §2.1. but now we replace the condition (2-1-4iv) by the following one [Tarz1]

$$\begin{cases} -k_2 \dfrac{\partial \theta_2}{\partial n}/\Gamma_1 = \alpha(k_2\,\theta_2 - B) & \text{if} \quad \theta/\Gamma_1 > 0, \\ -k_1 \dfrac{\partial \theta_1}{\partial n}/\Gamma_1 = \alpha(k_1\theta_1 - B) & \text{if} \quad \theta/\Gamma_1 < 0, \end{cases} \tag{1}$$

where α=Const.>0 represents a heat transfer coefficient on Γ_1.

We are interested in studying the temperature $\theta=\theta_\alpha$, represented in Ω by (2-1-3), which satisfies the conditions

$$\begin{cases} (2\text{-}1\text{-}4 \text{ i, ii, iii, v, vi}), \\ (1). \end{cases} \tag{2}$$

We shall use the same notations, given in Chapter 2.

The goal of this Chapter is to determine necessary and/or sufficient conditions for the data b=Const.>0 (or B=Const.>0) on Γ_1, α=Const.>0 on Γ_1 and q=Const.>0 on Γ_2 to be (2) a free boundary elliptic problem, that is $\mathcal{E}_\alpha \neq \phi$ [TaTa, GoTa2].

3.2 - VARIATIONAL FORMULATION AND SOME PROPERTIES

Following [Tarz1] (See also § 2.2.), we obtain

Theorem 3.2.1. If we define the function u (the new unknown function) as follows

$$u = k_2\,\theta^+ - k_1\theta^- \quad \text{in } \Omega, \tag{1}$$

where θ^+ and θ^- represent the positive and the negative parts of function θ respectively, then problem (3-1-2) is transformed into

$$\begin{cases} \text{i) } \Delta u = 0 \text{ in } D'(\Omega), \\ \text{ii) } -\frac{\partial u}{\partial n}/\Gamma_2 = q \quad , \qquad \text{iv) } \frac{\partial u}{\partial n}/\Gamma_3 = 0, \\ \text{iii) } -\frac{\partial u}{\partial n}/\Gamma_1 = \alpha(u-B) \ , \end{cases} \tag{2}$$

whose variational formulation is given by the following EVE

$$\begin{cases} d_\alpha(u,v) = L_{\alpha qB}(v), \qquad \forall v \in V \ , \\ u \in V \ , \end{cases} \tag{3}$$

where

$$\begin{cases} V = H^1(\Omega), \qquad a(u,v) = \int_\Omega \nabla u.\nabla v \ dx, \\ a_\alpha(u,v) = a(u,v) + \alpha \int_{\Gamma_1} uv \ d\gamma \ , \\ L_{\alpha qB}(v) = L_q(v) + \alpha \int_{\Gamma_1} Bv \ d\gamma \ , \\ L_q(v) = -\int_{\Gamma_2} qv \quad d\gamma \ . \end{cases} \tag{4}$$

Proof. Exercise 3.2.1.

Theorem 3.2.2. i) Under the hypotheses $L_q \in V'$ (e.g. $q \in L^2(\Gamma_2)$) and $b \in H^{1/2}(\Gamma_1)$, there exists a unique solution to (3).

ii) The solution $u = u_{\alpha qB}$ of (3) is characterized by the following minimization problem

$$\begin{cases} G(u_{\alpha qB}) \leq G(v), & \forall v \in V, \\ u_{\alpha qB} \in V, \end{cases} \tag{5}$$

where

$$G(v)=G_{\alpha qB}(v) = \frac{1}{2} \quad a_{\alpha}(v,v) - L_{\alpha qB}(v). \tag{6}$$

Proof. Exercise 3.2.2.

Exercise 3.2.3. The bilinear form a_1 is coercive on V, i.e.,

$$\exists \quad \lambda_1 > 0 / \quad a_1(v,v) = (v,v) + \int_{\Gamma_1} v^2 \, d\gamma \geq \lambda_1 \| v \|_V^2, \ \forall v \in V. \tag{7}$$

Moreover, so it is the bilinear form a_α and we have

$$\begin{cases} a_{\alpha}(v,v) \geq \lambda_{\alpha} \quad \| v \|_V^2 , & \forall v \in V, \\ \lambda_{\alpha} = \lambda_1 \quad \operatorname{Min}(1,\alpha). \end{cases} \tag{8}$$

Lemma 3.2.3. Under the hypotheses of Theorem 3.2.2. we have that (we denote $u_{\alpha qB}=u_{\alpha}$)

$$\lim_{\alpha \to +\infty} u_{\alpha} = u_{\infty} \quad \text{strongly in } V, \tag{9}$$

where u_{∞} is the solution to the EVE (2-2-10).

Proof. If we choose $v=u_{\alpha}-u_{\infty}$ V in (3) and use (7), we obtain

$$\lambda_1 \quad \| \varepsilon_{\alpha} \|^2 + (\alpha-1) \int_{\Gamma_1} \varepsilon_{\alpha}^2 \, d\gamma \leq C_1 \| \varepsilon_{\alpha} \| , \tag{10}$$

where $_{\alpha}=u_{\alpha}-u_{\infty}$ and C_1 is a positive constant independent of α. From (10), we deduce

$$\| \xi_\alpha \| \leq C_2, \quad (\alpha-1) \int_{\Gamma_1} \xi_\alpha^2 \, d\gamma \leq C_3, \tag{11}$$

where $C_2=C_1/\lambda_1$ and $C_3=C_1C_2$ are two constants independent of α, and this implies that u_α is bounded in V. We can extract from (u_α) a subsequence, still denoted by (u_α), such that

$$\begin{cases} u_\alpha \longrightarrow u^* \quad \text{weakly in } V, \\ \varliminf_{\alpha\to+\infty} \int_{\Gamma_1} (u_\alpha-B)^2 d\gamma = 0. \end{cases} \tag{12}$$

If we take into account the weak lower semi continuity property of the application

$$v\varepsilon V \longrightarrow \int_{\Gamma_1} v^2 \, d\gamma , \tag{13}$$

we deduce that

$$0 = \int_{\Gamma_1} (u^*-B)^2 \, d\gamma \leq \varliminf_{\alpha\to+\infty} \int_{\Gamma_1} (u_\alpha-B)^2 \, d\gamma=0, \tag{14}$$

that is $u^*\varepsilon K$.

If we choose $v\varepsilon V_0 \subset V$ in (3) and we take the limit $\alpha\to+\infty$, we obtain

$$a(u^*,v)=L_q(v), \quad \forall v\varepsilon V_0, \tag{15}$$

then u^* is a solution to (2-2-10), but this being unique we have $u^*=u_\infty$. The uniqueness property also implies that the whole sequence (u_α) converges weakly to u_∞.

For $\alpha>1$, we have

$$\lambda_1\| u_\alpha-u_\infty\|^2 \leq a_\alpha(u_\alpha-u_\infty, u_\alpha-u_\infty)=L_q(u_\alpha-u_\infty)- a(u_\infty,u_\alpha-u_\infty) \tag{16}$$

and from the weak convergence of u_α to u_∞, we deduce the strong convergence of u_α to u_∞.

Exercise 3.2.4. We have that

$$\lim_{\alpha\to+\infty} \|\theta_\alpha - \theta_\infty\|_{L^2(\Omega)} = 0, \tag{17}$$

where θ_α and θ_∞ are the temperatures corresponding to u_α (See (1)) and u_∞ (See (2-2-4)) respectively.

Remark 3.2.1. In this Chapter we consider that the domain Ω and the data b (or B) on Γ_1 and q on Γ_2 are sufficiently regular to have the regularity property $u_{\alpha qB} \in H^2(\Omega)\cap C^0(\bar{\Omega})$. In particular, we are interested in this property for the case b=Const.>0 and q= Const.>0. Moreover, in the three examples (See Appendix 1), the solution $u_{\alpha qB}$ satisfies this requirement.

3.3 - INEQUALITIES FOR THE CONSTANT HEAT FLUX AND THE HEAT TRANSFER COEFFICIENT TO OBTAIN A TWO-PHASE STEFAN PROBLEM (PART I)

From now on we shall consider the particular case B=Const.>0, α=Const.>0 and q=Const.>0 and use $u_\alpha=u_{\alpha q}=u_{\alpha qB}$ (solution of (3-2-3)) and $u_\infty=u_q=u_{qB}$ (solution of (2-2-10)) when it is necessary to simplify the notation.

Lemma 3.3.1. We have the following properties (for a given B>0):

$$\begin{cases} \text{i) } u_{\alpha qB} \leq B \text{ in } \Omega,\ \forall\alpha>0,\ \forall q>0, \\ \text{ii) } u_{\alpha qB} \leq u_{qB} \leq B \text{ in } \Omega,\ \forall\alpha>0,\ \forall q>0, \\ \text{iii) } u_{\alpha_1 q_1 B} \leq u_{\alpha_2 q_2 B} \text{ in } \Omega,\ \forall q_2 \leq q_1,\ \forall \alpha_1 \leq \alpha_2, \\ \text{iv) } M_2 \geq u_{\alpha qB} \leq M_1 \text{ in } \Omega, \end{cases} \tag{1}$$

where

$$M_2 = \underset{\Gamma_2}{\mathrm{Min}} \; u_{\alpha q B} \, , \qquad M_1 = \underset{\Gamma_1}{\mathrm{Max}} \; u_{\alpha q B} \tag{2}$$

<u>Proof.</u> We shall only prove iii) and iv) because the other cases are analogous <u>(Exercise 3.3.1.)</u>.

iii) We shall take into account the following equivalence:

$$u_1 \leq u_2 \text{ in } \Omega \iff w = 0 \text{ in } \Omega, \tag{3}$$

where $w=(u_1-u_2)^+$ in Ω and $u_i=u_{\alpha_i q_i B}$ (i=1,2).

If we use $v=w\in V$ in the EVE (2-2-3) corresponding to u_1, and $v= - w\in V$ in the one corresponding to u_2 and we later add them up, we have

$$a_1(w,w)+(q_1-q_2)\int_{\Gamma_2} w \, d\gamma + (\alpha_2-\alpha_1)\int_{\Gamma_1}(B-u_2)w \, d\gamma = 0, \tag{4}$$

that is, $w=0$ in Ω.

iv) Let w_1 and w_2 be the functions defined in the following way:

$$w_1=(u_{\alpha q B}-M_1)^+ \, , \quad w_2=(M_2-u_{\alpha q B})^+ \text{ in } \Omega. \tag{5}$$

If we use $v=w_1\in V$ and $v=w_2\in V$ in the EVE (3-2-3) and we take into account that $w_1/\Gamma_1=0$ and $w_2/\Gamma_2=0$, then we obtain

$$\begin{cases} a(w_1,w_1)+q\displaystyle\int_{\Gamma_2} w_1 \, d\gamma=0, \\ a(w_2,w_2)+\alpha\displaystyle\int_{\Gamma_1}(B-u_{\alpha q B})w_2 \, d\gamma=0, \end{cases} \tag{6}$$

that is, $w_1=w_2=0$ in Ω, i.e. (iv).

<u>Exercise 3.3.2.</u> We have the following properties:

i)

$$\underset{\overline{\Omega}}{\mathrm{Max}}\; u_{\alpha qB}=M_1 \quad , \quad \underset{\overline{\Omega}}{\mathrm{Min}}\; u_{\alpha qB}=M_2 , \tag{7}$$

where the elements M_1 and M_2 are defined in (2).

ii) The problem (2-2-3) is a two-phase Stefan problem in Ω iff

$$\exists x_1 \varepsilon \Gamma_1 \;,\; x_2 \varepsilon \Gamma_2 \;\; / \;\; u_{\alpha qB}(x_1)>0,\; u_{\alpha qB}(x_2)<0. \tag{8}$$

iii) If $u_{\alpha qB}$ satisfies the following condition

$$\int_{\Gamma_1} u_{\alpha qB}\, d\gamma>0,\; \int_{\Gamma_2} u_{\alpha qB}\, d\gamma<0 \tag{9}$$

then the problem (3-2-3) is a two-phase problem.

<u>Lemma 3.3.2.</u> For all $B>0$, we have the following expressions:

$$\int_{\Gamma_1} u_{\alpha qB}\, d\gamma = B\,|\Gamma_1| - \frac{q}{\alpha}\,|\Gamma_2|,\quad \forall\alpha,\; q>0\;, \tag{10}$$

$$a(u_{qB},u_{qB}) = L_q(u_{qB})+Bq|\Gamma_2|,\quad \forall q>0, \tag{11}$$

$$a(u_{\alpha qB},u_{qB})= a(u_{qB},u_{qB}),\quad \forall\alpha,\; q>0. \tag{12}$$

<u>Proof.</u> To prove (10) it is enough to choose $v=1\varepsilon V$ in (3-2-3).

<u>Exercise 3.3.3.</u> Prove (11) and (12) by using the EVE corresponding to $u_{\alpha qB}$ and u_{qB}.

When the heat flux $q>0$ on Γ_2 is such that $q>q_0(B)$, that is, the problem (2-2-10) is a two-phase Stefan problem in Ω, then we obtain the following result [TaTa].

Theorem 3.3.3. If $q>q_o(B)$, then (3-2-3) is a steady-state two-phase Stefan problem in Ω for all $\alpha>\alpha_o(q,B)$, where

$$\alpha_o(q,B) = \frac{q\,|\Gamma_2|}{B\,|\Gamma_1|} \tag{13}$$

Proof. As $q>q_o(B)$, we have that

$$\underset{\bar{\Omega}}{\mathrm{Min}}\; u_{qB} = \underset{\Gamma_2}{\mathrm{Min}}\; u_{qB}<0, \tag{14}$$

and therefore, by using (1ii), we deduce that $M_2<0$, $\forall\alpha>0$. Besides, by using (10), we have that

$$\int_{\Gamma_1} u_{\alpha qB}\, d\gamma>0 \iff \alpha>\alpha_o(q,B) \ , \tag{15}$$

the we obtain the thesis.

Corollary 3.3.4. In the case where, due to symmetry, we find that function $u_{\alpha q}$ is constant on Γ_1; then the sufficient condition, given by Theorem 3.3.3., is also necessary for problem (3-2-3) to be a two-phase Stefan problem.

Remark 3.3.1. For the three examples, given in Appendix 1, we can apply the above Corollary.

3.4 - INEQUALITIES FOR THE CONSTANT HEAT FLUX AND THE HEAT TRANSFER COEFFICIENT TO OBTAIN A TWO-PHASE STEFAN PROBLEM (PART II)

We shall continue with the paragraph § 3.3. for the general case $\alpha>0$, $q>0$ and $B>0$ (B or b is given arbitrary but positive constant).

