Numerical Analysis of a Mixed Elliptic Problem with Flux and Convective Boundary Conditions to Obtain a Discrete Solution of Nonconstant Sign

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We consider a material that occupies a convex polygonal bounded domain $\Omega \subset \mathbb{R}^n$, with regular boundary $\Gamma = \Gamma_1 \cup \Gamma_2$ (with $\Gamma_1^0 \cap \Gamma_2^0 = \emptyset$) with meas $(\Gamma_1) = |\Gamma_1| > 0$ and $|\Gamma_2| > 0$. We assume, without loss of generality, that the melting temperature is 0°C. We consider the following steady-state heat conduction problem in Ω:

$$\Delta u = 0 \text{ in } \Omega, \quad -\frac{\partial u}{\partial n}\Big|_{\Gamma_1} = \alpha(u - B), \quad -\frac{\partial u}{\partial n}\Big|_{\Gamma_2} = q,$$

with $\alpha, q, B = \text{Const} > 0$, and q and α represent the heat flux on Γ_2 and the heat transfer coefficient on Γ_1, respectively. In a previous article (Tabacman– Tarzia, J Diff Eq 77 (1989), 16– 37) sufficient and/or necessary conditions on data $\alpha, q, B, \Omega, \Gamma_1, \Gamma_2$ to obtain a temperature u of nonconstant sign in Ω (that is, a multidimensional steady-state, two-phase, Stefan problem) were studied. In this article, we consider a regular triangulation by finite element method of the domain Ω with Lagrange triangles of the type 1, with $h > 0$ the parameter of the discretization. We study sufficient (and/or necessary) conditions on data $\alpha, q, B, \Omega, \Gamma_1$, and Γ_2 to obtain a change of phase (steady-state, two-phase, discretized Stefan problem) in the corresponding discretized domain, that is, a discrete temperature of nonconstant sign in Ω. Moreover, error bounds as a function of the parameter h, are also obtained. Numer Methods Partial Differential Eq 15: 355–369, 1999

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I. INTRODUCTION

We consider material that occupies a convex polygonal bounded domain $\Omega \subset \mathbb{R}^n$, with regular boundary $\Gamma = \Gamma_1 \cup \Gamma_2(\Gamma_1^0 \cap \Gamma_2^0 = \emptyset)$ with meas $(\Gamma_1) = |\Gamma_1| > 0$ and $|\Gamma_2| > 0$ (meas denote the $(n-1)$-dimensional Lebesgue measure). We assume, without loss of generality, that

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the melting temperature is 0°C. We consider the following steady-state heat conduction problem in Ω [1, 2]:

$$\left| \begin{array}{l} \Delta u = 0 \text{ in } \Omega \\ -\frac{\partial u}{\partial n}|_{\Gamma_1} = \alpha(u - B) \text{ on } \Gamma_1, \\ -\frac{\partial u}{\partial n}|_{\Gamma_2} = q \text{ on } \Gamma_2, \end{array} \right. \tag{1}$$

with $\alpha, q, B = \text{Const} > 0$, and q and α represent the heat flux on Γ_2 and the heat transfer coefficient on Γ_1, respectively. The variational formulation of problem (1) is given by [1, 3]: Find u in V such that

$$a_\alpha(u, v) = L_{\alpha q B}(v), \quad \forall v \in V, \tag{2}$$

where the functional spaces, bilinear and linear forms used are defined by

$$V = H^1(\Omega), \quad V_o = \{v \in V / v|_{\Gamma_1} = 0\},$$

$$a_\alpha(u, v) = a(u, v) + \alpha \int_{\Gamma_1} uv d\gamma, \quad a(u, v) = \int_\Omega \nabla u, \nabla v dx,$$

$$L_{\alpha q B}(v) = L_q(v) + \alpha B \int_{\Gamma_1} v d\gamma, L_q(v) = -q \int_{\Gamma_2} v d\gamma, \tag{3}$$

where H^1 is the usual Sobolev space of order 1.

On the other hand, problem (2) is equivalent to the following minimization problem: Find u in V such that

$$G(u) \leq G(v), \quad \forall v \in V, \tag{4}$$

where the functional G is defined by

$$G(v) = \frac{1}{2} a_\alpha(v, v) - L_{\alpha q B}(v). \tag{5}$$

The unique solution $u = u(\alpha) = u(\alpha, q, B)$ of problem (2) can be obtained by

$$u(\alpha, q, B) = B - qU(\alpha) \text{ in } \Omega, \tag{6}$$

where $U = U(\alpha)$ is the unique solution of the variational equality:
Find $U(\alpha)$ in V such that

$$a_\alpha(U(\alpha), v) = \int_{\Gamma_2} v d\gamma, \quad \forall v \in V. \tag{7}$$

We suppose that Ω and Γ have the necessary regularity for $U(\alpha)$ to be a continuous function in $\bar{\Omega}$ as in [4] ([2, 5] give three examples in which this condition is satisfied). An evolution Stefan problem with mixed boundary conditions is considered in [6, 7]. In [5] the following results are obtained.

