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The one-phase supercooled Stefan problem

Abstract: We consider the supercooled one-phase Stefan problem with convective boundary condition at the fixed face. We analyse the relation between the heat transfer coefficient and the possibility of continuing the solution for arbitrarily large time intervals.

1. Introduction.

The classical one-dimensional Stefan problem has been studied since 1831(see [11]); it models conductive heat transfer on either side of a phase boundary in pure material on the assumptions (i) that the temperature at the phase boundary is constant, say zero, (ii) that there is a release of latent heat at the boundary on solidification, and an uptake on melting, and (iii) that the material on the solid and liquid sides of the phase boundary has negative and positive temperature, respectively.

With these assumptions the problem has a weak formulation and a global solution is known to exist ([4]). If the data are such that just one phase boundary exists the problem has also been shown to be well-posed in the classical sense [5,7].

But if the initial and/or boundary data violate the sign requirement (iii), i.e., if the liquid is supercooled or the solid is superheated, a solution still may exist, at least formally, but the result is generally only local in time and finite time blow-up can easily occur ([8]).

In this paper we consider this kind of problem in the following setting:

Problem I:

Find $\theta(y,\tau)$ the temperature and $r(\tau)$ the free-boundary such that:

$r(\tau)$ is Lipschitz continuous for $\tau \geq 0$;

$\dot{r}(\tau)$ is continuous for $\tau > 0$;

$\theta(y,\tau)$ is continuous for $\tau > 0$ and $0 \leq y \leq r(\tau)$;

$\theta_\tau(y,\tau)$, $\theta_{yy}(y,\tau)$ are continuous for $\tau > 0$ and $0 < y < r(\tau)$;

$\theta_y(y,\tau)$ is continuous for $\tau > 0$, $0 \leq y \leq r(\tau)$;

$r(\tau)$ and $\theta(y,\tau)$ obey the conditions:

$$\theta_\tau = \alpha\theta_{yy} \quad 0 < y < r(\tau), \quad 0 < \tau < \tau_0$$
$$\theta(r(\tau),\tau) = 0 \quad 0 < \tau < \tau_0$$
$$k\theta_y(r(\tau),\tau) = -\rho\lambda\dot{r}(\tau), \quad 0 < \tau < \tau_0$$
$$k\theta_y(0,\tau) = h(\theta(0,\tau) - g(\tau)) \quad 0 < \tau < \tau_0$$
$$\theta(y,0) = \theta_0(y) \quad 0 \leq y \leq b$$
$$r(0) = b$$

The parameters are
$\alpha = \frac{k}{\rho c}$ material thermal diffusivity (m^2/s)
$k =$ material thermal conductivity (KJs^0C/m)
$\rho =$ material density (Kg/m^3)
$\lambda =$ latent heat of melting (KJ/Kg)
$h =$ fluid to material surface heat transfer coefficient (KJs^0C/m^2)
$g(\tau) =$ ambient fluid temperature (^0C)
$c =$ specific heat (KJ^0C/Kg),

The melting front at time τ is $r(\tau)$ while $\theta(y,\tau)$ is the temperature at position y and time τ.

It is known that a solution to Problem I exists [1], for suitable τ_0 'sufficiently small'. This problem is often referred to as a mathematical scheme for the freezing of a supercooled liquid (although this simple scheme for such a non-equilibrium phenomenon is far from being satisfactory)[10].

The freezing of a supercooled liquid is due to convective heat transfer from a fluid with ambient temperature $g(\tau)$ flowing across the face $x = 0$.

This problem has been studied in [2],[3],[6] and [12].

The adimensional problem is obtained by the following transforms

$$x = \frac{y}{b} \qquad t = \frac{k\tau}{\rho c b^2}$$
$$z(x,t) = \frac{c}{\lambda}\theta(y,\tau) \qquad s(t) = \frac{r(\tau)}{b}$$

Then the variables (T, s, z) satisfy the problem:

Problem II:
(1.1) $z_{xx} = z_t$, in D_T;
(1.2) $s(0) = 1$;
(1.3) $z(s(t),t) = 0$, $0 < t < T$;
(1.4) $z_x(s(t),t) = -\dot{s}(t)$, $0 < t < T$;
(1.5) $z(x,0) = \varphi(x)$, $0 < x < 1$;

