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The Asymptotic Behavior for the One-Phase Stefan Problem with a Convective Boundary Condition

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Abstract—We consider the one-phase Stefan problem with a convective boundary condition at the fixed face, given by the temperature of the external fluid ($G(t)$) depending on time. We study the asymptotic behavior of the corresponding free boundary $s_\beta(t)$ when the time goes to infinity and we obtain $\lim_{t\to\infty}\left(s_\beta(t)/\sqrt{2\int_0^t G(\tau)\,d\tau}\right)=1$ for all heat transfer coefficients $\beta>0$.

Keywords—One-phase Stefan problem, Phase change process, Asymptotic behavior, Melting, Free boundary problem.

In this paper, we study the asymptotic behavior when $t\to\infty$ of the following parabolic free boundary problem (one-phase Stefan problem with a convective boundary condition on the fixed boundary $x=0$):

Problem (P):

$$\begin{aligned}
z_{xx} &= z_t, && \text{in } D_T; && (1)\\
s(0) &= 1; && && (2)\\
z\left(s(t),t\right) &= 0, && 0<t<T; && (3)\\
z_x\left(s(t),t\right) &= -\dot{s}(t), && 0<t<T; && (4)\\
z(x,0) &= \varphi(x), && 0<x<1; && (5)\\
z_x(0,t) &= \beta\left[z(0,t)-G(t)\right], && 0<t<T, && (6)
\end{aligned}$$

where $D_T=\{(x,t)\mid 0<x<s(t),\ 0<t<T\}$, $\beta>0$, $\varphi(x)\ge 0$, $0<x<1$, $G(t)\ge 0$, $t>0$ and the compatibility conditions $\varphi'(0)=\beta\left[\varphi(0)-G(0)\right]$ and $\varphi(1)=0$.

Existence and uniqueness for Problem (P) is given in [1]. Asymptotic behaviors for the one-phase problem with temperature boundary condition on the fixed face are given by [2,3].

For the particular case $G(t)=\text{Const}>0$, the study of the asymptotic behavior is obtained by using the variational inequality for the multidimensional case [4,5] and in [6] for the one-dimensional case. A general boundary condition is considered in [7,8] by using a quasi-variational

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inequality for the one-dimensional case. The same problem for the supercooled Stefan problem ($\varphi(x) \leq 0$, $G(t) \leq 0$) is considered in [9].

THEOREM 1. *Let (T, s_β, z_β) be a solution of Problem (P) satisfying the following hypotheses on φ and G.*

$$\varphi'(x) \leq 0 \qquad \text{for } 0 \leq x \leq 1; \tag{H1}$$

$$\dot{G}(t) \geq 0, \qquad \text{for } t > 0; \tag{H2}$$

$$\max_{[0,1]} \varphi(x) \leq G(0); \tag{H3}$$

then

(a) $z_\beta(x,t) \leq z_\infty(x,t)$ *in* D_T, $s_\beta(t) \leq s_\infty(t)$ $\forall \beta > 0$, $t > 0$.
(b) $\beta_1 \leq \beta_2$, *then* $z_{\beta_1}(x,t) \leq z_{\beta_2}(x,t)$ *in* D_T *and* $s_{\beta_1}(t) \leq s_{\beta_2}(t)$.

PROOF. This is obtained by using the maximum principle.

LEMMA 2. *Problem (P) depends monotonically on G.*

PROOF. This is obtained by using the maximum principle.

THEOREM 3.

(a) *If* $\int_0^\infty G(\tau)\,d\tau < \infty$, *then* $\lim_{t\to\infty} s(t) = s_\infty$ *where* $s_\infty = (\sqrt{1+2\beta A} - 1)/\beta$ *is the unique positive solution of the equation*

$$x\left(1 + \frac{\beta}{2}x\right) = A(\beta, \varphi, G), \qquad x > 1,$$

where $A(\beta,\varphi,G) = 1 + \beta/2 + \int_0^1 (1+\beta\xi)\varphi(\xi)\,d\xi + \beta\int_0^\infty G(\tau)\,d\tau$.

