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THE ASYMPTOTIC BEHAVIOR FOR THE TWO-PHASE STEFAN PROBLEM WITH A CONVECTIVE BOUNDARY CONDITION

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ABSTRACT: We consider a two-phase Stefan problem in a semi-infinite material, when a convective condition is assigned on the fixed face $x = 0$. We demonstrate the monotone dependence of the solution with respect to the data and with respect to the thermal transfer coefficient H. We study the asymptotic behavior of the solution when $H \to \infty$.

We also study the asymptotic behavior of the free boundary when $t \to \infty$ and we obtain an explicit expression with the same kind of behavior for the free boundary as in the one-phase Stefan problem with a convective boundary condition which corresponds to the case with a temperature boundary condition at the fixed face. We obtain some results for the disappearance of a phase. Finally we analyze the case when the liquid phase is a supercooled liquid.

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1. INTRODUCTION

In this paper we consider the two-phase unidimensional Stefan problem for a semi-infinite material with a convective boundary condition at the fixed boundary, $x = 0$.

Specifically the mathematical problem consists of determining two functions, $u^H(x,t)$ and $v^H(x,t)$, a function $x = s^H(t)$, called the free boundary, and a time T such that (u^H, v^H, s^H, T) satisfy the following equations, boundary and initial conditions. For each positive H we consider:

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Problem P_H :

$$\rho c_2 u_t^H - k_2 u_{xx}^H = 0, \quad D_2 = \{(x,t) : 0 < x < s^H(t), 0 < t < T\}, \tag{1.1}$$

$$\rho c_1 v_t^H - k_1 v_{xx}^H = 0, \quad D_1 = \{(x,t) : x > s^H(t), 0 < t < T\}, \tag{1.2}$$

$$u^H(x,0) = \varphi(x) \geq 0, \quad 0 < x < s^H(0) = b^H, \tag{1.3}$$

$$v^H(x,0) = \psi(x) \leq 0, \ x > b^H, \quad v^H(\infty,t) = \psi(\infty) \leq 0, \ t > 0, \tag{1.4}$$

$$k_2 u_x^H(0,t) = H(u^H(0,t) - f(t)), \quad 0 < t < T, \tag{1.5}$$

$$v^H(s^H(t),t) = u^H(s^H(t),t) = 0, \quad 0 < t < T, \tag{1.6}$$

$$k_1 v_x^H(s^H(t),t) - k_2 u_x^H(s^H(t),t) = \rho l \dot{s}^H(t), \quad 0 < t < T, \tag{1.7}$$

where the phase-change temperature is assumed to be zero and H is the thermal transfer coefficient ($H > 0$).

Very general results about the existence of classical solutions to the two-phase Stefan problem have been obtained in Fasano and Primicerio [7], Friedman [9], Cannon and Primicerio [4], Cannon and Primicerio [5]. The asymptotic behavior for the one-phase Stefan problem with temperature and flux conditions on the fixed boundary $x = 0$ are considered in Cannon and Hill [2], and Cannon and Primicerio [3], respectively.

In Solomon et al [13], the behavior of the solution with respect to the heat transfer coefficient H and the asymptotic behavior of the free boundary in the two-phase case are studied for the constant case $f(t) = T_L > 0$. In Tarzia and Turner [14], we generalized this result for the one-phase problem in the case when $f(t)$ is not a constant. There it was considered the one-phase Stefan problem with a convective boundary condition at the fixed face, given by the temperature of the external fluid $f(t)$ depending on time. It was proved that the asymptotic behavior of the free boundary is the same that for the case where the temperature boundary condition $f(t)$ is given at $x = 0$. Moreover, the explicit limit expression is also given. In this paper we study the asymptotic behavior of the corresponding free boundary $s^H(t)$ when the time goes to infinity and we give an explicit expression for this behavior. In Knaber [11], and in Aiki [1], a two-phase Stefan problem with very general boundary condition at $x = 0$ are studied. In Fasano and Primicerio [8], a one-phase Stefan problem for the supercooled liquid with a zero flux at the fixed face was considered. In Comparini et al [6], this problem was studied for a general flux $g(t)$. In Marangunic and Turner [12], and in Turner [16], the two-phase Stefan problem for the supercooled liquid with flux and temperature boundary conditions at the fixed faces $x = 0$ and $x = 1$ was analyzed.

This paper is organized in five sections. In the first part of Section 1 we reformulate the free boundary problem and we prove some preliminary results. In the second part we show the monotone dependence of the solution with respect to the data and with respect to the thermal transfer coefficient H. In the second section we consider a two-phase Stefan problem with a temperature boundary condition on the fixed face $x = 0$ and we prove that the solution of the two-phase Stefan problem with convective boundary condition is bounded by the solution of the two-phase Stefan problem with temperature condition. In the third section we obtain the convergency of the free boundary when $t \to \infty$ and we prove the convergency of the solution of the problem with convective condition when $H \to \infty$ to the solution of the problem with temperature condition. For the multidimensional two-phase Stefan problem through a variational inequality this asymptotic behavior was analized in Tarzia [15], where an explicit expression is given. Analogously we can find a result for the unidimensional two-phase Stefan problem, see e.g. Friedman [10]. In this sense, this work generalizes to two-phases the results obtained in Tarzia and Turner [14], where we prove the asymptotic behavior of the free boundary of the one-phase Stefan problem. In Section 4 we discuss the relation between the disappearance of a phase and the total energy supplied to the media. In the last section we consider a two-phase Stefan problem analyzing the relation between the initial data, boundary data and the possibility of continuing the solution for arbitrarily large time intervals.

In order to have existence and uniqueness of the solution we require the following assumptions on the initial and boundary data:

i) Let $\varphi = \varphi(x)$ and $\psi = \psi(x)$ be positive and negative respectively piecewise continuous functions.

ii) Let $f = f(t)$ be a positive bounded piecewise continuous function.

iii) Compatibility conditions: $f(0) > \varphi(x)$ in $(0, b^H)$, $\varphi(b^H) = \psi(b^H) = 0$, $k_2\varphi'(0) = H(\varphi(0) - f(0))$.

1.1. SOME PRELIMINARY RESULTS AND REFORMULATION OF THE FREE BOUNDARY PROBLEM

Lemma 1. *The temperatures $u^H(x,t)$ and $v^H(x,t)$, under the above hypotheses on the data, satisfy the following inequalities:*

i) $v^H \leq 0$,

ii) $u^H \geq 0$.

Proof. i) Since $v^H(x,0) = \psi(x) \leq 0$ for $x > b^H$ and $v^H(s(t),t) = 0$, then using the maximum principle we obtain $v^H(x,t) \leq 0$.

ii) Since $u^H(x,0) = \varphi(x) \geq 0$ and $u^H(s^H(t),t) = 0$, we will prove that $u^H(0,t) > 0, t > 0$. Therefore, let us suppose that there exists a first time $t_0 > 0$ such that $u^H(0,t_0) = 0$, then $u^H(0,t_0) = \min_{D_0} u^H(x,t)$, where $D_0 = D_2 \cap \{t \leq t_0\}$. Now by the maximum principle we obtain that $u_x^H(0,t_0) > 0$, but this contradicts the boundary condition

$$u_x^H(0,t_0) = \frac{H}{k_2}(u^H(0,t_0) - f(t_0)) = -\frac{H}{k_2}f(t_0) \leq 0.$$

Then $u^H(0,t) \geq 0,\ t > 0$, and $u^H(x,t) \geq 0$.

