A NOTE ON THE EXISTENCE OF A WAITING TIME
FOR A TWO-PHASE STEFAN PROBLEM

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Abstract. We consider a slab, represented by the interval $0 < x < x_0$, at the initial temperature $\theta_0 = \theta_0(x) \geq 0$ (or $\phi_0 = \phi_0(x) \geq 0$) having a heat flux $q = q(t) > 0$ (or convective boundary condition with a heat transfer coefficient h) on the left face $x = 0$ and a temperature condition $b(t) > 0$ on the right face $x = x_0$ (x_0 could be also $+\infty$, i.e., a semi-infinite material). We consider the corresponding heat conduction problem and assume that the phase-change temperature is $0°C$.

We prove that certain conditions on the data are necessary or sufficient in order to obtain the existence of a waiting-time at which a phase-change begins.

I. Introduction. We consider the following heat conduction problems ($0 < x_0 \leq +\infty$):

$$
\begin{aligned}
&\text{(i)} \quad \rho c\theta_t - k\theta_{xx} = 0, \qquad 0 < x < x_0, \ t > 0; \\
&\text{(ii)} \quad \theta(x, 0) = \theta_0(x) > 0, \qquad 0 \leq x \leq x_0; \\
&\text{(iii)} \quad k\theta_x(0, t) = q(t), \qquad t > 0; \\
&\text{(iv)} \quad \theta(x_0, t) = b(t), \qquad t > 0;
\end{aligned}
\tag{1}
$$

and

$$
\begin{aligned}
&\text{(i)} \quad \rho c\phi_t - k\phi_{xx} = 0, \qquad 0 < x < x_0, \ t > 0; \\
&\text{(ii)} \quad \phi(x, 0) = \phi_0(x) > 0, \qquad 0 \leq x \leq x_0; \\
&\text{(iii)} \quad k\phi_x(0, t) = h(D + \phi(0, t)), \qquad t > 0; \\
&\text{(iv)} \quad \phi(x_0, t) = b(t), \qquad t > 0,
\end{aligned}
\tag{1'}
$$

where ρ is the density, k is the thermal conductivity, c is the specific heat, h is the convective heat transfer coefficient from a fluid with ambient temperature $-D < 0$ flowing across the face $x = 0$. The function $b(t)$ represents the temperature at the face $x = x_0 > 0$, and θ_0 and ϕ_0 are the initial temperatures for problems (1) and (1') respectively.

We take, without loss of generality, the phase-change temperature as $0°C$ and replace condition (1)(iv) by $\theta(+\infty, t) = \theta_0(+\infty) > 0$, $t > 0$ for the case $x_0 = +\infty$

(idem for Problem $(1')$). We assume that the data satisfy the hypotheses that ensure the existence and uniqueness property of the solution of (1) and $(1')$.

We consider the following possibilities:

(a) the heat conduction problem is defined for all $t > 0$ (waiting-time $t^* = +\infty$);

(b) there exists a time $t^* < +\infty$ such that another phase (i.e., the solid phase) appears for $t \geq t^*$ (waiting-time $0 \leq t^* < +\infty$) and then we have a two-phase Stefan problem for $t > t^*$. In this case, there exists a free boundary $x = s(t)$, which separates the liquid and solid phases with $s(t^*) = 0$.

We will separate the cases waiting-time $t^* = 0$ (i.e., there exists an instantaneous change of phase) and $0 < t^* \leq +\infty$. These possibilities depend on the data θ_0, q, b for Problem (1) and the data ϕ_0, h, b for Problem $(1')$. We try to clarify this dependence by finding necessary or sufficient conditions on θ_0, q, b and ϕ_0, h, b in order to have the different possibilities.

In [5, 8, 9] the one-phase Stefan problem with prescribed flux or convective boundary condition at $x = 0$ is studied.

This paper was motivated by [10, 12, 13] (see also [14]) and the term *waiting-time* was motivated by its correspondence to the term as used in the porous medium equation (see, for instance [1]).

In Sec. II we analyse problem (1) with a flux boundary condition at $x = 0$ and in Sec. III we study the problem $(1')$ with a convective boundary condition at $x = 0$.