(1.6) $z_x(0,t) = \beta z(0,t) - G(t)]$, $0 < t < T$.
where $\beta = \frac{h}{kb}$ is an adimensional parameter, and

$$D_T = \{(x,t)|0 < x < s(t), 0 < t < T\}$$

$$G(t) = \frac{c}{\lambda} g\left(\frac{b^2 \rho c t}{k}\right).$$

2. The one-phase supercooled Stefan problem

In this section we consider the following hypotheses

$$\varphi(x) \leq 0,\ 0 < x < 1 \quad \text{and} \quad G(t) \leq 0,\ t > 0$$

and the compatibility condition

$$\varphi'(0) = \beta[\varphi(0) - G(0)].$$

The first simple properties of the solution of (1.1)-(1.6) are summarized in the following proposition :

Proposition 2.1. *If (T, s, z) is a solution of Problem II, then*
i) $z \leq 0$ in D_T.
ii) $\dot{s}(t) < 0$, $t > 0$.
iii) $\dot{G}(t) \leq 0$, $\varphi(x) \geq G(0) = \max_{t>0} G(t)$, then $z \geq G(t)$ in D_T.
iv) $\varphi' \geq 0$, then $z_x \geq 0$ in D_T.
v) $\dot{G} \geq 0$, $\varphi'' > 0$ then $z_t > 0$ in D_t.
vi)

$$s(t)\left[1 + \frac{\beta}{2}s(t)\right] = 1 + \frac{\beta}{2} + \int_0^1 (1 + \beta x)\varphi(x)\,dx + \int_0^t \beta G(\tau)\,d\tau - \int_0^{s(t)} (1 + \beta x) z(x,t)\,dx \tag{2.1}$$

Proof.
The proof is obtained by using the maximum principle and Green's identity.

Remark 1: In the following sections we denote

$$Q(t) = 1 + \frac{\beta}{2} + \int_0^1 (1 + \beta x)\varphi(x)\,dx + \int_0^t \beta G(\tau)\,d\tau \tag{2.2}$$

If $\varphi(1) = 0$, $\varphi(x)$ is Hölder continuous for $x = 1$ and $G(t)$ is a piecewise continuous on every interval $(0,t)$, $t > 0$, this problem possess one solution for suitable T

"sufficiently small" (see [1], [5], [6] where uniqueness and continuous dependence are also discussed).

Moreover, if a solution exists, then three cases can occur (see [6], Theorem 8 and [2]).

(A) The problem has a solution with arbitrarily large T.

(B) There exists a constant $T_B > 0$ such that $\lim_{t\to T_B} s(t) = 0$.

(C) There exists a constant $T_C > 0$ such that $\inf_{t\in(0,T_C)} s(t) > 0$ and $\lim_{t\to T_C} \dot{s}(t) = -\infty$.

We shall investigate the occurrence of these cases in conection with the behavior of the initial data φ, the adimensional temperature G of the external fluid and the adimensional coefficient β, (see [12]).

Our next aim will be to look for some conditions on φ, G and β giving an a priori caracterization of cases (A), (B) and (C).

Proposition 2.2. *If $\dot{G} \leq 0$, $\varphi(x) \geq G(0)$ and the solution (T, s, z) of Problem II is case (B), then $Q(T_B) = 0$.*

Proof. Setting $t \longrightarrow T_B$ in (2.3) and using the boundedness of z obtained in Proposition 2.1 we conclude the result. □

Proposition 2.3. *If (T, s, z) is a solution of problem P II, and the initial and boundary data satisfy the following hypotheses:*

i) $\varphi(x) \geq M(x-1)$, $\quad 0 \leq x \leq 1$, $\quad 0 < M < 1$;

ii) $G(t) \geq -M$

and it exists a time T_B such that $Q(T_B) = 0$ then the solution (T_B, s, z) is case (B).

Proof. First we prove that $z(x,t) \geq M(x-1)$. This is easily followed from the maximum principle applied to $w = z - M(x-1)$.

We replace this inequality in (2.1) for $t = T_B$, then $s(T_B)$ satisfies the following inequality

$$s(T_B)\left[(1-M) + s(T_B)\left[\frac{\beta(1-M)+M}{2}\right] + \beta s^2(T_B)\frac{M}{3}\right] \leq 0.$$

The quadratic form in brackets has coefficients $1 - M > 0$ and $\frac{\beta(1-M)+M}{2} > 0$, then $s(T_B) = 0$. □

Proposition 2.4. *Suppose that, $t_0 < T$ and $\lim_{t\to t_0} s(t) > 0$. φ satisfies the hypotheses iv) of Proposition 2.1. Moreover $Q(t) > 0$ for all $t \leq t_0$. Then if we define a function*

$$\eta(t) = \begin{cases} \max\{x \in [0, s(t)] | z(x,t) \leq -1\} \\ 0 \quad \text{if } z(x,t) > -1, \ x \in [0, s(t)] \end{cases}$$

then it follows

$$\lim_{t\to t_0} \eta(t) < \lim_{t\to t_0} s(t)$$

Proof. The proof is similar that of Proposition 2.3 in [2].

