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A TWO-PHASE STEFAN PROBLEM IN A SEMI-INFINITE DOMAIN WITH A CONVECTIVE BOUNDARY CONDITION AT THE FIXED FACE

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Abstract We studied a two-phase Stefan problem in a semi-infinite material, when a convective condition is assigned on the fixed face $x = 0$.

We demonstrate the monotone dependence of the solution with respect to the data and with respect to the thermal transfer coefficient H. We also studied the asymptotic behavior of the solution when $H \to \infty$.

1. INTRODUCTION

In this paper we consider the two-phase Stefan problem for a semi-infinite material with a convective boundary condition at the fixed boundary, $x = 0$.

Specifically the mathematical problem consists of determining two functions, $u^H(x,t)$ and $v^H(x,t)$, a function $x = s^H(t)$, called the free-boundary, and the time T such that (u^H, v^H, s^H, T) satisfy the following equations, boundary and initial conditions. For each positive H we consider:

Problem $P_H : (H > 0)$

(1.1) $$\rho c_2 u_t^H - k_2 u_{xx}^H = 0, \quad D_2 = \{(x,t) : 0 < x < s^H(t), 0 < t < T\}$$

(1.2) $$\rho c_1 v_t^H - k_1 v_{xx}^H = 0, \quad D_1 = \{(x,t) : x > s^H(t), 0 < t < T\}$$

(1.3) $$u^H(x,0) = \varphi(x) \geq 0, \quad 0 < x < s^H(0) = b^H$$

(1.4) $$v^H(x,0) = \psi(x) \leq 0, \quad x > b. \ v^H(\infty,t) = \psi(\infty), t > 0$$

(1.5) $$k_2 u_x^H(0,t) = H(u^H(0,t) - f(t)), \quad 0 < t < T$$

(1.6) $$v^H(s^H(t),t) = u^H(s^H(t),t) = 0, \quad 0 < t < T$$

(1.7) $$k_1 v_x^H(s^H(t),t) - k_2 u_x^H(s^H(t),t) = \rho l \dot{s}^H(t), \quad 0 < t < T.$$

where the phase-change temperature is zero and H is the thermal transfer coefficient $(H > 0)$.

Very general results about the existence of classical solutions to the two-phase Stefan problem have been obtained in [4],[5],[7],[9]. The asymptotic behavior for the one-phase Stefan problem with temperature and flux conditions on the fixed boundary $x = 0$ are considered in [2]and [3] respectively.

In [12] the behavior of the solution with respect to the heat transfer coefficient H and the asymptotic behavior of the free boundary are studied for the constant case $f(t) = T_L > 0$. In [13] we generalized this result for the case when $f(t)$ is not constant. There it was considered the one-phase Stefan problem with a convective boundary condition at the fixed face, given by the temperature of the external fluid ($f(t)$) depending on time. We study the aymptotic behavior of the corresponding free boundary $s_H(t)$ when the time goes to infinity. In [1] and [10] a two phase Stefan problem with very general boundary condition at $x = 0$ is studied. In [8] is considered a one-phase Stefan problem for the supercooled liquid with a zero flux at the fixed face. In [6] this problem is studied for a general flux $g(t)$. In [11] and [15] is analyzed the two-phase Stefan problem for the supercooled liquid with flux and temperature boundary conditions at the fixed faces $x = 0$ and $x = 1$.

In this paper we show monotone dependence of the solution with respect to the data and some asymptotic properties of the free boundary.

A complete version of this paper with all the proofs and the behavior of the free boundary when $t \to \infty$ and the corresponding study when the liquid phase is overcooled and the solid phase is overheated will appear in [14].

In order to have existence and uniqueness of the solution we require the following assumptions upon the initial and boundary data:

i) Let $\varphi = \varphi(x)$ and $\psi = \psi(x)$ be positive and negative respectively piecewise bounded continuous functions .

ii) Let $f = f(t)$ be a positive bounded piecewise continuous function.

iii) Compatibility conditions: $f(0) > \varphi(x)$ in $(0, b)$, $k_2\varphi'(0) = H(\varphi(0) - f(0))$, $\varphi(b) = \psi(b)$.