Following [TaTa] (See also 2.3.), let $g:(\mathbb{R}^+)^3\to\mathbb{R}$ be the real function defined by

$$g(\alpha,q,B) = G_{\alpha qB}(u_{\alpha qB}) \tag{1}$$

which is equivalent to the following expressions (Exercise 3.4.1.)

$$\begin{aligned} g(\alpha,q,B) &= -\frac{1}{2}\, a_\alpha(u_{\alpha qB},u_{\alpha qB}) = -\frac{1}{2} L_{\alpha qB}(u_{\alpha qB}) = \\ &= \frac{q}{2}\int_{\Gamma_2} u_{\alpha qB}\, d\gamma - \frac{\alpha B}{2}\int_{\Gamma_1} u_{\alpha qB}\, d\gamma \leq 0. \end{aligned} \tag{2}$$

Theorem 3.4.1. Function g has partial derivatives with respect to variables α, q and B, and they are given by the following expressions for all $\alpha,q,B>0$:

$$\frac{\partial g}{\partial \alpha}(\alpha,q,B) = \int_{\Gamma_1} \left(\frac{1}{2}u^2_{\alpha qB} - B\, u_{\alpha qB}\right) d\gamma \,, \tag{3}$$

$$\frac{\partial g}{\partial q}(\alpha,q,B) = \int_{\Gamma_2} u_{\alpha qB}\, d\gamma \,, \tag{4}$$

$$\frac{\partial g}{\partial B}(\alpha,q,B) = -\alpha\int_{\Gamma_1} u_{\alpha qB}\, d\gamma \,. \tag{5}$$

Proof. Exercise 3.4.2. For example, for the partial derivative of g with respect to α, we can obtain:

$$\begin{cases} \| \Delta_h u(\alpha) \|_V \leq E_1 h \,, \\ \| \Delta_h u(\alpha) \|_{L^2(\Gamma_1)} \leq E_2 h, \end{cases} \tag{6}$$

where

$$E_1 = \frac{\|\gamma_0\|}{\lambda_\alpha} \| B-u_{\alpha qB}\|^2_{L^2(\Gamma_1)} \,, \qquad E_2 = E_1 \|\gamma_0\| \tag{7}$$

and $\Delta_h u(\alpha)$ is defined by (h=Const.>0).

$$\Delta_h u(\alpha) = u_{\alpha+h,qB} - u_{\alpha qB} \in V \ . \tag{8}$$

Moreover, we can obtain

$$\begin{cases} \dfrac{\Delta_h g(\alpha)}{h} = \dfrac{1}{2} \displaystyle\int_{\Gamma_1} \left[u_{\alpha+h,qB}\, u_{\alpha qB} - B(u_{\alpha+h,qB} + u_{\alpha qB}) \right] d\gamma \ , \\ \lim\limits_{h\to 0^+} \displaystyle\int_{\Gamma_1} u_{\alpha+h,qB}\, u_{\alpha qB}\, d\gamma = \int_{\Gamma_1} u^2_{\alpha qB}\, d\gamma \ , \end{cases} \tag{9}$$

where $\Delta_h g(\alpha)$ is defined by

$$\Delta_h g(\alpha) = g(\alpha+h,q,B) - g(\alpha,q,B). \tag{10}$$

We can give an analogous definition for $\Delta_h u(q)$, $\Delta_h g(q)$ and $\Delta_h u(B)$, $\Delta_h g(B)$ [TaTa].

<u>Theorem 3.4.2.</u> There exists a function $A=A(\alpha)$, defined for $\alpha>0$, such that

$$g(\alpha,q,B) = -\frac{A(\alpha)}{2}\, q^2 + Bq|\Gamma_2| - \frac{B^2 d}{2}|\Gamma_1|, \ \forall \alpha,q,B>0. \tag{11}$$

<u>Proof.</u> By using (3-3-10), (2) and (4), we obtain (we note $g_q = \frac{\partial g}{\partial q}$)

$$g(\alpha,q,B) = \frac{q}{2}\, g_q(\alpha,q,B) + \frac{Bq}{2}|\Gamma_2| - \frac{\alpha B^2}{2}|\Gamma_1| \ . \tag{12}$$

If we differentiate (12) with respect to variable q, we deduce

$$q\, g_{qq} - g_q = -B\, |\Gamma_2|, \ g_{qqq} \equiv 0. \tag{13}$$

Then function g can be written in the form

$$g(\alpha,q,B) = -\frac{A(\alpha,B)}{2}\, q^2 + A_1(\alpha,B)\, q + A_2(\alpha,B). \tag{14}$$

By some manipulations with (12)-(14), we obtain

$$\begin{cases} \frac{\partial A}{\partial B}(\alpha,B)=0, \quad A_1(\alpha,B)=B\,|\Gamma_2|\,, \\ A_2(\alpha,B) = -\frac{B^2\alpha}{2}\,|\Gamma_1|\,, \end{cases} \tag{15}$$

that is (11).

<u>Exercise 3.4.3.</u> We have

i) $$\int_{\Gamma_2} u_{\alpha q B}\, d\gamma = B\,|\Gamma_2| - A(\alpha)q\,, \quad \forall \alpha,q,B>0, \tag{16}$$

ii) $$g(\alpha,q,B)<0, \quad \forall \alpha,q,B>0, \tag{17}$$

iii) Function $A=A(\alpha)$ is a decreasing positive function of α which verifies

$$A(\alpha) > \frac{|\Gamma_2|^2}{|\Gamma_1|}\,\frac{1}{\alpha}\,, \quad \forall \alpha>0, \tag{18}$$

iv) Let $q_m=q_m(\alpha,B)$ and $q_M=q_M(\alpha,B)$ be real functions, defined for $\alpha,B>0$ by the following expressions

$$q_m(\alpha,B) = \frac{B\,|\Gamma_2|}{A(\alpha)}\,, \quad q_M(\alpha,B) = \frac{B\alpha\,|\Gamma_1|}{|\Gamma_2|}\,. \tag{19}$$

Then, the set $S^{(2)}(B)$ is not empty for all $B>0$, where (See Figure 3.4.1.)

$$S^{(2)}(B) = \{(\alpha,q)\in(R^+)^2 \;/\; q_m(\alpha,B)<q<q_M(\alpha,B), \quad \alpha>0\}. \tag{20}$$

<u>Theorem 3.4.3.</u> If $(\alpha,q)\in S^{(2)}(B)$, then (3-2-2) or (3-2-3) is a steady-state two-phase Stefan problem.

Proof. By using (3-3-10) and (16), we deduce the following equivalences

$$\begin{cases} \text{i)} \displaystyle\int_{\Gamma_1} u_{\alpha q B}\, d\gamma > 0 \iff q < q_M(\alpha,B) , \\ \text{ii)} \displaystyle\int_{\Gamma_2} u_{\alpha q B}\, d\gamma < 0 \iff q > q_m(\alpha,B) , \end{cases} \tag{21}$$

which are the thesis, applying Exercise (3.3.2.iii).

Corollary 3.4.3. In the case where, due to symmetry, we find that $u_{\alpha q B}$ is constant on Γ_1 and Γ_2 respectively, then the sufficient condition, fiven by Theorem 3.4.3. is also necessary for problem (3-2-3) to be a two-phase Stefan problem.

Moreover, for the three examples given in Appendix 1, we can consider this fact.

Exercise 3.4.4. We have:

i)

$$\lim_{\alpha\to+\infty} A(\alpha) = C>0 , \tag{22}$$

where C is the constant defined by (2-3-22) or (2-3-28), which is independent of q,B>0.

ii) Function $q_m=q_m(\alpha,B)$ is an increasing monotone function of the parameter α and verifies

$$q_m(o^+,B)=0, \quad \lim_{\alpha\to+\infty} q_m(\alpha,B) = q_o(B), \ \forall B>0, \tag{23}$$

where $q_o(B)$ is defined by (2-3-29).

Exercise 3.4.5. Give a new proof of the result of Theorem 2.3.7 by passing to the limit $\alpha\to+\infty$ the above results, that is, if q,B>0 are such that $q>q_o(B)$ then problem (2-1-4) or (2-2-8) or (2-2-10) is a steady-state two-phase Stefan problem in Ω.

Exercise 3.4.6. i) Prove the following properties of the function $A=A(\alpha)$ ($' = \frac{d}{d\alpha}$)

$$[\alpha A(\alpha)]' = \frac{1}{q^2} \; a(u_{\alpha q B}, u_{\alpha q B}) \;, \; \forall \alpha>0 \;, \tag{24}$$

$$\lim_{\alpha\to+\infty} \alpha A'(\alpha) = 0, \tag{25}$$

ii) For all B>0, function $q_m = q_m(\alpha, B)$ verifies

$$\begin{cases} \dfrac{\partial q_m}{\partial \alpha}(\alpha,B) = - B\left|\Gamma_2\right| \dfrac{A'(\alpha)}{A(\alpha)} > 0, \quad \forall \alpha>0, \\ \lim\limits_{\alpha\to+\infty} \dfrac{\partial q_m}{\partial \alpha}(\alpha,B) = 0. \end{cases} \tag{26}$$

Remark 3.4.1. In the plane α,q (for a given B>0), we represent with dashes the region where a two-phase Stefan problem is obtained for the problem (3-2-2) or (3-2-3).

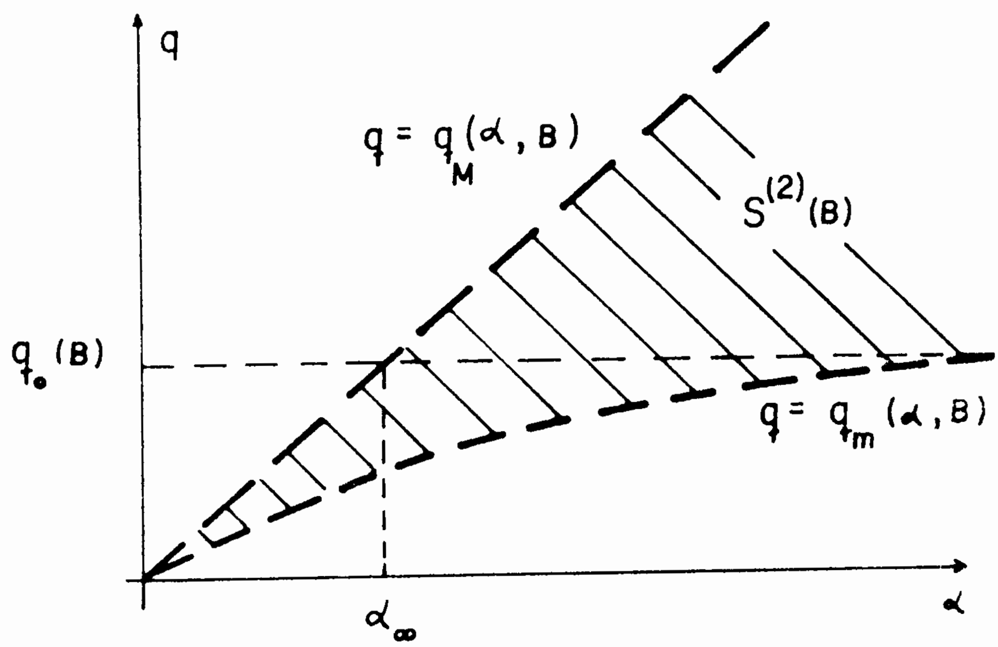


Figure 3.4.1.

The number α_{oo} is defined by

$$\alpha_{oo} = \frac{|\Gamma_2|^2}{c|\Gamma_1|} \tag{27}$$

which is the α- component of the intersection point between the two straight lines $q=q_o(B)$ and $q=q_M(\alpha,B)$. We remark that α_{oo} is independent of B.

Remark 3.4.2. In [SaTa], the numerical results in Figure 3.4.1. were obtained by using the software MODULEF [Bern] (finite element code).

The function $A=A(\alpha)$, defined for $\alpha>0$, is not explicitly known but has the properties (11), (18), (22), (24) and (25). In the paragraph § 3.6. we shall consider a particular case for which we can obtain more information about the expression of $A(\alpha)$.

3.5 - INEQUALITIES FOR THE CONSTANT HEAT FLUX AND THE HEAT TRANSFER COEFFICIENT TO OBTAIN A TWO-PHASE STEFAN PROBLEM (PART III)

The goal of this paragraph is to generalize Theorem 3.4.3. so that (3-2-2) or (3-2-3) is a steady-state two-phase Stefan problem [GoTa2] (We always consider the case $\alpha,q,B>0$).

Theorem 3.5.1. Problem (3-2-2) or (3-2-3) represents a steady-state two-phase Stefan problem if and only if the heat flux q verifies the following inequalities

$$q_1(\alpha,B)<q<q_2(\alpha,B), \quad \alpha>0, \quad B>0, \tag{1}$$

where $q_1=q_1(\alpha,B)$ and $q_2=q_2(\alpha,B)$ are given by (6) and (8) respectively.

Proof. Function $u_{\alpha qB}$ can be expressed as follows

$$u_{\alpha qB} = B - q\, U_\alpha \quad \text{in} \quad \Omega, \tag{2}$$

where U_α is defined by

$$\begin{cases} \Delta U_\alpha = 0 \text{ in } \Omega, \\ -\frac{\partial U_\alpha}{\partial n}/\Gamma_1 = \alpha\, U_\alpha, \\ \frac{\partial U_\alpha}{\partial n}/\Gamma_2 = 1 \;, \quad \frac{\partial U_\alpha}{\partial n}/\Gamma_3 = 0, \end{cases} \tag{3}$$

whose variational formulation is given by

$$\begin{cases} a_\alpha(U_\alpha, v) = \int_{\Gamma_2} U_\alpha \, d\gamma, \quad \forall v \in V, \\ U_\alpha \in V, \end{cases} \tag{4}$$

and verifies that $U_\alpha > 0$ in $\overline{\Omega}$ (<u>Exercise 3.5.1.</u>).

By using (3-3-2), (3-3-7) and (2), we obtain the thesis by virtue of the following equivalences:

i)

$$\begin{aligned} &u_{\alpha q B} \geq 0 \text{ in } \overline{\Omega} \iff u_{\alpha q B} \geq 0 \text{ on } \Gamma_2 \iff \\ &\quad q \leq \frac{B}{U_\alpha} \text{ on } \Gamma_2 \iff q \leq q_1(\alpha, B) \;, \end{aligned} \tag{5}$$

where

$$q_1(\alpha, B) = \operatorname*{Min}_{\Gamma_2}\left(\frac{B}{U_\alpha}\right) . \tag{6}$$

ii)

$$\begin{aligned} &u_{\alpha q B} \leq 0 \text{ in } \overline{\Omega} \iff u_{\alpha q B} \leq 0 \text{ on } \Gamma_1 \iff \\ &\quad q \geq \frac{B}{U_\alpha} \text{ on } \Gamma_1 \iff q \geq q_2(\alpha, B), \end{aligned} \tag{7}$$

where

$$q_2(\alpha,B) = \underset{\Gamma_1}{\text{Max}} \frac{B}{U_\alpha} \tag{8}$$

We can obtain a relationship between the Theorems 3.4.3. and 3.5.1. as follows

Lemma 3.5.2. i) Function U_α verifies the following properties:

$$\int_{\Gamma_1} U_\alpha \, d\gamma = \frac{|\Gamma_2|}{\alpha} \quad , \quad \forall\alpha>0 \ , \tag{9}$$

$$\int_{\Gamma_2} U_\alpha \, d\gamma = A(\alpha), \ \forall\alpha>0. \tag{10}$$

ii) We have the following inequalities

$$q_1(\alpha,B) \leq q_m(\alpha,B) < q_M(\alpha,B) \leq q_2(\alpha,B), \ \forall\alpha,B>0. \tag{11}$$

Moreover, we have that (for all B>0).

$$\begin{cases} q_1(\alpha,B) = q_m(\alpha,B) \iff U_\alpha/\Gamma_2 = \text{Const.} \left(= \dfrac{A(\alpha)}{|\Gamma_2|}\right) , \\[2ex] q_2(\alpha,B) = q_M(\alpha,B) \iff U_\alpha/\Gamma_1 = \text{Const.} \left(= \dfrac{|\Gamma_2|}{\alpha|\Gamma_1|}\right) . \end{cases} \tag{12}$$

Proof. Exercise 3.5.2.

Remark 3.5.1. In [GoTa2], for the general case $q=q(x)$ on Γ_2 and $B=B(x)$ on Γ_1, we can state the following optimization problems:

$$\begin{aligned} &\text{i)} \ \underset{q}{\text{Sup}} \int_{\Gamma_2} q \, d\gamma \quad \text{such that} \quad u_{\alpha qB} \geq 0 \ \text{in} \ \bar{\Omega}, \\ &\text{ii)} \ \underset{q}{\text{Inf}} \int_{\Gamma_2} q \, d\gamma \quad \text{such that} \quad u_{\alpha qB} \leq 0 \ \text{in} \ \bar{\Omega}. \end{aligned} \tag{13}$$

3.6 - A PARTICULAR CASE AND THE EXPLICIT EXPRESSION OF FUNCTION $A=A(\alpha)$

We consider the particular case when $u_{\alpha qB}$ verifies the condition [TaTa]

$$\frac{1}{q^2}\, a(u_{\alpha qB}, u_{\alpha qB}) = \text{Const. } (=\text{Const. } (\alpha,q,B)),\ \forall \alpha,q,B>0 \tag{1}$$

or in a equivalent way

$$[\alpha A(\alpha)]' = A(\alpha) + \alpha A'(\alpha) = \text{Const.},\ \forall \alpha>0, \tag{2}$$

due to (3-4-24).

