ISTITUTO
DI
ANALISI NUMERICA
del

CONSIGLIO NAZIONALE DELLE RICERCHE
Corso C. Alberto, 5 - 27100 PAVIA (Italy)

PAVIA
1989

PUBBLICAZIONI
N. 728

D.A. Tarzia

Mixed Elliptic and Parabolic Free Boundary Problems related to the Two-Phase Stefan Problem

MIXED ELLIPTIC AND PARABOLIC FREE BOUNDARY PROBLEMS RELATED TO THE TWO-PHASE STEFAN PROBLEM (*)

Domingo Alberto TARZIA

Instituto de Matemática "Beppo Levi",
Facultad de Ciencias Exactas, Ing. y Agr.,
Universidad Nacional de Rosario,
Av. Pellegrini 250,
(2000) Rosario, ARGENTINA.

ABSTRACT: We study some mixed elliptic and parabolic free boundary problems with phase-change, i.e. with solutions of non-constant sign as functions of the Dirichlet and Neumann data.

KEY WORDS: Stefan problem, free boundary problems, phase-change problems, variational inequalities, optimization problems, Mixed elliptic problem, Neumann solution, phase-change waiting-time.

1985 AMS Classification Subject : 35R35.

(*) Talk held on 14 July 1988 at Pavia Numerical Analysis Institute.

I. MIXED ELLIPTIC PROBLEMS WITH OR WITHOUT PHASE CHANGE.

We consider a heat conducting material occuping Ω, a bounded domain of $\mathbb{R}^n$ ($n = 1, 2, 3$ in practice), with a sufficiently regular boundary $\Gamma = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3$ (with $\mathrm{meas}(\Gamma_1) \equiv |\Gamma_1| > 0$, $|\Gamma_2| > 0$ and $|\Gamma_3| \geq 0$) . We assume, without loss of generality, that the phase-change temperature is 0°C. We impose a temperature $b = b(x) > 0$ on Γ_1 and an outcoming heat flux $q = q(x) > 0$ on Γ_2 ; we also suppose that the portion of the boundary Γ_3 (when it exists) is a wall impermeable to heat, i.e. the heat flux on Γ_3 is null. If we consider in Ω a steady-state heat conduction problem, then we are interested in finding sufficient and/or necessary conditions for the heat flux q on Γ_2 to obtain a change of phase in Ω, that is, a steady-state two-phase Stefan problem in Ω. Following [Ta1] we study the temperature $\theta = \theta(x)$, defined for $x \in \Omega$. The set Ω can be expressed in the form

$$(1) \qquad \Omega = \Omega_1 \cup \Omega_2 \cup \mathcal{L} \ .$$

where

$$\Omega_1 = \left\{ x \in \Omega \ / \ \theta(x) < 0 \right\} ,$$

$$(2) \qquad \Omega_2 = \left\{ x \in \Omega \ / \ \theta(x) > 0 \right\} , \qquad \mathcal{L} = \left\{ x \in \Omega \ / \ \theta(x) = 0 \right\} ,$$

are the solid phase, the liquid phase and the free boundary (e.g. a surface in $\mathbb{R}^3$) that separates them respectively. The temperature θ can be represented in Ω in the following way :

$$(3) \qquad \theta(x) = \begin{cases} \theta_1(x) < 0 , & x \in \Omega_1 , \\ 0 , & x \in \mathcal{L} , \\ \theta_2(x) > 0 , & x \in \Omega_2 , \end{cases}$$

and satisfies the conditions below :

$$(4) \qquad \begin{array}{l} \text{i) } \Delta\theta_i = 0 \quad \text{in } \Omega_i \ (i = 1, 2) , \\ \text{ii) } \theta_1 = \theta_2 = 0 , \qquad k_1 \dfrac{\partial\theta_1}{\partial n} = k_2 \dfrac{\partial\theta_2}{\partial n} \quad \text{on } \mathcal{L} , \\ \text{iii) } \theta_2 |_{\Gamma_1} = b , \qquad \text{iv) } \dfrac{\partial\theta}{\partial n} |_{\Gamma_3} = 0 , \\ \text{v) } \begin{cases} - k_2 \dfrac{\partial\theta_2}{\partial n} |_{\Gamma_2} = q & \text{if } \theta |_{\Gamma_2} > 0 , \\ - k_1 \dfrac{\partial\theta_1}{\partial n} |_{\Gamma_2} = q & \text{if } \theta |_{\Gamma_2} < 0 , \end{cases} \end{array}$$