II. On some conduction problems with a flux boundary condition. We consider the following properties for the problem (1).

THEOREM 1. If the data $q = q(t)$, $\theta_0 = \theta_0(x)$, and $b = b(t)$ verify conditions

(i) $0 < q(t) \leq q_0$, $0 < t \leq t_0$ with $t_0 > 0$;

(ii) $\theta_0'(x) \geq 0$ and $\beta_1 \geq \theta_0(x) \geq \beta_0 > 0$, $0 \leq x \leq x_0$ with $\beta_0 \leq \beta_1$;   (2)

(iii) $b(t) \geq \beta_1$ and $\dot{b}(t) \geq 0$, $t > 0$;

then there exists a waiting-time $t^* > 0$ for problem (1), (i.e., another phase could appear at $t \geq t^*$), where t^* verifies the following inequality:

$$t^* \geq \text{Min}(t_0, t_0^*), \quad \text{where } t_0^* = \pi k \rho c \beta_0^2 / 4 q_0^2. \qquad (3)$$

Proof. It is sufficient to prove that $\theta(x, t) \geq 0$ for $0 \leq x \leq x_0$ and $0 \leq t \leq t_0^*$. For the semi-infinite material $x > 0$, with the same thermal coefficients, we consider the following two problems:

$$\begin{aligned}
\rho c T_t - k T_{xx} &= 0, & x > 0, \ 0 < t < t_0; \\
k T_X(0, t) &= q(t), & t > 0; \\
T(x, 0) &= T_0(x), & x \geq 0,
\end{aligned} \qquad (4)$$

with

$$T_0(x) = \begin{cases} \theta_0(x), & 0 \leq x \leq x_0, \\ \theta_0(x_0), & x > x_0, \end{cases} \qquad (5)$$

and

$$\rho c V_t - k V_{xx} = 0, \qquad x > 0, \ t > 0;$$
$$k V_x(0, t) = q_0 > 0, \qquad t > 0; \tag{6}$$
$$V(x, 0) = \beta_0 > 0, \qquad x \geq 0,$$

whose solutions are given respectively by [3, 4]

$$T(x, t) = \int_0^{+\infty} N(x, t; \xi, 0) T_0(\xi)\, d\xi - \frac{2a}{k} \int_0^t K(x, t; 0, \tau) q(\tau)\, d\tau, \tag{7}$$

and

$$V(x, t) = \beta_0 - \frac{2q_0 a}{k} \sqrt{t}\, \text{ierfc} \left(\frac{x}{2a\sqrt{t}} \right), \tag{8}$$

where

$$a = \left(\frac{k}{\rho c} \right)^{1/2}; \qquad K(x, t; \xi, \tau) = \frac{\exp(-(x - \xi)^2 / 4a^2(t - \tau))}{2a\sqrt{\pi(t - \tau)}};$$

$$N(x, t; \xi, \tau) = K(x, t; \xi, \tau) + K(-x, t; \xi, \tau); \qquad \text{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x \exp(-t^2)\, dt;$$

$$\text{erfc}(x) = 1 - \text{erf}(x); \qquad \text{ierfc}(x) = \frac{\exp(-x^2)}{\sqrt{\pi}} - x\, \text{erfc}(x). \tag{9}$$

By the maximum principle [6, 7] we obtain

$$V(x, t) \leq T(x, t), \qquad x \geq 0, \ 0 \leq t \leq t_0,$$
$$T(x, t) \leq \theta(x, t), \qquad 0 \leq x \leq x_0, \ 0 \leq t \leq t_0, \tag{10}$$

because $T(x_0, t) \leq \beta_1 \leq b(t)$ for $0 \leq t \leq t_0$.