Exercise 3.6.1. We have that

$$\text{Const. } (\alpha,q,B) = C>0,\ \forall \alpha,q,B>0, \tag{3}$$

where C is the constant defined before by (2-3-22) or (2-3-28). Moreover, function U_α verifies

$$a(U_\alpha, U_\alpha) = C,\ \forall \alpha>0. \tag{4}$$

Lemma 3.6.1. We have the following equivalence

$$u_{qB} - u_{\alpha qB} \text{ is constant in } \Omega \iff [\alpha A(\alpha)]' = C. \tag{5}$$

Proof. By (3-2-12), we obtain

$$u_{qB} - u_{\alpha qB} \text{ is constant in } \Omega \iff a(u_{qB} - u_{\alpha qB}, u_{qB} - u_{\alpha qB}) = 0$$

$$\iff a(u_{\alpha qB}, u_{\alpha qB}) = a(u_{qB}, u_{qB}) \iff a(u_{\alpha qB}, u_{\alpha qB}) = Cq^2$$

$$\iff [\alpha A(\alpha)]' = C.$$

Theorem 3.6.2. For all $q,B>0$, we have that the following propositions are equivalent $(\alpha,\beta>0)$:

$$u_{qB}-u_{\alpha qB}=d_1(\text{Const.}) \text{ in } \Omega, \tag{6i}$$

$$u_{qB}-u_{\alpha qB}=\frac{q|\Gamma_2|}{\alpha|\Gamma_1|} \text{ in } \Omega, \tag{6ii}$$

$$u_{\beta qB}-u_{\alpha qB}=\frac{\beta-\alpha}{\beta\alpha}\, q\, \frac{|\Gamma_2|}{|\Gamma_1|} \text{ in } \Omega, \tag{6iii}$$

$$u_{\beta qB}-u_{\alpha qB}= d_2(\text{Const.}) \text{ in } \Omega, \tag{6iv}$$

$$\frac{\partial u_{\beta qB}}{\partial n}/\Gamma_1 = \frac{\partial u_{\alpha qB}}{\partial n}/\Gamma_1 \text{ on } \Gamma_1, \tag{6v}$$

$$u_{\alpha qB}/\Gamma_1 = B - \frac{q|\Gamma_2|}{\alpha|\Gamma_1|} \text{ on } \Gamma_1, \tag{6vi}$$

$$\frac{\partial u_{\alpha qB}}{\partial n}/\Gamma_1 = \frac{q|\Gamma_2|}{|\Gamma_1|} \text{ on } \Gamma_1, \tag{6vii}$$

$$\frac{\partial u_{\alpha qB}}{\partial n}/\Gamma_1 = d_3(\text{Const.}) \text{ on } \Gamma_1, \tag{6viii}$$

$$\frac{\partial u_{qB}}{\partial n} = d_3(\text{Const.}) \text{ on } \Gamma_1, \tag{6ix}$$

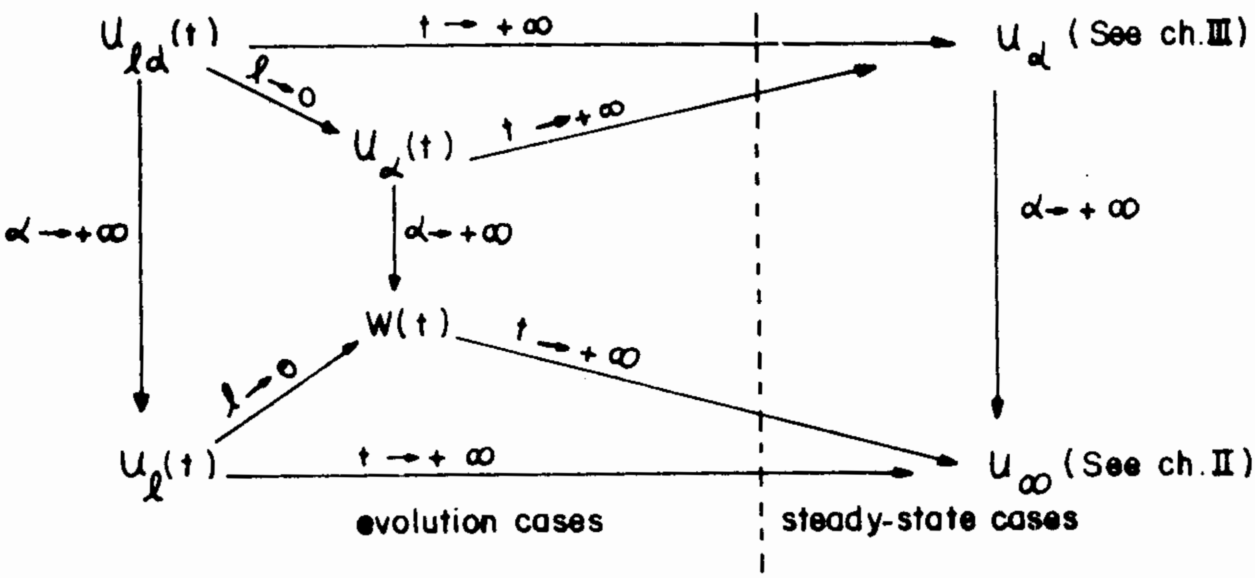
$$A(\alpha) = C + \frac{1}{\alpha}\frac{|\Gamma_2|^2}{|\Gamma_1|}, \ \forall\alpha>0, \tag{6x}$$

Proof. Exercise 3.6.2.

Corollary 3.6.3. For the particular case (1) or (2), the expression of function $A=A(\alpha)$ is given by (6x), or in an equivalent way

$$A(\alpha)=C\ (1+\frac{\alpha_{00}}{\alpha}). \tag{7}$$

Remark 3.6.1. (Importance of chapters 2 and 3) The two elliptic variational equalities (2-2-10) and (3-2-3), studied in chapter 2 and 3 respectively, appear if we consider the asymptotic behavior when the time $t \to +\infty$ in four parabolic variational inequalities of type 2, defined in [Tarz1], for the evolution two-phase Stefan problem according to the following diagram (ℓ: latent heat of fusion):



The goal of chapters 2 and 3 is to answer the following question: i) When do the functions u_∞ and u_α really represent a steady-state two-phase Stefan problem?

We have presented several problems, some of them related to the function u_∞ (See chapter 2) and some other ones related to the function u_α (See chapter 3) [Tarz 7].

The parabolic variational inequality formulation for the two-phase Stefan problem was firstly given in [Duva2, Frem]. In [Daml], it was proved the equivalence between Duvaut's and Frémond's methods. Other theoretical results were given in [Coll, Donn, HNPS, Mage, Pawl, Sagu, Visi].

Another formulation for the evolution two-phase Stefan problem is the so-called weak formulation (enthalpy method). It was firstly given in [Kame, Olei, Rose] (See also [ElOc] and its asymptotic behavior was given in [Frie1].

More information about the Stefan problem through the variational inequality theory is given in Appendix 3 (See also [Rodr]).

Open problem: Generalize chapters 2 and 3 for the evolution case.

In the next chapter 4, we shall study a simple evolution case for a semi-infinite material which is initially in the liquid phase and we can give a necessary and sufficient condition for a heat flux on the fixed face $x=0$ to have a two-phase Stefan problem.

CHAPTER 4

AN EVOLUTION TWO-PHASE STEFAN PROBLEM FOR A SEMI-INFINITE MATERIAL

4.1 - INTRODUCTION

We consider a semi-infinite material, represented by $\Omega=(0,+\infty)$, with an initial uniform temperature $\theta_0>0$. On the fixed face $x=0$ the body can have a temperature $-D<0$ (solidification problem; See §4.2) or an outward heat flux $q(t)>0$ (See §4.3) for all instant $t>0$.

We enlarge the problem by taking into account the effect of the density change during the phase change. Moreover, the material has constant thermal coefficients, e.g.:

$$\begin{cases} k_i>0 : \text{thermal conductivity of the phase } i \\ c_i>0 : \text{specific heat of the phase } i \\ \rho_i>0 : \text{mass density of the phase } i \\ \ell>0 : \text{latent heat of fusion} \\ \alpha_i = a_i^2 = \dfrac{k_i}{\rho_i c_i} >0 : \text{thermal diffusivity of the phase } i \end{cases} \tag{1}$$

where $i=1$ and $i=2$ represent the solid and liquid phase respectively.

Without loss of generality, we take null phase-change temperature (case: ice-water).

The problem consists in finding the function $x=s(t)>0$ (free boundary), defined for $t>0$ with $s(0)=0$, and the temperature

$$\theta(x,t)=\begin{cases} \theta_1(x,t)<0 & \text{if} \quad 0<x<s(t), \quad t>0, \\ 0 & \text{if} \quad x=s(t), \quad t>0, \\ \theta_2(x,t)>0 & \text{if} \quad x>s(t), \quad t>0, \end{cases} \tag{2}$$

defined for x>0 and t>0, such that they satisfy the following conditions [CaJa, Rubi]:

$$
\text{((3bis))} \quad
\begin{cases}
\alpha_1 \theta_{1_{xx}} = \theta_{1_t}, & 0<x<s(t),\ t>0 \\
\alpha_2 \theta_{2_{xx}} + \frac{\rho_1 - \rho_2}{2}\, \dot{s}(t)\, \theta_{2_x} = \theta_{2_t}, & x>s(t),\ t>0, \\
s(0) = 0, & \\
\theta_1(s(t),t) = \theta_2(s(t),t) = 0, & t>0 \\
k_1\, \theta_{1_x}(s(t),t) - k_2\, \theta_{2_x}(s(t),t) = \rho_1\, \ell\, \dot{s}(t), & t>0, \\
\theta_2(x,0) = \theta_2(+\infty,t) = \theta_0 > 0, & x>0,\ t>0, \\
\theta_1(0,t) = -D<0 \quad (k_1\, \theta_{1_x}(0,t) = q(t)>0), & t>0.
\end{cases}
\quad (3)
$$

In §4.2 we give the explicit Neumann solution to problem (3) and in §4.3 we analyse the solution of problem (3bis) for different heat fluxes q(t). We shall, prove that there is not always solution of the Neumann type for the problem (3bis), i.e., problem (3bis) does not always represent an evolution two-phase Stefan problem; the cases considered will be [BaTa, SWA, Tarz4]

$$
\begin{cases}
q(t) = q_0\, t^{n/2} \quad (q_0>0),\ t>0, \\
n = -1, 0, 1, \ldots
\end{cases}
\quad (4)
$$

For the case n=-1 we instantaneouly have a two-phase Stefan problem (evolution case) if and only if the coefficient q_0 verifies the following inequality [BaTa (for $\rho_1 \neq \rho_2$), Tarz4 (for $\rho_1 = \rho_2$)]

$$
q_0 > \frac{k_2\, \theta_0}{a_2 \sqrt{\pi}} = \theta_0 \sqrt{\frac{k_2\, \rho_2 c_2}{\pi}}. \quad (5)
$$

For the cases n=0,1,... with $\rho_1=\rho_2$ solidification does not immediately begin at t=0 because the material temperature in (0,t) must be raised from θ_0 to 0 before solidification begins and a waiting time t_n is necessary, where [SWA]

$$t_n = \left[\frac{k_2 \, \Gamma\left(\frac{3}{2} + \frac{n}{2}\right)}{a_2 q_0 \, \Gamma\left(1+\frac{n}{2}\right)} \theta_0 \right]^{\frac{2}{n+1}} \tag{6}$$

In [Bole], a melting problem with a waiting time is analyzed.

A review paper on explicit solutions for the unidimensional case was given in [Tarz6].

4.2 - THE NEUMAN SOLUTION FOR THE TWO-PHASE STEFAN PROBLEM

We have [BaTa, CaJa, Rubi]

LEMMA 4.2.1 - The solution to the problem (1-3) (known as Neumann solution) is given by

$$\begin{cases} \theta_1(x,t) = A_1 + B_1 \; f\left(\dfrac{x}{2a_1\sqrt{t}}\right), \\ \theta_2(x,t) = A_2 + B_2 \; f\left(\delta_1 + \dfrac{x}{2a_2\sqrt{t}}\right) \\ s(t) = 2 \, \gamma \sqrt{t} \quad (\gamma > 0), \end{cases} \tag{1}$$

where

$$
\begin{cases}
f(x) = \dfrac{2}{\sqrt{\pi}} \int_0^x \exp(-u^2)\, du \quad (=\mathrm{erf}(x)), \\
A_1(\gamma) = -D, \qquad B_1(\gamma) = \dfrac{D}{f(\frac{\gamma}{a_1})}, \\
A_2(\gamma) = -\dfrac{\theta_0 f\left(\frac{\gamma}{a_0}\right)}{1-f\left(\frac{\gamma}{a_0}\right)}, \quad B_2(\gamma) = \dfrac{\theta_0}{1-f(\frac{\gamma}{a_0})}, \\
\varepsilon = \dfrac{\rho_1-\rho_2}{\rho_2}, \quad \delta_1 = \dfrac{\gamma}{a_2}\,|\varepsilon|, \quad a_0 = \dfrac{a_2}{1+|\varepsilon|},
\end{cases} \tag{2}
$$

and γ is the unique solution of the equation

$$
\begin{cases}
F(x) = x, \\
x > 0
\end{cases} \tag{3}
$$

with

$$
F(x) = \frac{k_1}{\ell\rho_1 a_1\sqrt{\pi}}\, B_1(x)\, \exp\left(\frac{-x^2}{a_1^2}\right) - \frac{k_2}{\ell\rho_1 a_2\sqrt{\pi}}\, B_2(x)\, \exp\left(\frac{-x^2}{a_0^2}\right), \tag{4}
$$

which satisfies the following properties

$$
F(0^+) = +\infty, \quad F(+\infty) = -\infty, \quad F' < 0. \tag{5}
$$

PROOF - Exercise 4.2.1

4.3 - AN INEQUALITY FOR THE HEAT FLUX ON THE FIXED FACE TO OBTAIN AN INSTANTANEOUS TWO-PHASE STEFAN PROBLEM AND SOME RELATED PROBLEMS

We have [BaTa]

THEOREM 4.3.1 - i) When the heat flux is $q(t) = q_0\, t^{-1/2}$ $(t>0)$, then there exists a solution of the Neumann type for the problem (1-3bis) if and only if q_0 verifies the inequality (1-5). In this case, the solution of (1-3bis) is given by

$$\begin{cases} \theta_1(x,t) = C_1 + D_1\, f\left(\dfrac{x}{2a_1\sqrt{t}}\right), \\ \theta_2(x,t) = C_2 + D_2\, f\left(\delta_2 + \dfrac{x}{2a_2\sqrt{t}}\right), \\ s(t) = 2\omega\sqrt{t} \quad (\omega>0), \end{cases} \tag{1}$$

where

$$\begin{cases} C_1(\omega) = -\dfrac{a_1 q_0\sqrt{\pi}}{k_1}\, f\left(\dfrac{\omega}{a_1}\right), \quad D_1(\omega) = \dfrac{a_1 q_0\sqrt{\pi}}{k_1}, \\ C_2(\omega) = -\dfrac{\theta_0 f\left(\frac{\omega}{a_0}\right)}{1-f\left(\frac{\omega}{a_0}\right)}, \quad D_2(\omega) = \dfrac{\theta_0}{1-f\left(\frac{\omega}{a_0}\right)} \\ \delta_2 = \dfrac{\omega|\varepsilon|}{a_2} \end{cases} \tag{2}$$

and ω is the unique solution of the equation

$$\begin{cases} F_0(x) = x, \\ x>0 \end{cases} \tag{3}$$

with

$$F_0(x) = \frac{q_0}{\ell k_1} \exp\left(-\frac{x^2}{a_1^2}\right) - \frac{k_2\theta_0}{\ell\rho_1 a_2\sqrt{\pi}}\, \frac{\exp\left(-\frac{x^2}{a_0^2}\right)}{1-f\left(\frac{x}{a_0}\right)} \tag{4}$$

ii) If $q_0 \leq \frac{k_2\theta_0}{a_2\sqrt{\pi}}$ there is no solution for the solidification problem (1-3bis), we just have a problem of heat conduction in the initial liquid phase.

iii) The case $q_0 = \frac{k_2\theta_0}{a_2\sqrt{\pi}}$ corresponds to the limit case of problem (1-3) when $\ell \longrightarrow +\infty$.