where $k_i > 0$ is the thermal conductivity of phase i ($i = 1$: solid phase, $i = 2$: liquid phase), $b > 0$ is the temperature given on Γ_1 , and $q > 0$ is the heat flux given on Γ_2 . Problem (4) represents a free boundary elliptic problem (when $\mathcal{L} \neq \emptyset$) where the free boundary $\mathcal{L}$ (unknown a priori) is characterized by the three conditions (4ii). Following the idea of [Ba, Du1, Du2, Fre, Ta1] we shall transform (4) into a new elliptic problem but now without a free boundary. If we define the function u in Ω as follows

$$(5) \qquad u = k_2\,\theta^+ - k_1\,\theta^- \quad \left(\theta = \frac{1}{k_2}\,u^+ - \frac{1}{k_1}\,u^-\right) \qquad \text{in } \Omega \ ,$$

where θ^+ and θ^- represent the positive and the negative parts of the function θ respectively, then problem (4) is transformed into

$$(6) \qquad \begin{array}{l} \text{i) } \Delta u = 0 \quad \text{in } D'(\Omega), \\ \text{ii) } u\,|_{\Gamma_1} = B \ \ (B = k_2\,b > 0\,), \quad \text{iii) } -\frac{\partial u}{\partial n}\,|_{\Gamma_2} = q \ , \ \ \frac{\partial u}{\partial n}\,|_{\Gamma_3} = 0 \ , \end{array}$$

whose variational formulation is given by

$$(7) \qquad a(u,v-u) = L(v-u) \quad , \quad \forall\, v \in K \ , \quad u \in K \ ,$$

where

$$(8) \qquad \begin{array}{l} V = H^1(\Omega) \quad , \quad V_0 = \left\{ v \in V \,/\, v\,|_{\Gamma_1} = 0 \right\} \ , \\ K = K_B = \left\{ v \in V \,/\, v\,|_{\Gamma_1} = B \right\} \ , \\ a(u,v) = \int_\Omega \nabla u \,.\, \nabla v \ dx \ , \qquad L(v) = L_q(v) = -\int_{\Gamma_2} q\,v\ d\gamma \ . \end{array}$$

Under the hypotheses $L \in V_0'$ (e.g. $q \in L^2(\Gamma_2)$) and $B \in H^{1/2}(\Gamma_1)$, there exists a unique solution of (7) which is characterized by the following minimization problem [BC, KS, Ro, Ta3]

$$(9) \qquad u \in K \quad , \quad J(u) \leq J(v) \quad , \quad \forall\, v \in K \ ,$$

where

$$(10) \qquad J(v) = J_q(v) = \frac{1}{2}\,a(v,v) - L(v) = \frac{1}{2}\,a(v,v) + \int_{\Gamma_2} q\,v\,d\gamma \ .$$

LEMMA 1: If $u = u_{qB}$ is the unique solution of problem (7) for data q on Γ_2 and $B > 0$ on Γ_1, then we have the monotony property :

$$(11) \qquad B_1 \leq B_2 \text{ on } \Gamma_1 \text{ and } q_2 \leq q_1 \text{ on } \Gamma_2 \ \Rightarrow\ u_{q_1B_1} \leq u_{q_2B_2} \text{ in } \overline{\Omega}\ .$$

Moreover,

$$(12) \qquad q > 0 \text{ on } \Gamma_2 \quad \Rightarrow \quad u_{qB} \leq \max_{\Gamma_1} B \ \text{ in } \overline{\Omega} \ ,$$

and function $u = u_{qB}$ satisfies the equality

$$(13) \qquad a(u^-,u^-) = \int_{\Gamma_2} q\,u^-\,d\gamma \ .$$

COROLLARY 2. From (13), we deduce the equivalence

$$(14) \qquad u^- \neq 0 \text{ in } \overline{\Omega} \ \Leftrightarrow\ u^- \neq 0 \text{ on } \Gamma_2 \ ,$$

where $q > 0$ and $B > 0$.