Lemma 2. *If (v^H, u^H, s^H, T) is a solution of Problem P_H', and $\psi, x\psi \in L^1(b^H, \infty)$ then, setting $s^H = s$ and $b^H = b$ for convenience in the notation, we have the following equality:*

$$\begin{aligned}\rho l s(t)\left(1 + \frac{H}{k_2}\frac{s(t)}{2}\right) &= \rho l b\left(1 + \frac{H}{k_2}\frac{b}{2}\right) + \int_0^b \frac{(k_2 + xH)}{\alpha_2}\varphi(x)\,dx \\ &+ \int_b^\infty \left(k_1 + k_1 x\frac{H}{k_2}\right)\frac{\psi(x)}{\alpha_1}\,dx + \int_0^t Hf(\tau)\,d\tau \\ &- \int_0^{s(t)} \frac{(k_2 + xH)}{\alpha_2}u^H(x,t)\,dx - \int_{s(t)}^\infty \left(k_1 + x\frac{H}{k_2}k_1\right)\frac{v^H(x,t)}{\alpha_1}\,dx,\end{aligned} \tag{1.8}$$

where $\alpha_i = \frac{k_i}{\rho c_i}, i = 1, 2$.

Proof. Consider the Green's identity

$$\iint_{D_t} (wLz - zL^*w)\,dx\,d\tau = \oint_{\partial D_t} (wz_x - zw_x)\,d\tau + \frac{wz}{\alpha}\,dx,$$

where L denotes the heat operator $Lu = u_{xx} - \frac{1}{\alpha}u_t$, $\alpha = \frac{k}{\rho c}$, and L^* its adjoint.

We take $z = u^H$ and $w = 1$ in D_1, $z = v^H$ and $w = 1$ in $D_c = \{(x,t) : s(t) < x < c,\ t > 0\}$, with $c > 0$.

We replace in the Green identity, take the limit when $c \to \infty$, and obtain the first integral representation:

$$\begin{aligned}\rho l s(t) = \rho l b &+ \int_0^b \frac{k_2}{\alpha_2}\varphi(x)\,dx + \int_b^\infty \frac{k_1}{\alpha_1}\psi(x)\,dx \\ &- \int_0^t k_2 u_x^H(0,\tau)\,d\tau \\ &- \int_0^{s(t)} \frac{k_2}{\alpha_2}u^H(x,t)\,dx - \int_{s(t)}^\infty \frac{k_1}{\alpha_1}v^H(x,t)\,dx.\end{aligned} \tag{1.9}$$

Now we set $z = u^H, w = x$ in D_1, $z = v^H$, $w = x$ in D_c. We replace in the Green identity, take the limit when $c \to \infty$, then obtain the following integral representation:

$$\rho l \frac{s^2(t)}{2} = \rho l \frac{b^2}{2} + \int_0^b \frac{k_2}{\alpha_2} x\varphi(x)\,dx$$

$$+ \int_b^\infty \frac{k_1}{\alpha_1} x\psi(x)\,dx + \int_0^t k_2 u^H(0,\tau)\,d\tau \tag{1.10}$$

$$- \int_0^{s(t)} x\frac{k_2}{\alpha_2} u^H(x,t)\,dx - \int_{s(t)}^\infty x\frac{k_1}{\alpha_1} v^H(x,t)\,dx.$$

Now if we consider (1.9) + $\frac{H}{k_2}$(1.10), obtain the relation (1.8). □

Lemma 3. *The temperatures $v^H(x,t)$ and $u^H(x,t)$ satisfy the following inequalities:*

i) $u^H(x,t) \le f(t)$ in D_2,

ii) $v^H(x,t) \ge \psi(x)$ in D_1,

under the following hypotheses on the data: $\dot f(t) \ge 0$, $\psi'(x) \le 0$, and $\psi''(x) \ge 0$.

Proof. i) We define an auxiliary function

$$V(x,t) = f(t) - u^H(x,t).$$

The function V satisfies the following problem in D_2:

$$LV = \rho c_2 V_t - k_2 V_{xx} = \rho c_2 \dot f(t) \ge 0$$

$$V(x,0) = f(0) - \varphi(x) \ge 0$$

$$V(s(t),t) = f(t) \ge 0$$

$$k_2 V_x(0,t) = -H(u^H(0,t) - f(t)) = HV(0,t).$$

Since $LV \ge 0$, by the maximun principle we obtain $V(x,t) \ge 0$.

ii) In the same way we define the auxiliary function

$$W(x,t) = v^H(x,t) - \psi(x)$$

which satisfies the following problem in D_1:

$$\rho c_1 W_t - k_1 W_{xx} = k_1\psi''(x) \ge 0$$

$$W(x,0) = v^H(x,0) - \psi(x) = 0$$

$$W(s(t),t) \ge 0.$$

We obtain $W \ge 0$ in D_1 by using the minimum principle. □

1.2. MONOTONE DEPENDENCE OF THE SOLUTION WITH RESPECT TO THE DATA

Lemma 4. *If $(v_i^H, u_i^H, s_i^H, T), i = 1, 2$, are solutions of the Stefan problem (1.1)-(1.7) corresponding to the data f_i, φ_i, ψ_i and b_i, and if $f_1 \le f_2$, $\varphi_1 \le \varphi_2$, $\psi_1 \le \psi_2$ and $b_1 \le b_2$, then $s_1^H(t) \le s_2^H(t)$ and $u_1^H \le u_2^H$, $v_1^H \le v_2^H$ in their corresponding common domains.*

Proof. The proof is a straight-forward application of the maximun principle to the auxiliary functions $V = v_2^H - v_1^H$ and $U = u_2^H - u_1^H$. □

1.3. MONOTONE DEPENDENCE OF THE SOLUTION WITH RESPECT TO THE THERMAL TRANSFER COEFFICIENT H

Theorem 1. *If $(u^{H_i}, v^{H_i}, s^{H_i}, T), i = 1, 2$, are solutions of the Stefan problem (1.1)-(1.7) corresponding to the data H_1 and H_2 with $H_1 \le H_2$, and $\dot{f} > 0$, then $v^{H_1} \le v^{H_2}$, $u^{H_1} \le u^{H_2}$, $s^{H_1}(t) \le s^{H_2}(t)$ in the common domains where they are defined.*

Proof. Suppose first that $s^{H_1}(0) = b^{H_1} < b^{H_2} = s^{H_2}(0)$, then $s^{H_1}(t) < s^{H_2}(t)$. If not, there exists a first positive time t_0 such that $s^{H_1}(t_0) = s^{H_2}(t_0)$ and $\dot{s}^{H_1}(t_0) \ge \dot{s}^{H_2}(t_0)$. We define the functions:

$$U = u^{H_2} - u^{H_1}, \qquad \text{in } 0 < x < s^{H_1}(t), \quad 0 < t < t_0,$$

$$V = v^{H_2} - v^{H_1}, \qquad \text{in } s^{H_2}(t) < x < \infty, 0 < t < t_0.$$

These functions satisfy the following equations and conditions:

$$\rho c_2 U_t - k_2 U_{xx} = 0, \qquad 0 < x < s^{H_1}(t), \quad 0 < t < t_0,$$

$$U(x,0) = 0, \qquad 0 < x < b^{H_1},$$

$$U(s^{H_1}(t), t) = u^{H_2}(s^{H_1}(t), t) \ge 0, \qquad 0 < t < t_0,$$

$$k_2 U_x(0,t) = (H_2 - H_1)(u^{H_2}(0,t) - f(t)) + H_1 U(0,t), \qquad 0 < t < t_0,$$

and

$$\rho c_1 V_t - k_1 V_{xx} = 0, \qquad s^{H_2}(t) < x < \infty, \quad 0 < t < t_0,$$

$$V(x,0) = 0, \qquad b^{H_2} < x < \infty,$$

$$V(s^{H_2}(t), t) = -v^{H_1}(s^{H_2}(t), t) \ge 0, \qquad 0 < t < t_0.$$

Using the maximum principle we obtain $V \geq 0$.

