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A STEFAN PROBLEM FOR A NON-CLASSICAL HEAT EQUATION*

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Abstract

We review some recent results concerning to the heat equation with a heat source depending on the heat flux occurring at the fixed x=0 of a semi-infinite material. We also present a new free boundary problem (one-phase Stefan-like problem) for a non-classical heat equation, and we obtain the temperature and the free boundary (the phase-change interface) through the solution of a system of two Volterra integral equations.

Resumen: Se da una revisión de algunos recientes resultados concernientes a la ecuación del calor con una fuente que depende del flujo de calor que ocurre en la frontera fija x=0 de un cuerpo semi-infinito. También se presenta un nuevo problema de frontera libre (problema de tipo Stefan a una fase) para una ecuación no clásica para la cual se obtiene la temperatura y la frontera libre (la frontera de cambio de fase) a través de la solución de un sistema de dos ecuaciones integrales de Volterra.

Key words: Non-classical heat equation, asymptotic behavior, Stefan problem, phase-change problem, free boundary problem, Volterra integral equation.

Palabras claves: Ecuación del calor no-clásico, comportamiento asintótico, problema de Stefan, problema de cambio de fase, problemas de frontera libre, ecuación integral de Volterra.

AMS Subject classification: 35R35, 80A22, 35K05, 45D05, 35B40.

1 Introduction

The following non-classical heat conduction problem for a semi-infinite material was studied in [17]

$$\begin{cases} u_t(x,t) - u_{xx}(x,t) = \Phi(x)F(u_x(0,t)), & x>0, \quad t>0, \\ u(0,t) = g(t), & t>0, \\ u(x,0) = h(x), & x>0, \end{cases} \tag{1}$$

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where Φ, g, h are real functions defined on $\mathbb{R}^+$ and F is defined on $\mathbb{R}^+ \times \mathbb{R}$ which depends on the heat flux at the extremum $x = 0$. Non-classical problems like (1) are motivated by the modelling of a system of temperature regulation in isotropic media and the source term $\Phi(x)\, F(u_x(0,t))$ describes a cooling or heating effect depending on the properties of F which are related to the evolution of the heat flux $u_x(0,t)$. It is called the thermostat problem. Related problems are considered in [4],[6],[9]. Under suitable assumptions on data, existence, uniqueness and monotone-continuous dependence on the data are established in [17] for problem (1).

It was consider in [2] the simple instance of problem (1) given by

$$\begin{cases} u_t - u_{xx} = -F(u_x(0,t)), & x > 0, \quad t > 0, \\ u(0,t) = 0, & t > 0, \\ u(x,0) = h(x), & x > 0, \end{cases} \tag{2}$$

where $h(x)$, $x > 0$, and $F(v)$, $v \in \mathbb{R}$, are continuous functions. The function F, referred as *control function*, was assumed to fulfill the following condition:

A) $v\,F(v) \geq 0, F(0) = 0,$

which intuitively means that the control attempts to stabilize the process at every time.

As it is shown in [18] (see also [17]), the solution to problem (2) can be represented by

$$u(x,t) = u_0(x,t) - \int_0^t \operatorname{erf}\left(\frac{x}{2\sqrt{t-\tau}}\right) F\left(V(\tau)\right) d\tau, \tag{3}$$

where $u_0 = u_0(x,t)$, defined by

$$u_0(x,t) = \int_0^{+\infty} G(x,t;\xi,0)\, h(\xi)\, d\xi, \tag{4}$$

is the solution to problem (2) with null source term $F = 0$. Function $V = V(t)$ in (3) represents the heat flux at the extremum of the slab, i.e.

$$V(t) = u_x(0,t), \ t > 0, \tag{5}$$

and it satisfies the following Volterra integral equation

$$V(t) = V_0(t) - \int_0^t \frac{F\left(V(\tau)\right)}{\sqrt{\pi(t-\tau)}}\, d\tau, \tag{6}$$

where the forcing function $V_0(t)$ is given by

$$V_0(t) = \frac{1}{2\sqrt{\pi}\, t^{\frac{3}{2}}} \int_0^{+\infty} \xi \exp\left(-\frac{\xi^2}{4t}\right) h(\xi)\, d\xi, \ t > 0. \tag{7}$$

Function G in (4) denotes the Green function of the heat equation in the quarter plane and, as it is well-known, it can be written as $G(x,t;\xi,\tau) = K(x,t;\xi,\tau) - K(-x,t;\xi,\tau)$, $x,\xi > 0$, $t > \tau > 0$, where

$$K(x,t;\xi,\tau) = \frac{1}{\sqrt{4\pi(t-\tau)}} \exp\left(-\frac{(x-\xi)^2}{4(t-\tau)}\right),$$

is the one-dimensional heat kernel. Moreover, we also define the Neumann function of the heat equation in the quarter plane as $N(x,t;\xi,\tau) = K(x,t;\xi,\tau) + K(-x,t;\xi,\tau)$, $x,\xi > 0$, $t > \tau > 0$.