PROOF - Exercise 4.3.1 (Function F_0 verifies the following properties

$$F_0(0^+) = \frac{1}{\ell\rho_1}\left(q_0 - \frac{k_2\theta_0}{a_2\sqrt{\pi}}\right), \quad F_0(+\infty) = -\infty, \quad F_0' < 0). \tag{5}$$

REMARK 4.3.1 - The inequality (1-5) was previously obtained in [Tarz4] for the case $\rho_1 = \rho_2$.

REMARK 4.3.2 - We can also obtain the inequality (1-5) as follows: We state a conduction problem for the semi-infinite material $x>0$, that is initially in liquid phase, at a constant temperature $\theta_0>0$, that is

$$\begin{cases} \alpha_2 T_{xx} = T_t, & x>0, \quad t>0, \\ T(x,0) = \theta_0 > 0, & x>0, \\ k_2 T_x(0,t) = q_0 t^{-1/2}, & t>0. \end{cases} \tag{6}$$

The solution to (6) is given by (Exercise 4.3.2)

$$T(x,t) = \theta_0 - \frac{q_0 a_2\sqrt{\pi}}{k_2}\left[1 - f\left(\frac{x}{2a_2\sqrt{t}}\right)\right], \quad x>0, \quad t>0. \tag{7}$$

Since the temperature calculated on the fixed face $x=0$, is

$$T(0,t) = \theta_0 - \frac{q_0 a_2\sqrt{\pi}}{k_2} \tag{8}$$

which is constant in time, the semi-infinite material will undergo a phase change if and only if $T(0,t)<0$ which is the inequality (1-5). This method, given in [SWA] for the case $\rho_1=\rho_2$, allows us to obtain the desired inequality but not the solution to problem (1-3bis).

REMARK 4.3.3 - Another method to obtain the inequality (1-5) consists in introducing the mass as a space variable by [Quil]

$$\mu=\begin{cases} \rho_1 x & \text{if} \quad x\leq s(t), \\ \rho_1 s(t)+\rho_2(x-s(t)) & \text{if} \quad x>s(t), \end{cases} \tag{9}$$

If we define the following changes of variables

$$\begin{cases} e(t)=\rho_1\, s(t), \\ V_1(\mu,t)=\theta_1\left(\frac{\mu}{\rho_1},\, t\right) & \text{if } 0<\mu<e(t), \quad t>0, \\ V_2(\mu,t)=\theta_2\left(\frac{\rho_2-\rho_1}{\rho_2}\, s(t)+\frac{\mu}{\rho_2},\, t\right) & \text{if } \mu>e(t), \quad t>0, \end{cases} \tag{10}$$

then problem (1-3bis) is transformed into

$$\begin{cases} D_1\, V_{1_{\mu\mu}}=V_{1_t}, & 0<\mu<e(t), \quad t>0, \\ D_2\, V_{2_{\mu\mu}}=V_{2_t}, & e(t)<\mu, \quad t>0, \\ e(0)=0, \\ V_1(e(t),t)=V_2(e(t),t)=0 & t>0, \\ K_1\, V_{1_\mu}(e(t),t)-K_2\, V_{2_\mu}(e(t),t)=\ell\, \dot{e}(t), & t>0, \\ K_1\, V_{1_\mu}(0,t)=q(t)=q_0\, t^{-1/2}, & t>0, \end{cases} \tag{11}$$

where

$$K_i = k_i \rho_i , \quad D_i = \frac{k_i \rho_i}{C_i} \quad (i=1,2), \tag{12}$$

are the new thermal conductivity and diffusion coefficients respectively.

We remark that (11) represents a phase-change problem with the same densities (in our case equal to 1) and then by using the inequality obtained for the case $\rho_1 = \rho_2$ [Tarz4] we deduce that problem (11) (or in an equivalent way problem (1-3bis) has a solution of a Neumann type if and only if q_0 satisfies the inequality

$$q_0 > \frac{\theta_0 K_2}{\sqrt{D_2 \pi}} = \frac{\theta_0 k_2}{a_2 \sqrt{\pi}} , \tag{13}$$

which is the inequality (1-5).

Since the temperature θ_1, defined in (1), verifies that $\theta_1(0,t) = C_1(\omega) < 0$, then we can consider the problem (1-3) for $D = -C_1(\omega)$ and so we obtain the following

<u>LEMMA 4.3.2</u> - If the condition (1-5) is valid and we take $D = -C_1(\omega)$ in problem (1-3), we have:

i) $\gamma = \omega$,

ii) the coefficient γ, which characterizes the free boundary of the Neumann solution for the problem (1-3), verifies the following inequality

$$f\left(\frac{\gamma}{a_1}\right) < \frac{D}{\theta_0} \frac{\rho_1 C_1 k_1}{\rho_2 C_2 k_2} . \tag{14}$$

PROOF - Exercise 4.3.2 (We have $F(\omega)= F_0(\omega)$).

REMARK 4.3.2 - When q_0 verifies the inequality (1-5) we have that problem (1-3) is equivalent to problem (1-3 bis) with

$$D= \frac{a_1 q_0 \sqrt{\pi}}{k_1} \; f(\frac{\omega}{a_1}). \tag{15}$$

REMARK 4.3.5 - When we consider an overspecified condition on the fixed face x=0, i.e., we give the temperature and the heat flux on x=0, we can obtain formulas for the determination of one or two unknown thermal coefficients in [StaTa] and [StoTa] for the cases $\rho_1=\rho_1$ and $\rho_1\neq\rho_2$ respecively. Moreover, we can also deduce other inequalities for the coefficient γ which characterizes the Neumann solution for the two-phase Stefan problem.

We shall now consider problem (1-3 bis) when the heat flux on x=0 is given by $q(t)=q_0>0$ [SWA].

LEMMA 4.3.3 - If $q(t)= q_0>0$ there exists a waiting time $t_0>0$, given by

$$t_0= \frac{\pi k_2^2 \ \theta_2^2}{4 \ \alpha_2 q_0^2}, \tag{16}$$

such that the material temperature is positive for all instant $t<t_0$ (and for all x>0), that is, problem (1-3 bis) does not represent a phase-change process (or a two-phase Stefan problem).

PROOF - We consider the following heat conduction problem for the initial liquid phase

$$\begin{cases} \alpha_2 T_{xx} = T_t, & x>0, \quad t>0, \\ T(x,0) = \theta_0 > 0, & x>0, \\ k_2 T_x(0,t) = q_0 > 0, & t>0. \end{cases} \tag{17}$$

The solution of (17) is given by (Exercise 4.3.3)

$$T(x,t) = \theta_0 - \frac{2q_0}{K_2}\sqrt{\alpha_2 t}\ \mathrm{ierfc}\left(\frac{x}{2\sqrt{\alpha_2 t}}\right),\ x>0,\ t>0, \tag{18}$$

which verifies that

$$T(0,t) = \theta_0 - \frac{2q_0}{k_2}\sqrt{\frac{\alpha_2 t}{\pi}}\ ,\ t>0 \tag{19}$$

where

$$\mathrm{ierfc}(x) = \frac{\exp(-x^2)}{\sqrt{\pi}} - x(1-\mathrm{erf}(x)). \tag{20}$$

From (18), we deduce that the temperature on the fixed face x=0 satisfies the following conditions

$$\begin{cases} T(0,t) = 0 \iff t = t_0, \\ T(0,t) > 0, \quad \forall t<t_0, \\ T(0,t) < 0, \quad \forall t>t_0, \end{cases} \tag{21}$$

where to is given by (16).

Exercise 4.3.4 - If $q(t) = q_0 t^{n/2}$ (with $q_0>0$ and $n=1,2,\ldots$) then we have the same conclusion as in Lemma 4.3.3., but with a waiting time $t_n>0$, given by (1-6).

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NOTE:

A large bibliography with approximately 2500 references on moving and free boundary problems for the heat or diffusion equation, particularly regarding the Stefan problem is given in:

D.A. TARZIA, "A bibliography on moving-free boundary problems for the heat-diffusion equation. The Stefan problem", Technical Report of the Instituto Matematico "U. Dini", Univ. di Firenze, Florence (1987).

APPENDIX I

THREE STEADY-STATE EXAMPLES WITH EXPLICIT SOLUTION

Abstract:

We present three domains $\Omega \subset \mathbb{R}^n$ (n=2,3) with the explicit solution corresponding to several problems studied in chapters II and III.

We shall give three examples in which the solution is explicitly known [Tarzia, Math. Notae, 28(1980/81), 73-89] (We use $B=k_2\, b>0$ in the problems analyzed in the Table below):

Example 1 - We consider the following data

$$\begin{cases} n=2, \quad \Omega=(0,x_0)\times(0,y_0),\ x_0>0,\ y_0>0, \\ \Gamma_1=\{0\}\times[0,y_0],\quad \Gamma_2=\{x_0\}\times[0,y_0], \\ \Gamma_3=(0,x_0)\times\{0\}\ \cup\ (0,x_0)\times\{y_0\}. \end{cases} \tag{1}$$

Example 2 - Next we consider

$$\begin{cases} n=2, \quad 0<r_1<r_2, \quad \Gamma_3=\emptyset, \\ \Omega:\ \text{annulus of radius } r_1 \text{ and } r_2, \text{ centred in } (0,0), \\ \Gamma_1:\ \text{circunference of radius } r_1 \text{ and centre } (0,0), \\ \Gamma_2:\ \text{circunference of radius } r_2 \text{ and centre } (0,0). \end{cases} \tag{2}$$

For the numerical approximation and due to the symmetry of the problem, it is convenient to solve it for a quarter of the annulus (the one corresponding to the first quadrant), bearing in mind that in this case a new portion Γ_3 of the boundary appears, which is given by

$$\Gamma_3=\{0\}\times(1,2)\ \cup\ (1,2)\times\{0\}.$$

Therefore the values for $|\Gamma_2|$, $|\Gamma_1|$ and C are modified by a $\frac{1}{4}$ factor, but the expression of q_0 and q_c, which are the values of our interest, does not vary.

Example 3 - Finally, we take into account the same information of Example 2 but now for the case $n=3$.

REMARK - We remark that for the three above examples we can directly verify all the theoretical results obtained in this work.

ELEMENT (defined in)	EXAMPLE 1	EXAMPLE 2	EXAMPLE 3
Variables	x,y	$x,y;\ r=(x^2+y^2)^{1/2}$	$x,y,z;r=(x^2+y^2+z^2)^{1/2}$
$\lvert\Gamma_1\rvert$	y_0	$2\pi r_1$	$4\pi r_1^2$
$\lvert\Gamma_2\rvert$	y_0	$2\pi r_2$	$4\pi r_2^2$
$u_\infty=u_q=u_{qB}$ (II-2-8)or(II-2-10)	$B-qx$	$B-qr_2\log\frac{r}{r_1}$	$B-qr_2^2(\frac{1}{r_1}-\frac{1}{r})$
$u_\alpha=u_{\alpha qB}$ (III-2-2)or(III-2-3)	$B-\frac{q}{\alpha}-qx$	$B-\frac{qr_2}{\alpha r_1}-qr_2\log\frac{r}{r_1}$	$B-\frac{qr_2^2}{\alpha r_1^2}-qr_2^2(\frac{1}{r_1}-\frac{1}{r})$
U_α (III-5-3)	$\frac{1}{\alpha}+x$	$r_2(\frac{1}{\alpha r}+\log\frac{r}{r_1})$	$r_2^2(\frac{1}{\alpha r_1^2}+\frac{1}{r_1}-\frac{1}{r})$
w (II-4-2)	$-B(\frac{x}{x_0}-1)$	$B\frac{\log\frac{r_2}{r}}{\log\frac{r_2}{r_1}}$	$\frac{Br_1r_2}{r_2-r_1}(\frac{1}{r}-\frac{1}{r_2})$
u_3 (II-3-30)or(II-5-21)	x	$r_2\log\frac{r}{r_1}$	$r_2^2(\frac{1}{r_1}-\frac{1}{r})$
u_2 (II-5-7)	$-qx$	$-qr_2\log\frac{r}{r_1}$	$-qr_2^2(\frac{1}{r_1}-\frac{1}{r})$
u_1 (II-5-6)or(II-5-20)	B	B	B
Z_0 (II-5-17)	$-x$	$-r_2\log\frac{r}{r_1}$	$-r_2^2(\frac{1}{r_1}-\frac{1}{r})$
M_1 (III-3-2)	$B-\frac{q}{\alpha}$	$B-\frac{qr_2}{\alpha r_1}$	$B-\frac{qr_2^2}{\alpha r_1^2}$
M_2 (III-3-2)	$B-\frac{q}{\alpha}-qx_0$	$B-\frac{qr_2}{\alpha r_1}-qr_2\log\frac{r_2}{r_1}$	$B-\frac{qr_2^2}{\alpha r_1^2}-\frac{qr_2}{r_1}(r_2-r_1)$

ELEMENT (defined in)	EXAMPLE 1	EXAMPLE 2	EXAMPLE 3
$P_1 \epsilon \Gamma_1$ (II-4-5)	$(0,0)$	$(r_1,0)$	$(r_1,0,0)$
$P_2 \epsilon \Gamma_2$ (II-4-5)	$(x_0,0)$	$(r_2,0)$	$(r_2,0,0)$
hyperplane π (II-4-5)	$B-\frac{B}{x_0}x$	$\frac{Br_2}{r_2-r_1}-\frac{Bx}{r_2-r_1}$	$\frac{Br_2}{r_2-r_1}-\frac{Bx}{r_2-r_1}$
q_s (II-4-6)	$\frac{B}{x_0}$	$\frac{B}{r_2-r_1}$	$\frac{B}{r_2-r_1}$
C (II-3-22)or (II-3-29)	x_0y_0	$2\pi r_2^2 \log \frac{r_2}{r_1}$	$4\pi \frac{r_2^3(r_2-r_1)}{r_1}$
$q_0(B)$ (II-3-31)	$\frac{B}{x_0}$	$\frac{B}{r_2 \log \frac{r_2}{r_1}}$	$\frac{Br_1}{r_2(r_2-r_1)}$
q_c (II-4-1)	$\frac{B}{x_0}$	$\frac{B}{r_2 \log \frac{r_2}{r_1}}$	$\frac{Br_1}{r_2(r_2-r_1)}$
$\overline{q}$ (II-5-10)	$\frac{B}{x_0}$	$\frac{B}{r_2 \log \frac{r_2}{r_1}}$	$\frac{Br_1}{r_2(r_2-r_1)}$
q_i (II-4-3)	$\frac{B}{x_0}$	$\frac{B}{r_2 \log \frac{r_2}{r_1}}$	$\frac{Br_1}{r_2(r_2-r_1)}$
q_M^0 (II-5-19)	$\frac{B}{x_0}$	$\frac{B}{r_2 \log \frac{r_2}{r_1}}$	$\frac{Br_1}{r_2(r_2-r_1)}$
α_{00} (III-4-27)	$\frac{1}{x_0}$	$\frac{1}{r_1 \log \frac{r_2}{r_1}}$	$\frac{r_2}{r_1(r_2-r_1)}$

ELEMENT (defined in)	EXAMPLE 1	EXAMPLE 2	EXAMPLE 3
$\alpha_0(q,B)$ (III-3-13)	$\frac{q}{B}$	$\frac{qr_2}{Br_1}$	$\frac{qr_2^2}{Br_1^2}$
$A(\alpha)$ (III-4-11)or(III-5-10)	y_c [illegible] $\frac{1}{\alpha})$	$2\pi r_2^2(\frac{1}{\alpha r_1} + \log\frac{r_2}{r_1})$	$4\pi r_2^4(\frac{1}{\alpha r_1^2} + \frac{1}{r_1} - \frac{1}{r_2})$
$q_m(\alpha,B)$ (III-4-19)	$\frac{B}{x_0 + \frac{1}{\alpha}}$	$\frac{B}{r_2(\frac{1}{\alpha r_1} + \log\frac{r_2}{r_1})}$	$\frac{B}{r_2^2(\frac{1}{\alpha r_1^2} + \frac{1}{r_1} - \frac{1}{r_2})}$
$q_1(\alpha,B)$ (III-5-6)	$\frac{B}{x_0 + \frac{1}{\alpha}}$	$\frac{B}{r_2(\frac{1}{\alpha r_1} + \log\frac{r_2}{r_1})}$	$r_2^2(\frac{1}{\alpha r_1^2} + \frac{1}{r_1} - \frac{1}{r_2})$
$q_M(\alpha,B)$ (III-4-19)	$B\alpha$	$\frac{Br_1\alpha}{r_2}$	$\frac{Br_1^2\alpha}{r_2^2}$
$q_2(\alpha,B)$ (III-5-8)	$B\alpha$	$\frac{Br_1\alpha}{r_2}$	$\frac{Br_1^2\alpha}{r_2^2}$
$\mathcal{E} = \{u_m = 0\}$ (III-1-2) (when it exists)	$x = \frac{B}{q}$	$r = r_1 \exp(\frac{B}{qr_2})$	$r = \frac{qr_2^2 r_1}{qr_2^2 - Br_1}$
$\mathcal{E}_\alpha = \{u_\alpha = 0\}$ (when it exists)	$x = \frac{B}{q} - \frac{1}{\alpha}$	$r = r_1 \exp(\frac{B}{qr_2} - \frac{1}{\alpha r_1})$	$r = \frac{1}{\frac{1}{\alpha r_1^2} + \frac{qr_2^2 - Br_1}{qr_2^2 r_1}}$

APPENDIX 2 (*)

APPROXIMATE, NUMERICAL AND THEORETICAL METHODS TO SOLVE STEFAN-LIKE PROBLEMS

Abstract:

We present a short review on the approximate, numerical and theoretical methods to solve free boundary problems for the heat-diffusion equation of Stefan type.