Theorem 1. (i) *If* $(\alpha, q) \in S^2(B)$, *then we obtain a steady-state, two-phase Stefan problem with*

$$\left| \begin{array}{l} S^2(B) = \{(\alpha, q) \in (\mathbb{R}^+)^2 / q_m(\alpha, B) < q < q_M(\alpha, B)\}, \\ q_m(\alpha, B) = \frac{B|\Gamma_2|}{A(\alpha)}, q_M(\alpha, B) = \frac{B\alpha|\Gamma_1|}{|\Gamma_2|}, \end{array} \right. \tag{8}$$

where $A = A(\alpha)$ is a decreasing function in variable α, which verifies the following properties:

$$\left| \begin{array}{l} A(\alpha) > \frac{|\Gamma_2|^2}{|\Gamma_1|}\frac{1}{\alpha}, \lim_{\alpha\to+\infty} A(\alpha) = C > 0 \\ \lim_{\alpha\to+\infty} \alpha A'(\alpha) = 0, (\alpha A(\alpha))' = \frac{1}{q^2} a(u_{\alpha qB}, u_{\alpha qB}), \\ A(\alpha) = \int_{\Gamma_2} U(\alpha) d\gamma = a_\alpha(U(\alpha), U(\alpha)), \end{array} \right. \tag{9}$$

where "′" represents the derivative with respect to α. On the other hand, the constant $C > 0$ is given by

$$C = \int_{\Gamma_2} u_3 d\gamma = a(u_3, u_3) > 0, \tag{10}$$

where u_3 is the unique solution of the variational equality:

Find u_3 in V_0 such that

$$a(u_3, v) = \int_{\Gamma_2} v d\gamma, \quad \forall v \in V_0, \tag{11}$$

(ii) *For the continuous particular case, defined by the condition*

$$\frac{1}{q^2} a(u(\alpha, q, B), u(\alpha, q, B)) = \textit{ Const. } (= C > 0), \tag{12}$$

we have that the Const in (12) *is C, defined in* (10), *and the function $A = A(\alpha)$ is given explicitly by the expression*

$$A(\alpha) = C + \frac{|\Gamma_2|^2}{|\Gamma_1|}\frac{1}{\alpha}. \tag{13}$$

■

In this article, we consider τ_h, a regular triangulation of nonnegative type [8] of the polygonal domain Ω with Lagrange triangles of type 1, constituted by affine-equivalent finite element of class C^o, where $h > 0$ is a parameter that goes to zero. We can take h equal to the longest side of the triangles $T \in \tau_h$ and we can approximate V by [9, 10]:

$$V_h = \{v_h \in C^0(\bar{\Omega}) / v_h|_T \in P_1(T), \forall T \in \tau_h\} \subset V, \tag{14}$$

where P_1 is the set of the polynomials of degree less than or equal to 1. Let Π_h be the corresponding linear interpolation operator.

We consider the following finite dimensional approximate variational problem, corresponding to the continuous variational problem (2), given by:

Find $u_h = u_h(\alpha, q, B)$ in V_h such that

$$a_\alpha(u_h, v_h) = L_{\alpha qB}(v_h), \quad \forall v_h \in V_h, \tag{15}$$

and then we obtain the following results.

Theorem 2. (i) *There exists a unique solution $u_h(\alpha, q, B) \in V_h$ of the discretized problem* (15). *Moreover, $u_h(\alpha, q, B)$ is given by*

$$u_h(\alpha, q, B) = B - qU_h(\alpha), \tag{16}$$

where $U_h(\alpha)$ is the unique solution of the following variational equality: *Find $U_h(\alpha)$ in V_h such that*

$$a_\alpha(U_h(\alpha), v_h) = \int_{\Gamma_2} v_h d\gamma, \quad \forall v_h \in V_h. \tag{17}$$

(ii) *Function $U_h(\alpha)$ satisfies the following property*:

$$\int_{\Gamma_1} U_h(\alpha)d\gamma = \frac{|\Gamma_2|}{\alpha}, U_h(\alpha) > 0 \text{ in } \Omega. \tag{18}$$

(iii) *Function $u_h(\alpha, q, B)$ satisfies the following properties*:

$$\left| \begin{array}{l} u_h(\alpha, q, B) \leq B \text{ in } \bar{\Omega}, u_h(\alpha, q, B) \leq u_h(q, B) \text{ in } \bar{\Omega}, \\ u_h(\alpha, q, B) \to u_h(q, B) \text{ in } V \text{ when } \alpha \to +\infty, \\ Min_{\Gamma_2} u_h(\alpha, q, B) \leq u_h(\alpha, q, B) \leq Max_{\Gamma_1} u_h(\alpha, q, B) \text{ in } \bar{\Omega}, \end{array} \right. \tag{19}$$

where $u_h(q, B) = B - qu_{3h}$, being u_{3h} the unique solution of the following discrete variational equality: Find u_3 in V_{oh} such that

$$a(u_{3h}, v_h) = \int_{\Gamma_2} v_h d\gamma, \quad \forall v_h \in V_{oh}, \tag{20}$$

where

$$V_{oh} = \{v_h \in C^0(\bar{\Omega})/v_h|_T \in P_1(T), \forall T \in \tau_h, v_h|_{\Gamma_1} = 0\}. \tag{21}$$

(iv) *We have the equalities*

$$a(u_h(\alpha, q, B), u_h(q, B)) = a(u_h(q, B), u_h(q, B)). \tag{22}$$

$$\begin{aligned} a(u_h(\alpha, q, B) - u_h(q, B), u_h(\alpha, q, B) - u_h(q, B)) \\ = a(u_h(\alpha, q, B), u_h(\alpha, q, B)) - a(u_h(q, B), u_h(q, B)). \end{aligned} \tag{23}$$

(v) We have the following monotony property:

$$\alpha_1 \leq \alpha_2, q_2 \leq q_1 \Rightarrow u_h(\alpha_1, q_1, B) \leq u_h(\alpha_2, q_2, B) \text{ in } \bar{\Omega}, \quad \forall B > 0. \tag{24}$$

Proof. (i) The discretized variational equalities (15) and (16) have a unique solution $u_h(\alpha, q, B) \in V_h$ and $U_h(\alpha) \in V_h$, respectively, because the Lax–Milgram Theorem [3] and the fact that a_α is a bilinear, symmetric, continuous, and coercive form with

$$\left| \begin{array}{l} a_\alpha(v, v) \geq \lambda_\alpha \|v\|^2, \forall v \in V, \\ \lambda_\alpha = \lambda_1 \text{ Inf } (1, \alpha), \forall \alpha > 0, \end{array} \right. \tag{25}$$

where $\lambda_1 > 0$ is the coercive constant of the bilinear form a_1 (a_1 is the particular case of a_α when $\alpha = 1$), that is

$$a_1(v, v) \geq \lambda_1 \|v\|^2, \quad \forall v \in V. \tag{26}$$

Moreover, $u_h(\alpha, q, B)$ is the solution of (15) if and only if $U_h(\alpha)$ is the unique solution of (17) because of the linearity and the uniqueness of (15) and (17).