From now on, we suppose that h is a non-negative and non-identically null function which, in view of (7), implies $V_0(t) > 0$, $t > 0$. When the control function F satisfies condition (A) and, moreover, the initial temperature h is non-negative, then the solution $u(x,t)$ to problem (2) tends to zero when $t \to +\infty$ (see [17], [18]). In [2] was studied the problem of "controlling" problem (2) through F so that, by the stabilizing effect of the control, its solution should converge to zero (when the time goes to infinity) faster than that corresponding to problem (2) in absence of control; i.e.$\lim_{t\to+\infty} u(x,t)/u_0(x,t) = 0$. The heat flux $w(x,t) = u_x(x,t)$ satisfies a classical heat conduction problem with a nonlinear convective condition at $x = 0$. The first papers in this direction are [10] and [13]. Other related problems are considered in [1], [8] and [12]. In [2], a general study of the above stated control problem for (2) was done finding spatially uniform bounds for the quotient $u(x,t)/u_0(x,t)$ which depend on the solution $V(t)$ to integral equation (6), from which becomes apparent that condition (A) is not sufficient to attain the objective of the control; i.e., to obtain $\lim_{t\to+\infty} u(x,t)/u_0(x,t) = 0$. In Section 2, for linear control functions $F(v) = \lambda\, v$, we give an example to illustrate that there exists an exact solution to problem (6) providing $u(x,t)/u_0(x,t) \cong 1/(2\,\lambda^2 t)$, $t \to +\infty$. In Section 3, we present a one-phase Stefan problem for a semi-infinite material for a non-classical heat equation with a source term F which depends of the evolution of the heat flux at the extremum $x = 0$. Its solution is given by the solution of a system of two Volterra integral equations [3],[7],[11].

2 Constant initial temperature and their control function

We shall consider the instance of problem (2) corresponding to a constant initial temperature $h(x) = h_0 > 0$, $x \geq 0$. The solution to problem (2) is represented by (3) with

$$u_0(x,t) = h_0 \operatorname{erf}\left(\frac{x}{2\sqrt{t}}\right), \; x > 0, \; t > 0, \tag{8}$$

while $V = V(t)$ becomes the solution to the Volterra integral equation

$$V(t) = \frac{h_0}{\sqrt{\pi t}} - \int_0^t \frac{F(V(\tau))}{\sqrt{\pi(t-\tau)}}\, d\tau, \; t > 0. \tag{9}$$

Therefore, the following inequalities

$$\frac{\sqrt{\pi t}}{h_0} V(t) \leq \frac{u(x,t)}{u_0(x,t)} \leq \frac{1}{h_0} \int_0^t \frac{V(\tau)}{\sqrt{\pi(t-\tau)}} d\tau = 1 - \frac{1}{h_0} \int_0^t F(V(\tau))\, d\tau, \tag{10}$$

hold for $x > 0,\ t > 0$ [2].

From now on we suppose the case of linear controls: i.e.,

$$F(v) = \lambda v,\ (\lambda > 0), \tag{11}$$

and in order to obtain the explicit solutions u and V of the problems (2) and (9) respectively, we define the real function $Q(x) = \sqrt{\pi} x \exp(x^2) \operatorname{erfc}(x)$, defined for $x > 0$ [15] which satisfies the following properties: $Q(0) = 0,\ Q(+\infty) = 1,\ Q'(x) > 0,\ x > 0$. The most important facts on the behavior of the solution $V(t)$ to equation (9) corresponding to a linear control (11) are collected in the following result (See [2]).