(b) *Let (s,z) be a solution of Problem (P) with* $\int_0^\infty G(\tau)\,d\tau = \infty$. *For each* $t_0 \geq 0$, *let* (σ, v) *be the solution of the following problems:*
   (i) $v_{xx} = v_t$, $0 < x < \sigma(t)$, $t > t_0$;
   (ii) $v_x(0,t) = \beta[v(0,t) - G(t)]$, $t > t_0$;
   (iii) $v(\sigma(t),t) = 0$, $t > t_0$;
   (iv) $\sigma(t_0) = 0$;
   (v) $\dot{\sigma}(t) = -v_x(\sigma(t),t)$, $t > t_0$.

Then we obtain

$$1 \leq \left(\frac{s(t)}{\sigma(t)}\right)^2 \leq 1 + \frac{C(t_0)}{\sigma^2(t)}, \qquad t > t_0,$$

where

$$C(t_0) = s^2(t_0) + \frac{2s(t_0)}{\beta} + \frac{2\int_0^{s(t_0)}(1+\beta x)z(x,t_0)\,dx}{\beta},$$

and

$$\lim_{t\to\infty} \frac{s(t)}{\sigma(t)} = 1.$$

PROOF. (a) The solution of the Problem (P) satisfies

$$s(t)\left(1 + \frac{\beta}{2}s(t)\right) = Q(t) - \int_0^{s(t)} z(x,t)\,dx \leq Q(t) \leq A(\beta,\varphi,G),$$

where $Q(t) = 1 + \beta/2 + \int_0^1 (1+\beta x)\varphi(x)\,dx + \beta\int_0^t G(\tau)\,d\tau$.

Thus we obtain $s(t) \leq s_\infty$ for $t \geq 0$.

When the function G has compact support, let W be the solution of the following problems:

(i) $W_t = W_{xx}$, $0 < x < s_\infty$;

(ii) $W(x,0) = \begin{cases} \varphi(x) & \text{if } 0 < x < 1, \\ 0 & \text{if } 1 < x < s_\infty; \end{cases}$

(iii) $W(s_\infty, t) = 0$, $t > 0$;

(iv) $W_x(0,t) = \beta[W(0,t) - G(t)]$, $t > 0$.

Using the maximum principle, we obtain $z(x,t) \le W(x,t)$ in D_T and we deduce that

$$\lim_{t\to\infty} \int_0^{s(t)} (1+\beta x) z(x,t)\, dx = 0.$$

Then the proposition holds.

We have to complete the proof for general G not necessarily with compact support. Let

$$G_n(t) = \begin{cases} G(t), & 0 < t < n, \\ 0, & t > n. \end{cases}$$

For each G_n, we have a problem noted P_n for z_n and s_n. Since G_n has compact support $\lim_{t\to\infty} s_n(t) = s_{n\infty}$. Using monotonicity, it follows that $s_n < s_m$, for all $n < m$ (since $G_n < G_m$), and $s_{n\infty} \le s_{m\infty}$ and $\lim_{n\to\infty} s_{n\infty} = s_\infty$ ($\lim_{n\to\infty} G_n = G$).

(b) Using the maximum principle and the fact that $\sigma(t) < s(t)$ for $t > t_0$, we obtain $z(x,t) > v(x,t)$, $0 < x < \sigma(t)$, $t > t_0$. Now, we use an integral representation associated to Problem (P), with an adequate initial condition at $t = t_0$ and we get

$$\begin{aligned} s(t) + \frac{\beta s^2(t)}{2} &= s(t)\left(1 + \frac{\beta}{2} s(t)\right) \\ &\le \frac{\beta C(t_0)}{2} + \int_0^t \beta G(\tau)\, d\tau - \int_0^{\sigma(t)} (1+\beta x)\, v(x,t)\, dx \\ &= \sigma(t)\left(1 + \sigma(t)\frac{\beta}{2}\right) + \frac{\beta C(t_0)}{2} \le s(t) + \frac{\beta\sigma(t)^2}{2} + \frac{\beta C(t_0)}{2}; \end{aligned}$$

then $\sigma^2(t) \le s^2(t) \le \sigma^2(t) + C(t_0)$, $t > t_0$, from which we obtain the result. ∎

THEOREM 4. *Let (T, s_β, z_β) be a solution of Problem (P) with the hypothesis (H3); then if*

$$\int_0^t G(\tau)\, d\tau = \infty, \quad \int_{t_0}^t G(\tau)\, d\tau < \infty, \qquad \text{for all } t \text{ and } t_0,$$

and $\lim_{t_0\to\infty} \max_{[t_0,\infty)} G(\tau) = \lim_{t_0\to\infty} \|G\|_{[t_0,\infty)} = 0$, we have

$$\lim_{t\to\infty} \frac{s_\beta(t)}{\sqrt{2\int_0^t G(\tau)\, d\tau}} = 1 \qquad \text{for all } \beta > 0.$$

PROOF. We will use the definition of the function $v(x,t)$ of Theorem 3(b).