(*) This appendix presents a summarized version of the lecture delivered by the author at the Sixth Latin American Congress on Computational Methods held in Paraná and Santa Fe (Argentina) on October 15-18, 1985 and published in Mecánica Computacional, Vol. 5, L.A. Godoy (Ed.), AMCA, Santa Fe (1986), 241-265.

The present paper presents a summarized version of the lecture delivered on occasion of the Sixth American Congress on Computational Methods held in Paraná and Santa Fe (Argentina) on October 15-18, 1985.

The free boundary problems that belong to Mathematical Physics, particularly those related to the heat and/or diffusion equation, have been of great concern for mathematicians, physicists and engineers due to the wide variety of different processes which involve a mathematical model of this kind; some problems that deserve special consideration are those where a phase-change problem occurs, which are better known as Stefan problem.

The free boundary problems of Mathematical Physics are connected with various branches of Mathematics, Physics and Engineering, in particular, with continuous mechanics, heat transfer, ordinary and partial differential equations, functional analysis, elliptic and evolution variational inequalities, numerical analysis, etc. Among other free boundary problems, we may mention: dam problem [Bai, BaCa], semiconductors problem [HuNa], obstacle problem [BaCa, KiSt], Bingham fluid [Cl], semi-permeable wall problem or "black body" [Duli], Stefan problem [Ru]. A survey of all these problems with a mathematical analysis through variational inequalities was done in [Ta1]; see also [BaCa, Cr2, ElOc, F_r2, K_iSt, Lun, Ma1, Oz, Ru1, Sa, Tay].

Free boundary problems for the heat and/or diffusion equation may be classified into:

i) of explicit type: when $\dot{S}$ appears explicitly in, at least, one of the two conditions on the free boundary, e.g., the Stefan problem [LaCl].

ii) of implicit type: when $\dot{S}$ does not appear explicitly on neither condition on the free boundary, e.g., the problem of diffusion-consumption of oxygen in living tissues [CrCu].

In general, the free boundary problems of explicit and implicit type are mutually related [Fa, Sc].

A discussion on fixed, moving and free boundary problems for the heat or diffusion equation of the explicit or implicit type is given in [Ta4] and several long bibliographies on moving or free boundary problems are given in [Cr2, Pr, Ru1, Ta2, WiSoTr].

Among the free boundary problems for the heat or diffusion equation we may cite: Stefan problem [LaCl, St1, St2, We], diffusion-consumption of oxygen in a living tissue [CrCu], noncatalytic gas-solid diffusion-reaction problem [We], continuous casting problem [Rod], optimal control [Sa], solidification processing [Fl], metals solidification [Bi1], solidification of roads [AgFr, AgFrCo], ablation by melting [SzTh], the welding of two steel plates [Tay], the shape of laser melt-pools [Tay], ablation by a high power laser [AnAt], and several other applications in [Cr2, FaPr1, Lun, Ma2, Oc tto, WiSoBo], i.e., electromechanical machining, Hele-Shaw flow, solidification of binary alloys, storage of solar energy, etc. A relation among different free boundary problems is analyzed in [Ro].

A discussion on the conflict among physical reality, mathematical rigour and engineering applications is analyzed in [Sz].

In order to have an idea of the importance of the methods and applications related to free and moving boundary problems for the heat and/or diffusion equation, we may mention:

i) Meetings, conferences or seminars completely devoted to the subject [AlCoHo, BoDaFr, FaPr1, Fu1, Ho, Ma2, OcHo, Ta3, WiSoBo]. In addition, there exist several other meetings where papers on this subject, which are not specified here, are presented.

ii) Books completely devoted to this subject [Ca, Cr2, Da, ElOc, Fr2, Ru1, Ta1].

iii) Review papers on the subject, both from the theoretical and/or numerical point of view, such as [Ban, Cr3, Cr4, CrId, Du4, Fo, Fu2, GaSa, Ma1, Me3, MuSu, Nie1, No1, Pr1, Pr2, Se2, SoBi, Ta2].

iv) Books that devote several chapters to the subject, for example [CaJa, Cr1, EcDr, Fr1, Go, Je, KiSt, Lui, Lun, Oz, SzTh]. In addition, there are various other books where the subject is treated to a lesser extent which are not specified here.

Some of these papers have been used as background to the present lecture thus citing, in the majority of the cases, the original reference or review paper on the subject.

The outline of what will be further given consists in a selection of theoretical, numerical and approximate methods that have been used in free boundary problems of the Stefan type, such as:

I) Exact solutions
II) Approximate methods or models
III) Integral formulation methods
IV) Front-tracking methods
V) Front-fixing methods
VI) Fixed-domain methods.

We suggest readers to refer to the original papers and the references thereby mentioned in order to have a more extensive information on the different methodologies used, in particular, considering the shortness of the present classification.

I. EXACT SOLUTIONS

i) The mathematical model for the fusion of a semi-infinite material $x > 0$, initially in the solid phase at the melting temperature $T_m = 0°C$, is given by [LaCl]: Find the functions $T = T(x,t)$ and $S = S(t)$ such that

i) $\rho c\, T_t - k\, T_{xx} = 0 \;,\; 0 < x < S(t) \;,\; t > 0$

ii) $T(S(t),t) = 0 \;,\; t > 0$

iii) $k\, T_x(S(t),t) = -\rho l\, \dot{S}(t) \;,\; t > 0$

iv) $T(o,t) = B > 0 \;,\; t > 0$

v) $S(0) = 0$

where k is the thermal conductivity coefficient, ρ is the mass density, c is the specific heat, l is the latent heat of fusion and $B > 0$ is the temperature on the fixed face $x = 0$. The condition (1iii) is known as the Stefan condition.

The solution of (1) if given by (Lamé-Clapeyron solution):

$$T(x,t) = B - \frac{B}{\operatorname{erf}(\xi)} \operatorname{erf}\left(\frac{x}{2a\sqrt{t}}\right) , \quad 0 \le x \le s(t) , \quad t \ge 0$$

$$s(t) = 2a\,\xi\,\sqrt{t} \tag{2}$$

where $a^2 = k/\rho c$ is the thermal diffusivity coefficient and ξ is the unique solution of the equation:

$$x \exp(x^2) \operatorname{erf}(x) = \frac{Ste}{\sqrt{\pi}} , \quad x > 0 \tag{3}$$

where $Ste = \frac{c\,B}{l} > 0$ is the Stefan number.

ii) A mathematical model for the solidification of a semi-infinite material $x > 0$ (initially in the liquid phase at the temperature $C > 0$ and for $t > 0$ with a temperature $-B < 0$ on the fixed face $x = 0$, considering the jump density at the solid-liquid interphase) is given by [CaJa, We]: Find the functions $T_1 = T_1(x,t)$, $T_2 = T_2(x,t)$ and $S = S(t)$ such that

$$\rho_1 c_1 T_{1_t} - k_1 T_{1_{xx}} = 0 , \quad 0 < x < S(t) , \quad t > 0$$

$$\rho_2 c_2 T_{2_t} + c_2(\rho_1 - \rho_2)\dot{S}(t)\, T_{2_x} - k_2 T_{2_{xx}} = 0 , \quad S(t) < x, t > 0$$

$$T_1 (S(t),t) = T_2 (S(t),t) = 0, t > 0$$

$$k_1 T_{1_x} (S(t),t) - k_2 T_{2_x} (S(t),t) = \rho_1\, l\, \dot{s}(t) , \quad t > 0 \tag{4}$$

$$T_2(x,0) = T_2 (+\infty, t) = C > 0 , \quad x > 0, t > 0$$

$$T_1(0,t) = -B < 0 , \quad t > 0$$

$$S(0) = 0$$

where $i = 1$ represents the solid phase and $i = 2$ the liquid phase.

The solution of (4) is given by (Neumann solution):

$$T_1(x,t) = -B + \frac{B}{\operatorname{erf}(\sigma/a_1)} \operatorname{erf}\left(\frac{x}{2a_1\sqrt{t}}\right)$$

$$T_2(x,t) = \frac{-C}{\operatorname{erfc}(\sigma/a_0)} \left[\operatorname{erf}(\sigma/a_0) - \operatorname{erf}\left(\delta + \frac{x}{2a_2\sqrt{t}}\right)\right] \tag{5}$$

$$S(t) = 2\,\sigma\,\sqrt{t}$$

where

$$a_i = \left(\frac{k_i}{\rho_i\, c_i}\right)^{1/2} \; (i = 1,2) \; , \qquad \delta = \frac{\sigma}{a_2}\,|\varepsilon|$$

$$a_o = \frac{a_2}{1 + |\varepsilon|} \quad , \quad \varepsilon = \frac{\rho_1 - \rho_2}{\rho_2} \tag{6}$$

$$\frac{k_1\, B}{l\, \rho_1\, a_1\, \sqrt{\pi}}\, \frac{\exp(-\sigma^2/a_1^2)}{\operatorname{erf}(\sigma/a_1)} - \frac{k_2\, C}{l\, \rho_1\, a_2\, \sqrt{\pi}}\, \frac{\exp(-\sigma^2/a_o^2)}{\operatorname{erfc}(\sigma/a_o)} = \sigma \tag{7}$$

$\sigma > 0$

iii) The solutions previously presented are included within a more general methodology known as the similarity method which consists in finding the solution of the form $T(x,t) = u(\eta)$ where $\eta = x/\sqrt{t}$ is the similarity variable [Bou].

Other exact solutions are given in [CaJa, ChSu, Rog, TaS, Ti].

II. APPROXIMATE METHODS OR MODELS

i) Quasi-stationary method.

It consists in replacing the heat equation (1i) by [St1, St2]:

$$T_{xx} = 0 \; , \; 0 < x < S(t) \; , \; t > 0 \tag{1ibis}$$

If the boundary condition (1iv) on the fixed face $x = 0$ varies with time, i.e., $B = B(t)$, then the solution of problem (1ibis, 1ii-v) is given by:

$$T(x,t) = B(t) - \frac{B(t)}{S(t)}\, x \; , \; 0 \le x \le S(t) \; , \; t > 0$$

$$S(t) = \left[\frac{2k}{\rho l} \int_0^t B(C)\, dc\right]^{1/2} \quad , \quad t \ge 0 \tag{8}$$

If the data $B(t) = B > 0$ is constant and the Stefan number $Ste = BC/l \ll 1$, then the Lamé-Clapeyron and the Quasi-stationary solutions are very close. A recent analysis with a convection-type condition:

$$- k\, T_x(0,t) = h\, [T_L - T(0,t)] \; , \; t > 0 \tag{9}$$

on the fixed face $x = 0$, is given in [SoWiA1].

ii) Balance integral method.

It is based on the physical concept of the thermal layer. It is assumed that the temperature is propagated in a bounded interval $[0,\delta(t)]$ ($\delta(t)$ represents the thermal layer) and that outside that interval the temperature remains equal to its initial value. The method consists in assuming that $\delta(t)$ coincides with the free boundary $S(t)$ and in replacing conditions (1ii) and (1iii) by (1itris) and (1iiibis) respectively, where [Goo]:

(1itris) $\quad \frac{d}{dt} \int_0^{S(t)} T(x,t)\, dx = \frac{k}{\rho c} [T_x (S(t),t) - T_x (o,t)]$, $t>0$,

(1iiibis) $\quad k\, T_x^2 (S(t),t) = \rho l\, T_t (S(t),t)$, $t > 0$.

Then, we suggest a polynomial distribution in the x variable for the temperature $T(x,t)$ of the form

$$T(x,t) = \alpha(t)\,(x-S(t)) + \beta(t)\,(x - S(t))^2 , \tag{10}$$

α and β being in function of S and then S being a solution of a Cauchy problem.

iii) Biot's variational methods.

We introduce the vectorial heat displacement field $H = H(x,t)$ such that:

$$\dot{H} = - k\, \nabla T\ , \ \operatorname{div} H = - \rho c T\ . \tag{11}$$

If we consider that H is also a function of a given number of generalized coordinates, $H = H\,(x,t,q_1,\dots,q_n)$ suitably chosen, we have the following Lagrange equations (similar to those of analytical mechanics) [Bio]:

$$\frac{\partial V}{\partial q_i} + \frac{\partial D}{\partial \dot{q}_i} = Q_i \quad , \quad i = 1,\dots,n \tag{12}$$

where

$V = \frac{1}{2} \int\int\int c\, T^2\, dx \qquad$: thermal potential

$D = \frac{1}{2k} \int\int\int |\dot{H}|^2\, dx \qquad$: dissipation function

$Q_i = - \int\int T \frac{\partial H}{\partial q_i} \cdot n\, d\gamma \qquad$: generalized thermal forces

In general, for $n = 1$, we take $q_1 = S$ (free boundary) and we suggest for the temperature a polynomial distribution, then finding for s an ordinary differential equation.

iv) Series expansion.

If the initial and boundary data can be developed in a series, then the temperature (of both phases) and the free boundary can be suggested also as a series [RuFaPr, RuSh, Tao].

v) Perturbation series expansion.

We choose a significant parametre of the system (e.g., the Stefan number) and we do a perturbation series expansion with respect to that parametre [Ji, PeDo].

III. INTEGRAL FORMULATIONS METHODS

i) Rubinstein-Friedman Method.