Let W be the function $W = \theta_x$, which satisfies the following heat conduction problem:

$$\rho c W_t - k W_{xx} = 0, \qquad 0 < x < x_0, \ t > 0;$$
$$W(x, 0) = \theta_0'(x), \qquad 0 \leq x \leq x_0; \tag{11}$$
$$W(0, t) = \frac{q(t)}{k}, \qquad W_x(x_0, t) = \frac{\rho c}{k} b(t), \ t > 0.$$

By the maximum principle we have $W = \theta_x \geq 0$ for $0 \leq x \leq x_0$, $t \geq 0$. Then, we deduce that

$$\theta(x, t) \geq \theta(0, t) \geq V(0, t) = \beta_0 - \frac{2q_0 a}{k} \sqrt{\frac{t}{\pi}} \geq 0 \quad \text{for } 0 \leq t \leq t_0^*, \tag{12}$$

where t_0^* is defined by (3) proving our assertion.

REMARK 1. When the data verify conditions (2), problem (1) represents a heat conduction problem for the initial phase (in our case, the liquid phase) for $t \leq t^*$.

REMARK 2. We can see that t_0^* does not depend on the length of the slab $x_0 > 0$.

COROLLARY 2. Under the hypotheses (2)(ii),(iii), a necessary condition in order to have an instantaneous change of phase (i.e., $t^* = 0$) for problem (1) is given by

$$q(0^+) = +\infty. \tag{13}$$

REMARK 3. If we consider the case

$$x_0 = +\infty, \qquad \theta_0(x) \ge \beta_0 > 0, \quad \forall x \ge 0,$$

$$q(t) \le q_0(t) = \frac{\beta_0 k}{a\sqrt{\pi t}}, \qquad \forall t > 0, \tag{14}$$

then problem (1) is a heat conduction problem for the liquid phase for all $t > 0$ i.e., there is not a phase-change process for any $t > 0$ because we have

$$\theta(x, t) \ge \theta q_0(x, t), \qquad x \ge 0, \ t > 0, \tag{15}$$

where θq_0 is the solution of (1) with data: heat flux $q_0(t)$ at $x = 0$, $x_0 = +\infty$, and initial temperature β_0. It is given by [12]

$$\theta q_0(x, t) = \beta_0 \mathrm{erf}\left(\frac{x}{2a\sqrt{t}}\right) \ge 0, \qquad x \ge 0, \ t > 0. \tag{16}$$

Moreover, the particular case

$$q(t) = \frac{\beta_0 k}{a\sqrt{\pi t}} = q_0(t), \qquad t > 0, \tag{17}$$

shows us that condition (13) is not sufficient in order to have an instantaneous change of phase for problem (1).

REMARK 4. If $x_0 = +\infty$ and $\theta_0(x) \ge \beta_0 > 0$ for $x \ge 0$, then a necessary condition for problem (1) to have an instantaneous change of phase (i.e., the waiting-time is $t^* = 0$) is for there to exist a $t_0 > 0$ such that

$$q(t_0) > \frac{\beta_0 k}{a\sqrt{\pi t_0}}. \tag{18}$$

THEOREM 3. If the data verify the conditions

$$x_0 = +\infty; \qquad 0 \le \theta_0(x) \le \beta_1 \quad \text{for } x \ge 0,$$

$$q(t) \ge \frac{q_0}{t^\beta}, \qquad 0 < t < 1, \text{ with } q_0 > 0 \text{ and } \tfrac{1}{2} < \beta < 1, \tag{19}$$

then an instantaneous phase-change occurs, that is, the waiting-time is $t^* = 0$.

Proof. Let $U = U(x, t)$ be the solution of the following heat conduction problem:

$$\begin{aligned} \rho c U_t - k U_{xx} &= 0, & x > 0, \ t > 0; \\ U(x, 0) &= \beta_1, & x \ge 0; \\ k U_x(0, t) &= \frac{q_0}{t^\beta}, & t > 0, \end{aligned} \tag{20}$$

which is given by [3]

$$U(x, t) = \beta_1 - \frac{2aq_0}{k}\int_0^t \frac{K(x, t; 0, \tau)}{\tau^\beta}\, d\tau. \tag{21}$$