(ii) By choosing $v_h = 1 \in V_h$ in (17), we get

$$|\Gamma_2| = \int_{\Gamma_2} 1 d\gamma = a_\alpha(U_h(\alpha), 1) = \alpha \int_{\Gamma_1} U_h(\alpha) d\gamma \tag{27}$$

that is (18).

(iii) and (v). They follow by an analogous method to the one used in [5].

(iv) If we choose $v = u_h(q, B) \in B + V_{0h} \subseteq V_h$ in the variational equality (15), we obtain

$$a_\alpha(u_h(\alpha, q, B), u_h(q, B)) = L_{\alpha q B}(u_h(q, B)) = -q \int_{\Gamma_2} u_h(q, B) d\gamma + \alpha B^2 |\Gamma_1|, \quad (28)$$

where, by definition, we have

$$a_\alpha(u_h(\alpha, q, B), u_h(q, B)) = a(u_h(\alpha, q, B), u_h(q, B)) + \alpha B \int_{\Gamma_1} u_h(\alpha, q, B) d\gamma. \quad (29)$$

On the other hand, we have

$$a(u_h(q, B), B - u_h(q, B)) = -q \int_{\Gamma_2} (B - u_h(q, B)) d\gamma,$$

that is

$$a(u_h(q, B), u_h(q, B)) = qB|\Gamma_2| - q \int_{\Gamma_2} u_h(q, B) d\gamma. \quad (30)$$

Therefore, from (18), (28)– (30) we get

$$\begin{aligned} a(u_h(\alpha, q, B), u_h(q, B)) &= -q \int_{\Gamma_2} u_h(q, B) d\gamma + \alpha B^2 |\Gamma_1| - \alpha B \int_{\Gamma_1} u_h(\alpha, q, B) d\gamma \\ &= a(u_h(q, B), u_h(q, B)) - qB|\Gamma_2| + \alpha B^2 |\Gamma_1| \\ &\quad - \alpha B \left(B|\Gamma_1| - \frac{q|\Gamma_2|}{\alpha} \right) \\ &= a(u_h(q, B), u_h(q, B)), \end{aligned} \quad (31)$$

that is (22).

Moreover, (23) holds, because

$$\begin{aligned} &a(u_h(\alpha, q, B) - u_h(q, B), u_h(\alpha, q, B) - u_h(q, B)) \\ &= a(u_h(\alpha, q, B), u_h(\alpha, q, B)) - 2a(u_h(\alpha, q, B), u_h(q, B)) \\ &\quad + a(u_h(q, B), u_h(q, B)) \\ &= a(u_h(\alpha, q, B), u_h(\alpha, q, B)) - a(u_h(q, B), u_h(q, B)). \end{aligned} \quad (32)$$

■

We define, for each $h > 0$, the following real functions:

$$\left| \begin{array}{l} A_h(\alpha) = \int_{\Gamma_2} U_h(\alpha) d\gamma = a_\alpha(U_h(\alpha), U_h(\alpha)) > 0, \forall \alpha > 0, \\ q_{m_h}(\alpha, q) = \frac{B|\Gamma_2|}{A_h(\alpha)}, \forall \alpha, q > 0. \end{array} \right. \quad (33)$$

The goal of this article is to consider the discrete equivalent of the inequalities (8), which define the set $S^2(B)$. We shall obtain sufficient (and/or necessary) conditions on data $\alpha, q, B, h, \Omega, \Gamma_1$, and Γ_2 to get that the discrete solution $u_h(\alpha, q, B)$ is of nonconstant sign in Ω (that is, a steady-state, two-phase, discretized Stefan problem). We shall obtain error bounds for $A_h(\alpha) - A(\alpha)$ and for $q_{m_h}(\alpha, B) - q_m(\alpha, B)$ as functions of the parameter h. We also will analyze the discrete particular case corresponding to the continuous particular case (12).

In other words, we shall obtain for the solution of the mixed elliptic discretized problem (15), defined as $u_h(\alpha, q, B)$, analogous conditions to the ones obtained for the corresponding continuous problem [5], defined by $u(\alpha, q, B)$. For the corresponding numerical analysis, we use ideas developed recently in [11].

II. CONDITIONS FOR THE EXISTENCE OF A DISCRETE SOLUTION OF NONCONSTANT SIGN

For each $q > 0, \alpha > 0, B > 0$, we consider the functions $u(\alpha, q, B) \in V$ and $u_h(\alpha, q, B) \in V_h$, respectively, as the unique solution of the variational equalities (2) (continuous problem) and (15) (discrete problem). Therefore, we obtain the following properties.