Now we will show some preliminary results, the reformulation of the free boundary problem and the monotone dependence of the solution with respect to the data (ϕ, ψ, f, b) and with respect to the thermal transfer coefficient H.

Lemma 1. *Under the above hypothese on the data, the temperatures $u^H(x,t)$ and $v^H(x,t)$, satisfy the inequalities: $v^H \leq 0$ and $u^H \geq 0$.*

Moreover $u^H(x,t) \leq f(t)$ in D_2 and $v^H(x,t) \geq \psi(x)$ in D_1 when $\dot{f}(t) \geq 0$, $\psi'(x) \leq 0, \psi''(x) \geq 0$.

Lemma 2. *If (v^H, u^H, s^H, T) is a solution of Problem P_H, and $\psi, x\psi \in L^1(b,\infty)$ then, setting $s^H = s$, we have the following equality:*

(1.8)

$$\rho l s(t)\left(1 + \frac{H}{k_2}\frac{s(t)}{2}\right) = \rho l b\left(1 + \frac{H}{k_2}\frac{b}{2}\right) + \int_0^b \frac{(k_2 + xH)}{\alpha_2}\varphi(x)\,dx$$
$$+ \int_b^\infty \left(k_1 + k_1 x\frac{H}{k_2}\right)\frac{\psi(x)}{\alpha_1}\,dx + \int_0^t Hf(\tau)\,d\tau$$
$$- \int_0^{s(t)} \frac{(k_2 + xH)}{\alpha_2}u^H(x,t)\,dx - \int_{s(t)}^\infty \left(k_1 + x\frac{H}{k_2}k_1\right)\frac{v^H(x,t)}{\alpha_1}\,dx$$

where $\alpha_i = \frac{k_i}{\rho c}, i = 1, 2$

Lemma 3. *If $(v_i^H, u_i^H, s_i^H, T), i = 1, 2$, are solutions of the Stefan problem P_H corresponding to the data f_i, φ_i, ψ_i and b_i, and if $f_1 \leq f_2$, $\varphi_1 \leq \varphi_2$, $\psi_1 \leq \psi_2$ and $b_1 \leq b_2$, then $s_1^H(t) \leq s_2^H(t)$ and $u_1^H \leq u_2^H$, $v_1^H \leq v_2^H$ in their corresponding common domains.*

Theorem 1. *If $(u^{H_i}, v^{H_i}, s^{H_i}, T), i = 1, 2$, are solutions of the Stefan problem (1.1)-(1.7) corresponding to the data H_1 and H_2 with $H_1 \leq H_2$, and $\dot{f} > 0$ then*
$v^{H_1} \leq v^{H_2}$, $u^{H_1} \leq u^{H_2}$, $s^{H_1}(t) \leq s^{H_2}(t)$ in the common domains where they are defined.

2. The case where the thermal transfer coefficient approaches to infinity

We consider the following two-phase Stefan problem for a semi-infinite material with a temperature boundary condition on the fixed face $x = 0$. We call this problem:

Problem P_∞

(2.1) $$\rho c_2 u_t - k_2 u_{xx} = 0, \qquad 0 < x < s(t)$$
(2.2) $$\rho c_1 v_t - k_1 v_{xx} = 0, \qquad s(t) < x < \infty$$
(2.3) $$u(x, 0) = \varphi(x) \geq 0, \qquad 0 \leq x \leq b$$
(2.4) $$v(x, 0) = \psi(x) \leq 0, \qquad b < x < \infty$$
(2.5) $$v(\infty, t) = \psi(\infty), t > 0$$
(2.6) $$u(0, t) = f(t) \geq 0, \qquad 0 < t < T$$
(2.7) $$u(s(t), t) = v(s(t), t) = 0, \qquad 0 < t < T$$
(2.8) $$k_1 v_x(s(t), t) - k_2 u_x(s(t), t) = \rho l \dot{s}(t), \quad 0 < t < T.$$