Lemma 1 *If F is given by (11), then we have*

$$0 < V(t) = \frac{h_0}{\sqrt{\pi t}} \left[1 - Q(\lambda\sqrt{t})\right] < \frac{h_0}{\sqrt{\pi t}}, \tag{12}$$

$$1 - \frac{1}{h_0} \int_0^t F(V(\tau))\, d\tau = \exp(\lambda^2 t) \operatorname{erfc}(\lambda\sqrt{t}), \tag{13}$$

for all $t > 0$ and $\lim_{t\to+\infty} u(x,t)/u_0(x,t) = 0$, uniformly in $x > 0$. Furthermore, we have the estimates

$$\frac{1}{\pi\lambda^2 t} \leq \frac{u(x,t)}{u_0(x,t)} \leq \frac{1}{\lambda\sqrt{\pi t}}\ , \tag{14}$$

as $t \to +\infty$. Moreover, the temperature u is given by

$$u(x,t) = h_0 \exp(\lambda^2 t) \left[\operatorname{erfc}\left(\lambda\sqrt{t}\right) - \exp(\lambda x)\ \operatorname{erfc}\left(\lambda\sqrt{t} + \frac{x}{2\sqrt{t}}\right)\right], \tag{15}$$

and a more accurate estimation $\dfrac{u(x,t)}{u_0(x,t)} \sim 1/(2\lambda t^2)$, when $t \to +\infty$, uniformly in $x > 0$ is also obtained.

3 A Stefan problem for a non-classical heat equation.

We consider the following free boundary problem (one-phase Stefan problem) for the temperature $u = u(x,t)$ and the free boundary $x = s(t)$ (see [16]) with a control function F which depends on the evolution of the heat flux at the extremum $x = 0$ given by the following conditions:

$$\begin{cases} u_t - u_{xx} = -F(u_x(0,t)), & 0 < x < s(t), 0 < t < T, \\ u(0,t) = f(t) \geqslant 0, & 0 < t < T, \\ u(s(t),t) = 0, u_x(s(t),t) = -\dot{s}(t)\ , & 0 < t < T, \\ u(x,0) = h(x), & 0 \leqslant x \leqslant b = s(0)\ . \end{cases} \tag{16}$$

Theorem 2 *The solution of the free boundary problem (16) is given by*

$$
\begin{aligned}
u(x,t) &= \int_0^b G(x,t;\xi,0)h(\xi)d\xi + \int_0^t G_\xi(x,t;0,\tau)f(\tau)d\tau + \int_0^t G_\xi(x,t;s(\tau),\tau)v(\tau)d\tau \\
&\quad - \iint_{D(t)} G(x,t;\xi,\tau)F(V(\tau))d\xi d\tau \ , \\
s(t) &= b - \int_0^t v(\tau)d\tau
\end{aligned}
$$

where $D(t) = \{(x,\tau)/\ 0 < x < s(\tau), 0 < \tau < t\}$, *and* $v(t) = u_x(s(t),t) = -\ \dot{s}\ (t)$ *and* $V(t) = u_x(0,t)$ *must satisfy the following system of two Volterra integral equations*

$$
\begin{aligned}
v(t) &= 2\left[h(0) - f(0)\right] N(s(t),t;0,0) + \int_0^b N(s(t),t;\xi,0)h'(\xi)d\xi \\
&\quad -2\int_0^t N(s(t),t;0,\tau)\ \dot{f}\ (\tau)d\tau + 2\int_0^t G_x(s(t),t;s(\tau),\tau)v(\tau)d\tau \\
&\quad +2\int_0^t \left[N(s(t),t;s(\tau),\tau) - N(s(t),t;0,\tau)\right] F(V(\tau))d\tau \ , \\
V(t) &= \left[h(0) - f(0)\right] N(0,t;0,0) + \int_0^b N(0,t;\xi,0)h'(\xi)d\xi - \int_0^t N(0,t;0,\tau)\ \dot{f}\ (\tau)d\tau \\
&\quad + \int_0^t G_x(0,t;s(\tau),\tau)v(\tau)d\tau + \int_0^t \left[N(0,t;s(\tau),\tau) - N(0,t;0,\tau)\right] F(V(\tau))d\tau \ ,
\end{aligned}
$$

where G and N are the Green and Neumann functions of the heat equation in the quarter plane, defined previously in Section 1.

Proof. *We compute* $u_x(x,t)$, *and their corresponding limits as* $x \longrightarrow 0^+$ *and* $x \longrightarrow s(t)^-$. *By using the jump relations [5], [14] the system of two Volterra integral equations holds.*

The corresponding study of the existence and uniqueness of the solution will be given in a forthcoming paper.

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