In order to see that $U \geq 0$, first we suppose that there exists a $t_1 < t_0$ such that $U(0,t_1) < 0$ then $U_x(0,t_1) > 0$. But this contradicts the boundary condition $k_2 U_x(0,t) = (H_2 - H_1)(u^{H_2}(0,t) - f(t)) + H_1 U(0,t) < 0$, since $H_2 \geq H_1$ and $u^{H_2} \leq f(t)$. Then we conclude that $U \geq 0$.

Now, we shall prove that $s^{H_1}(t) < s^{H_2}(t)$. We compute U at the point $(s^{H_1}(t_0), t_0)$, namely:

$$U(s^{H_1}(t_0), t_0) = u^{H_2}(s^{H_2}(t_0), t_0) - u^{H_1}(s^{H_1}(t_0), t_0) = u^{H_2}(s^{H_2}(t_0), t_0) = 0.$$

Then the point $(s^{H_1}(t_0), t_0) = (s^{H_2}(t_0), t_0)$ is a minimum for the function U in $0 < x < s_{H_1}(t)$, $0 < t < t_0$, and by the maximum principle $U_x(s^{H_1}(t_0), t_0) < 0$.

In the same way we conclude that $(s(t_0), t_0)$ is a minimum point for the function V at its domain, and $V_x(s(t_0), t_0) > 0$. We plug these inequalities in the Stefan condition for the free boundary and we have :

$$0 > k_2 U_x(s^{H_1}(t_0), t_0) - k_1 V_x(s^{H_1}(t_0), t_0)$$

$$= (k_2 u_x^{H_2} - k_1 v_x^{H_2})(s^{H_2}(t_0), t_0) - (k_2 u_x^{H_1} - k_1 v_x^{H_1})(s^{H_1}(t_0), t_0)$$

$$= -\rho l \dot{s}^{H_2}(t_0) + \rho l \dot{s}^{H_1}(t_0) \geq 0,$$

which is a contradiction. Then $s^{H_1}(t) < s^{H_2}(t), t > 0$.

The case $b^{H_1} = b^{H_2}$ is analyzed by taking the limit of the previous result when $b^{H_2} > b^{H_1}$ and we take $b^{H_2} \to b^{H_1}$. □

2. THE CASE WHERE THE THERMAL TRANSFER COEFFICIENT APPROACHES TO INFINITY

We consider the following two-phase Stefan problem for a semi-infinite material with a temperature boundary condition on the fixed face $x = 0$.

We call this Problem P_∞ which is given by:

$$\rho c_2 u_t - k_2 u_{xx} = 0, \qquad 0 < x < s(t), \tag{2.1}$$

$$\rho c_1 v_t - k_1 v_{xx} = 0, \qquad s(t) < x < \infty, \tag{2.2}$$

$$u(x,0) = \varphi(x) \geq 0, \qquad 0 \leq x \leq b, \tag{2.3}$$

$$v(x,0) = \psi(x) \leq 0, \qquad b < x < \infty, \tag{2.4}$$

$$v(\infty,0) = \psi(\infty), \qquad 0 < t, \tag{2.5}$$

$$u(0,t) = f(t) \geq 0, \qquad 0 < t < T, \tag{2.6}$$

$$u(s(t),t) = v(s(t),t) = 0, \qquad 0 < t < T, \tag{2.7}$$

$$k_1 v_x(s(t),t) - k_2 u_x(s(t),t) = \rho l \dot{s}(t), \quad 0 < t < T. \tag{2.8}$$

Theorem 2. *The solution (u,v,s,T) of Problem P_∞ and the solution (u^H, v^H, s^H, T) of Problem P_H satisfy the following inequalities:*

i) $s^H(t) < s(t)$, $t > 0$,

ii) $u^H < u$, $0 < x < s^H(t)$, $0 < t < T$,

iii) $v^H < v$, $s(t) < x < \infty$, $0 < t < T$,

provided $\dot{f} \geq 0$, $b^H < b$, *and* $H > 0$.

Proof. Since $s^H(0) = b^H < s(0) = b$, then $s^H(t) < s(t)$, $t > 0$. We denote by t_0 the first positive time such that $s^H(t_0) = s(t_0)$ and $\dot{s}(t_0) \leq \dot{s}^H(t_0)$. We define the functions:

$$W_2 = u - u^H, \qquad 0 < x < s^H(t), \quad 0 < t < t_0,$$

$$W_1 = v - v^H, \qquad s(t) < x, \quad 0 < t < t_0.$$

The functions W_2 and W_1 satisfy the following equations:

$$\rho c_2 (W_2)_t - k_2 (W_2)_{xx} = 0, \qquad 0 < x < s^H(t), \quad 0 < t < t_0,$$

$$W_2(x,0) = 0, \qquad 0 < x < b^H,$$

$$W_2(s^H(t),t) = u(s^H(t),0) > 0, \qquad 0 < t < t_0,$$

$$W_2(0,t) = u(0,t) - u^H(0,t) \geq f(t) - f(t)) = 0,$$

since $u^H(0,t) \leq f(t)$, $(\dot{f} \geq 0)$. Then $W_2 > 0$ in $0 < x < s^H(t), 0 < t < t_0$. Since $W_2(s^H(t_0), t_0) = W_2(s(t_0), t_0) = u(s(t_0), t_0) = 0$, the point $(s^H(t_0), t_0)$ is the minimum point for the function W_2, then by the maximum principle we obtain

$$(W_2)_x(s^H(t_0), t_0) = (W_2)_x(s(t_0), t_0) < 0.$$

In the same way we obtain that $W_1 > 0$ in the domain $s(t) < x < \infty, t < t_0$, and the point $(s(t_0), t_0)$ is the minimum for W_1, then by the maximum principle we conclude

$$(W_1)_x(s^H(t_0), t_0) = (W_1)_x(s(t_0), t_0) > 0.$$

Now we compute

$$0 > k_2(W_2)_x - k_1(W_1)_x(s^H(t_0), t_0) = \rho l(\dot{s}^H(t_0) - \dot{s}(t_0)) > 0,$$

which is a contradiction. Then $s^H(t) < s(t), t > 0, H > 0$. □

3. ASYMPTOTIC BEHAVIOR OF THE FREE BOUNDARY

We will study the asymptotic behavior of the free boundary $s^H(t)$ when $t \to \infty$ or $H \to \infty$. In Tarzia and Turner [14], was considered the global existence in a general Stefan-like problem.