By using the maximum principle we have that $\theta(x, t) \le U(x, t)$ for $x \ge 0$, $t > 0$. Therefore, we obtain

$$\theta(0, t) \le U(0, t) \le \beta_1 - \frac{aq_0}{k\sqrt{\pi}}\int_0^t \frac{d\tau}{\tau^\beta\sqrt{t - \tau}} \tag{22}$$

and, for $0 < \varepsilon = t/2 < t < 1$,

$$\int_0^t \frac{d\tau}{\tau^\beta \sqrt{t-\tau}} \geq \int_0^\varepsilon \frac{d\tau}{\varepsilon^\beta \sqrt{t-\tau}} + \int_\varepsilon^t \frac{d\tau}{\tau^\beta \sqrt{t-\varepsilon}} = C_\beta \left(\frac{2}{t}\right)^{\beta-1/2}. \tag{23}$$

Moreover, the temperature on the fixed face $x = 0$ verifies the inequality

$$\theta(0, t) \leq \beta_1 - \frac{aq_0}{k\sqrt{\pi}} C_\beta \left(\frac{2}{t}\right)^{\beta-1/2} < 0, \tag{24}$$

for all $t < \min(1, t_\beta)$, where

$$t_\beta = 2 \left(\frac{aq_0}{k\beta_1\sqrt{\pi}} C_\beta\right)^{1/(\beta-1/2)} > 0,$$

$$C_\beta = \frac{1}{1-\beta}[2^{1-\beta} - 1 + 2(1-\beta)(\sqrt{2}-1)] > 0, \tag{25}$$

that is, the thesis is achieved.

REMARK 5. If we consider the density jump under the phase of change, that is, $\rho_1 \neq \rho_2$, and the data verify the conditions

$$x_0 = +\infty; \qquad 0 \leq \theta_0(x) \leq \beta_1 \quad \text{for } x \geq 0,$$

$$q(t) \geq \frac{q_0}{\sqrt{t}} \quad \text{for } t > 0 \text{ with } q_0 > \frac{\beta_1 k_2}{a_2\sqrt{\pi}}, \tag{26}$$

where k_i, c_i, ρ_i, $a_i = (k_i/\rho_i c_i)^{1/2}$ are the corresponding thermal coefficients for the phase i ($i = 2$: liquid phase, $i = 1$: solid phase), then the temperature $\theta = \theta_{q,\theta_0}$, solution of problem (1), verifies the inequality $\theta_{q,\theta_0}(x, t) \leq T_{q_0,\beta_1}(x, t)$, $x \geq 0$, $t > 0$, where T_{q_0,β_1} is the solution of (1) with initial constant temperature β_1 and a flux condition of type $q_0/\sqrt{t}$ on $x = 0$. Therefore, we obtain [2]

$$\theta_{q,\theta_0}(0, t) \leq T_{q_0,\beta_1}(0, t) = \beta_1 - \frac{q_0 a_2 \sqrt{\pi}}{k_2} < 0, \qquad t > 0, \tag{27}$$

that is, the waiting-time is $t^* = 0$ (i.e., we have an instantaneous two-phase Stefan problem) for data q and θ_0. Moreover, its free boundary $x = s_{q,\theta_0}(t)$ verifies $s_{q,\theta_0}(0) = 0$ and it is characterized by $\theta_{q,\theta_0}(s_{q,\theta_0}(t), t) = 0$ for all $t > 0$.

The free boundary $x = s_{q_0,\beta_1}(t)$ corresponding to the temperature T_{q_0,β_1} is given by [2]

$$s_{q_0,\beta_1} = 2\omega\sqrt{t}, \tag{28}$$

where ω is the unique solution of the equation

$$F_0(x) = x, \qquad x > 0 \tag{29}$$

with

$$F_0(x) = \frac{q_0}{h\rho_1} \exp\left(\frac{-x^2}{a_1^2}\right) - \frac{k_2 \beta_1}{h\rho_1 a_2 \sqrt{\pi}} \frac{\exp(-x^2/a_2)}{\text{erfc}(x/a_2)}, \tag{30}$$

where $h > 0$ is the latent heat. Owing to

$$\theta_{q,\theta_0}(s_{q_0,\beta_1}(t), t) \leq T_{q_0,\beta_1}(s_{q_0,\beta_1}(t), t) = 0, \quad \text{for all } t > 0, \tag{31}$$

it follows that

$$s_{q,\theta_0}(t) \geq s_{q_0,\beta_1}(t) = 2\omega\sqrt{t}, \qquad t > 0. \quad \blacksquare \qquad (32)$$