We shall give three problems, with their corresponding solutions, which are related to problem (6) or (7).

Problem 1: For the constant case $B > 0$ and $q > 0$, find a constant $q_0 = q_0(B) > 0$ such that for $q > q_0(B)$ we have a steady-state two-phase Stefan problem in Ω, that is the solution u of (7) is a function of non-constant sign in Ω.

Remark 1: From (14) we deduce that an answer to problem 1 is the element q for which u takes negative values on the boundary Γ_2 .

LEMMA 3: Let $u = u_q$ be the unique solution of the variational equality (7) for $q > 0$ (for a given $B > 0$). Then : i) The mappings

(15) $$q > 0 \rightarrow u_q \in V \quad \text{and} \quad q > 0 \rightarrow \int_{\Gamma_2} u_q \, d\gamma \in \mathbb{R}$$

are strictly decreasing functions.

ii) For all $q > 0$ and $h > 0$ we have the following estimates :

(16) $$\| \tfrac{1}{h} (u_{q+h} - u_q) \|_V \leq C_1 = \frac{\| \gamma_0 \|}{\alpha_0} \, | \Gamma_2 |^{1/2} ,$$

(17) $$\| \tfrac{1}{h} (u_q - u_{q+h}) \|_{L^2(\Gamma_2)} \leq C_2 = C_1 \, \| \gamma_0 \| ,$$

where γ_0 is the trace operator (linear and continuous, defined on V), and $\alpha_0 > 0$ is the coercivity constant on V_0 of the bilinear a , i.e. :

(18) $$a(v,v) \geq \alpha_0 \, \| v \|_V^2 , \ \forall v \in V_0 .$$

iii) For all $q > 0$ and $h > 0$ we have

(19) $$0 < \int_{\Gamma_2} u_q \, d\gamma - \int_{\Gamma_2} u_{q+h} \, d\gamma \leq C_3 \, h \quad (C_3 = C_2 \, | \Gamma_2 |^{1/2} > 0)$$

and therefore the function $q > 0 \rightarrow \int_{\Gamma_2} u_q \, d\gamma$ is continuous.

Let $f : \mathbb{R}^+ \rightarrow \mathbb{R}$ be the real function defined by

(20) $$f(q) = J(u_q) = \tfrac{1}{2} \, a(u_q,u_q) + q \int_{\Gamma_2} u_q \, d\gamma .$$

Remark 2. To solve Problem 1 it is sufficient to find a value $q > 0$ for which we have $f(q) < 0$. We shall further see that this technique can still be improved.

THEOREM 4. i) The function f is differentiable. Moreover, f' is a continuous and strictly decreasing function, and it is given by the following expression

$$f'(q) = \int_{\Gamma_2} u_q \, d\gamma \quad . \tag{21}$$

ii) There exists a constant $C = C(\Omega,\Gamma_1,\Gamma_2) > 0$ such that

$$a(u_q,u_q) = C\, q^2 \ , \qquad f(q) = -\frac{C}{2} q^2 + B \mid \Gamma_2 \mid q \ . \tag{22}$$

iii) If $q > q_0(B)$, then we obtain a two-phase steady-state Stefan problem in Ω (i.e. u_q is a function of non-constant sign in Ω), where

$$q_0(B) = \frac{B \mid \Gamma_2 \mid}{C} \ . \tag{23}$$

Remark 3. The sufficient condition $f(q) < 0$, to solve Problem 1, was improved by the condition $f'(q) < 0$, which is optimal (see examples more later). In the case where, because of symmetry, we find that the function u_q is constant on Γ_2 , the sufficient condition, given by (Th.2-iii), is also necessary to have a steady-state two-phase Stefan problem.