Theorem 2. *The solution (u, v, s, T) of Problem P_∞ and the solution (u^H, v^H, s^H, T) of Problem P_H satisfy the following inequalities,*

i) $s^H(t) < s(t)$, $t > 0$,
ii) $u^H < u$, $0 < x < s^H(t)$, $0 < t < T$,
iii) $v^H < v$, $s(t) < x < \infty$, $0 < t < T$.

if $\dot{f} \geq 0$, $b_H < b$, and $H > 0$ are provided.

Proof. The proof is obtained by using the maximun principle to the functions $W_2 = u - u_H$ and $W_1 = v - v_H$ in the corresponding domains.

□

3. Asymptotic behavior of the free boundary

We will study the asymptotic behavior of the free boundary $s^H(t)$ when $t \to \infty$ or $H \to \infty$. In [1] is consider the global existence in a general Stefan-like problem.

Theorem 3. *If (u^H, v^H, s^H, T) is the solution of Problem P_H, and $\psi, x\psi \in L^1(b, \infty)$ then we have the following properties:*

i) *If $\int_0^\infty f(\tau) d\tau < \infty$ and $\lim_{t \to \infty} f(t) = 0$, then $\lim_{t \to \infty} s^H(t) = s_H^\infty$,*
where s_H^∞ satisfies the equation of second order given by

$$\rho l x\left(1+\frac{H}{2k_2}x\right)=\rho l b_H\left(1+\frac{b_H H}{2k_2}\right)+\int_0^{b_H}(k_2+yH)\frac{\phi(y)}{\alpha_2}\,dy$$
$$+\int_{b_H}^{\infty}\left(k_1+\frac{yHk_1}{k_2}\right)\frac{\psi(y)}{\alpha_1}\,dy+\int_0^{\infty}Hf(\tau)\,d\tau$$

ii) *If* $\int_0^\infty f(\tau)d\tau=\infty$,. *then* $\lim_{t\to\infty}\frac{s^H(t)}{\sigma(t)}=1$, *where* $\sigma(t)$ *is the free-boundary of the following problem:*

For each $t_0\geq 0$, *let* (σ, V_1, V_2) *be the solution to the following problem* P_{t_0}

$$\begin{aligned}
\rho c_2(V_2)_{xx}&=k_2(V_2)_t, && 0<x<\sigma(t), t\geq t_0\\
\rho c_1(V_1)_{xx}&=k_1(V_1)_t, && x>\sigma(t), t\geq t_0\\
V_1(x,0)&=0, && 0\leq x\leq s^H(t_0)\\
V_1(x,0)&=v^H(x,t_0), && x\geq s^H(t_0)\\
V_1(\infty,t)&=v^H(\infty,t), && t\geq t_0\\
k_2(V_2)_x(0,t)&=H(V_2(0,t)-f(t)) && t\geq t_0\\
V_2(\sigma(t),t)&=V_1(\sigma(t),t)=0, && t\geq t_0\\
\sigma(t_0)&=0\\
k_1(V_1)_x(\sigma(t),t)-k_2(V_2)_x(\sigma(t),t)&=\rho l\dot{\sigma}(t), && t>t_0.
\end{aligned}$$

Proof.