It consists in finding an equivalent integral formulation to the problem under study. For example [Fr3]: (u,s) is the solution of the problem:

$$\begin{aligned}
&\text{i)} \quad u_{xx} - u_t = 0 \ , \ 0 < x < S(t) \ , \ 0 < t < T \\
&\text{ii)} \quad u(0,t) = f(t) \ , \ 0 < t < T \\
&\text{iii)} \quad u(x,0) = h(x) \ , \ 0 \le x \le b \\
&\text{iv)} \quad S(0) = b \ge 0 \\
&\text{v)} \quad u(S(t),t) = 0 \ , \ 0 < t < T \\
&\text{iv)} \quad u_x(S(t),t) = -\dot{S}(t) \ , \ 0 < t < T \ ,
\end{aligned} \tag{14}$$

if the function $v(t) = u_x\,(S(t),t)$ satisfies the integral equation

$$v = R(v) \tag{15}$$

with

$$\begin{aligned}
R(v)(t) = {} & 2\,[h(0) - f(0)]\, N(S(t)\ ,\ t\ ;\ 0,0) + \\
& + 2 \int_0^b h'\,(\xi)\, N(S(t)\ ,\ t\ ;\ \xi,0)\, d\xi - \\
& - 2 \int_0^t \dot{f}(c)\, N(S(t)\ ,\ t\ ;\ 0,c)\, dc + \\
& + 2 \int_0^t v(c)\, G_x(S(t)\ ,\ t\ ;\ S(c),c)\, dc
\end{aligned} \tag{16}$$

$$S(t) = b - \int_0^t v(c)\, dc \tag{17}$$

where

$$K(x,t;\xi,c) = \frac{1}{2\sqrt{\pi(t-c)}} \exp\left[-\frac{(x-\xi)^2}{4(t-c)}\right] \ : \ \text{fundamental solution}$$

$$N(x,t;\xi,c) = K(x,t;\xi,c) + K(-x,t;\xi,c) \ : \ \text{Neumann function}$$

$$G(x,t;\xi,c) = K(x,t;\xi,c) - K(-x,t;\xi,c) \ : \ \text{Green function} \ .$$

The operator R is a contraction application on the Banach space

$$V = \{v \in C^{\circ}[0,c] \ / \ \| v \| = \underset{t \in [0,c]}{\text{Max}} |v(t)| \le M\} \tag{19}$$

for a happy choice of c , $M > 0$. Moreover, the procedure can be extended in time, thus obtaining the solution for each arbitrary

$T > 0$, and so the unique fixed point can be obtained as the limit of the succession v_n , defined by:

a) $v_o \in V$, arbitrarily chosen.

(20)

b) If v_n is known, then $v_{n+1} = R(v_n)$.

Other general methods are given in [Ru1, Ru2].

ii) Other methods.

We can cite:

a) Green function [ChSz, Ko].
b) Embedding technique [Bo11, Bo12].
c) Boundary element method [BrFuTa, BrTeWr, HoUmKi, TaOnKuHa, Wr].

IV. FRONT-TRACKING METHODS

These methods compute, at each time step, the position of the free boundary.

i) Fixed finite-difference grid.

When the grid is fixed in the domain space-time the free boundary will be, in general, between two points of the grid at each time step and thus we shall need special formulas to compute T_x and $\dot{S}$ in a neighbour of the free boundary [C_r2, Fu 2].

ii) Modified grids.

The grid is modified with the passing of time, e.g.;

a) Finite-difference grid with variable time step: We determine as part of the solution, a variable time step such that the free boundary coincides with a grid line in space at each time level. To do that we use for $S(t)$ an integral equation equivalent to the Stefan condition [DoGa].

b) Finite difference grid with variable space step: Since the number of space intervals between the fixed boundary $x = 0$ and the free boundary $S(t)$ are taken constant for all time, and the free boundary is always on the same grid line, then the space step is different in each time step [MuLa].

c) Space-time finite elements: We use a spatial discretization which we adapt in each time step for building quadrilateral finite elements in space-time [BoJa, Ja].

d) Moving finite elements: We use a finite difference procedure in time with finite elements in space which are adapted at each time step to fit the new position of the free boundary [AlMaMi, MiMoBa].

iii) Method of lines.

The time variable is discretized and the partial differential equation is replaced by a sequence of ordinary differential equations at discrete time levels. The position of the free boundary is calculated at each time step [Me1, Me2].

iv) The Polygonal Method.

To solve problem (14) we divide the interval $[0,T]$ in n sub-intervals of $\theta = T/n$ length. For $t \in [0,\theta]$ we put $S_\theta(t) = b - h'(b)t$ and we determine $u_\theta = u_\theta(x,t)$, defined in $0 < x < S_\theta(t)$ and $0 < t < \theta$, as the solution to problem (14i-v). Then, we calculate $a_{1_\theta} = u_{\theta_x}(S_\theta(\theta),\theta)$. For $t \in [\theta, 2\theta]$ we put:

$$\dot{S}_\theta(t) = S_\theta(\theta) - a_{1_\theta} \cdot (t - \theta) \tag{21}$$

and we determine $u_\theta = u_\theta(x,t)$ for $0 < x < S_\theta(t)$, $\theta \le t < 2\theta$, and so forth. Thus we obtain a polygonal $S_\theta = S_\theta(t)$, defined for $t \in [0,T]$, and a function $u_\theta = u_\theta(x,t)$, defined in $0 \le x \le S_\theta(t)$ and $0 \le t \le T$. When $\theta \to 0$ we obtain as limit the solution (s,u) for the problem (14).

v) Retarded argument method.

To solve problem (14) for $b > 0$ we built, for each $\theta \in (0,b)$, a succession $S_\theta(t)$ and $u_\theta = u_\theta(x,t)$ defined in the following way: For $t \in [0,\theta]$ we put $S_\theta(t) = b$ and we determine $u_\theta = u_\theta(x,t)$ in $0 < x < S_\theta(t)$ and $0 \le t < \theta$ as the solution of the problem (14i-v). Then, for $\theta \le t \le 2\theta$, we calculate

$$S_\theta(t) = b - \int_0^t u_{\theta_x}(S_\theta(\eta-\theta), \eta-\theta)\, d\eta \tag{22}$$

and we determine $u_\theta = u_\theta(x,t)$ in $0 < x < S_\theta(t)$ and $\theta \le t < 2\theta$ as the solution of problem (14i-v), and so forth. When $\theta \to 0$ we obtain as limit the solution (s,u) of problem (14) [Ca, CaHi].

vi) Equivalent Stefan Condition Method.

For the case $b = 0$ and condition

$$u_x(0,t) = -g(t), \quad 0 < t < T \tag{23}$$

on the fixed face $x = 0$ instead of (14ii), the Stefan condition (14vi) is equivalent to:

$$S(t) = \int_0^t g(\tau)\, d\tau - \int_0^{S(t)} u(x,t)\, dx, \quad 0 \le t < T \tag{24}$$

Thus, we can define the operator

$$R_1 : S(t) \to r(t) \tag{25}$$

where $r(t)$ is defined as the second member of (24) with u the solution of problem (14i,v), (23) with $b = 0$.

Then, we can define a succession (S_n, u_n) $(n \ge 0)$ in the following way: If S_n is known, we calculate u_n as the solution of problem (14i,ii,v) with $b = 0$ and then we compute $S_{n+1} = R_1(S_n)$. In [Ev, Se1], we study the case $g(t) \equiv 1$ with $S_0(t) \equiv t$, thus obtaining the following results:

a) $\dot{S}_n \geq 0$, $S_{2n-1} \leq S_{2n+1} \leq S_{2n} \leq S_{2n-2}$ (26)

b) S_n and u_n are convergent to the solution of problem (14i,v,vi), (23) and $S(0) = 0$.

With this methodology we can see that the free boundary problem can be interpreted as the limit to a succession of moving boundary problems.

V. FRONT-FIXING METHODS

A Method to track the free boundary is to fix it by a suitable choice of new variables:

i) Inmobilization method.

The Landau transformation

$$\xi = \frac{x}{S(t)} \tag{27}$$

fixes the free boundary $x = S(t)$ at $\xi = 1$ for all time $t > 0$ [La, Cr5]. By using this methodology, in [Co, FaPr3, FaPr4] we prove the general existence and unicity results. Some complementary variants have been done, e.g.:

a) In [Nit], we use the double transformation

$$\xi = \frac{x}{S(t)} \quad , \quad \tau = \int_0^t \frac{d\eta}{S^2(\eta)} \tag{28}$$

for a one-phase Stefan problem.

b) In [Fu2], we use the transformation

$$\xi_1 = \frac{x - l_1}{S(t) - l_1} \quad , \quad \xi_2 = \frac{l_2 - x}{l_2 - S(t)} \tag{29}$$

for a two-phase Stefan problem in the interval $[l_1, l_2]$.

ii) Isotherm migration method.

It is a curvilinear transformation in which the dependent temperature variable u is exchanged with one of the space variables. In the one dimensional case, $u = u(x,t)$ becomes $x = x(u,t)$. The expression $x = x(u,t)$ indicates how a specific temperature u moves through the medium, i.e., how isotherms migrate (in particular, the free boundary) [Ch].

VI. FIXED-DOMAIN METHODS

It consists in reformulating the problem, over the whole fixed domain occupied by the two phases, such that the Stefan condition is included within the new formulation of the equations and the initial and boundary conditions of the problem. Moreover, the position of the free boundary will later reappear, as a consequence of our knowledge of the solution of the new problem posed.

i) Enthalphy method.

The enthalpy function is defined by:

$$H(T) = \int_{T_o}^{T} [\rho(\xi)\, c(\xi) + l\, \rho(\xi)\, \delta(\xi - T_m)]\, d\xi \qquad (.\ddot{\upsilon})$$

where T_o ($< T_m$) is a fixed temperature, T_m is the melting temperature and δ is the Dirac distribution. The enthalpy function has incorporated the heat jump ρl at the free boundary.

For the two-phase multidimensional Stefan problem the problem is reduced to the following differential equation in the distributional sense:

$$\frac{\partial H(T)}{\partial t} = \nabla \cdot (K\, \nabla T) \qquad (31)$$

with the corresponding initial and fixed boundary conditions, where

$$K(T) = \begin{cases} K_1(T) & \text{if } T < T_m \\ K_2(T) & \text{if } T > T_m \end{cases} \qquad (32)$$

With the Kirchoff transformation

$$v = \int_{T_o}^{T} K(t)\, dt \qquad (33)$$

equation (31) is reduced to

$$\frac{\partial H(v)}{\partial t} = \Delta v \qquad (34)$$

In trying to avoid the difficulties that the H jump produces, we have the following possibilities:

a) Weak solution: It satisfies a suitable integral form of the differential equation in which the derivatives of H and T (or v) do not appear [At, Cro, ElOc, Fr4, Ka, Ol, Ros].

b) Regularization enthalpy method:

bi) Due to the discontinuous jump of $H = H(T)$ at the melting temperature $T = T_m$, we suggest to make it regular over a small temperature zone $T_m - \varepsilon \leq T \leq T_m + \varepsilon$ where $\varepsilon > 0$, i.e. $H_\varepsilon = H_\varepsilon(T)$ is a continuous function. In [Je, JeRo, No2] we find error estimates $\| T - T_{\varepsilon h c} \|$ in terms of the regularization parametre ε, time step c and spatial step h.

bii) We replace the enthalpy jump at the phase-change interphase by an equivalent heat capacity $\tilde{C}(T)$, expressing the problem in the following [BoCoFaPr]:

$$\tilde{C}(T)\, T_t = \nabla \cdot (K(T)\, \nabla T) \qquad (31bis)$$

c) Various numerical methods are used, e.g., implicit and explicit finite-differences, implicit, explicit and space-time finite elements, regularization method with implicit finite-differences or finite-elements, for example [At, BuSoUs, Ci, VoCr]. For numerous references on this subject see [Cr2, ElOc, Lun, Ma3, Ta2].

d) In [At] we describe an enthalpy method to solve a welding problem in which a mushy region appears.

e) The convergence for the approximate free boundary is given in [No3].

f) A semigroup approach is given in [BeBrRo, MaVeVi].

ii) Variational inequalities.

We consider the following one-phase fusion problem:

$$
\begin{aligned}
&\frac{\partial\theta}{\partial t} - \Delta\theta = 0 \ , \ t > l(x) \\
&\theta \equiv 0 \quad , \quad t \leq l(x) \\
&- \nabla\theta \cdot \nabla l = 1 \quad \text{on} \quad t = l(x) \\
&\left.\frac{\partial\theta}{\partial n}\right|_{\Gamma_2} = 0 \quad , \quad \theta\Big|_{\Gamma_3} = 0 \qquad\qquad (35) \\
&- \left.\frac{\partial\theta}{\partial n}\right|_{\Gamma_1} = b(\theta-\theta_1) \\
&\theta(x,0) = 0
\end{aligned}
$$

where $b > 0$, $\theta_1 > 0$, $t = l(x)$ represents the solid-liquid interphase and $\Gamma_1 \cup \Gamma_2 \cup \Gamma_3$ is the boundary of the phase-change material Ω . In [Du1, Du2] we realize the change of the function unknown (similar to [Bai] for the dam problem):

$$
u(x,t) = \begin{cases} \int_{l(x)}^{t} \theta(x,c)\, dc & \text{if} \quad t > l(x) \\ 0 & \text{if} \quad t \leq l(x) \end{cases} \qquad (36)
$$

which satisfies the following variational inequality (see also [FrKi]):

$$
\int_\Omega u_t\,(v-u)\,dx + \int_\Omega \nabla u \cdot \nabla (v-u)\,dx + b\int_{\Gamma_1} (u-\theta_1 t)\,(v-u)\,d\gamma \geq
$$

$$
\geq - \int_\Omega (v-u)\,dx \ , \quad \forall v \in K \ , \qquad (37)
$$

$u(t) \in K$, $u(0) = 0$

where

$$V = \{v \in H^1(\Omega) / v/\Gamma_3 = 0\} \quad ,$$
$$K = \{v \in V / v \geq 0 \quad \text{in} \quad \Omega\} \quad . \qquad (38)$$

a) Other problems have been studied with an analogous formulation, for example: electrochemical machining [El], diffusion-consumption of oxygen in a living tissue [Du5], injection of fluid into a laminar cell under the assumption of Hele-Shaw flow [ElJa], simulation and control in a Stefan problem [Sa].

b) For the two-phase Stefan problem, the new function [AgFr, Du3, Fr, Pa, Ta6]:

$$u(x,t) = \int_o^t [k_2\ \theta^+(x,c) - k_1\ \theta^-(x,c)]\ dc \qquad (39)$$

is introduced to obtain a variational inequality formulation.

c) A numerical analysis corresponding to the variational inequality formulation is given in [Fe, IcKi, KaSa, KiIc, Nie2, PiVe, TiTi].

d) In [Fr5], a quasi-variational inequality formulation is given for the unidimensional one-phase Stefan problem.

iii) Alternating phase truncation method.

It consists in solving a heat conduction problem in either the liquid or the solid phase alternatively in successive time steps [BeCiRo, RoBeCi].

<u>Note</u>: Numerous references (other than the ones hereby cited) can be found in recent publications and in the references within them, e.g., [BoDaFr, Ca, Cr2, Ta2, WiSoTr].

NOMENCLATURE

$a^2 = \frac{k}{\rho c}$: thermal diffusivity

a_o : coefficient defined in (6)

$-B < 0$: temperature on the fixed face

c : specific heat

$\tilde{c}$: equivalent heat capacity defined in (31 bis)

$C > 0$: initial temperature

D : dissipation function defined in (12)

f : function defined in (14ii)

g : function defined in (23)

G : Green function defined in (18)

h : function defined in (14iii)

h : coefficient defined in (9)

H : enthalpy function defined by (30)

$\mathbf{H}$: heat displacement field defined in (11)

k : thermal conductivity

K	: fundamental solution defined in (18)
K	: convex set defined in (38)
ℓ	: latent heat of fusion
N	: Neumann function defined in (18)
q_i	: generalized coordinates
Q_i	: generalized thermal forces defined in (12)
R	: operator defined by (16)
R_1	: operator defined by (25)
S	: solid-liquid interface
Ste	: Stefan number
t	: time variable
T	: temperature
T_L	: coefficient defined in (9)
T_m	: melting temperature
u	: transformation defined by (36) and (39)
u	: function defined by problem (14)
v	: Kirchoff transformation defined by (33)
v	: function defined by (15)
V	: thermal potential defined in (12)
V	: Banach space defined by (19)
V	: Hilbert space defined in (38)
x	: spatial variable

Greek Letters

α,β	: coefficients defined in (10)
ϵ	: dimensionless parametre defined in (6)
δ	: dimensionless parametre defined in (6)
σ	: coefficient which characterizes the boundary S by (5)
ρ	: mass density
θ	: function defined by problem (35)
ξ	: dimensionless parametre defined in (2)
ξ	: dimensionless parametre defined by the Landau transformation in (27).
ξ_1,ξ_2	: dimensionless parametres defined in (29)

Subscripts

i=1	: solid phase
i=2	: liquid phase

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APPENDIX 3 (**)

"EL PROBLEMA DE STEFAN A TRAVES DE LA TEORIA DE LAS INECUACIONES VARIACIONALES"

(in Spanish)

Abstract:

We present a short review on the Stefan problem (heat conduction problems with change on phase) through elliptic (steady-state case) and parabolic (evolution case) variational inequalities and their numerical approximations.