Lemma 1. *We have*:

(i) *Function* $A_h(\alpha)$ *is also given by the expression*:

$$A_h(\alpha) = a_\alpha(U(\alpha), U_h(\alpha)). \tag{34}$$

(ii) *We have*

$$A(\alpha) - A_h(\alpha) = a_\alpha(U_h(\alpha) - U(\alpha), U_h(\alpha) - U(\alpha)) \geq 0. \tag{35}$$

(iii) *Functions* q_m *and* q_{m_h} *are related by the following inequality*:

$$q_m(\alpha, B) \leq q_{m_h}(\alpha, B). \tag{36}$$

(iv) *We have the following integral expressions*:

$$\int_{\Gamma_1} u_h(\alpha, q, B)d\gamma = \frac{|\Gamma_2|}{\alpha}[q_M(\alpha, B) - q], \quad \forall h > 0, \tag{37}$$

$$\int_{\Gamma_2} u_h(\alpha, q, B)d\gamma = A_h(\alpha)[q_{m_h}(\alpha, B) - q], \quad \forall h > 0. \tag{38}$$

Proof. (i) If we choose $v = U_h(\alpha) \in V_h \subset V$ in the variational equality (17) we have

$$a_\alpha(U(\alpha), U_h(\alpha)) = \int_{\Gamma_2} U_h(\alpha)d\gamma = a_\alpha(U_h(\alpha), U_h(\alpha)) = A_h(\alpha), \tag{39}$$

that is (34).

(ii) We have

$$\begin{aligned} 0 &\leq a_\alpha(U_h(\alpha) - U(\alpha), U_h(\alpha) - U(\alpha)) \\ &= a_\alpha(U_h(\alpha), U_h(\alpha)) + a_\alpha(U(\alpha), U(\alpha)) - 2a_\alpha(U_h(\alpha), U(\alpha)) \\ &= A_h(\alpha) + A(\alpha) - 2A_h(\alpha) = A(\alpha) - A_h(\alpha). \end{aligned} \tag{40}$$

(iii) (36) follows from the fact that $A(\alpha) \geq A_h(\alpha)$.

(iv) Taking into account (18) we obtain

$$\begin{aligned} \int_{\Gamma_1} u_h(\alpha, q, B)d\gamma &= \int_{\Gamma_1} (B - qU_h(\alpha))d\gamma = B|\Gamma_1| - q\int_{\Gamma_1} U_h(\alpha)d\gamma \\ &= B|\Gamma_1| - q\frac{|\Gamma_2|}{\alpha} = \frac{|\Gamma_2|}{\alpha}(q_M(\alpha, B) - q), \quad \forall h > 0, \end{aligned} \tag{41}$$

that is (37). On the other hand, (38) holds, because

$$\begin{aligned}\int_{\Gamma_2} u_h(\alpha,q,B)d\gamma &= B|\Gamma_2| - q\int_{\Gamma_2} U_h(\alpha)d\gamma = B|\Gamma_2| - qA_h(\alpha)\\ &= A_h(\alpha)[q_{m_h}(\alpha,B)-q], \quad \forall h>0. \end{aligned} \tag{42}$$

■

Remark 1. From (37) and (38) we obtain the following equivalences:

$$\int_{\Gamma_2} u_h(\alpha,q,B)d\gamma < 0 \Leftrightarrow q > q_{m_h}(\alpha,B), \tag{43}$$

$$\int_{\Gamma_1} u_h(\alpha,q,B)d\gamma > 0 \Leftrightarrow q < q_M(\alpha,B). \tag{44}$$

■

For each $h>0$, we define the real function $g_h : (\mathbb{R}^+)^3 \to \mathbb{R}$ in the following way:

$$\begin{aligned} g_h(\alpha,q,B) &= G_{\alpha qB}(u_h(\alpha,q,B)) = -\frac{1}{2}L_{\alpha qB}(u_h(\alpha,q,B))\\ &= -\frac{1}{2}a_\alpha(u_h(\alpha,q,B),u_h(\alpha,q,B)) < 0, \quad \forall \alpha,q,B>0. \end{aligned} \tag{45}$$

Remark 2. Owing to (45) we deduce

$$\begin{aligned} g_h(\alpha,q,B) &= -\frac{1}{2}L_{\alpha qB}(u_h(\alpha,q,B))\\ &= \frac{q}{2}\int_{\Gamma_2} u_h(\alpha,q,B)d\gamma - \frac{\alpha B}{2}\int_{\Gamma_1} u_h(\alpha,q,B)d\gamma\\ &= \frac{q}{2}[B|\Gamma_2| - qA_h(\alpha)] - \frac{\alpha B}{2}\left[B|\Gamma_1| - \frac{q|\Gamma_2|}{\alpha}\right]\\ &= -\frac{A_h(\alpha)}{2}q^2 + Bq|\Gamma_2| - \frac{\alpha B^2}{2}|\Gamma_1| < 0, \quad \forall \alpha,q,B>0. \end{aligned} \tag{46}$$

■

Corollary 1. *From* (46) *we have*

$$A_h(\alpha) > \frac{2B|\Gamma_2|}{q} - \frac{\alpha B^2|\Gamma_1|}{q^2}, \quad \forall \alpha,h>0, \forall q,B>0, \tag{47}$$

and, therefore, we obtain

$$A_h(\alpha) > \frac{|\Gamma_2|^2}{\alpha|\Gamma_1|}, \quad \forall \alpha,h>0, \tag{48}$$

by choosing adequate numbers q and B. ■

Theorem 3. (i) *We have the following inequalities*:

$$q_m(\alpha,B) \le q_{m_h}(\alpha,B) < q_M(\alpha,B), \quad \forall \alpha,B>0; \tag{49}$$

therefore, the set $S_h^2(B)$ is nonempty, where

$$S_h^2(B) = \{(\alpha,q)\in(\mathbb{R}^+)^2/q_{m_h}(\alpha,B) < q < q_M(\alpha,B)\} \ne \emptyset. \tag{50}$$