Proposition 2.5. *Let (T, s, z) be a solution of Problem II such that $S_T = \inf_{t\in(0,T)} s(t) > 0$. If there exist two constants $d \in (0, S_T)$, $z_0 \in (0,1)$ such that $Md \geq z_0$, and*

$$z(s(t) - d, t) \geq -z_0, \quad 0 \leq t \leq T,$$

then

$$\dot{s}(t) \geq \frac{\ln(1 - z_0)}{d}.$$

Proof. It is the same that of the Lemma 2.4 in [2] (See also [6]). □

Proposition 2.6. *Let be (T, s, z) a solution of Problem II and φ satisfies the hypotheses of Proposition 2.1 iv), then if the solution is case (C), then $Q(T_C) \leq 0$.*

Proof. Suppose $Q(T_C) > 0$, then from the Proposition 2.5 the isotherm $z = -1$ is separated from the free-boundary. Using the Proposition 2.5 $\dot{s}$ has a lower bound, which contradicts the case (C). □

Corollary 2.7. *If (T, s, z) is a solution of Problem II and φ, G satisfy the following hypotheses:*

i) $\varphi(x) \geq M(x - 1), \quad 0 \leq x \leq 1;$
ii) $G(t) \geq -M, \quad 0 < M < 1.$
iii) $\dot{\varphi}(x) \geq 0, \quad 0 \leq x \leq 1.$
And the solution is case (C), then $Q(T_C) < 0$.

Proof. It follows from Propositions 2.3 and 2.6. □

Proposition 2.8. *Let (T, s, z) be a solution of Problem II, φ and G satisfy the following hypotheses:*

i) $\varphi(x) \geq M(x - 1), \quad M > 0, \quad 0 \leq x \leq 1;$
ii) $G \in L^1(0, \infty).$
If the solution is case (A), then $Q(t) \geq 0, t > 0$. Moreover, if $G(t) \geq -M, (M > 0), \forall t > 0$, then case (A) implies that $Q(t) > 0, \forall t > 0$.

Proof. This proof can be seen in [12].

3. Asymptotic behavior of the solution

Proposition 3.1. *Let (T, s, z) be a solution of Problem II of case (A) under the hypotheses of Proposition 2.9 and (iii) of Proposition 2.1. Moreover, we assume that the limit of $G(t)$ when $t \to \infty$ exists. If we denote $Q_\infty = \lim_{t\to\infty} Q(t)$ and $s_\infty = \lim_{t\to\infty} s(t)$, then s_∞ is given by*

$$s_\infty = \frac{-1 + \sqrt{1 + 2\beta Q_\infty}}{\beta} \tag{3.1}$$

Proof. The existence of the limit of $G(t)$ when $t \to \infty$ and $G \in L^1(0,\infty)$ assure that $\lim_{t\to\infty} G(t) = 0$.

We denote z_∞ the limit of z when t tends to infinity. The existence of $\lim_{t\to\infty} z(x,t)$ is due to Proposition 2.1 and [6,Chapter 6]. The function z_∞ satisfies: $z''_\infty = 0$ in $(0, s_\infty)$, $z_\infty(s_\infty) = 0$, $z'_\infty(0) = \beta z_\infty(0)$, then $z_\infty(x) = 0$, $0 < x < s_\infty$.

Taking limit when $t \longrightarrow \infty$ in (2.3), then

$$s_\infty \left[1 + \beta \frac{s_\infty}{2}\right] - Q_\infty = 0$$

That means that $s_\infty \in (0,1)$ is the root of the above equation, that is 3.1.

Moreover, we have $s_\infty < 1$ since

$$s_\infty < 1 \Longleftrightarrow 1 + 2\beta Q_\infty < (1+\beta)^2 \Longleftrightarrow 2Q_\infty - 2 - \beta < 0.$$

By taking limit when $t \longrightarrow \infty$ in (2.3) the last inequality holds always due to the following expresion

$$2Q_\infty - 2 - \beta = 2\int_0^1 (1+\beta x)\varphi(x)\,dx - 2\beta \|G\|_1 < 0$$

where $\|G\|_1 = -\int_0^\infty G(\tau)d\tau$ $\square$

4 The oxygen-comsumption problem

As in [8] we are interested in the dependence on the heat transfer coefficient h or its adimensional coefficient β.If, in Problem II we perform the classical transformation

$$u(x,t) = \int_x^{s(t)} \left\{ \int_\gamma^{s(t)} [1 + z(\alpha,t)]\,d\alpha \right\} d\gamma$$

then we obtain the following oxygen-comsumption problem.