(**) This appendix presents a summarized version of the lecture delivered by the author at the 4º Encuentro Nacional de Investigadores Y Usuarios del Método de Elementos Finitos (ENIEF'86), held at San Carlos de Bariloche (Argentina) on June 23-27, 1986 and published in Mecánica Computacional, Vol. 5, L. A. Godoy (ed.), AMCA, Santa Fe (1986), 213-240.

I. INTRODUCCION

Las inecuaciones variacionales (I.V.) son un conjunto de desigualdades o de igualdades que reemplazan las ecuaciones de Euler-Lagrange del cálculo de variaciones clásico cuando éstas no son más válidas. Estas I.V. aparecen en numerosos problemas, a saber: cálculo de variaciones con restricciones, mecánica del continuo (problema del obstáculo, torsión elasto-plástica, fluido de Bingham, dique poroso, cambio de fase), teoría de control (tiempo final óptimo, sistemas a parámetros distribuidos), programación matemática, física del plasma, etc.

La teoría de las I.V. comenzó a cobrar importancia con el trabajo [Sta1] y sobre todo con [LiSt].

A partir de ese momento se hicieron nemerosos trabajos sobre el tema, entre los cuales merecen destacarse diferentes aplicaciones en la teoría del control [Ba1, BeLi1, BeLi2, Li1, Li2], en la mecánica y la física [DuLi, Li3], en el análisis numérico [Ci, Gl, GlLiTr], en problemas de frontera libre [BaCa, ElOc, Fri1, Ta1, Ta2], y aquellos de fundamentación matemática [Br, KiSt, LeSt, St2].

La teoría de las I.V., que venía cumpliendo un papel importante, cobró una mayor relevancia en el año 1971 al resolverse, previo cambio de función incógnita, un problema no-trivial de frontera libre en Hidráulica, conocido en la literatura como el problema del dique poroso [Ba1, Ba2].

Con respecto al problema de Stefan (conducción del calor con cambio de fase), la teoría de las I.V. fue aplicada en 1973 al problema a una fase en [Du1, Du2], y posteriormente al de dos fases en [Du3, Du4, Fre1, Fre2, Pa1, Ta3], y en [Ma1] para el caso unidimensional. A partir de ese momento se realizaron numerosos trabajos sobre la teoría de las I.V. aplicados al problema de Stefan, ya sea desde un punto de vista teórico, numérico o de las aplicaciones; por ejemplo se han realizado los siguientes congresos [AlCoHo, BoDaFr, FaPr, Fu, GoHo, Ho, Ma2, OcHo, Ta6, WiSoBo], libros [Ca, Cr, ElOc, Fri, Je1, KiSt, Ru] y trabajos de revisión con una extensa bibliografía [Cry, Ma3, Pr, Ta5, WiSoTr]. Estos trabajos pueden ser utilizados para obtener información general sobre el tema.

A continuación analizaremos suscintamente el problema de Stefan multidimensional a una y dos fases, el caso estacionario correspondiente al de dos fases y algunas de sus aproximaciones numéricas.

II. PROBLEMA DE STEFAN MULTIDIMENSIONAL A UNA FASE

Se considerará el trabajo [Du1, Du2] en el cual se estudia un bloque de hielo a 0 C que ocupa el dominio acotado $\Omega\subset\mathbb{R}^3$ con frontera $\Gamma=\partial\Omega$ regular. Se supone que Γ está compuesta de tres porciones Γ_1, Γ_2 y Γ_3 sin puntos en común. Se supone además que Γ_1 y Γ_3 no tienen frontera común y que Γ_1 tiene una medida de superficie positiva $(med(\Gamma_1)>0)$.

El problema consiste en hallar la evolución del bloque de hielo cuando la frontera Γ_2 es una pared impermeable al calor, Γ_3 es mantenida a 0 C y sobre Γ_1 existe un flujo de calor del tipo Ley de Newton (con coeficiente de transferencia de calor α y temperatura exterior $\theta_1>0$).

Se supone que todos los coeficientes térmicos son iguales a la unidad y se designa con $\mathcal{F}(t)$ a la superficie de separación de las fases sólida (a temperatura 0 C) y líquida (a temperatura $\theta>0$), conocida como la frontera libre del problema de fusión en análisis.

Si se supone, por monotonía del problema planteado, que la frontera libre está definida por la ecuación

$$t= \ell(x) \qquad (x= (x_1,x_2,x_3) \in \mathbb{R}^3) \tag{1}$$

entonces el problema consiste en hallar $\ell=\ell(x)$, $T>0$ y $\theta= \theta(x,t)$ con $x\in\Omega$ y $t\in(0,T)$ de manera que se satisfagan las siguientes condiciones:

$$\begin{cases} \text{i)} & \dfrac{\partial\theta}{\partial t} - \Delta\theta = 0 \quad , \quad t>\ell(x) \\ \text{ii)} & \theta \equiv 0 \quad , \quad t\leq\ell(x) \\ \text{iii)} & -\nabla\theta.\nabla\ell = 1 \quad , \quad x=\ell(x) \\ \text{iv)} & -\dfrac{\partial\theta}{\partial n}\Big|_{\Gamma_1} = \alpha(\theta-\theta_1) \quad , \quad \text{v)} \quad \dfrac{\partial\theta}{\partial n}\Big|_{\Gamma_2} = 0 \\ \text{vi)} & \theta\big|_{\Gamma_3} = 0 \quad , \quad \text{vii)} \quad \theta(x,0) = 0 \quad , \quad x\in\Omega. \end{cases} \tag{2}$$

Si se introduce la nueva función incógnita $u=u(x,t)$, definida de la siguiente manera:

$$u(x,t)= \begin{cases} \displaystyle\int_{\ell(x)}^{t} \theta(x,s)\cdot ds & \text{si} \quad t>\ell(x) \\ 0 & \text{si} \quad t\leq\ell(x) \end{cases} \tag{3}$$

el problema (2) se transforma en el siguiente

$$\begin{cases} \text{i)} & \dfrac{\partial u}{\partial t} - \Delta u = -1 \quad , \quad t>\ell(x) \\ \text{ii)} & u \equiv 0 \quad , \quad t<\ell(x) \\ \text{iii)} & u=0 \quad , \quad \nabla u=0 \quad , \quad t=\ell(x) \\ \text{iv)} & -\dfrac{\partial u}{\partial n}\Big|_{\Gamma_1} = \alpha(u-\theta_1 t) \quad , \quad \text{v)} \quad \dfrac{\partial u}{\partial n}\Big|_{\Gamma_2} = 0 \\ \text{vi)} & u\big|_{\Gamma_3} = 0 \quad , \quad \text{vii)} \quad u(x,0) = 0 \quad , \quad x\in\Omega. \end{cases} \tag{4}$$

Además, de (4), se deduce que la aplicación $u=u(t)$ (función de la variable espacial x) satisface la siguiente inecuación variacional parabólica:

$$\begin{cases} (u'(t),v-u(t)) + a_\alpha(u(t),v-u(t)) \geq L_\alpha(v-u(t)) \quad , \quad \forall v \in K, \\ u(t) \in K \quad , \quad u(0)=0' \end{cases} \tag{5}$$

donde

$$\begin{cases} V= \left\{ v \in H^1(\Omega) \; / \; v|_{\Gamma_2} = 0 \right\} \\ K= \left\{ v \in V \; / \; v \geq 0 \text{ en } \Omega \right\} \\ a_\alpha(u,v)= \int_\Omega \nabla u.\nabla v \, dx + \alpha \int_{\Gamma_1} uv \, d\gamma \\ L_\alpha(v)= - \int_\Omega v \, dx + \alpha\theta_1 t \int_{\Gamma_1} v \, d\gamma \\ (u,v)= \int_\Omega uv \, dx \end{cases} \tag{6}$$

En [Du1,Du2] se estudia la existencia y unicidad de la solución de la inecuación variacional (5), como así también la convergencia cuando el parámetro α tiende a infinito, obteniéndose de este modo la solución del problema (2) reemplazando la condición (2iv) por

$$\theta|_{\Gamma_1} = \theta_1 \tag{2ivbis}$$

Puede notarse que el conjunto convexo que se obtiene en este caso depende del tiempo.

Esta formulación y algunas de sus generalizaciones han sido estudiadas desde un punto de vista teórico en [Ca, CaFr, DiSh, FrKi1, Ga, Li4-5, Ro2-Ro4] y desde el punto de vista de su cálculo o análisis numérico en [El1, Fe1, Je1, Je2, OdKi, PiVe, Sa3]. Además, se han realizado aplicaciones al problema de la colada continua (solidificación de metales) [Br1, Br2, ChRo, Ro1-Ro3, Ro6], a la teoría de la homogeneización [BoDa, Da3, Li5, Li6, Ro5], a la teoría cuasi-estacionaria (Hele-Shaw flow y electrochemical machinig) [El1, El3, ElJa2] con su correspondiente análisis numérico [El2, ElJa1, ElJa3], y a la teoría de control óptimo [Ba 2, HoSa, Sa3, Sa4, Sa6].

Con respecto al caso unidimensional del problema de Stefan a una fase pueden mencionarse desde un punto de vista teórico [Co, Fri2, Ga, Ma1, Sa2, Sa5, Ya], del cálculo o análisis numérico [Dj, KaSa, Ni-Ni6, Sa1, Sa2, Sa5], del control óptimo [Sa1, Sa5]. Por otra parte, el problema es estudiado a través de una inecuación cuasi-variacional en [Fri3, FrKi2].

Se considerará a continuación el análisis numérico del Problema de Stefan multidimensional a una fase siguiendo [Fe1]:

$$\begin{cases} (u',v-u) + a(u,v-u) \geq (f,v-u) \quad , \quad \forall v \in K(t), \\ u=u(t) \in K(t) \quad , \quad u(0)=0 \end{cases} \tag{7}$$

donde

$$\begin{cases} V=\left\{ v \in H^1(\Omega) \ / \ v|_{\Gamma_2}=0 \right\} \\ K(t)=\left\{ v \in V \ / \ v \geq 0 \text{ en } \Omega, \ v|_{\Gamma_1}=\psi(x,t) \right\} \\ a(u,v)=\int_\Omega \nabla u . \nabla v \, dx \quad , \quad (u,v)=\int_\Omega uv \, dx \quad , \quad f \in L^2(\Omega) \end{cases} \tag{8}$$

Si se penaliza la condición de Dirichlet sobre Γ_1 se obtiene la siguiente inecuación variacional:

$$\begin{cases} (u'_\varepsilon,v-u_\varepsilon) + a_\varepsilon(u_\varepsilon,v-u_\varepsilon) \geq \langle f_\varepsilon,v-u_\varepsilon \rangle \quad , \quad \forall v \in K_0 \\ u_\varepsilon=u_\varepsilon(t) \in K_0 \quad , \quad u_\varepsilon(0)=0 \end{cases} \tag{9}$$

donde

$$\begin{cases} K_0=\left\{ v \in V \ / \ v \geq 0 \text{ en } \Omega \right\} \quad , \\ a_\varepsilon(u,v)=a(u,v) + \frac{1}{\varepsilon} \int_{\Gamma_1} uv \, d\gamma \quad , \\ \langle f_\varepsilon,v \rangle=(f,v) + \frac{1}{\varepsilon} \int_{\Gamma_1} \psi v \, d\gamma \quad . \end{cases} \tag{10}$$

obteniéndose que

$$\lim_{\varepsilon \to 0} ||u_\varepsilon - u||_{L^2(0,T;V)} = 0 \tag{11}$$

Para hallar la solución de la inecuación variacional (9) puede utilizarse la metodología dada en [GlLiTr]: diferencias finitas en la variable tiempo $t \in [0,T]$ y elementos finitos en las variables espaciales $x=(x_1,x_2,x_3) \in \Omega$.

El intervalo $[0,T]$ se divide en N sub-intervalos de amplitud k y el conjunto Ω se triangulariza ($T \in \tau_h$) con elementos finitos regulares, afín equivalentes de clase C^0 con parámetro h, obteniéndose un conjunto V_h de dimensión finita que aproxima a V. Sea

$$K_h=\left\{ v_h \in V_h \ / \ v_h(b) \geq 0, \ \forall b \text{ nodo de la triangulación} \right\} \tag{12}$$

el conjunto que aproxima a K.

Si la función $u=u(x,t)$ se aproxima por

$$\left\{\begin{array}{l} u_{h,k}(x,t)=\sum_{i=0}^{N-1} u_h^i(x)\ \chi_k^i(t) \quad , \quad u_h^i \in V_h \ , \\ \chi_k^i \ : \text{función característica de } [ik,(i+1)k] \end{array}\right. \tag{13}$$

entonces se define la aproximación de la inecuación variacional (9) de la siguiente manera (se sigue notando u en lugar de u_ε, por conveniencia):

$$\left\{\begin{array}{l} \left(\frac{u_h^{i+1} - u_h^i}{k}, v_h-u_h^{i+1}\right) + a_\varepsilon(u_h^{i+\theta}, v_h-u_h^{i+1}) \geq \\ \geq \langle f_{\varepsilon h}^{i+1}, v_h-u_h^{i+1}\rangle \quad , \quad \forall v_h \in K_h \\ u_h^{i+1} \in K_h \end{array}\right. \tag{14}$$

donde

$$\left\{\begin{array}{l} \langle f_{\varepsilon h}^{i+1}, v_h\rangle = \frac{1}{k}\int_{ik}^{(i+1)k} \langle f_\varepsilon(t), v_h\rangle\, dt \\ u_h^{i+\theta} = u_h^i + \theta(u_h^{i+1} - u_h^i) \\ \theta \in [0,1] \quad (\theta=1 \ : \text{implícito}, \ \theta=0 \ : \text{explícito}, \\ \qquad \theta=\frac{1}{2} \ : \text{Crank-Nicholson}). \end{array}\right. \tag{15}$$

La inecuación variacional elíptica (14) con incógnita u_h^{i+1} puede expresarse de la siguiente forma:

$$\left\{\begin{array}{l} (u_h^{i+1}, v-u_h^{i+1}) + k\theta\ a_\varepsilon(u_h^{i+1}, v-u_h^{i+1}) \geq \\ \geq (u_h^i, v-u_h^{i+1}) - k(1-\theta)\ a_\varepsilon(u_h^i, v-u_h^{i+1}) + \\ + k \langle f_h^{i+1}, v-u_h^{i+1}\rangle \quad , \quad \forall v \in K_h \\ u_h^{i+1} \in K_h \ . \end{array}\right. \tag{16}$$

Teniendo en cuenta la relación existente entre inecuaciones variacionales elípticas con forma bilineal simétrica y minimización de funcionales, se tiene que u_h satisface el siguiente problema de mínimo:

$$\begin{cases} J_i(u_h^{i+1}) \leq J_i(v) & , \quad \forall v \in K_h \\ u_h^{i+1} \in K_h \end{cases} \tag{17}$$

donde

$$J_i(v) = \frac{1}{2}\left[(v,v) + k\theta\, a_\varepsilon(v,v)\right] - \left[(u_h^i,v) - k(1-\theta)\, a_\varepsilon(u_h^i,v) + k\, \langle f_h^{i+1},v\rangle\right] \tag{18}$$

con lo cual en cada paso de tiempo se debe hallar la solución de un problema de mínimo para el funcional J_i en K_h ($i=0,\dots,N-1$).

En [Fe1], se dan condiciones para que la aproximación utilizada sea convergente.