(ii) *If* $(\alpha, q) \in S_h^2(B)$, *for a given* $B > 0$, *then the discrete temperature* $u_h(\alpha, q, B)$ *is of nonconstant sign in* Ω, *that is, we have a steady-state, two-phase Stefan problem in the corresponding discretized domain.*

Proof. (i) The left and right inequalities in (49) are obtained from (36) and (47), respectively, because

$$q_{m_h}(\alpha, q) = \frac{B|\Gamma_2|}{A_h(\alpha)} < \frac{B\alpha|\Gamma_1|}{|\Gamma_2|}. \tag{51}$$

(ii) From (43) and (44) we get

$$\int_{\Gamma_2} u_h(\alpha, q, B)d\gamma < 0 \text{ and } \int_{\Gamma_1} u_h(\alpha, q, B)d\gamma > 0. \tag{52}$$

This means that $u_h(\alpha, q, B)$ is a function of nonconstant sign in Ω, that is, we have a steady-state, two-phase Stefan problem in the corresponding discretized domain. ∎

Lemma 2. (i) *We have the following estimates*:

$$\left| \begin{array}{l} u_{3h} \leq U_h(\alpha) \text{ in } \bar{\Omega}, \\ C_h \leq A_h(\alpha), \\ q_{m_h}(\alpha, B) \leq q_{o_h}(B), \end{array} \right. \tag{53}$$

where

$$C_h = \int_{\Gamma_2} u_{3h} d\gamma, \quad q_{o_h}(B) = \frac{B|\Gamma_2|}{C_h}. \tag{54}$$

(ii) *The function* $A_h = A_h(\alpha)$ *is a decreasing function in* α *and verifies*

$$\lim_{\alpha \to +\infty} A_h(\alpha) = C_h, \quad \forall h > 0. \tag{55}$$

(iii) *The function* $q_{m_h} = q_{m_h}(\alpha, B)$ *is an increasing function in* α *and verifies the following properties*:

$$q_{m_h}(0^+, B) = 0, \quad q_{m_h}(+\infty, B) = q_{o_h}(B), \quad \forall B, h > 0. \tag{56}$$

Proof. From (6), (16), and (19), we obtain $B - qU_h(\alpha) \leq B - qu_{3h}$ in $\bar{\Omega}$, that is, $u_{3h} \leq U_h(\alpha)$ in $\bar{\Omega}$. This implies that

$$C_h = \int_{\Gamma_2} u_{3h} d\gamma \leq \int_{\Gamma_2} U_h(\alpha) d\gamma = A_h(\alpha) \tag{57}$$

and

$$q_{m_h}(\alpha, B) \leq \frac{B|\Gamma_2|}{A_h(\alpha)} \leq \frac{B|\Gamma_2|}{C_h} = q_{o_h}(B). \tag{58}$$

(ii) From (24), we get that $u_h(\alpha, q, B)$ is an increasing function in α, i.e., $U_h(\alpha)$ and then $A_h(\alpha)$ are decreasing functions in α. This means that $q_{m_h}(\alpha, q, B)$, defined by (23), is an increasing function in α.

Moreover, from (19) we obtain that $U_h(\alpha) \to u_{3h}$ in V when $\alpha \to +\infty$, that is

$$\lim_{\alpha \to +\infty} A_h(\alpha) = \lim_{\alpha \to +\infty} \int_{\Gamma_2} U_h(\alpha) d\gamma = \int_{\Gamma_2} u_{3h}(\alpha) d\gamma = C_h, \quad \forall h > 0. \tag{59}$$

(iii) From (ii) and (33), we get that $q_{m_h}(\alpha, B)$ is an increasing function in α and satisfies

$$q_{m_h}(+\infty, B) = \lim_{\alpha\to+\infty} q_{m_h}(\alpha, B) = \frac{B|\Gamma_2|}{C_h} = q_{o_h}(B), \quad \forall B, h > 0. \tag{60}$$

■

Theorem 4. *If $q > q_{o_h}(B)$, then $u_h(\alpha, q, B)$ is a function of non-constant sign in Ω when*

$$\alpha > \alpha_o(q, B) = \frac{q|\Gamma_2|}{B|\Gamma_1|}. \tag{61}$$

Proof. If $q > q_{o_h}(B)$, then the corresponding discrete problem for $u_h(q, B)$ (i.e., $\alpha = +\infty$ for $u_h(\alpha, q, B)$) is a two-phase one [11] and, therefore, $u_h(q, B) < 0$ in some part of Γ_2, that is, $u_h(\alpha, q, B) < 0$ on some part of Γ_2.

On the other hand, from (37) we get the following equivalence:

$$\int_{\Gamma_1} u_h(\alpha, q, B)d\gamma > 0 \Leftrightarrow q_M(\alpha, B) - q > 0 \Leftrightarrow \frac{B\alpha|\Gamma_1|}{|\Gamma_2|} - q > 0 \Leftrightarrow \alpha > \alpha_o(q, B), \tag{62}$$

then the proof is completed. ■

Theorem 5. (i) *The function $g_h = g_h(\alpha, q, B)$ satisfies the following properties*:

$$\frac{\partial g}{\partial q}(\alpha, q, B) = \int_{\Gamma_2} u_h(\alpha, q, B)d\gamma, \tag{63}$$

$$\frac{\partial g}{\partial B}(\alpha, q, B) = -\alpha \int_{\Gamma_1} u_h(\alpha, q, B)d\gamma, \tag{64}$$

$$\frac{\partial g}{\partial \alpha}(\alpha, q, B) = \frac{1}{2}\int_{\Gamma_1} \left[u_h^2(\alpha, q, B) - Bu_h(\alpha, q, B)\right] d\gamma. \tag{65}$$