COROLLARY 5. If we consider the general case $b = b(x) > 0$ on Γ_2 , we obtain : If function q satisfies the inequality

$$\operatorname*{Inf}_{x \in \Gamma_2} q(x) > \frac{k_2 \mid \Gamma_2 \mid}{C} \operatorname*{Sup}_{x \in \Gamma_1} b(x) \tag{24}$$

then we have a two-phase steady-state Stefan problem in Ω, that is function $u = u_{qb}$ is of non-constant sign in Ω.

Let $q_c > 0$ be the critical heat outgoing flux which characterizes a steady-state two-phase Stefan problem, that is

$$\begin{aligned} & q > q_c \ \Leftrightarrow \ \exists \text{ 2-phases,} \\ & q \leq q_c \ \Leftrightarrow \ \exists \text{ 1-phase (the liquid phase).} \end{aligned} \tag{25}$$

We shall give now some estimates for the critical flux q_c [BST].

LEMMA 6. i) Let w denote the solution to

$$\Delta w = 0 \ \text{in}\ \Omega \ , \quad w \mid_{\Gamma_1} = B \ , \ w \mid_{\Gamma_2} = 0 \ , \ \frac{\partial w}{\partial n} \mid_{\Gamma_3} = 0 \ . \tag{26}$$

If we define

$$q_i = \operatorname*{Min}_{\Gamma_2} \left(-\frac{\partial w}{\partial n} \mid_{\Gamma_2}\right) \tag{27}$$

then $u_q \geq w \geq 0$ in $\overline{\Omega}$, $\forall q \leq q_i$. Moreover, we have $q_i \leq q_c$.

ii) Let $P_2 \in \Gamma_2$ and the affine function π such that

$$(28) \qquad \pi \mid_{\Gamma_1} \geq B \ , \ \pi(P_2) = 0 \ , \ \pi \mid_{\Gamma_2} \geq 0 \ , \ \frac{\partial \pi}{\partial n} \mid_{\Gamma_3} \geq 0 \ .$$

If we define

$$(29) \qquad q_s = \underset{\Gamma_2}{\mathrm{Max}} \left(-\frac{\partial \pi}{\partial n} \mid_{\Gamma_2}\right)$$

then $u_q \leq \pi$ in Ω, $\forall q \geq q_s$. Moreover, we have $u_q(P_2) < 0$, $\forall q > q_s$ and then $q_c \leq q_s$.

iii) On the other hand $w \leq \pi$ in $\overline{\Omega}$ and if $w \not\equiv \pi$ we have $q_i < q_s$.

Remark 4. A sufficient condition for such π to exist is the existence of supporting hyperplanes σ to Ω at $P_2 \in \Gamma_2$ which are a positive distance away from $\overline{\Gamma}_1$: construct an affine function π vanishing on σ (and at P_2), such that $\pi \mid_{\Gamma_1} \geq B$ and there is $P_1 \in \overline{\Gamma}_1$ with $\pi(P_1) = B$. The optimal q_s can be obtained by selecting P_2, $\sigma = \sigma_{P_2}$ such that $\mathrm{dist}(\sigma, \overline{\Gamma}_1)$ is the largest. This construction fails if Γ_2 is a flat portion of Γ, e.g. the side of a triangle $\Omega \subset \mathbf{R}^2$, Γ_1 being formed by the other two sides and $\Gamma_3 = \emptyset$. The fact that $u_q(P_2) < 0$ suggests that the second phase appears at $P_2 \in \Gamma_2$, the point "farthest" from Γ_1. In many cases (c.f. [BST]) the function π can be obtained by satisfying (28) and $\pi(P_1) = B$, where $P_1 \in \overline{\Gamma}_1$, $P_2 \in \Gamma_2$ and $\mathrm{dist}(P_1, P_2) = \underset{x \in \Gamma_2}{\mathrm{Sup}} \ \mathrm{dist}(x, \overline{\Gamma}_1)$. There is no uniqueness in general for the points $P_1 \in \overline{\Gamma}_1$ and $P_2 \in \Gamma_2$. For instance, in Example 1 (see below) there are many $P_1 = (0,y)$ and $P_2 = (x_0,y)$, with $y \in [0,y_0]$.