From now on we consider the particular case of constant temperature $b(t) = b > 0$, $t > 0$ at $x = x_0$ and constant heat flux $q(t) = q > 0$, $t > 0$ at $x = 0$ for problem (1). The steady-state solution is given by

$$\theta_\infty(x) = \frac{q}{k}(x - x_0) + b, \qquad (33)$$

and a necessary and sufficient condition in order to have a two-phase steady-state Stefan problem is given by

$$q > kb/x_0, \qquad (34)$$

where k is the thermal conductivity of the liquid phase [11]. (See [13, 14] for the general steady-state case for an n-dimensional domain).

Using the fact that $\theta = \theta(x, t)$, the solution of problem (1) with data $q > 0$ and $b > 0$, converges to $\theta_\infty = \theta_\infty(x)$ when t goes to $+\infty$ [6], for any initial temperature $\theta_0 = \theta_0(x)$, we can formulate the following problem: Find the relation between the heat flux $q > 0$ on $x = 0$ and a time t_1 such that another phase appears for $t \geq t_1$, and then we can reformulate problem (1) in a two-phase Stefan problem for $t \geq t_1$.

We obtain the following result.

THEOREM 4. Suppose the initial temperature verifies the conditions $b \geq \theta_0 \geq 0$ in $[0, x_0]$ and $\theta_0(x_0) = b$. If the time $t_1 > 0$ and the constant heat flux $q > 0$ verify the inequality

$$q > \frac{bk}{x_0(1 - \exp(-\alpha\pi^2 t_1/4x_0^2))}, \qquad \alpha = \frac{k}{\rho c}, \qquad (35)$$

then another phase (the solid phase) appears for $t \geq t_1$. Moreover, $\theta(0, t) < 0$ for all $t \geq t_1$ and the free boundary $x = s(t)$ begins at a point $(0, t')$ with $0 \leq t' < t_1$.

Proof. The temperature $\theta(x, t)$ is given by

$$\theta(x, t) = \theta_\infty(x) + \sum_{n=0}^{\infty} C_n \cos(\sqrt{\lambda_n}x) \exp(-\alpha t \lambda_n), \qquad (36)$$

where

$$\lambda_n = \left(n + \frac{1}{2}\right)^2 \frac{\pi^2}{x_0^2}, \qquad n = 0, 1, 2, \ldots, \qquad (37)$$

$$C_n = \frac{2}{x_0} \int_0^{x_0} [\theta_0(x) - \theta_\infty(x)] \cos(\sqrt{\lambda_n}x) \, dx. \qquad (38)$$

Therefore, the temperature at $x = 0$ is given by

$$\theta(0, t) = b - \frac{qx_0}{k}(1 + S(t)) + S_0(t) \qquad (39)$$

with

$$S(t) = \frac{2}{x_0^2} \sum_{n=0}^{\infty} \exp(-\alpha t \lambda_n) \int_0^{x_0} (x - x_0) \cos(\sqrt{\lambda_n}x) \, dx = -\frac{2}{\pi^2} \sum_{n=0}^{+\infty} \frac{\exp(-\alpha t \lambda_n)}{(n + \frac{1}{2})^2} \qquad (40)$$

and

$$S_0(t) = \frac{2}{x_0} \sum_{n=0}^{+\infty} \exp(-\alpha t \lambda_n) \int_0^{x_0} (\theta_0(x) - b) \cos(\sqrt{\lambda_n} x) \, dx. \tag{41}$$

We get that $S_0(t) = v(0, t) \leq 0$ for all $t \geq 0$ because of the maximum principle, where the function $v = v(x, t) \leq 0$ is the solution of the problem

$$
\begin{aligned}
\rho c v_t - k v_{xx} &= 0, & 0 < x < x_0, \ t > 0; \\
v_x(0, t) &= 0, & t > 0; \\
v(x_0, t) &= 0, & t > 0; \\
v(x, 0) &= \theta_0(x) - b \leq 0, & 0 \leq x \leq x_0.
\end{aligned}
\tag{42}
$$