(ii) *The function $A_h = A_h(\alpha)$ satisfies the following properties*:

$$A_h'(\alpha) = \frac{dA_h}{d\alpha}(\alpha) = \frac{B^2|\Gamma_1|}{q^2} - \frac{2B|\Gamma_2|}{\alpha q} - \frac{1}{q^2}\int_{\Gamma_1} u_h^2(\alpha, q, B)d\gamma, \tag{66}$$

$$A_h'(\alpha) \geq -\frac{2B|\Gamma_2|}{\alpha q}, \quad \forall h > 0, \tag{67}$$

$$\frac{d}{d\alpha}[\alpha A_h(\alpha)] = \frac{1}{q^2} a(u_h(\alpha, q, B), u_h(\alpha, q, B)), \tag{68}$$

$$\lim_{\alpha\to+\infty} \alpha A_h'(\alpha) = 0, \quad \forall h > 0, \tag{69}$$

$$\lim_{\alpha\to+\infty} \frac{\partial q_{m_h}}{\partial \alpha}(\alpha, B) = 0, \quad \forall B > 0. \tag{70}$$

Proof. (i) We use (45) and the definition of the derivative as the limit of the incremental quotient to obtain (65). In order to obtain (63) and (64), we differentiate (45) with respect to q and B, respectively, taking into account (16).

(ii) (66) follows from (46) and (63). (67) is obtained from (19) and (66). (68) follows from (45) and (66). Moreover, (69) is a consequence of (19), (67), and (68). At the end, (70) follows from (33) and (69). ■

III. PARTICULAR DISCRETE CASE

We define a particular discrete case (PDC), similarly to the continuous case (see (V.1) in [5]), as the problem in variables h, α, q, B, which verifies the condition

$$\frac{1}{q^2}a(u_h(\alpha,q,B),u_h(\alpha,q,B)) = \textit{Const}. \tag{71}$$

Necessarily the constant must be $\textit{Const} = C_h > 0$ taking $\alpha \to +\infty$ in (51). On the other hand, taking into account (68), we have the equivalence:

$$(PDC) \Leftrightarrow \frac{1}{q^2}a(u_h(\alpha,q,B),u_h(\alpha,q,B)) = C_h \Leftrightarrow \frac{d}{d\alpha}[\alpha A_h(\alpha)] = C_h. \tag{72}$$

Since (16) and (71), we deduce

$$C_h = a(U_h(\alpha),U_h(\alpha)) = A_h(\alpha) - \alpha\int_{\Gamma_1} U_h^2(\alpha)d\gamma, \tag{73}$$

that is

$$(PDC) \Leftrightarrow A_h(\alpha) = C_h + \alpha\int_{\Gamma_1} U_h^2(\alpha)d\gamma. \tag{74}$$

Theorem 6. *The following propositions are all equivalent to the particular discrete case* (71) *or* (72):

$$u_h(q,B) - u_h(\alpha,q,B) = \textit{Const in } \Omega, \tag{75}$$

$$u_h(q,B) - u_h(\alpha,q,B) = \frac{q|\Gamma_2|}{\alpha|\Gamma_1|} \textit{ in } \Omega, \tag{76}$$

$$u_h(\alpha,q,B)|_{\Gamma_1} = B - \frac{q|\Gamma_2|}{\alpha|\Gamma_1|} \textit{ on } \Gamma_1, \quad \left(\textit{or } U_h(\alpha)|_{\Gamma_1} = \frac{|\Gamma_2|}{\alpha|\Gamma_1|} \textit{ on } \Gamma_1\right), \tag{77}$$

$$A_h(\alpha) = C_h + \frac{|\Gamma_2|^2}{\alpha|\Gamma_1|}, \tag{78}$$

$$\frac{d}{d\alpha}[\alpha A_h(\alpha)] = C_h. \tag{79}$$

Proof. (75) $\Rightarrow$ (76). If $u_h(q,B) - u_h(\alpha,q,B) = \textit{Const}$ in Ω, then the constant must necessarily be given by $\textit{Const} = q|\Gamma_2|/\alpha|\Gamma_1|$, because we integrate equality (75) on Γ_1 and we use (37) and the fact that $u_h(q,B)|_{\Gamma_1} = B$.

(76) $\Rightarrow$ (77). It follows from the fact that $u_h(q,B)|_{\Gamma_1} = B$.

(77) $\Rightarrow$ (78). From (66) we get

$$A_h'(\alpha) = \frac{B^2|\Gamma_2|}{q^2} - \frac{2B|\Gamma_2|}{\alpha q} - \frac{1}{q^2}\left(B - \frac{q|\Gamma_2|}{\alpha|\Gamma_1|}\right)^2 |\Gamma_1| = -\frac{|\Gamma_2|^2}{\alpha|\Gamma_1|}. \tag{80}$$

Therefore, integrating the differential Eq. (80) with the initial condition $A_h(+\infty) = C_h$, we obtain

$$A_h(\alpha) = C_h - \frac{|\Gamma_2|^2}{|\Gamma_1|}\int_{+\infty}^{\alpha}\frac{dt}{t^2} = C_h + \frac{|\Gamma_2|^2}{\alpha|\Gamma_1|}, \tag{81}$$

that is (78).

(78) $\Rightarrow$ (79). By differentiating the expression (78) with respect to α, we obtain

$$\frac{d}{d\alpha}[\alpha A_h(\alpha)] = \alpha\left(\frac{-|\Gamma_2|^2}{\alpha^2|\Gamma_1|}\right) + C_h + \frac{|\Gamma_2|^2}{\alpha|\Gamma_1|} = C_h, \tag{82}$$

i.e., (79).