We shall consider $q_c = q_c(\Omega)$ as a function of the domain Ω. Let Ω_1 and Ω_2 be two bounded domains, with regular boundaries, such that [BST] :

$$(30) \qquad \Omega_1 \subset \Omega_2 \ , \quad \partial(\Omega_1) = \Gamma_1^{(1)} \cup \Gamma_2 \cup \Gamma_3 \ , \quad \partial(\Omega_2) = \Gamma_1^{(2)} \cup \Gamma_2 \cup \Gamma_3 \ ,$$

where the boundary conditions on $\Gamma_1^{(i)}$ $(i = 1, 2)$, Γ_2 and Γ_3 are of the same type as the ones defined before. Let u_i $(i = 1, 2)$ be the solution to problem (7) for the domain Ω_i with data $B = B(x) > 0$ on $\Gamma_1^{(i)}$ and $q_i = q_i(x)$ on Γ_2 $(i = 1, 2)$, that is

$$(31) \qquad u_i \in K_i \quad (i = 1, 2) \ , \quad a_i(u_i, v - u_i) = -\int_{\Gamma_2} q_i \, (v - u_i) \, d\gamma \ , \ \forall v \in K_i \ ,$$

where

$$(32) \qquad a_i(u,v) = \int_{\Omega_i} \nabla u \, . \, \nabla v \, dx \ \ (i = 1, 2) \ , \quad K_i = \left\{ v \in H^1(\Omega_i) \ / \ v \mid_{\Gamma_1^{(i)}} = B \right\} .$$

THEOREM 7. Under the above hypotheses, we obtain the following property :

$$(33) \qquad q_1 \leq q_2 \ \text{ on } \Gamma_2 \ \Rightarrow \ u_2 \leq u_1 \ \text{ in } \overline{\Omega}_1 \ .$$

Moreover, we have that $q_c(\Omega_2) \leq q_c(\Omega_1)$, that is, $q_c = q_c(\Omega)$ is a non-increasing function of the domain Ω where the order is represented by conditions (30).

We shall give now another estimate for q_c, by using Poincaré type barriers. Let $\xi \in \Gamma_2$ be such that there exists $x_0 \notin \overline{\Omega}$, with

(34) $$\| x_0 - \xi \| = a > 0 \quad , \quad \left\{ x \,/\, \| x - x_0 \| \leq a \right\} \cap \overline{\Omega} = \left\{ \xi \right\} ,$$

where a is a positive parameter and $\| \cdot \|$ is the euclidean norm in $\mathbb{R}^n$. The Poincaré barriers at $\xi \in \Gamma_2$ are [Ke] :

(35) $$V_{D,a}(x,\xi) = V(x,\xi) = \begin{cases} D \left\{ \dfrac{1}{a^{n-2}} - \dfrac{1}{\| x - x_0 \|^{n-2}} \right\} , & (n \geq 3) \\ D \log \left(\dfrac{\| x - x_0 \|}{a} \right) , & (n = 2) \end{cases}$$

where D is a another positive parameter. Let $P_1 \in \overline{\Gamma}_1$, $P_2 \in \Gamma_2$ be such that

(36) $$d = \operatorname*{Sup}_{x \in \Gamma_2} \operatorname{dist}(x,\overline{\Gamma}_1) = \| P_2 - P_1 \| > 0 \ .$$

Let $\xi = P_2 \in \Gamma_2$ be . Then we have

THEOREM 8. We assume the following hypotesis :

(37) $$V \,|_{\Gamma_1} \geq B \iff V(P_1,\xi) \geq B \ .$$

Let q_V be defined by

(38) $$q_V = \operatorname*{Inf}_{V(P_1,\xi)=B} \left(\frac{D}{a^{n-1}} \right) .$$

Then we obtain that

(39) $$V(x,\xi) > u_q(x) \ , \ \forall x \in \Omega \ , \ \forall q > q_V \ ,$$ (40) $$q_c < q_V \ .$$

Remark 5. Equivalence (37) is an immediate consequence of the monotonicity of V on $\|x - x_0\|$, for special domains.