By some manipulations, it follows that

$$0 < |S(t)| = -S(t) \leq \exp\left[-\frac{\alpha \pi^2 t}{4 x_0^2}\right] < 1 \quad \text{for all } t \geq t_1. \tag{43}$$

Therefore, if q and t_1 verify (35), we obtain

$$\theta(0, t) \leq b - \frac{q x_0}{k}(1 + S(t)) < 0 \quad \text{for all } t \geq t_1, \tag{44}$$

i.e., the thesis is achieved.

REMARK 6. If $\theta_0(x) = b$ in $[0, x_0]$, we deduce $S_0(t) = 0$ for $t > 0$. We remark that inequality (35) was obtained for this particular initial temperature because $S_0(x) \leq 0$ in $(0, +\infty)$ for $\theta_0(x) \leq b$ in $[0, x_0]$.

COROLLARY 5. If we consider the t, q plane and define the following set

$$Q = \{(t, q) \mid q > f(t), \ t > 0\}, \qquad f(t) = \frac{bk}{x_0[1 - \exp(-\alpha \pi^2 t / 4 x_0^2)]}, \tag{45}$$

then we have a two-phase problem for all $(t, q) \in Q$.

III. On some conduction problems with a convective boundary condition. Now we consider the same kind of techniques used in Sec. 2 for problem $(1')$ corresponding to a heat conduction problem with a convective boundary condition at $x = 0$.

THEOREM 6. If the data $\phi_0 = \phi_0(x)$, $b = b(t)$, and D verify the conditions

$$
\begin{aligned}
&\text{(i)} \quad \phi_0'(x) \geq 0 \quad \text{and} \quad \beta_1 \geq \phi_0(x) \geq \beta_0 > 0, \qquad 0 \leq x \leq x_0; \\
&\text{(ii)} \quad b(t) \geq \beta_1 \quad \text{and} \quad \dot{b} \geq 0, \qquad t > 0; \\
&\text{(iii)} \quad D > 0,
\end{aligned}
\tag{46}
$$

then there exists a waiting-time $t^* > 0$ for problem $(1')$, where t^* verifies the inequality

$$t^* \geq t_1^*, \quad \text{where } t_1^* = \frac{kc\rho}{h^2}\left(F^{-1}\left(1 + \frac{\beta_0}{D}\right)\right)^2, \tag{47}$$

where F^{-1} is the inverse function of

$$F(x) = \frac{\exp(-x^2)}{\text{erfc}(x)}, \qquad x > 0. \tag{48}$$

Proof. By using the maximum principle we get that

$$\phi_x(x, t) \geq 0, \qquad 0 \leq x \leq x_0, \ t \geq 0,$$
$$\phi(x, t) \geq z(x, t), \qquad 0 \leq x \leq x_0, \ t \geq 0, \tag{49}$$

where $z = z(x, t)$ is the solution of the following heat conduction problem:

$$\begin{align}
\text{(i)} \quad & \rho c z_t - k z_{xx} = 0, \qquad x > 0, \ t > 0; \\
\text{(ii)} \quad & z(x, 0) = \beta_0, \qquad x > 0; \\
\text{(iii)} \quad & k z_k(0, t) = h(D + z(0, t)), \qquad t > 0,
\end{align} \tag{50}$$

which is given by $(a^2 = k/\rho c)$:

$$z(x, t) = (\beta_0 + D)\left[\operatorname{erfc}\left(\frac{x}{2a\sqrt{t}}\right) + \exp\left(\frac{hx}{k} + \eta^2\right)\right]\operatorname{erfc}\left(\frac{x}{2a\sqrt{t}} + \eta\right), \tag{51}$$
$$x \geq 0, \ t \geq 0,$$

where

$$\eta = \frac{ha\sqrt{t}}{k}. \tag{52}$$

Taking into account (47) and (49) we get

$$\phi(x, t) \geq \phi(0, t) \geq z(0, t) = -D + (\beta_0 + D)\exp(\eta^2)\operatorname{erfc}(\eta) \geq 0, \qquad t \leq t_1^*, \tag{53}$$

because the function $F(x)$ verifies the conditions

$$F(0) = 1, \qquad F(+\infty) = +\infty, \qquad F' > 0 \quad \text{in } R^+. \tag{54}$$