Theorem 3. *If (u^H, v^H, s^H, T) is the solution of Problem P_H, and $\psi, x\psi \in L^1(b^H, \infty)$, then we have the following properties:*

i) If $\int_0^\infty f(\tau)d\tau < \infty$ and $\lim_{t\to\infty} f(t) = 0$, then $\lim_{t\to\infty} s^H(t) = s_H^\infty$,
where s_H^∞ is the unique positive solution of the equation of second order given by

$$\rho l x \left(1 + \frac{H}{2k_2}x\right) = D,$$

where

$$D \equiv \rho l b_H \left(1 + \frac{b_H H}{2k_2}\right) + \int_0^{b^H} (k_2 + yH)\frac{\varphi(y)}{\alpha_2}\,dy + \int_{b^H}^\infty \left(k_1 + \frac{yHk_1}{k_2}\right)\frac{\psi(y)}{\alpha_1}\,dy + \int_0^\infty Hf(\tau)\,d\tau,$$

provided that $D > 0$.

ii) If $\int_0^\infty f(\tau)d\tau = \infty$*, then* $\lim_{t\to\infty} \dfrac{s^H(t)}{\sigma(t)} = 1$*, where* $\sigma(t)$ *is the free boundary of the following problem:*

For each $t_0 \geq 0$*, let* (σ, V_1, V_2) *be the solution to the following problem* P_{t_0}*:*

$$
\begin{aligned}
\rho c_2 (V_2)_t &= k_2 (V_2)_{xx}, && 0 < x < \sigma(t),\ t > t_0,\\
\rho c_1 (V_1)_t &= k_1 (V_1)_{xx}, && x > \sigma(t),\ t > t_0,\\
V_1(x,0) &= 0, && 0 \leq x \leq s^H(t_0),\\
V_1(x,0) &= v^H(x,t_0), && x \geq s^H(t_0),\\
V_1(\infty,t) &= v^H(\infty,t_0), && t > t_0,\\
k_2(V_2)_x(0,t) &= H(V_2(0,t) - f(t)), && t > t_0,\\
V_2(\sigma(t),t) = V_1(\sigma(t),t) &= 0, && t > t_0,\\
\sigma(t_0) &= 0, &&\\
k_1(V_1)_x(\sigma(t),t) - k_2(V_2)_x(\sigma(t),t) &= \rho l \dot{\sigma}(t), && t > t_0.
\end{aligned}
$$

Proof. i) First we find bounds for the functions u^H and v^H.

We define the auxiliary function U which satisfies the following:

Problem P_U

$$
\begin{aligned}
\rho c_2 U_t - k_2 U_{xx} &= 0, \qquad 0 < x < \infty, \quad t > 0,\\
k_2 U_x(0,t) &= H(U(0,t) - f(t)), \qquad t > 0,\\
U(x,0) &= \begin{cases} \varphi(x), & 0 \leq x \leq b^H,\\ 0, & b^H \leq x. \end{cases}
\end{aligned}
$$

Using the maximum principle we can prove that $u^H(x,t) \leq U(x,t)$, for $0 < x < s^H(t)$, $t > 0$. Moreover if $\lim_{t\to\infty} f(t) = 0$, then $\lim_{t\to\infty} U(x,t) = 0$, by Friedman [9].

Now we define the function V such that Problem P_V is satisfied:

Problem P_V

$$\rho c_1 V_t - k_1 V_{xx} = 0, \qquad 0 < x < \infty, \quad t > 0,$$

$$V(0,t) = 0, \qquad t > 0,$$

$$V(\infty,t) = \psi(\infty), \qquad t > 0,$$

$$V(x,0) = \begin{cases} \psi(b^H) = 0, & 0 \le x \le b^H, \\ \psi(x), & b^H \le x. \end{cases}$$

It follows that $V(x,t) \le 0$ and $V(x,t) \le v^H(x,t)$ in $s^H(t) < x < \infty$. It is known that $\lim_{t\to\infty} V(x,t) = 0$ by Friedman [9].

Using the integral representation (1.8) and the bounds for u^H and v^H we obtain the following inequalities:

$$\rho l b^H \left(1 + \frac{b^H H}{2k_2}\right) + \int_0^{b^H} (k_2 + xH)\frac{\varphi(x)}{\alpha_2}\,dx + \int_{b^H}^{\infty} \left(k_1 + x\frac{k_1}{k_2}H\right)\frac{\psi(x)}{\alpha_1}\,dx$$
$$+ \int_0^t Hf(\tau)\,d\tau - \int_0^{s^H(t)} (k_2 + xH)\frac{U(x,t)}{\alpha_2}\,dx$$
$$\le \rho l s^H(t)\left(1 + \frac{H}{2k_2}s^H(t)\right) \tag{3.1}$$

$$\le \rho l b^H \left(1 + \frac{b^H H}{2k_2}\right) + \int_0^{b^H} (k_2 + xH)\frac{\varphi(x)}{\alpha_2}\,dx + \int_{b^H}^{\infty} \left(k_1 + x\frac{k_1}{k_2}H\right)\frac{\psi(x)}{\alpha_1}\,dx$$
$$+ \int_0^t Hf(\tau)\,d\tau - \int_{s^H(t)}^{\infty} (k_1 + xH\frac{k_1}{k_2})\frac{V(x,t)}{\alpha_1}\,dx.$$

From the above inequality we can conclude that the $\lim_{t\to\infty} s^H(t)$ exists, then taking limit when $t \to \infty$ in (3.1) we obtain:

$$\lim_{t\to\infty} \rho l s^H(t)\left(1 + \frac{H}{2k_2}s^H(t)\right) = \rho l b^H\left(1 + \frac{b^H H}{2k_2}\right) + \int_0^{b^H} (k_2 + xH)\frac{\varphi(x)}{\alpha_2}\,dx$$
$$+ \int_{b^H}^{\infty} \left(k_1 + x\frac{k_1}{k_2}H\right)\frac{\psi(x)}{\alpha_1}\,dx + \int_0^{\infty} Hf(\tau)\,d\tau.$$

ii) The proof is the following. Using the maximum principle we can prove that $\sigma(t) < s^H(t)$, $t > t_0$ and

$V_2(x,t) < u^H(x,t)$, $V_1(x,t) < v^H(x,t)$ in the corresponding domains, for $t > t_0$.

Now, we use the integral representation (1.8) with the adequate initial condition at $t = t_0$ and we get

$$\begin{aligned}
&\rho l s^H(t)\left(1+\frac{H}{2k_2}s^H(t)\right) = \rho l s^H(t_0)\left(1+\frac{s^H(t_0)H}{2k_2}\right)\\
&+\int_0^{s^H(t_0)}\left(k_1+x\frac{k_1}{k_2}H\right)\frac{u^H(x,t_0)}{\alpha_2}\,dx \;+\int_{s^H(t_0)}^{\infty}(k_2+xH)\frac{v^H(x,t_0)}{\alpha_1}\,dx\\
&+\int_{t_0}^{t} Hf(\tau)\,d\tau \;-\int_0^{s^H(t)}\left(k_1+x\frac{k_1}{k_2}\right)\frac{u^H(x,t)}{\alpha_2}\,dx\\
&-\int_{s^H(t)}^{\infty}(k_2+xH)\frac{v^H(x,t)}{\alpha_1}\,dx \le \rho l s^H(t_0).\left(1+\frac{s^H(t_0)H}{2k_2}\right)\\
&+\int_0^{s^H(t_0)}(k_2+xH)\frac{u^H(x,t_0)}{\alpha_2}\,dx \;+\int_{s^H(t_0)}^{\infty}\left(k_1+x\frac{k_1}{k_2}H\right)\frac{v^H(x,t_0)}{\alpha_1}\,dx\\
&+\int_{t_0}^{t} Hf(\tau)\,d\tau \;-\int_0^{\sigma(t)}(k_2+xH)\frac{V_2}{\alpha_2}\,dx-\int_{\sigma(t)}^{\infty}\left(k_1+x\frac{k_1}{k_2}\right)\frac{V_1}{\alpha_1}\,dx\\
&= C(t_0)+\rho l\sigma(t)\left(1+\frac{H}{2k_2}\sigma(t)\right),
\end{aligned}$$

where

$$C(t_0) = \rho l s^H(t_0)\left(1+\frac{s^H(t_0)H}{2k_2}\right)+\int_0^{s^H(t_0)}(k_2+xH)\frac{u^H(x,t_0)}{\alpha_2}\,dx.$$