Remark 6. Let

(40) $$\Omega = \left\{ (x,y) \in \mathbb{R}^2 \,/\, -E \leq x \leq E \,,\, -h \leq y \leq h \right\} , \quad E > 0 \ , \ h > 0 \ .$$

Let Γ_1 be the top and bottom sides of this rectangle, and let Γ_2 be the two vertical sides. We maintain a temperature $b > 0$ on Γ_1 ($B=k_2 b > 0$) and ask for the minimum heat flux q on Γ_2 for which the zone $\left\{ (x,y) \in \Omega \,/\, u(x,y) > 0 \right\}$ (whose boundary obviously contains Γ_1) is disconnected, a region where $u < 0$ joins the two components of Γ_2 . By introducing a variant of the Poincaré barriers

(35) we obtain that [BST]

$$(41) \qquad q > \frac{2\,e\,B\,E}{h^2 - E^2} \quad (\text{with } h > E) \quad \Rightarrow \quad \left\{ u > 0 \right\} \text{ is disconected.}$$

Problem 2 : For the general case $b = b(x) > 0$ on Γ_1 and $q = q(x)$ on Γ_2 , we consider the following optimization problem : Find $q \in Q^+$ that produces the maximum heat flux on Γ_2 , without change of phase within Ω , i.e. [GT] :

$$(42) \qquad \underset{q \in Q^+}{\text{Max}} \; F(q)$$

where

$$(43) \qquad \begin{aligned} & F : Q \to \mathbb{R} \;/\; F(q) = \int_{\Gamma_2} q \, d\gamma , \\ & Q = H^{1/2}(\Gamma_2) \;,\; S = \left\{ v \in K \;/\; \Delta v = 0 \text{ in } \Omega \,,\, \frac{\partial v}{\partial n} \mid_{\Gamma_3} = 0 \right\} , \\ & S^+ = \left\{ v \in S \;/\; v \geq 0 \text{ in } \Omega \right\} , \quad Q^+ = T^{-1}(S^+) = \left\{ q \in Q \;/\; u_q \geq 0 \text{ in } \Omega \right\} . \end{aligned}$$

The application $T : Q \to S$ is defined by $T(q) = u$ where $u = u_q$ is the unique solution of (7). We consider that the domain Ω and the data B on Γ_1 (e.g. $B \in H^{3/2}(\Gamma_1)$) and q on Γ_2 (e.g. $q \in Q$) are sufficiently regular to have the regularity property $u \in H^2(\Omega) \cap C^0(\overline{\Omega})$ (for $n \leq 3$, $H^2(\Omega) \subset C^0(\overline{\Omega})$) . Moreover, in the three examples given below, the solution satisfies this condition for the constant case. Therefore, we have that there will not exist a phase change in Ω for any heat flux $q \in Q^+$.

THEOREM 9. (i) The operator T is an affine and monotone increasing operator, that is, there exist $u_1 \in S$ and two new operator T_1 and T_2 so that $T = T_1 + T_2$, where

$$(44) \qquad \begin{aligned} & T_1 : Q \to S \;/\; T_1(q) = u_1 \in S \;,\; \forall q \in Q \;, \\ & T_2 : Q \to V_0 \;/\; T_2 \text{ is linear and continuous.} \end{aligned}$$

(ii) Q^+ is a convex set and F is a linear (then, convex) functional.

(iii) There exists a unique $\overline{q} \in Q^+$ such that

$$(45) \qquad F(\overline{q}) = \underset{q \in Q^+}{\text{Max}} \; F(q) \;.$$

Moreover, the element $\overline{q}$ is defined by $\overline{q} = -\frac{\partial \omega}{\partial n} \mid_{\Gamma_2}$ where ω is given by (26).