(79) $\Rightarrow$ (75). We deduce the following equivalence:

$$\begin{aligned} u_h(q,B) - u_h(\alpha,q,B) &= \textit{Const in } \Omega \\ &\Leftrightarrow a(u_h(q,B) - u_h(\alpha,q,B), u_h(q,B) - u_h(\alpha,q,B)) = 0 \\ &\Leftrightarrow a(u_h(\alpha,q,B), u_h(\alpha,q,B)) \\ &= a(u_h(q,B), u_h(q,B)) \Leftrightarrow (51), \end{aligned} \tag{83}$$

by using (22) and (13). ∎

Remark 3. For the particular discrete case, we have obtained an analytical expression for the function $A_h(\alpha)$, given by (78), and, therefore, the description of the set $S_h^2(B)$ is complete.

IV. ERROR BOUNDS AS FUNCTIONS OF THE PARAMETER h

If we take into account the following interpolation result [9, 10],

$$\|v - \Pi_h v\|_V \leq C_o h^{r-1}\|v\|_{r,\Omega}, \quad \forall v \in H^r(\Omega)(C_o > 0), \tag{84}$$

and we suppose the regularity property $U(\alpha) \in H^r(\Omega)$ with $r > 1$ [4, 12– 14] (Refs. [5] and [2] give us three examples in which the function $U(\alpha)$ is C^∞) then we deduce the following approximation results for the function of the discretization parameter h.

Theorem 7. (i) *We have*

$$\begin{aligned} A(\alpha) - A_h(\alpha) &= a_\alpha(U_h(\alpha) - U(\alpha), U_h(\alpha) - U(\alpha)) \\ &= \frac{1}{q^2} a_\alpha(u(\alpha,q,B) - u_h(\alpha,q,B), u(\alpha,q,B) - u_h(\alpha,q,B)) \geq 0. \end{aligned} \tag{85}$$

(ii) *We have*

$$a_\alpha(U(\alpha) - U_h(\alpha), U_h(\alpha)) = 0, \tag{86}$$

$$a_\alpha(U(\alpha) - U_h(\alpha), v_h) = 0, \quad \forall v_h \in V_h. \tag{87}$$

(iii) *We have that* $U_h(\alpha) = P_{V_h}^\alpha(U(\alpha))$ *is the projection of* $U(\alpha)$ *over the space* V_h *with respect to the norm associated to the scalar product* a_α *given by*

$$\|v\|_\alpha = \sqrt{a_\alpha(v,v)}, \quad \forall v \in V. \tag{88}$$

(iv) *We have the following abstract estimate*:

$$\|U(\alpha) - U_h(\alpha)\|_V \leq M_\alpha \inf_{v_h \in V_h} \|U(\alpha) - v_h\|_V \tag{89}$$

with

$$M_\alpha = \frac{\|a_\alpha\|}{\lambda_\alpha} \le \frac{1+\alpha\|\gamma_\alpha\|^2}{\lambda_1 \inf(1,\alpha)}, \tag{90}$$

being $\gamma_o : V \to L^2(\Gamma)$ *the trace operator.*

(v) *We have the following abstract estimate*:

$$0 < A(\alpha) - A_h(\alpha) \le \inf_{v_h \in V_h} a_\alpha(v_h - U(\alpha), v_h - U(\alpha)). \tag{91}$$

(vi) *If we have the regularity property* $U(\alpha) \in H^r(\Omega)$ *with* $r > 1$, *then we deduce the following estimates*:

$$\|U(\alpha) - U_h(\alpha)\|_V \le C_1(\alpha) h^{r-1}, \tag{92}$$

$$0 < A(\alpha) - A_h(\alpha) \le C_2(\alpha, r) h^{2(r-1)}, \tag{93}$$

$$0 < q_{m_h}(\alpha, B) - q_m(\alpha, B) \le \frac{C_2(\alpha, r)}{A(\alpha)} q_{m_h}(\alpha, B) h^{2(r-1)}, \tag{94}$$

where

$$C_1(\alpha) = C_o M_\alpha \|U(\alpha)\|_{r,\Omega}, \quad C_2(\alpha, r) = C_o^2 \|a_\alpha\| \|U(\alpha)\|_{r,\Omega}. \tag{95}$$

Proof. (i) and (ii) follow from the definitions of $U(\alpha), U_h(\alpha), A(\alpha)$, and $A_h(\alpha)$. Moreover, from (87) we deduce (iii) [3, 10].

(iv) Taking into account that the bilinear form a_α is coercive with a coercive constant λ_α (i.e., (25)), for $v_h \in V_h$ we have

$$\begin{aligned} \lambda_\alpha \|U(\alpha) - U_h(\alpha)\|_V^2 &\le a_\alpha(U(\alpha) - U_h(\alpha), U(\alpha) - U_h(\alpha)) \\ &= a_\alpha(U_\alpha - U_h(\alpha), U(\alpha)) \\ &= a_\alpha(U_\alpha - U_h(\alpha), U(\alpha) - v_h) \\ &\le \|a_\alpha\| \|U(\alpha) - U_h(\alpha)\|_V \|U(\alpha) - v_h\|_V, \end{aligned} \tag{96}$$

which implies (89). On the other hand, we have

$$\begin{aligned} |a_\alpha(u,v)| \le |a(u,v)| + \alpha \int_{\Gamma_1} |u||v| d\gamma \le \|u\|_V \|v\|_V + \alpha \|u\|_{L^2(\Gamma_1)} \|v\|_{L^2(\Gamma_1)} \\ \le (1 + \alpha \|\gamma_o\|^2) \|u\|_V \|v\|_V, \end{aligned} \tag{97}$$

that is (90).