Then we have

$$\sigma^2(t) \le s_H^2(t) \le \sigma^2(t) + C(t_0)\frac{2k_2}{\rho l H}.$$

Taking the limit when $t \to \infty$ in the above inequalities and using the fact that $\lim_{t\to\infty}\sigma(t) = \infty$ (since $\int_0^\infty f(\tau)\,d\tau = \infty$), then $\lim_{t\to\infty}\frac{s^H(t)}{\sigma(t)} = 1$.

□

Lemma 5. *If (u^H, v^H, s^H, T) is a solution of Problem P_H, with $\dot{f} > 0$ and $H > 0$, then*

$$\int_0^t (u^H(0,\tau) - f(\tau))\,d\tau \le \frac{\left[s(t)(\rho l + \frac{k_2}{\alpha_2}\|f\|_{(0,t))}) + C\right]}{H},$$

where

$$C = -\int_b^\infty \frac{k_1}{\alpha_1}\psi(x)\,dx > 0$$

and $s(t)$ is the free boundary of the problem P_∞.

Proof. Using the integral representation (1.9) we can write

$$H\int_0^t (f(\tau)-u^H(0,\tau))d\tau = \rho l s^H(t) - \rho l b - \int_0^b \frac{k_2}{\alpha_2}\varphi(x)\,dx$$

$$-\int_b^\infty \frac{k_1}{\alpha_1}\psi(x)\,dx + \int_0^{s^H(t)} \frac{k_2}{\alpha_2}u^H(x,t)\,dx$$

$$+\int_{s^H(t)}^\infty \frac{k_1}{\alpha_1}v^H(x,t)\,dx \le \rho l s^H(t) - \int_b^\infty \frac{k_1}{\alpha_1}\psi(x)\,dx + s^H(t)\frac{k_2}{\alpha_2}\max_{(0,t)} f(\tau)$$

$$\le s^H(t)[\rho l + \frac{k_2}{\alpha_2}\|f\|_{(0,t))}] - \int_b^\infty \frac{k_1}{\alpha_1}\psi(x)\,dx,$$

that is the thesis. □

Lemma 6. *If (u^H, v^H, s^H, T) is a solution of Problem P_H and the data satisfy $\dot f \ge 0$, then (u^H, v^H, s^H, T) and (u, v, s, T) satisfy the following inequality*

$$0 \le \frac{\rho l(s^2(t) - s_H^2(t))}{2} + \int_0^{s^H(t)} \frac{xk_2(u(x,t)-u^H(x,t))}{\alpha_2}\,dx$$

$$+\int_{s(t)}^\infty \frac{xk_1(v(x,t)-v^H(x,t))}{\alpha_1}\,dx \le \int_0^t k_2(f(\tau)-u^H(0,\tau))\,d\tau\,.$$

Proof. We use the integral representation (1.10) for both pairs: (u^H, v^H, s^H, T) and (u, v, s, T) and we obtain

$$\rho l\frac{s_H^2(t)}{2} = \rho l\frac{b^2}{2} + \int_0^b \frac{k_2}{\alpha_2}x\varphi(x)\,dx$$

$$+\int_b^\infty \frac{k_1}{\alpha_1}x\psi(x)\,dx + \int_0^t k_2u^H(0,\tau)\,d\tau \tag{3.2}$$

$$-\int_0^{s^H(t)} x\frac{k_2}{\alpha_2}u^H(x,t)\,dx - \int_{s^H(t)}^\infty x\frac{k_1}{\alpha_1}v^H(x,t)\,dx,$$

$$\rho l\frac{s^2(t)}{2} = \rho l\frac{b^2}{2} + \int_0^b \frac{k_2}{\alpha_2}x\varphi(x)\,dx$$

$$+\int_b^\infty \frac{k_1}{\alpha_1}x\psi(x)\,dx + \int_0^t k_2f(\tau)\,d\tau \tag{3.3}$$

$$-\int_0^{s(t)} x\frac{k_2}{\alpha_2}u(x,t)\,dx - \int_{s(t)}^\infty x\frac{k_1}{\alpha_1}v(x,t)\,dx.$$

Then we substract s_H^2 from s^2 and using the fact that $u \ge 0$ and $v^H \le 0$ we obtain the thesis. □

Theorem 4. *(Convergency when $H \to \infty$). If (u^H, v^H, s^H, T) is a solution of Problem P_H, (u, v, s, T) is a solution of Problem P_∞ and the data satisfies $\dot{f} \geq 0$, then*

i) $\lim_{H\to\infty} s^H(t) = s(t)$,

ii) $\lim_{H\to\infty} u^H(x,t) = u(x,t)$ *and* $\lim_{H\to\infty} v^H(x,t) = v(x,t)$ *for all compact sets included in their corresponding domains.*

Proof. Using Lemmas 5 and 6 we can write the following inequality:

$$0 \leq \frac{\rho l(s^2(t) - s_H^2(t))}{2} + \int_0^{s^H(t)} \frac{x k_2(u(x,t) - u^H(x,t))}{\alpha_2}\, dx$$
$$+ \int_{s^H(t)}^{\infty} \frac{x k_1(v(x,t) - v^H(x,t))}{\alpha_1}\, dx \leq k_2 \frac{[s(t)(\rho l + \frac{k_2}{\alpha_2}\|f\|_{(0,t)}) + C]}{H}.$$

Since $s^H \leq s$, $u^H \leq u$ and $v^H \leq v$ in their corresponding domains for all positive H, then the three left hand side members of the inequality are positive, and we obtain:

$$0 \leq \frac{\rho l(s^2(t) - s_H^2(t))}{2} \leq k_2 \frac{[s(t)(\rho l + \frac{k_2}{\alpha_2}\|f\|_{(0,t)}) + C]}{H},$$

for all H. Let us tend H to infinity for each $t > 0$, then

$$\lim_{H\to\infty} s^H(t) = s(t), \quad \forall t > 0.$$

We can do the same with the difference $u - u^H$ and $v - v^H$. □

It is interesting to study the asymptotic behavior of the free boundary $s(t)$. In this direction we have the following results.