REMARK 7. We can see that t_1^* does not depend on the length of the slab $x_0 > 0$.
∎

From now on we consider the particular case of constant temperature $b(t) = b > 0$, $t > 0$ at $x = x_0$ for problem $(1')$. The corresponding steady-state solution is given by

$$\phi_\infty(x) = b - \frac{h(D + b)}{k + hx_0}(x_0 - x), \qquad 0 \leq x \leq x_0, \tag{55}$$

and a necessary and sufficient condition, in order to have a two-phase steady-state Stefan problem is given by

$$h > \frac{kb}{Dx_0}. \tag{56}$$

We consider the following problem related to problem $(1')$: Find the relation between the heat transfer coefficient h and a time t_2 such that another phase appears for $t \geq t_2$, and then we can reformulate problem $(1')$ in a two-phase Stefan problem for $t \geq t_2$. We obtain the following result.

THEOREM 7. Suppose the initial temperature verifies the conditions $b \geq \theta_0 \geq 0$ in $[0, x_0]$ and $\theta_0(x_0) = b$. If the time $t_2 > 0$ and the constant heat transfer coefficient $h > 0$ verify the inequality

$$h > g(t_2), \tag{57}$$

then another phase (the solid phase) appears for $t \geq t_2$, where the function $g = g(t)$ is defined implicitly by the equation

$$\psi(t, g(t)) = 0, \qquad t > 0, \tag{58}$$

with

$$\psi(t, h) = -D + \frac{k(D + b)}{k + hx_0} + \frac{2k(D + b)}{hx_0} \gamma(t), \qquad t > 0, \ h > 0, \tag{59}$$

$$\gamma(t) = \sum_{n=1}^{\infty} \exp\left[-\frac{(2n - 1)^2 \pi^2 a^2 t}{x_0^2}\right], \qquad t > 0. \tag{60}$$

Proof. The solution of problem $(1')$ is given by

$$\phi(x, t) = \phi_\infty(x) + \sum_{n=1}^{\infty} B_n \exp(-\mu_n^2 a^2 t) \left[\text{Sin}(\mu_n x) + \frac{k\mu_n}{h} \text{Cos}(\mu_n x)\right],$$
$$0 \leq x \leq x_0, \ t > 0, \tag{61}$$

where

$$B_n = \frac{2}{x_0} \int_0^{x_0} [\phi_0(x) - \phi_\infty(x)] \left[\text{Sin}(\mu_n x) + \frac{k\mu_n}{h} \text{Cos}(\mu_n x)\right] dx, \tag{62}$$

and $\mu_n = \omega_n/x_0$, where ω_n is the nth root of the eigenvalue equation

$$tg(\omega) = -\frac{k}{hx_0}\omega, \qquad \omega > 0. \tag{63}$$

Moreover, we get that

$$(2n - 1)\tfrac{\pi}{2} < \omega_n < n\pi, \qquad n \in \mathbb{N}. \tag{64}$$

After some manipulation, we deduce that the temperature at $x = 0$ is bounded by

$$\phi(0, t) \leq \psi(t, h), \qquad t > 0, \tag{65}$$

where the function ψ has been defined before.