(v) We have

$$\begin{aligned} A(\alpha) - A_h(\alpha) &= a_\alpha(U(\alpha) - U_h(\alpha), U(\alpha) - U_h(\alpha)) \\ &= \|U(\alpha) - U_h(\alpha)\|_\alpha^2 \le \|U(\alpha) - v_h\|_\alpha^2 \\ &= a_\alpha(U(\alpha) - v_h, U(\alpha) - v_h), \quad \forall v_h \in V_h, \end{aligned} \tag{98}$$

that is (91).

(vi) Taking into account the interpolation result (84) and the fact that $\Pi_h(U(\alpha)) \in V_h$, then from (89) and (91) we deduce (92) and (93), respectively. On the other hand, we have

$$q_{m_h}(\alpha, B) - q_m(\alpha, B) = \frac{B|\Gamma_2|}{A_h(\alpha)} \frac{A(\alpha) - A_h(\alpha)}{A(\alpha)} \leq \frac{C_2(\alpha, r)}{A(\alpha)} q_{m_h}(\alpha, B) h^{2(r-1)}, \quad (99)$$

that is (94). ∎

Remark 4. If $U(\alpha) \in V$ (not necessarily $U(\alpha) \in H^r(\Omega)$ with $r > 1$), then, when $h \to 0$, we have [9, 10]

$$0 < A(\alpha) - A_h(\alpha) \leq \|U(\alpha) - \Pi_h(U(\alpha))\|_\alpha^2 \to 0, \quad \text{when } h \to 0. \quad (100)$$

∎

We define the function

$$Z(\alpha, h, r) = 1 - \frac{C_2(\alpha, r) h^{2(r-1)}}{A(\alpha)} < 1, \quad \forall \alpha, h > 0, \quad (101)$$

and we have the following equivalence (for any $0 < \epsilon < 1$):

$$\epsilon < Z(\alpha, h, r) < 1 \Leftrightarrow h < h_r(\epsilon, \alpha), \quad (102)$$

where

$$h_r(\epsilon, \alpha) = \sqrt{\frac{(1-\epsilon) A(\alpha)}{C_o C_2(\alpha, r) \|U(\alpha)\|_{r,\Omega}^2}}, \quad 0 < \epsilon < 1, \alpha > 0. \quad (103)$$

Theorem 8. *Let be $h, B > 0$, and $0 < \epsilon < 1$ (ϵ is a parameter to be chosen arbitrarily). Then we have the following estimates as functions of the parameter h :*

$$q_{m_h}(\alpha, B) \leq \frac{1}{\epsilon} q_m(\alpha, B), \quad \forall h \leq h_r(\epsilon, \alpha), \quad (104)$$

$$0 < q_{m_h}(\alpha, B) - q_m(\alpha, B) \leq \frac{B|\Gamma_2| C_2(\alpha, r)}{\epsilon A^2(\alpha)} h^{2(r-1)}, \quad \forall h \leq h_r(\epsilon, \alpha). \quad (105)$$

Proof. From (94) we deduce

$$Z(\alpha, h, r) q_{m_h}(\alpha, B) \leq q_m(\alpha, B) \quad (106)$$

and, therefore, (104), because of the equivalence (102). From (94) and (104) we obtain (105). ∎

Corollary 2. *We have the following limit*:

$$\lim_{h \to 0^+} q_{m_h}(\alpha, B) = q_m(\alpha, B), \quad \forall \alpha, B > 0. \quad (107)$$

∎

Lemma 3. *If the continuous and discrete cases are particular cases, then we have*

$$0 < A(\alpha) - A_h(\alpha) = C - C_h \leq C_0^2 |u_3|_{r,\Omega}^2 h^{2(r-1)}, \quad (108)$$

$$q_{m_h}(\alpha, B) < \frac{B\alpha|\Gamma_1|}{|\Gamma_2|} = q_M(\alpha, B), \quad (109)$$

$$q_{m_h}(\alpha, B) - q_m(\alpha, B) \le \frac{BC_o^2|\Gamma_1|}{|\Gamma_2|}\|u_3\|_{r,\Omega}^2 \frac{\alpha}{A(\alpha)} h^{2(r-1)}. \tag{110}$$

Proof. If the continuous and discrete cases are particular cases, then $A(\alpha)$ and $A_h(\alpha)$ are given explicitly by (13) and (78), respectively, and we obtain that $A(\alpha) - A_h(\alpha) = C - C_h$. The right-hand side inequality is deduced by (72) of [11]. On the other hand, we have

$$q_{m_h}(\alpha, B) = \frac{B|\Gamma_2|}{C_h + \frac{|\Gamma_2|^2}{\alpha|\Gamma_1|}} < \frac{B\alpha|\Gamma_1|}{|\Gamma_2|} = q_M(\alpha, B), \tag{111}$$

that is (109). Moreover, from (99), (108), and (109), we obtain

$$\begin{aligned} q_{m_h}(\alpha, B) - q_m(\alpha, B) &= \frac{B|\Gamma_2|}{A_h(\alpha)} \frac{A(\alpha) - A_h(\alpha)}{A(\alpha)} \\ &= q_{m_h}(\alpha, B)\frac{C - C_h}{A(\alpha)} \le \frac{B\alpha|\Gamma_1|}{|\Gamma_2|}\frac{C - C_h}{A(\alpha)} \\ &\le \frac{B\alpha|\Gamma_1|}{|\Gamma_2|A(\alpha)} C_0^2 |u_3|_{r,\Omega}^2 h^{2(r-1)}, \end{aligned} \tag{112}$$

that is (110). ∎

Remark 5. If $U(\alpha) \in H^2(\Omega) \cap C^o(\bar{\Omega})$, then the convergence of $A_h(\alpha)$ to $A(\alpha)$ and $q_{m_h}(\alpha, B)$ to $q_m(\alpha, B)$ is of the order h^2 when $h \to 0^+$.

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