Theorem 5. *If (u^H, v^H, s^H, T) is a solution of Problem P_H, for $H > 0$ and the following conditions:*

i) $\int_0^\infty f(\tau)\, d\tau = \infty$, $\int_{t_0}^t f(\tau)\, d\tau < \infty \quad \forall t, t_0$ *and* $\lim_{t\to\infty} \sqrt{t} f(t) = \infty$,

ii) $\lim_{t_0\to\infty} \max_{[t_0,\infty)} f(\tau) = \lim_{t_0\to\infty} \|f\|_{[t_0,\infty)} = 0$

are satisfied, then

$$\lim_{t\to\infty} \frac{\rho l s^H(t)(1 + \frac{H}{2k_2} s^H(t))}{H \int_0^t f(\tau)\, d\tau} = 1,$$

for all $H > 0$.

Proof. We will use the definition of the functions V_1 and V_2 of Theorem 1. We write the integral representation (1.8) for (V_1, V_2, σ):

$$\rho l\sigma(t)\left(1+\frac{H}{2k_2}\sigma(t)\right) = -\int_0^{\sigma(t)} \frac{(k_2+xH)}{\alpha_2} V_2(x,t)\,dx + H\int_{t_0}^{t} f(\tau)d\tau$$
$$-\int_{\sigma(t)}^{\infty} \frac{1}{\alpha_1}\left(k_1+x\frac{k_1}{k_2}H\right) V_1(x,t)\,dx + \int_{s^H(t_0)}^{\infty} (k_1+xH\frac{k_1}{k_2})\frac{v^H(x,t_0)}{\alpha_1}dx.$$

Using the maximum principle and the fact that $V_2(x,t) \leq \max_{[t_0,\infty)} f(\tau)$, we have:

$$\begin{aligned}\rho l\sigma(t)\left(1+\frac{H}{2k_2}\sigma(t)\right) &\geq -\int_0^{\sigma(t)} \frac{(k_2+xH)}{\alpha_2}\|f\|_{[t_0,t)}\,dx \\ &\quad+\int_{s^H(t_0)}^{\infty}(k_1+xH\frac{k_1}{k_2})\frac{v^H(x,t_0)}{\alpha_1}dx + H\int_{t_0}^{t} f(\tau)d\tau \\ &= -\|f\|_{[t_0,t)}\frac{(k_2\sigma(t)+\frac{H}{2}\sigma(t)^2)}{\alpha_2} \\ &\quad+\int_0^{\infty}(k_1+xH\frac{k_1}{k_2})\frac{v^H(x,t_0)}{\alpha_1}dx + H\int_{t_0}^{t} f(\tau)d\tau\end{aligned} \tag{3.4}$$

and then

$$\sigma(t)(1+\frac{H\sigma(t)}{2k_2}) \geq \frac{\int_{t_0}^{t} Hf(\tau)d\tau + \int_0^{\infty}(k_1+xH\frac{k_1}{k_2})\frac{V(x,t_0)}{\alpha_1}dx}{(\rho l+\frac{\|f\|_{[t_0,t]}k_2}{\alpha_2})},$$

since $v^H \geq V_1 \geq V$, $x \geq \sigma(t)$, $t \geq t_0$.

Now for (u^H, v^H, s^H, T) we have the following relations obtained from (1.8)

$$\begin{aligned}\rho l s^H(t)\left(1+\frac{H}{2k_2}s^H(t)\right) &= D(H,b,\varphi,\psi) + \int_0^t Hf(\tau)\,d\tau \\ &\quad-\int_0^{s^H(t)}\frac{(k_2+xH)}{\alpha_2}u^H(x,t)\,dx - \int_{s^H(t)}^{\infty}\frac{1}{\alpha_1}\left(k_1+x\frac{Hk_1}{k_2}\right)v^H(x,t)\,dx \\ &\leq D(H,b,\varphi,\psi) + \int_0^t Hf(\tau)\,d\tau - \int_0^{\infty}\frac{1}{\alpha_1}\left(k_1+x\frac{Hk_1}{k_2}\right)V(x,t)\,dx,\end{aligned}$$

where

$$D(H,b,\varphi,\psi) = \rho lb\left(1+\frac{Hb}{k_2}\right) + \int_0^b \frac{(k_2+xH)}{\alpha_2}\varphi(x)\,dx + \int_b^{\infty}\frac{1}{\alpha_1}\left(k_1+\frac{xk_1}{k_2}\right)\psi(x)\,dx.$$

Since we prove $\sigma(t) \le s^H(t)$, then dividing by $H \int_0^t f(\tau)\,d\tau$ we have

$$\frac{\int_{t_0}^t Hf(\tau)\,d\tau - E(t)}{\left(\rho l + \frac{k_2}{\alpha_2}\|f\|_{[t_0,t]}\right) H \int_0^t f(\tau)\,d\tau} \le \frac{\rho l s^H(t)\left(1+\frac{H}{2k_2}s^H(t)\right)}{H\int_0^t f(\tau)\,d\tau}$$

$$\le 1 + \frac{D(H,b,\varphi,\psi)+E(t)}{H\int_0^t f(\tau)\,d\tau},$$

where

$$E(t) = -\int_0^\infty \frac{1}{\alpha_1}\left(k_1 + x\frac{Hk_1}{k_2}\right) V(x,t)\,dx.$$

We have

$$E(t) \le \frac{k_1}{\alpha_1}\|\psi\|_{L^1_{(b,\infty)}} + 2\frac{Hk_1}{\alpha_1 k_2}\sqrt{\pi t}\|x\psi\|_{L^1_{(b,\infty)}},$$

then

$$0 \le \lim_{t\to\infty}\frac{E(t)}{\int_0^t f(\tau)d\tau} \le \frac{2Hk_1\sqrt{\pi}}{\alpha_1 k_2}\|x\psi\|_{L^1_{(b,\infty)}} \lim_{t\to\infty}\frac{\sqrt{t}}{\int_0^t f(\tau)d\tau} = 0.$$

First we take limit when $t\to\infty$, and then when $t_0\to\infty$. Therefore we obtain the following inequality:

$$1 \le \lim_{t\to\infty}\frac{\rho l s^H(t)\left(1+\frac{H}{2k_2}s^H(t)\right)}{H\int_0^t f(\tau)\,d\tau} \le 1,$$

that is the thesis. □

Corollary 1. *Under the hypotheses of Theorem 5 we have the following result for all* $H>0$:

$$\lim_{t\to\infty}\frac{s^H(t)}{\sqrt{\frac{2k_2}{\rho l}\int_0^t f(\tau)d\tau}} = 1,$$

which gives an explicit formula for the asymptotic behavior for the free boundary s^H *which is independent of the positive heat transfer coefficient* H.

Remark 1. The hypotheses for f in Theorem 5 are satisfied when f verifies the asymptotic condition $f \sim t^{-\alpha}$ with $0<\alpha<\frac{1}{2}$ when $t\to\infty$.

Moreover, in this case the free boundary $s^H(t)$ has the asymptotic behavior $s^H(t) \simeq t^\mu$ as $t\to\infty$ independently of H, where the coefficient μ is given by $\mu = \dfrac{1-\alpha}{2} \in (\frac{1}{4},\frac{1}{2})$.

Remark 2. The result of Corollary 1 shows that the free boundary $s^H(t)$ has the same kind of behavior as for the one-phase Stefan problem as it was proved in Tarzia and Turner [14]. We emphasize that the limit is independent of the thermal transfer coefficient H.