We notice that the function $g = g(t)$ is well defined since the functions $\gamma = \gamma(t)$ and $\psi = \psi(t, h)$ satisfy the properties

$$\gamma(0^+) = +\infty, \qquad \gamma(+\infty) = 0, \qquad \gamma'(t) < 0, \ \forall t > 0, \tag{66}$$

$$\begin{array}{lll}
\text{(a)} & \psi(t, 0^+) = +\infty, & \psi(t, +\infty) = -D < 0, \qquad t > 0; \\
\text{(b)} & \dfrac{\partial \psi}{\partial h}(t, h) < 0, & \dfrac{\partial \psi}{\partial t}(t, h) < 0, \qquad t > 0, \ h > 0.
\end{array} \tag{67}$$

Therefore, the function $g = g(t)$ satisfies the conditions

$$g(0^+) = +\infty, \qquad g(+\infty) = \frac{kb}{Dx_0}, \qquad g'(t) < 0, \ \forall t > 0. \tag{68}$$

By using the inequality (65) we get the thesis.

COROLLARY 8. We consider in the plane t, h the following set:

$$R_2 = \{(t, h) \mid h > g(t), \ t > 0\};$$ (69)

then we have a two-phase problem for all $(t, h) \in R_2$.

REMARK 8. If the initial temperature is given by $\phi_0(x) = b > 0$ in $[0, x_0]$, then we have a heat conduction problem for the initial phase for all $(t, h) \in R_1$, where

$$R_1 = \left\{ (t, h) \mid 0 < h < \text{Max}\left(\frac{kb}{Dx_0}, F^{-1}\left(1 + \frac{b}{D}\right)\sqrt{\frac{k\rho c}{t}} \right), \ t > 0 \right\}.$$ (70)

REFERENCES

[1] D. G. Aronson, *The porous medium equation*, Nonlinear Diffusion Problems (A. Fasano and M. Primicerio, eds.), Lecture Notes in Math., Vol. 1224, Springer-Verlag, Berlin, 1986, pp. 1–46

[2] A. B. Bancora and D. A. Tarzia, *On the Neumann solution for the two-phase Stefan problem including the density jump at the free boundary*, Lat. Am. J. Heat Mass Transfer **9**, 215–222 (1985)

[3] J. R. Cannon, *The One-dimensional Heat Equation*, Addison-Wesley, Menlo Park, California, 1967

[4] H. S. Carslaw and J. C. Jaeger, *Conduction of Heat in Solids*, Clarendon Press, Oxford, 1959

[5] A. Fasano and M. Primicerio, *General free boundary problems for the heat equation.* I, J. Math. Anal. Appl. **57**, 694–723 (1977)

[6] A. Friedman, *Partial Differential Equations of Parabolic Type*, Prentice-Hall, Englewood Cliffs, 1964

[7] M. H. Protter and H. F. Weinberger, *Maximum Principles in Differential Equations*, Prentice-Hall, Englewood Cliffs, 1967

[8] B. Sherman, *General one-phase Stefan problems and free boundary problems for the heat equation with Cauchy data prescribed on the free boundary*, SIAM J. Appl. Math. **20**, 555–570 (1971)

[9] A. D. Solomon, V. Alexiades, and D. G. Wilson, *The Stefan problem with a convective boundary condition*, Quart. Appl. Math. **40**, 203–217 (1982)

[10] A. D. Solomon, D. G. Wilson, and V. Alexiades, *Explicit solutions to change problems*, Quart. Appl. Math. **41**, 237–243 (1983)

[11] D. A. Tarzia, *Sobre el caso estacionario del problema de Stefan a dos fases*, Math. Notae **28**, 73–89 (1980)

[12] D. A. Tarzia, *An inequality for the coefficient σ of the free boundary $s(t) = 2\sigma\sqrt{t}$ of the Neumann solution for the two-phase Stefan problem*, Quart. Appl. Math. **39**, 491–497 (1981–82)

[13] D. A. Tarzia, *An inequality for the constant heat flux to obtain a steady-state two-phase Stefan problem*, Engineering Analysis **5**, 177–181 (1988). See also *On heat flux in materials on free boundary problems: Theory and applications*, Irsee/Bavaria, 11–20 June 1987, Res. Notes in Math., No. 186, Pitman, London, 1990, pp. 703–709.

[14] D. A. Tarzia, *The two-phase Stefan problem and some related conduction problems*, Reuniões em Matemática Aplicada e Computacão Científica, Vol. 5, SBMAC-Soc. Brasileira Mat. Apl. Comput., Gramado, 1987