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Exact solution for a Stefan problem with convective boundary condition and density jump

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We consider the solidification of a semi-infinite material which is initially at its liquid phase at a uniform temperature T_i. Suddenly at time $t > 0$ the fixed face $x = 0$ is submitted to a convective cooling condition with a time-dependent heat transfer coefficient of the type $H(t) = ht^{-1/2}$ $(h > 0)$ The bulk temperature of the liquid at a large distance from the solid-liquid interface is T_∞, a constant temperature such that $T_\infty < T_f < T_i$ where T_f is the freezing temperature. We also consider the density jump between the two phases. We obtain that the corresponding phase-change (solidification) process has an explicit solution of a similarity type for the temperature of both phases and the solid-liquid interface, if and only if the coefficient h is large enough, that is $h > h_0 = \frac{k_l}{\sqrt{\pi\alpha_l}}\frac{T_i - T_f}{T_i - T_\infty}$ where k_l and α_l are the conductivity and diffusion coefficients of the initial liquid phase. Moreover, when $h \leq h_0$ we only have a heat conduction problem for the initial liquid phase and the corresponding change of phase does not occur. We do the mathematical analysis of a problem which was presented in S.M. Zubair - M.A. Chaudhry, Wärme-und-Stoffübertragung, 30 (1994), 77-81 and we generalized the results obtained in D.A. Tarzia, MAT-Serie A, 8 (2004), 21-27 when the density jump between the two phases was neglected.

1 Introduction

Heat transfer problems involving a change of phase due to melting or freezing processes are very important in science and technology [3], [5]. This kind of problems are generally refered as moving-free boundary problems which have been the subject of numerous theoretical, numerical and experimental investigations, e.g. we can see the large bibliography on the subject given in [7]. We consider the solidification of a semi-infinite material which is initially at its liquid phase at a uniform temperature T_i. Suddenly at time $t > 0$ the fixed face $x = 0$ is submitted to a convective cooling condition due to a sudden drop in the ambient temperature. The bulk temperature of the liquid at a large distance from the solid- liquid interface is T_∞, a constant such that $T_\infty < T_f < T_i$ where T_f is the freezing temperature. The density jump between the two phases are also considered. In order to solve the phase-change process with a convective condition at the fixed face $x = 0$, approximate method were used, for example in [1], [2], [4]. In [9] the problem was analyzed and a closed-form expression for the solid-liquid interface and both temperatures were found when the heat transfer coefficient h is time-dependent and proportional to $t^{-\frac{1}{2}}$. The solution is obtain graphically. The goal of this paper is to give the mathematical analysis of this problem, that is the solidification of a semi-infinite material which is initially at the constant temperature T_i and a convective cooling condition is impossed at the fixed boundary $x = 0$ for a time-dependent heat transfer coefficient of the type

$$H(t) = h/\sqrt{t},\ h > 0,\ t > 0, \tag{1}$$

and the density jump between the two solid and liquid phases are also considered. We prove that there exists an instantaneous phase-change process if and only if the coefficient h is large enough, that is

$$h > \frac{k_l}{\sqrt{\pi\alpha_l}}\frac{T_i - T_f}{T_i - T_\infty} \tag{2}$$

T_f

where k_i and $\alpha_i = k_i/\rho_i c_i$ are the conductivity and diffusion coefficients of the phase i; c_i and ρ_i are the specific heat and the mass density of the phase i ($i = s$: solid; $i = l$: liquid). Moreover, the explicit expression for the solid-liquid interface $s(t)$ and the temperatures of the solid $T_s(x,t)$ and liquid $T_l(x,t)$ phases respectively can be obtained if and only if the inequality (2) is verified for the coefficient h which characterizes the dependent-time heat transfer coefficient $H(t)$given by (1).