4. DISAPPEARANCE OF A PHASE

In this section we discuss the relation between the disappearance of a phase and the total energy supplied to the media. We use the following definitions:

$$\Phi(x) = \begin{cases} (k_2 + Hx)\dfrac{\varphi(x)}{\alpha_2}, & 0 < x < b_H, \\ \left(k_1 + \frac{k_1}{k_2}Hx\right)\dfrac{\psi(x)}{\alpha_1}, & b_H < x < \infty, \end{cases}$$

$$T_\delta = \inf\{t^*,\ t^* > 0,\ s^H(t^*) = \delta \text{ or } s^H(t^*) = \frac{1}{b_H - \delta}\},\ 0 < \delta < b_H$$

$$T_0 = \sup_{0<\delta<b_H} \{T_\delta\},$$

$$Q(t) = \rho l b^H \left(1 + \frac{b^H H}{2k_2}\right) + \int_0^\infty \Phi(x)\,dx + \int_0^t Hf(\tau)\,d\tau.$$

From now on, we write $s^H = s^H = s$ for convenience in the notation.

Theorem 6. *If $0 < Q(t) < \infty$ for all $t > 0$, then $T_0 = \infty$, which means that neither phase disappears in a finite time period.*

Proof. Suppose $T_0 < \infty$, then there exists a sequence $\{\delta_i\}$ with $\lim_{\delta_i \to 0} T_{\delta_i} = T_0$, such that $s(T_{\delta_i}) \to 0$ or $s(T_{\delta_i}) \to \infty$ as $\delta_i \to 0$ or $\delta_i \to b^H$. Suppose $s(T_{\delta_i}) \to 0$ as $\delta_i \to 0$, then using the integral representation of Lemma 2, we obtain

$$\begin{aligned} \rho l s(T_{\delta_i})\left(1 + \frac{H}{2k_2}s(T_{\delta_i})\right) &= \rho l b^H \left(1 + \frac{b^H}{2k_2}\right) + \int_0^\infty \Phi(x)\,dx + \int_0^{T_{\delta_i}} Hf(\tau)\,d\tau \\ &\quad - \int_0^{s(T_{\delta_i})} (k_2 + xH)\frac{u^H(x,t)}{\alpha_2}\,dx \\ &\quad - \int_{s(T_{\delta_i})}^\infty \left(k_1 + \frac{xk_1H}{k_2}\right)\frac{v^H(x,t)}{\alpha_1}\,dx \\ &\geq Q(T_{\delta_i}) - \int_0^{s(T_{\delta_i})} (k_2 + xH)\frac{u^H(x,t)}{\alpha_2}\,dx, \end{aligned}$$

since $v^H < 0$. Therefore, as $s(T_{\delta_i}) \to 0$ as $\delta_i \to 0$, then $0 \geq Q(T_0)$, which contradicts the assumption of the theorem.

For the case $s(T_{\delta_i}) \to \infty$ as $\delta_i \to b$, we now compute:

$$\rho l s(T_{\delta_i})(1 + \frac{Hs(T_{\delta_i})}{2k_2}) \leq Q(T_{\delta_i}) - \int_{s(T_{\delta_i})}^\infty (k_1 + \frac{xk_1H}{k_2})\frac{v^H(x,t)}{\alpha_1}dx,$$

since $u^H \geq 0$, therefore if $s(T_{\delta_i}) \to \infty$ as $\delta_i \to b$, then $Q(T_{\delta_i}) \geq \infty$, which contradicts the assumption of the theorem. □

Recalling Problems P_U and P_V in Theorem 3 we demonstrate the following result.

Theorem 7. *If there exists a T^{**} such that*

$$\rho lb\left(1+\frac{bH}{2k_2}\right)+\int_0^\infty \Phi(x)\,dx+\int_0^{T^{**}} Hf(\tau)\,d\tau - \int_0^\infty (k_1+xH\frac{k_1}{k_2})\frac{V(x,T^{**})}{\alpha_1}\,dx<0,$$

then $T_0 \le T^{**}$.

Proof. Suppose that the inequality holds and $T_0 > T^{**}$. Then we define

$$H(t)=\int_0^\infty \Phi(x)\,dx+\int_0^t Hf(\tau)\,d\tau-\int_0^{s(t)}(k_2+xH)\frac{u^H(x,t)}{\alpha_2}\,dx - \int_{s(t)}^\infty\left(k_1+\frac{xk_1H}{k_2}\right)\frac{v^H(x,t)}{\alpha_1}\,dx.$$

H is a continuous function and $H(0)=0$. We compute

$$H(T^{**})\le\int_0^\infty \Phi(x)\,dx+\int_0^{T^{**}} Hf(\tau)\,d\tau-\int_0^\infty\left(k_1+\frac{xk_1H}{k_2}\right)\frac{V}{\alpha_1}\,dx$$
$$\le -\rho lb\left(1+\frac{bH}{2k_2}\right).$$

Then there exists a $0<t_2<T^{**}$ such that

$$H(t_2)=-\rho lb\left(1+\frac{bH}{2k_2}\right)=-\rho ls(t_2)\left(1+\frac{Hs(t_2)}{2k_2}\right)-\rho lb(1+\frac{bH}{2k_2}),$$

that is $\rho ls(t_2)\left(1+\frac{s(t_2)H}{2k_2}\right)=0$, then $s(t_2)=0$ which is a contradiction. □

5. THE TWO-PHASE STEFAN PROBLEM FOR A SUPERCOOLED LIQUID AND A CLASSICAL SOLID WITH A CONVECTIVE BOUNDARY CONDITION

In this section we consider a two phase Stefan problem analyzing the relation between the initial and boundary data, and the possibility of continuing the solution for arbitrarily large time intervals.

Moreover, if the solution exists, then three cases can occur, see Cannon and Primicerio [3].

(A) The problem has a solution with arbitrarily large t.

(**B**) There exists a constant $T_B > 0$ such that $\liminf_{t\to T_B} s(t) = 0$ or $\limsup_{t\to T_B} s(t) = \infty$.

(C) There exists a constant $T_C > 0$ such that $\liminf_{t\to T_C} s(t) > 0$ or $\limsup_{t\to T_C} s(t) = \infty$ and $\limsup_{t\to T_C} |\dot{s}(t)| = \infty$.

We will consider two cases:

(i) Case I. The supercooled liquid. From now on we consider the case $\varphi \le 0$, $f \le 0$ and $\psi \le 0$. In this case the liquid is supercooled and the solid is classic. A first simple result is Lemma 8 below which uses the function $Q(t)$ which was defined in Section 4.

Lemma 7. *If (u^H, v^H, s, T) is a solution of Problem P_H, then:*

a) $u^H \le 0$, $v^H \le 0$, in their respective domains,

b) $u_x^H \ge 0$, $v_x^H \le 0$ on $x = s(t)$,

c) $s(t)$ is a decreasing function in $(0,T)$,

d) $Q(t) \le \rho l s(t)\left(1 + \frac{Hs(t)}{2k_2}\right) \le \rho l b(1 + \frac{Hb}{2k_2})$, $t > 0$,

e) $Q(t)$ is a decreasing function in $(0,T)$.