i) First we obtain the following bounds for the functions u^H and v^H:
(a) $u^H(x,t)\leq U(x,t)$, in $0<x<s^H(t)$, $t>0$.
(b) $V(x,t)\leq v^H(x,t)$ in $s^H(t)<x<\infty$, $t>0$.
where U and V are defined by the following problems:

$$\begin{aligned}
\rho c_2U_t-k_2U_{xx}&=0. && 0<x<\infty, t>0,\\
k_2U_x(0,t)&=H(U(0,t)-f(t)), && t>0,\\
U(x,0)&=\begin{cases}\varphi(x), & 0\leq x\leq b,\\ 0, & b\leq x.\end{cases}
\end{aligned}$$

and

$$\begin{aligned}
\rho c_1V_t-k_1V_{xx}&=0, && 0<x<\infty, t>0,\\
V(0,t)&=0, && t>0,\\
V(\infty,t)&=\psi(\infty), t>0,\\
V(x,0)&=\begin{cases}\psi(b), & 0\leq x\leq b,\\ \psi(x), & b\leq x.\end{cases}
\end{aligned}$$

Using the integral representation (1.8), the bounds for u^H and v^H and taking limit when $t \to \infty$ we obtain the thesis.

ii) The sketch of proof is the following:
Using the maximum principle we can prove that $\sigma(t) < s^H(t)$, $t > t_0$ and $V_2(x,t) < u^H(x,t)$, $V_1(x,t) < v^H(x,t)$ in the corresponding domains, for $t > t_0$. Now, we use an integral representation like in Lemma 2 with the adecuate initial condition at $t = t_0$ and we get

$$\rho l s^H(t)\left(1+\frac{H}{2k_2}s^H(t)\right) = \rho l s^H(t_0)\left(1+\frac{s^H(t_0)H}{2k_2}\right) + \int_0^{s^H(t_0)}(k_2+xH)u^H(x,t_0)\,dx$$
$$+\int_{s^H(t_0)}^{\infty}\left(k_1+x\frac{k_1}{k_2}H\right)v^H(x,t_0)\,dx + \int_{t_0}^{t} Hf(\tau)\,d\tau$$
$$-\int_0^{s^H(t)}\left(k_1+x\frac{k_1}{k_2}\right)u^H(x,t)\,dx - \int_{s^H(t)}^{\infty}(k_2+xH)v^H(x,t)\,dx$$
$$\leq C(t_0) + \rho l\sigma(t)\left(1+\frac{H}{2k_2}\sigma(t)\right),$$

where

$$C(t_0) = \rho l s^H(t_0)\left(1+\frac{s^H(t_0)H}{2k_2}\right) + \int_0^{s^H(t_0)}(k_2+xH)u^H(x,t_0)\,dx.$$

Then we have

$$\sigma^2(t) \leq s_H^2(t) \leq \sigma^2(t) + C(t_0)\frac{2k_2}{\rho l H}$$

Taking the limit when $t \to \infty$ in the above inequalities and using the fact that $\lim_{t\to\infty}\sigma(t) = \infty$ (since $\int_0^\infty f(\tau)\,d\tau = \infty$), then $\lim_{t\to\infty}\dfrac{s^H(t)}{\sigma(t)} = 1.$ □

We state two preliminary lemmas in order to prove the convergency when $H \to \infty$

Lemma 4. *If (u^H, v^H, s^H, T) is a solution of Problem P_H, with $\dot{f} > 0$ and $H > 0$ then*

$$\int_0^t (u^H(0,\tau) - f(\tau))\,d\tau \leq \frac{[s(t)(\rho l + \frac{k_2}{\alpha_2}\|f\|_{(0,t)}) + C]}{H},$$

where$C = -\int_b^\infty \frac{k_1}{\alpha_1}\psi(x)\,dx > 0$ and $s(t)$ is the free boundary of the problem P_∞.
□

Lemma 5. *If (u^H, v^H, s^H, T) is a solution of Problem P_H and the data satisfy $\dot{f} \geq 0$ then (u^H, v^H, s^H, T) and (u, v, s, T) satisfy the following inequality*

$$0 \leq \frac{\rho l(s^2(t) - s_H^2(t))}{2} + \int_0^{s^H(t)} \frac{xk_2(u(x,t) - u^H(x,t))}{\alpha_2}\,dx +$$
$$\int_{s^H(t)}^{\infty} \frac{xk_1(v(x,t) - v^H(x,t))}{\alpha_1}\,dx \leq \int_0^t k_2(f(\tau) - u(0,\tau)\,d\tau$$