2 Explicit solution for a convective boundary condition and density jump.

We consider the following free boundary problem: find the solid-liquid interface $x = s(t)$ and the temperature $T(x,t)$ defined by $T_s(x,t)$ if $0 < x < s(t), t > 0, T_f$ if $x = s(t), t > 0$ and $T_l(x,t)$ if $x > s(t), t > 0$ which satisfy the following equations and boundary conditions

$$T_{s_t} = \alpha_s T_{s_{xx}},\ 0 < x < s(t),\ t > 0 \tag{3}$$

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$$T_{l_t} \overset{-}{+} \frac{\rho_s - \rho_l}{\rho_l}\, \dot{s}\,(t)\, T_{l_x} = \alpha_l T_{l_{xx}},\ x > s\,(t)\,,\ t > 0 \tag{4}$$

$$T_s\,(s\,(t)\,, t) = T_l\,(s\,(t)\,, t) = T_f,\ x = s\,(t)\,,\ t > 0 \tag{5}$$

$$T_l\,(x, 0) = T_l\,(+\infty, t) = T_i,\ x > 0,\ t > 0 \tag{6}$$

$$k_s T_{s_x}\,(0, t) = \frac{h}{\sqrt{t}}\,(T_s\,(0, t) - T_\infty)\,,\ t > 0 \tag{7}$$

$$k_s T_{s_x}\,(s\,(t)\,, t) - k_l T_{l_x}\,(s\,(t)\,, t) = \rho_s l\, \dot{s}\,(t)\,,\ t > 0 \tag{8}$$

$$s\,(0) = 0 \tag{9}$$

where l is the latent heat of fusion, and $T_\infty < T_f < T_i$.

We obtain the following results:

Theorem 2.1 *(i) If the coefficient h verifies the inequality (2) then the free boundary problem (3)—(9) has the explicit solution of a similarity type given by*

$$s\,(t) = 2\xi\sqrt{\alpha_s t} \tag{10}$$

$$T_s\,(x, t) = T_\infty + \frac{(T_f - T_\infty)\left[1 + \frac{h\sqrt{\pi\alpha_s}}{k_s}\, erf\left(\frac{x}{2\sqrt{\alpha_s t}}\right)\right]}{1 + \frac{h\sqrt{\pi\alpha_s}}{k_s}\, erf\,(\xi)},\ 0 < x < s\,(t)\,,\ t > 0 \tag{11}$$

$erf\left((1+\varepsilon)\sqrt{\frac{\alpha_s}{\alpha_l}}\,\xi\right)$

$$T_l\,(x, t) = T_f + (T_i - T_f)\left[\, erf\left(\delta + \frac{x}{2\sqrt{\alpha_l t}}\right) - erf\left((1+\varepsilon)\sqrt{\frac{\alpha_s}{\alpha_l}}\xi\right)\right],\ x > s\,(t)\,,\ t > 0 \tag{12}$$

where erf is the error function and $erfc$ is the complementary error function defined by $erfc\,(z) = 1 - erf(z)$, and the dimensionless parameter $\xi > 0$ satisfies the following equation

$$G\,(x) = x,\ x > 0 \tag{13}$$

where function G and the coefficients δ and ε are given by

$$G\,(x) = b_1 \frac{\exp\left(-x^2\right)}{1 + b_2\, erf\,(x)} - b_3 \frac{\exp\left(-b^2 x^2\right)}{erfc\,(bx)} \tag{14}$$

$$\delta = \varepsilon\xi\sqrt{\frac{\alpha_s}{\alpha_l}},\quad \varepsilon = \frac{|\rho_s - \rho_l|}{\rho_l},\quad \alpha_0 = \frac{\alpha_l}{(1+\varepsilon)^2},\quad b = \sqrt{\frac{\alpha_s}{\alpha_0}} \tag{15}$$

$$b_1 = \frac{h\,(T_f - T_\infty)}{\rho_s l\sqrt{\alpha_l}},\quad b_2 = \cancel{1+}\frac{h}{k_s}\sqrt{\pi\alpha_s};\quad b_3 = \frac{k_l\,(T_i - T_f)}{\rho_s l\sqrt{\alpha_s\alpha_l\pi}}. \tag{16}$$

(ii) The Eq. (13) has a unique solution ξ if and only if the coefficient h satisfies the inequality (2). In this case, there exists an instantaneous solidification process.

Remark 2.2 (i) We note that the temperature at the fixed face $x = 0$ is given by $T_s\,(0, t) = Const. < T_f$.

(ii) We note that the inequality (2) for the coefficient h is of the type that it was obtained in [6] when a time-dependent heat flux condition on the fixed face is impossed.

(iii) When the jump density between the two phases is negleted, i.e. $\varepsilon = 0$, then we find the result obtained in [8].

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