Proof. (a),(b) and (c) : Since $\varphi \le 0$ and $f \le 0$, by using the maximum principle we can obtain $u^H \le 0$; in the same way, since $\psi \le 0$, then $v^H \le 0$. In both cases v^H and u^H have a maximum at $x = s(t)$, then $u_x^H(s(t),t) \ge 0$ and $v_x^H(s(t),t) \le 0$. Now using the above inequalities in the Stefan condition we obtain $\dot{s} \le 0$. (d) is obtained replacing (a) in the integral representation of Lemma 2. In order to prove (e) we get $\dot{Q}(t) = Hf(t) \le 0$. □

We proceed to characterize cases (A), (B) and (C) depending on the value of $Q(t)$.

Lemma 8. *Case (B)* $\Longrightarrow$ $Q(T_B) \le 0$.

Proof. If we take limit when $t \to T_B$ in the integral representation, we obtain:

$$Q(T_B) = \int_0^\infty \left(k_1 + x\frac{k_1}{k_2}H\right)\frac{v^H(x,t)}{\alpha_1}\,dx \le 0.$$

□

Lemma 9. *Let (u^H, v^H, s, T) be a solution of Problem P_H. We suppose $s_T = \lim_{t\to T} s(t) > 0$, and there exist $d \in (0, s(T)), d \le 1 - b, z_1 \in (0,1) and z_2 > 0$.*

If the solution (u^H, v^H, s, T) satisfies

$$u^H(s(t) - d, t) \le -z_1 \quad \textit{and} \quad v^H(s(t) + d, t) \ge -z_2 \quad \textit{in} \quad (0,T),$$

then

$$\dot{s}(t) \geq -k, \quad \textit{for some} \quad k > 0.$$

Proof. We can use a similar technique as in Marangunic and Turner [12] because the proof is independent of the value of u^H in the boundary. □

Corollary 2. If Case (C) occurs, the isotherm $u^H = -1$ exits and reaches the free boundary.

Lemma 10. *If* $f \in L^1(0,\infty)$, $\lim_{t\to\infty} f(t) = 0$, *then case (A)* $\Longrightarrow$ $0 \leq Q(t) \leq b\left(1 + \frac{bH}{2k_2}\right)$.

Proof. Letting $t \to \infty$ in the integral representation (1.8), and using $\lim_{t\to\infty} u^H(x,t) = 0 = \lim_{t\to\infty} v^H(x,t)$ uniformly in x, we obtain:

$$\lim_{t\to\infty} s(t)\left(1 + \frac{s(t)H}{2k_2}\right) = Q(t),$$

then

$$0 \leq Q(t) \leq b\left(1 + \frac{bH}{2k_2}\right).$$

□

Lemma 11. *Suppose* $t_0 \leq T$, *let* $\lim_{t\to t_0} s(t) > 0$ *and*

$$Q(t) = \rho l b\left(1 + \frac{bH}{2k_2}\right) + \int_0^{t_0} Hf(\tau)\,d\tau + \int_0^{\infty} \Phi(x)\,dx > 0.$$

If we define a function η *as*

$$\eta(t) = \begin{cases} \max\{x \in [0, s(t)] : u^H(x,t) \leq -1\} \\ 0, \ \textit{if}\ u^H(x,t) > -1, x \in [0, s(t)], \end{cases}$$

then $\limsup_{t\to t_0} \eta(t) < \lim_{t\to t_0} s(t)$.

Proof. The proof is the same as this done in Comparini et al [6]. □

Lemma 12. *Case (C)* $\Longrightarrow$ $Q(T_C) \leq 0$.

Proof. Suppose $Q(T_C) > 0$. Then by using Lemma 12, the free boundary should be separated from the isoterm $u^H = -1$, contradicting the hypothesis of case (C). □

Lemma 13. *If* $Q(t) > 0$ *for every* $t > 0$, *then we have case (A).*

Proof. If we have case (B), then $Q(T_B) \leq 0$, contradicting the hypothesis. If we have case (C), then $Q(T_C) \leq 0$, contradicting the hypothesis, then the only possibility is the case (A). □

Remark 3. Note that for $Q < 0$, it is impossible to characterize cases (B) and (C) only by the value of $Q(t)$, they depend on the initial configuration ψ, ϕ and on the boundary data f (see also Marangunic and Turner [12]), this in contrast to the one-phase problem in Fasano and Primicerio [8].

(ii) Case II: The superheated solid.

The initial temperature for the liquid will be positive, i.e. $u(x,0) = \varphi(x) \geq 0$ and the initial temperature for the solid will be positive, i.e. $v(x,0) = \psi(x) \geq 0$. In this case the solid is superheated and the liquid is classical.

In the same way as before we can obtain analogous results for this case.

REFERENCES

[1] V. Aiki, Two phase Stefan problems with dynamic boundary conditions, *Advances in Math. Sci. and Appl.*, **22** (1993), 253-270.

[2] J. R. Cannon and C. D. Hill, Remarks on a Stefan problem, *J. Math. Mech.*, **17** (1967), 433-441.

[3] J. R. Cannon and M. Primicerio, Remarks on the one-phase Stefan problem for the heat equation with the flux prescribed on the fixed boundary, *J. Math. Anal. Appl.*, **35** (1971), 361-373.

[4] J.R. Cannon and M. Primicerio, A two phase Stefan problem with boundary conditions, *Ann. Mat. Pura Appl.*, **88** (1971), 177-192.

[5] J.R. Cannon and M. Primicerio, A Stefan problem involving the appearence of a phase, *J. Math. Anal. Appl.*, **4** (1973), 141-149.

[6] E. Comparini, R. Ricci and D.A. Tarzia, Remarks on a one dimensional Stefan problem related to the diffusion-comsumption model, *Z. Angew. Math. Mech.*, **64** (1984), 543-550.

[7] A. Fasano and M. Primicerio, General free boundary problems for the heat equation, *J. Math. Anal. Appl.*, I; II **57** (1977), 694-723; **58** (1977), 202-231; III **57** (1977), 1-14.

[8] A. Fasano and M. Primicerio, New results in some classical parabolic free boundary problems, *Quarterly Appl. Math.*, **38** (1980), 439-460.

[9] A. Friedman, *Partial Differential Equations of Parabolic Type*, Prentice Hall, Englewood Cliffs, 1964.

[10] A. Friedman, One dimensional Stefan problems with monotone free boundary, *Trans. Amer. Math. Soc.*, **133** (1968), 89-114.

[11] P. Knaber, Global existence in general Stefan like problems, *J. Anal. Math. Appl.*, **115** (1986), 543-559.

[12] P.R. Marangunic and C.V. Turner, The behavior of the solutions of a two-phase Stefan problem and the value of an energy integral, *Bolletino de la UMI*, **C 12** (1986), 215-227.

[13] A.D. Solomon, D.G. Wilson and V. Alexiades, The Stefan problem with a convective boundary condition, *Quart. Appl. Math.*, **40** (1982), 203-217.

[14] D.A. Tarzia and C.V. Turner, The asymptotic behavior for the one-phase Stefan problem with a convective boundary condition, *Appl. Math. Letters*, **9** (1996), 21-24.

[15] D.A. Tarzia, Etude de l' inéquation variationnelle proposée par Duvaut pour le problème de Stefan à deux phases I, II, *Boll. Un. Mat. Italiana*, **1 B** (1982), 865-833; **2 B** (1983), 589-603.

[16] C. Turner, Remarks on a two phase Stefan problem with flux boundary conditions, *Comp. Math. Appl.*, 9 (1990), 79-86.