Problem 3 : For the general case $b = b(x) > 0$ on Γ_1 and $q = q(x) > 0$ on Γ_2, we consider the following optimization problem : Find the maximum upper bound for q such that there is no change of phase within Ω, i.e. [GT]

(46) Find $q_M^0 > 0 \;/\; u_q \geq 0$ in Ω , $\forall q = q(x) \leq q_M^0$ on Γ_2 .

THEOREM 10. (i) For the case $q =$ const. > 0 , we obtain that

$$q_M^0 = \operatorname*{Inf}_{x \in \Gamma_2} \frac{u_1(x)}{u_3(x)} , \tag{47}$$

where u_1 and u_3 are given respectively by

$$\Delta u_1 = 0 \text{ in } \Omega , \quad u_1 \,|\, \Gamma_1 = B \;, \frac{\partial u_1}{\partial n} \,|\, \Gamma_2 \cup \Gamma_3 = 0 \;, \tag{48}$$

$$\Delta u_3 = 0 \text{ in } \Omega , \quad u_3 \,|\, \Gamma_1 = 0 , \quad \frac{\partial u_3}{\partial n} \,|\, \Gamma_2 = 1 \;, \quad \frac{\partial u_3}{\partial n} \,|\, \Gamma_3 = 0 \;. \tag{49}$$

(ii) If $q = q(x) > 0$ on Γ_2 satisfies the condition $\operatorname*{Sup}_{x \in \Gamma_2} q(x) \leq q_M^0$, where q_M^0 is defined by (47), then $u_q \geq 0$ in Ω.

(iii) For the constant case, we have that $q_M^0 = q_c$, where q_c is the critical heat outgoing flux (25).

Now, we replace the condition (4iii) by the following one [Ta1] :

$$(50) \qquad \begin{aligned} -k_2 \frac{\partial \theta_2}{\partial n} \,|\, \Gamma_1 &= \alpha \,(k_2 \,\theta_2 - B) \qquad \text{if} \quad \theta \,|\, \Gamma_1 > 0 \;, \\ -k_1 \frac{\partial \theta_1}{\partial n} \,|\, \Gamma_1 &= \alpha \,(k_1 \,\theta_1 - B) \qquad \text{if} \quad \theta \,|\, \Gamma_1 < 0 \;, \end{aligned}$$

where $\alpha =$ const. > 0 represents a heat transfer coefficient on Γ_1 . We are interested in studying the temperature $\theta = \theta_\alpha$, represented in Ω by (3), which satisfies the conditions (4i,ii,iv,v) and (50) . If we define the function u_α in Ω by (5), then it is transformed into

$$(51) \qquad \begin{aligned} &\text{i) } \Delta u = 0 \quad \text{in } D'(\Omega), \qquad \text{iii) } -\frac{\partial u}{\partial n} \,|\, \Gamma_2 = q \;, \quad \frac{\partial u}{\partial n} \,|\, \Gamma_3 = 0 \;, \\ &\text{ii) } -\frac{\partial u}{\partial n} \,|\, \Gamma_1 = \alpha \,(u - B) \,, \; B = k_2 \, b > 0 \,, \end{aligned}$$

whose variational formulation is given by ($u = u_{\alpha q B}$) :

(52) $a_\alpha(u,v) = L_{\alpha q B}(v)$, $\forall\, v \in V$, $\quad u \in V$,

where

$$(53) \qquad a_\alpha(u,v) = a(u,v) + \alpha \int_{\Gamma_1} u\, v\, d\gamma \;, \qquad L_{\alpha qB}(v) = L_q(v) + \alpha \int_{\Gamma_1} B\, v\, d\gamma \,.$$

Under the hypotheses $L_{\alpha qB} \in V'$ (e.g. $q \in L^2(\Gamma_2)$ and $B \in H^{1/2}(\Gamma_1)$), there exists a unique solution of (52) which is characterized by the following minimization problem

$$(54) \qquad G(u) \leq G(v) \;, \quad \forall\, v \in V \;, \quad u \in V \;,$$

where

$$(55) \qquad G(v) = G_{\alpha qB}(v) = \tfrac{1}{2}\, a_\alpha(v,v) - L_{\alpha qB}(v) = J_q(v) + \tfrac{\alpha}{2} \int_{\Gamma_1} v^2\, d\gamma - \alpha \int_{\Gamma_1} Bv\, d\gamma \,.$$