If we write an integral representation for the pair $(\sigma, v) = (\sigma_\beta, v_\beta)$ and use the maximum principle, we obtain $v_\beta(x,t) \le \|G\|_{[t_o,t]}$ and then

$$\begin{aligned} \sigma_\beta(t)\left(1 + \frac{\beta}{2}\sigma_\beta(t)\right) &\ge \int_{t_0}^t \beta G(\tau)\, d\tau - \int_0^{\sigma_\beta(t)} (1+\beta x)\|G\|_{[t_o,t]}\, dx \\ &\ge \int_{t_0}^t \beta G(\tau)\, d\tau - \|G\|_{[t_0,t]}\sigma_\beta(t)\left(1 + \frac{\beta}{2}\sigma_\beta(t)\right). \end{aligned}$$

Thus we obtain

$$\frac{\beta\int_{t_0}^{t} G(\tau)\,d\tau}{1+\|G\|_{[t_0,t]}} \leq \sigma_\beta(t)\left(1+\frac{\beta}{2}\sigma_\beta(t)\right).$$

For (s_β, z_β), we have

$$s_\beta(t)\left(1+\frac{\beta}{2}s_\beta(t)\right) \leq Q(t) = D(\beta,\varphi) + \int_0^t \beta G(\tau)\,d\tau.$$

Since $\sigma_\beta(t) < s_\beta(t)$, dividing by $\beta\int_0^t G(\tau)\,d\tau$ and taking the limit when $t \longrightarrow \infty$ and the limit when $t_0 \longrightarrow \infty$, the inequality becomes

$$1 \leq \lim_{t\to\infty} \frac{s_\beta(t)\,(1+(\beta/2)s_\beta(t))}{\beta\int_0^t G(\tau)\,d\tau} \leq 1.$$

COROLLARY 5. *(Convergence when $\beta \longrightarrow \infty$.) If (s_β, z_β) is a solution of the Problem (P) and (s_∞, z_∞) is a solution of the Problem (P_∞), with the hypotheses (H1), (H2) and (H3), then*

(i) $\lim_{\beta\to\infty} s_\beta(t) = s_\infty(t)$ *for each* $t > 0$,

(ii) $\lim_{\beta\to\infty} z_\beta(x,t) = z_\infty(x,t)$ *for each* $0 \leq x < s_\infty(t)$, *for each* $t > 0$.

PROOF. The solutions z_β and z_∞ satisfy the following inequality for all β:

$$0 \leq \int_0^{s_\beta(t)} x\,(z_\infty(x,t) - z_\beta(x,t))\,dx + \frac{\left(s_\infty^2(t) - s_\beta^2(t)\right)}{2} \leq \frac{s_\infty(t)}{\beta}\,[1+\|G\|_t].$$

Using the fact that $s_\beta(t) \leq s_\infty(t)$ and $z_\beta \leq z_\infty$ for all β, the left-hand side terms of the inequality are positive. Thus,

$$0 \leq \frac{s_\infty^2(t) - s_\beta^2(t)}{2} \leq \frac{s_\infty(t)}{\beta}\,(1+\|G\|_t) \qquad \text{for all } \beta.$$

Letting β tend to infinity for each $t > 0$, then $\lim_{\beta\to\infty} s_\beta(t) = s_\infty(t)$ and $\lim_{\beta\to\infty}\int_0^{s_\infty(t)} x(z_\infty(x,t) - z_\beta(x,t))\,dx = 0$. Then we can conclude $\lim_{\beta\to\infty} z_\beta(x,t) = z_\infty(x,t)$ for each $0 \leq x < s_\infty(t)$, for each $t > 0$. ∎

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