Theorem 4. *(Convergency when $H \to \infty$) . If (u^H, v^H, s^H, T) is a solution of the problem P_H, (u, v, s, T) is a solution of Problem P_∞ and the data satisfies $\dot{f} \geq 0$ then*

i) $lim_{H\to\infty} s^H(t) = s(t)$
ii) $\lim_{H\to\infty} u^H(x,t) = u(x,t)$ *and* $\lim_{H\to\infty} v^H(x,t) = v(x,t)$ *for all compact sets included in their corresponding domains.*

Proof.
Using the Lemmas 5 and 6, Theorem 2 we have the following inequality:

$$0 \le \frac{\rho l(s^2(t) - s_H^2(t))}{2} \le k_2 \frac{[s(t)(\rho l + \frac{k_2}{\alpha_2}\|f\|_{(0,t))}) + C]}{H},$$

for all H.
Then for each $t > 0$, $lim_{H\to\infty} s^H(t) = s(t)$. We can do the same with the difference $u - u^H$ and $v - v^H$.
□

4. Disappearance of a phase

In this section we will discuss the relation between the disappearance of a phase and the total energy supplied to the media.
We will use the following definitions:

$$\Phi(x) = \begin{cases} (k_2 + Hx)\dfrac{\varphi(x)}{\alpha_2}, & 0 < x < b_H, \\ \left(k_1 + \dfrac{k_1}{k_2} Hx\right) \dfrac{\psi(x)}{\alpha_1}, & b_H < x < \infty, \end{cases}$$

$$T_\delta = \inf\{t^*, t^* > 0, s^H(t^*) = \delta \text{ or } s^H(t^*) = \frac{1}{b_H - \delta}\}$$

$$T_0 = \sup_{0<\delta<b_H} \{T_\delta\}$$

Theorem 5. *If* $0 < \rho l b^H \left(1 + \dfrac{b^H H}{2k_2}\right) + \int_0^\infty \Phi(x)\,dx + \int_0^t Hf(\tau)\,d\tau = Q(t) < \infty$ *for all* $t > 0$, *then* $T_0 = \infty$, *which means that neither phase disappears in a finite time period.*

Proof. Suppose $T_0 < \infty$, then there exists a sequence $\{\delta_i\}$ with $\lim_{\delta_i\to 0} T_{\delta_i} = T_0$, such that $s(T_{\delta_i}) \to 0$ or $s(T_{\delta_i}) \to \infty$ as $\delta_i \to 0$ or $\delta_i \to b^H$.
We consider the case $s(T_{\delta_i}) \to 0$ as $\delta_i \to 0$ then, using the integral representation of Lemma 2, we obtain

$$\begin{aligned} \rho l s(T_{\delta_i})\left(1 + \frac{H}{2k_2} s(T_{\delta_i})\right) &= \rho l b^H \left(1 + \frac{b_H}{2k_2}\right) + \int_0^\infty \Phi(x)\,dx + \int_0^{T_{\delta_i}} Hf(\tau)\,d\tau \\ &- \int_0^{s(T_{\delta_i})} (k_2 + xH)\frac{u^H(x,t)}{\alpha_2}\,dx - \int_{s(T_{\delta_i})}^\infty \left(k_1 + \frac{xk_1H}{k_2}\right)\frac{v^H(x,t)}{\alpha_1}\,dx \\ &\ge Q(T_{\delta_i}) - \int_0^{s(T_{\delta_i})} (k_2 + xH)\frac{u^H(x,t)}{\alpha_2}\,dx, \end{aligned}$$

since $v^H < 0$. Therefore, as $s(T_{\delta_i}) - 0$, as $\delta_i \to 0$, then $0 \ge Q(T_0)$, which contradicts the assumption of the theorem.

The case $s(T_{\delta_i}) \to \infty$, as $\delta_i \to b$ is similar to the previous case.
□

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