Problem III:

$u_{xx} - u_t = 1, \quad$ in D_t;

$s(0) = 1$;

$u(s(t),t) = u_x(s(t),t) = 0, \quad t > 0$;

$u(x,0) = H(x), \quad 0 \le x \le 1$;

$u_x(0,t) - H'(0) = \beta[u(0,t) - H(0) + \|G\|_{1,t}], \quad t > 0$,

where

$$H(x) = \int_x^1 \int_\gamma^1 (1 + \varphi(\alpha))\,d\alpha d\gamma$$

From now on, in this section, we consider the following hypotheses for φ

$$-1 < \varphi(x) \le 0, \quad 0 \le x \le 1.$$

then

$$H(x) > 0,\ 0 \le x \le 1;\ H'(x) < 0,\ 0 \le x \le 1;\ H''(x) > 0,\ 0 \le x \le 1.$$

We now address the question of how the solution to Problem III depends upon $G(t)$.

Proposition 4.1. *The solution (T, s, u) of Problem III depends monotonically on G. In particular if (T_i, s_i, u_i), $i = 1, 2$ are the solutions for G_1 and G_2 respectively, and if $G_1(t) < G_2(t)$, then $s_1(t) \le s_2(t)$ and $u_1(x,t) \le u_2(x,t)$ whatever they are both defined.*

Proof. This is seen by considering the difference

$$v(x,t) = u_2(x,t) - u_1(x,t)$$

at the points where they are both defined.

Let $t^* = \sup\{t > 0 | u_2(0,t) > u_1(0,t)\}$ and $t^{**} = \sup\{t > 0 | s_2(t) > s_1(t)\}$. Let us suppose that both t^* and t^{**} are finite. By definition v satisfies the following problem

$$v_{xx} = v_t, x \in (0, s_1(t)), t \in (0, t^{**});$$

$$v(x, 0) = 0;$$

$$v(s_1(t), t) = u_2(s_1(t), t) > 0;$$

$$v_x(0,t) = \beta \left[v(0,t) + (||G_2||_{1,t} - ||G_1||_{1,t}) \right].$$

Claim 1 :$t^* \neq t^{**}$.

In order to prove that t^* and t^{**} are different, let us suppose that they are equal, then

a) $s_1(t^*) = s_2(t^*)$

b) $\dot{s}_1(t^*) > \dot{s}_2(t^*)$

c) $v(s_1(t^*), t^*) = u_2(s_1(t^*), t^*) = u_2(s_2(t^*), t^*) = 0$

Morever $u_2(0,t) > u_1(0,t)$ for $t < t^*$, then

$$v(0,t) > 0, \quad t < t^*$$

and

$$v(s_1(t), t) = u_2(s_1(t), t) > 0.$$

Since v has the minimum value zero at $(s_1(t^*), t^*)$,the minimum principle to v in $D^1_{t^*}$, we get $v_x(s_1(t^*), t^*) < 0$ which is a contradiction by (a) to

$$v_x(s_1(t^*), t^*) = u_{2x}(s_1(t^*), t^*) = u_{2x}(s_2(t^*), t^*) = 0$$

Then $t^* \neq t^{**}$.

Claim 2 : $t^* < t^{**}$ is impossible:

On $[0, t^*]$, $s_1(t) < s_2(t)$, whence $v(s_1(t), t) > 0$. By definition $v(0, t) > 0$ for $t < t^*$ and $v(0, t^*) = 0$. That implies $v(0, t^*)$ is a minimum value up to time t^* whence $v_x(0, t^*) > 0$, which contradicts

$$v_x(0, t^*) = \beta\,[v(0, t^*) + (||G_2||_{1,t^*} - ||G_1||_{1,t^*})] = \beta\,[||G_2||_{1,t^*} - ||G_1||_{1,t^*}] < 0$$

Claim 3 : $t^{**} < t^*$ is impossible since:

Let be $t^{**} < t^*$, and since $v(0, t) > 0$, $v(s_1(t), t) = u_2(s_1(t), t) > 0$, for $t < t^{**}$, the point $(s_1(t^{**}), t^{**})$ is a minimum point for v because $v(s_1(t^{**}), t^{**}) = u_2(s_1(t^{**}), t^{**}) = u_1(s_1(t^{**}), t^{**}) = 0$.

By the corner minimum principle

$$v_x(s_1(t^{**}), t^{**}) < 0$$

which contradicts

$$v_x(s_1(t^{**}), t^{**}) = u_{2x}(s_2(t^{**}), t^{**}) = 0.$$

Thus the proposition is proved. □

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