III. PROBLEMA DE STEFAN MULTIDIMENSIONAL A DOS FASES

Se considera un dominio material $\Omega \subset \mathbb{R}^3$ a una temperatura inicial $\theta_0=\theta_0(x)$ ($x \in \Omega$), al cual se le aplica una temperatura $b=b(x)$ sobre Γ_1 y un flujo de calor $\bar{h}=\bar{h}(x,t)$ sobre Γ_2 a un instante de tiempo $t>0$. Se considera además, sin pérdida de generalidad, que la temperatura del cambio de fase es 0 C. Se estudia la temperatura $\theta=\theta(x,t)$, definida para $x \in \Omega$ y $t \in (0,T)$ con $T>0$, tiempo dado. A cada instante $t>0$, el conjunto Ω está dividido en dos regiones ocupadas por la fase sólida $\Omega_1(t)$ y la fase líquida $\Omega_2(t)$ las cuales se encuentran separadas por la frontera libre $\mathcal{L}(t)$ ($\mathcal{L}(0)$ es un dato del problema).

Se definen los siguientes conjuntos:

$$\begin{cases} Q_i = \bigcup\limits_{0<t<T} \Omega_i(t)\times(t) \\ \Sigma = \bigcup\limits_{0<t<T} \mathcal{L}(t)\times(t) \\ Q = \Omega\times(0,T) = Q_1 \cup Q_2 \cup \Sigma \end{cases} \tag{1}$$

con lo cual la temperatura θ puede expresarse en Q de la siguiente forma:

$$\theta(x,t) = \begin{cases} \theta_1(x,t)<0 & \text{si} \quad (x,t)\in Q_1 \\ 0 & \text{si} \quad (x,t)\in \Sigma \\ \theta_2(x,t)>0 & \text{si} \quad (x,t)\in Q_2 \end{cases} \tag{2}$$

debiendo satisfacer las siguientes condiciones:

$$\begin{cases} C_i \dfrac{\partial \theta_i}{\partial t} - k_i \Delta\theta_i = g \quad , \quad \text{en } Q_i \ (i=1,2) \\ \theta_1(x,t) = \theta_2(x,t) = 0 \quad , \quad x \in \mathcal{\phi}(t) \quad , \quad 0<t<T \\ k_1 \dfrac{\partial \theta_1}{\partial n} - k_2 \dfrac{\partial \theta_2}{\partial n} = L\, V.n \quad , \quad x \in \mathcal{\phi}(t) \quad , \quad 0<t<T \\ \theta|_{\Gamma_1} = b \\ -k_1 \dfrac{\partial \theta_1}{\partial n}|_{\Gamma_2} = \tilde{h} \quad \text{si} \quad \theta|_{\Gamma_2} < 0 \\ -k_2 \dfrac{\partial \theta_2}{\partial n}|_{\Gamma_2} = \tilde{h} \quad \text{si} \quad \theta|_{\Gamma_2} > 0 \\ \theta(x,0) = \theta_0(x) \quad , \quad x \in \Omega \end{cases} \tag{3}$$

donde $k_i>0$ es la conductividad térmica de la fase i, $C_i>0$ es el calor específico por unidad de volumen de la fase i, $g=g(x,t)$ es el aporte de energía por unidad de tiempo y de volumen, n es un vector normal a $\mathcal{\phi}(t)$ en $\mathbb{R}^3$, $L>0$ es el calor latente de fusión por unidad de volumen, $\Gamma=\Gamma_1\cup\Gamma_2$ es la frontera de Ω con $\Gamma_1\cap\Gamma_2=\emptyset$ y $med(\Gamma_1)>0$, $i=1$ y $i=2$ representan la fase sólida y líquida respectivamente.

Si se realiza el cambio de función incógnita

$$u(x,t) = \int_0^t [k_2\ \theta^+(x,s) - k_1\ \theta^-(x,s)]\, ds \tag{4}$$

entonces u satisface el problema (se simboliza con $u'(t)$ la aplicación que a cada $x\in\Omega$ le hace corresponder $u_t(x,t)$):

$$\begin{cases} \beta(u') - \Delta u = G - L\, \chi_{\{\theta>0\}} \quad , \quad \text{en } D'(Q) \\ u|_{\Gamma_1} = tb_0 \\ -\dfrac{\partial u}{\partial n}|_{\Gamma_2} = h(t) \equiv \displaystyle\int_0^t \tilde{h}(s)\, ds \\ u(0)=0 \end{cases} \tag{5}$$

y, en consecuencia, la inecuación variacional parabólica de tipo II siguiente:

$$\begin{cases} (\beta(u'(t),v-u'(t)) + a(u(t),v-u'(t)) + L\, j(v) - L\, j(u'(t)) \geq \\ \geq \langle f(t),v-u'(t)\rangle \quad , \quad \forall\, v\in K \quad , \quad t\in[0,T] \\ u'(t)\in K \quad , \quad \frac{u(t)}{t}\in K \quad , \quad u(0)=0 \end{cases} \tag{6}$$

donde

$$\begin{cases} V= H^1(\Omega) \quad , \quad K=\left\{v\in V \;/\; v|_{\Gamma_1} = b_0\right\} \\ G(t)= C_2\theta_0^+ - C_1\theta_0^- + L\, H_0(\theta_0) + \int_0^t g(s)\, ds \\ \beta(v)= \frac{C_2}{K_2} v^+ - \frac{C_1}{K_1} v^- \quad , \quad (u,v)= \int_\Omega uv\, dx \\ \langle f(t),v\rangle = \langle G(t),v\rangle - \int_{\Gamma_2} h(t)\, v\, d\gamma \quad , \quad j(v)= \int_\Omega v^+\, dx \, , \end{cases} \tag{7}$$

donde H es la función de Heaviside y χ_A representa la función característica del conjunto A.

Puede notarse que si $b=b(x,t)$ sobre Γ_1, entonces el conjunto convexo K depende del tiempo.

Observación 1:

Si en (5) se incorpora la función característica χ a la función β, se obtiene de este modo una función, llamada entalpía, que tiene un salto en el origen igual al calor latente. De esta nueva ecuación surge la formulación entálpica, que no se tratará aquí, y sobre la cual se han realizado numerosos trabajos, ya sea desde un punto de vista teórico, numérico y de las aplicaciones.

La formulación (6) y algunas de sus generalizaciones han sido estudiadas desde un punto de vista teórico en [AgFr, Co, Da1, Da2, Du3-Du6, Fre1, Fre2, Pa1, Pa2, Ta3, Ta7] y desde el punto de vista de su cálculo o análisis numérico [Bo, Fa2, Ic Ki, KiIc, MiPa, Pa6, PaMi, TiTi]. Además, se han realizado aplicaciones a la teoría de control [MeTi, Pa3-Pa5], a la solidificación de aleaciones binarias [Do], a la identificación de parámetros [MoMiPaSp]. En el caso unidimensional fue aplicado a la teoría de control en [Sa1].

IV. CASO ESTACIONARIO DEL PROBLEMA DE STEFAN MULTIDIMENSIONAL A DOS FASES Y SU ANALISIS NUMERICO

Se considera un material Ω, dominio acotado en $\mathbb{R}^3$, con frontera $\Gamma=\partial\Omega$ regular. Se supone que la temperatura del cambio de fase es 0 C y que Γ está compuesta de dos porciones Γ_1 y Γ_2 con $med(\Gamma_1)>0$. Se aplica una temperatura $b=b(x)$ sobro Γ_1 y un flujo de calor $q=q(x)$ sobre Γ_2.

El problema consiste en estudiar la temperatura $\theta=\theta(x)$, definida para $x\in\Omega$. El conjunto Ω puede expresarse de la forma $\Omega=\Omega_1\cup\Omega_2\cup L$, donde

$$\begin{cases} \Omega_1=\{x\in\Omega\ /\ \theta(x)<0\} \quad , \quad \Omega_2=\{x\in\Omega\ /\ \theta(x)>0\} \\ L=\{x\in\Omega\ /\ \theta(x)=0\} \end{cases} \tag{1}$$

son la fase sólida, la fase líquida y la frontera libre que las separa.

La temperatura θ puede ser representada en Ω de la siguiente manera:

$$\theta(x)=\begin{cases} \theta_1(x)<0 & \text{si} \quad x\in\Omega_1 \\ 0 & \text{si} \quad x\in L \\ \theta_2(x)>0 & \text{si} \quad x\in\Omega_2 \end{cases} \tag{2}$$

y satisface las siguientes condiciones:

$$\begin{cases} \Delta\theta_i=0 \quad , \quad \text{en } \Omega_i \quad (i=1,2) \\ \theta_1=\theta_2=0 \quad , \quad k_1\dfrac{\partial\theta_1}{\partial n}=k_2\dfrac{\partial\theta_2}{\partial n} \quad \text{sobre } L \\ \theta|_{\Gamma_1}=b \\ \quad -k_2\dfrac{\partial\theta_2}{\partial n}|_{\Gamma_2}=q \quad \text{si} \quad \theta|_{\Gamma_2}>0 \\ \quad -k_1\dfrac{\partial\theta_1}{\partial n}|_{\Gamma_2}=q \quad \text{si} \quad \theta|_{\Gamma_2}<0 \end{cases} \tag{3}$$

donde $k_i>0$ es la conductividad térmica de la fase i (i=1 :fase sólida, i=2 :fase líquida).

Si se define una nueva función incógnita [Ta4, Ta8]

$$u=k_2\ \theta^+ - k_1\ \theta^- \quad \text{en } \Omega\ , \tag{4}$$

donde θ^+ y θ^- representan la parte positiva y la parte negativa de la función θ respectivamente, entonces el problema (3) se transforma en

$$\begin{cases} \Delta u=0 \quad , \quad \text{en } D'(\Omega) \\ u|_{\Gamma_1}=b_0\equiv k_2\ b^+ - k_1\ b^- \end{cases} \tag{5}$$

$$\left\{ \quad - \frac{\partial u}{\partial n}\Big|_{\Gamma_2} = q \right. \tag{5}$$

cuya formulación variacional está dada por:

$$\left\{ \begin{array}{ll} a(u,v-u) = \langle f, v-u \rangle , & \forall\, v \in K \\ u \in K \end{array} \right. \tag{6}$$

donde

$$\left\{ \begin{array}{l} V = H^1(\Omega) \quad , \quad V_0 = \left\{ v \in V \;/\; v|_{\Gamma_1} = 0 \right\} , \\ K = \left\{ v \in V \;/\; v|_{\Gamma_1} = b_0 \right\} \quad , \quad H = L^2(\Omega) \quad , \\ a(u,v) = \int_\Omega \nabla u . \nabla v \, dx \quad , \quad \langle f, v \rangle = - \int_{\Gamma_2} qv \, d\gamma \; . \end{array} \right. \tag{7}$$

Bajo la hipótesis $b \in H^{1/2}(\Gamma_1)$ se tiene que existe $B \in V$ tal que $B/_{\Gamma_1} = b_0$. Entonces, si se define $U = u - B \in V_0$, se tiene que (6) es equivalente a (8), donde

$$\left\{ \begin{array}{ll} a(U,v) = \langle F, v \rangle , & \forall\, v \in V_0 \\ U \in V_0 \end{array} \right. \tag{8}$$

con

$$\langle F, v \rangle = \langle f, v \rangle - a(B,v) \; . \tag{9}$$

Si además se tiene la hipótesis $q \in L^2(\Gamma_2)$, entonces por el Teorema de Lax-Milgram se deduce que existe una única solución U de (8) y por ende una única solución u de (6).

Sean Ω un dominio poligonal convexo con frontera regular y τ_h una triangulación regular de Ω, donde $h>0$ es un parámetro destinado a tender a cero, formada por elementos finitos afín-equivalentes de clase C^0. Se toma h igual a la longitud del lado más grande de los triángulos $T \in \tau_h$ y se aproxima V_0 por:

$$V_h = \left\{ v_h \in C^0(\bar{\Omega}) \;/\; v_{h/_T} \in P_1 \; , \; \forall\, T \in \tau_h \; , \; v_{h/_{\Gamma_1}} = 0 \right\} \tag{10}$$

donde P_1 es el conjunto de los polinomios de grado menor o igual a 1.

Sea Π_h el operador de interpolación lineal correspondiente. Se considera el siguiente problema aproximado, en dimensión finita, del problema continuo (8):

$$\begin{cases} a(U_h,v_h)= \langle F,v_h\rangle \quad , \quad \forall\, v_h\in V_h \\ U_h\in V_h \end{cases} \qquad (11)$$

obteniéndose los siguientes resultados:

Propiedad 1:

Bajo las hipótesis anteriores, se tienen las siguientes propiedades:

i) Existe una única solución U_h de (11), que satisface

$$a(U-U_h,v_h)= 0 \quad , \quad \forall\, v_h\in V_h \qquad (12)$$

es decir que $U_h= P_{V_h}(U)$.

ii) La sucesión U_h está acotada en V; más aún, se tiene

$$||U_h||_V \leq \frac{||F||}{\alpha} \quad , \quad \forall\, h>0, \qquad (13)$$

donde α es la constante de la coercitividad de la forma bilineal a en V_*, es decir:

$$a(v,v)= ||v||^2_{V_*} \geq \alpha\, ||v||^2_V \quad , \quad \forall\, v\in V_* \ . \qquad (14)$$

iii) $U_h \to U$ en V débil cuando $h\to 0$.

iv) $U_h \to U$ en V fuerte cuando $h\to 0$, es decir:

$$\lim_{h\to 0} ||U_h-U||_V = 0 \quad . \qquad (15)$$

v) Se tiene la siguiente estimación:

$$||U_h-U||_V \leq \frac{1}{\alpha} \inf_{v_h\in V_h} ||U-v_h||_V \quad . \qquad (16)$$

vi) Existe una constante $C_1>0$ (independiente de h) de manera que

$$||U_h-U||_V \leq C_1\, h \ . \qquad (17)$$

Si se definen

$$\begin{cases} K_h= V_h + \Pi_h(B) \\ u_h= U_h + \Pi_h(B)\in K_h \quad , \quad \theta_h= \frac{1}{k_2}\, u_h^+ - \frac{1}{k_1}\, u_h^-\in V \\ u = U + B\in K \quad , \quad \theta = \frac{1}{k_2}\, u^+ - \frac{1}{k_1}\, u^-\in V \end{cases}$$

entonces se deducen los siguientes resultados:

Propiedad 2

i) Bajo las hipótesis anteriores, se tiene:

$$\lim_{h\to 0} ||u_h - u||_V = \lim_{h\to 0} ||\theta_h - \theta||_H = 0 \quad . \qquad (19)$$

ii) Si además, los datos del problema b y q son tales que se tenga la hipótesis suplementaria $\theta \in H^2(\Omega)$ (ver Propiedad siguiente), entonces existen dos constantes $C_2>0$, $C_3>0$ (independientes de h) de manera que:

$$\begin{cases} ||u_h - u||_V \leq C_2 h \\ ||\theta_h - \theta||_H < C_3 h \quad . \end{cases} \qquad (20)$$

iii) Bajo hipótesis adicionales, los resultados (20) pueden generalizarse con h^k tomando P_k (polinomios de grado $k\geq 1$) en lugar de P_1.

Propiedad 3

i) Si se nota con $u=u_{bq}$ la solución correspondiente a los datos b y q, entonces se tiene el resultado de comparación siguiente:

$$\left.\begin{array}{l} b_1 \leq b_2 \text{ sobre } \Gamma_1 \\ q_2 \leq q_1 \text{ sobre } \Gamma_2 \end{array}\right\} \Rightarrow \quad u_{b_1q_1} \leq u_{b_2q_2} \text{ en } \Omega. \qquad (21)$$

ii) Si $b=b(x)>0$ sobre Γ_1 y $q=q(x)>0$ sobre Γ_2 verifican la desigualdad (con $med(\Gamma_2)>0$):

$$\operatorname*{Inf}_{x\in\Gamma_2} q(x) > \frac{k_2 \, med(\Gamma_2)}{C} \cdot \operatorname*{Sup}_{x\in\Gamma_1} b(x)$$

donde C es una constante adecuada [Ta8], entonces se tiene un problema de Stefan a dos fases.

iii) Si los datos b y q son como en (ii), satisfacen (22) y $b\in H^{1/2}(\Gamma_1)$ entonces $\theta \in H^2(\Omega)$.

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