LEMMA 11: If $u = u_{\alpha qB}$ is the solution of problem (52) for data $q > 0$ on Γ_2, $B > 0$ on Γ_1 and $\alpha > 0$, then we have the following properties (for a given $B > 0$):

$$(56) \qquad \begin{array}{l} \text{(i) } u_{\alpha qB} \leq B \text{ in } \Omega \,,\ \forall\, \alpha > 0 \,,\ \forall\, q > 0 \,, \\ \text{(ii) } u_{\alpha qB} \leq u_{qB} \leq B \text{ in } \Omega \,,\ \forall\, \alpha > 0 \,,\ \forall\, q > 0 \,, \\ \text{(iii) } u_{\alpha_1 q_1 B} \leq u_{\alpha_2 q_2 B} \text{ in } \Omega \,,\ \forall\, \alpha_1 \leq \alpha_2 \,,\ \forall\, q_2 \leq q_1 \,, \\ \text{(iv) } M_2 \leq u_{\alpha qB} \leq M_1 \text{ in } \Omega \,,\ \forall\, \alpha > 0 \,,\ \forall\, q > 0 \,; \end{array}$$

where

$$(57) \qquad M_2 = M_2(\alpha,q,B) = \min_{\Gamma_2} u_{\alpha qB} \,, \qquad M_1 = M_1(\alpha,q,B) = \max_{\Gamma_1} u_{\alpha qB} \,.$$

Moreover, we have that $\lim_{\alpha \to +\infty} u_{\alpha qB} = u_{\alpha q}$ strongly in V , where $u_{\alpha q}$ is the solution of (7).

COROLLARY 12 : From (56), we deduce

$$(58) \qquad \max_{\overline{\Omega}} u_{\alpha qB} = M_1 \quad , \quad \min_{\overline{\Omega}} u_{\alpha qB} = M_2 \,.$$

where the elements M_1 and M_2 are defined in (57).

Now, we shall consider a problem (Problem 4) related to (51) or (52).

Problem 4. For the constant case $B > 0$, $q > 0$ and $\alpha > 0$, find conditions between α, q (for a given $B > 0$) to have a steady-state two-phase Stefan problem in Ω, that is the solution u of (52) is a function of non-constant sign in Ω.

We shall consider that the domain Ω and the data b (or B) on Γ_1 and q on Γ_2 are sufficiently regular to have the regularity property $u_{\alpha qB} \in H^2(\Omega) \cap C^0(\overline{\Omega})$. Moreover, in the three examples, the solution $u_{\alpha qB}$ satisfies this requirement.

Remark 8. (i) The problem (52) is a two-phase Stefan problem in Ω if and only if :

(59) $\quad \exists\, x_1 \in \Gamma_1 , x_2 \in \Gamma_2 \;/\; u_{\alpha q B}(x_1) > 0 \;,\; u_{\alpha q B}(x_2) < 0 \;.$

(ii) If $u_{\alpha q B}$ satisfies the following condition

(60) $$\int_{\Gamma_1} u_{\alpha q B}\, d\gamma > 0 \;,\; \int_{\Gamma_2} u_{\alpha q B}\, d\gamma < 0$$

then the problem (52) is a two-phase problem.

THEOREM 13. If $q > q_0(B)$, then (52) is a steady-state two-phase Stefan problem in Ω for all $\alpha > \alpha_0(q,B)$, where

(61) $$\alpha_0(q,B) = \frac{q\,|\,\Gamma_2\,|}{B\,|\,\Gamma_1\,|} \;.$$

Remark 8. In the case where, due to symmetry, we find that function $u_{\alpha q B}$ is constant on Γ_1, then the sufficient condition, given by Theorem 10, is also necessary for problem (52) to be a steady-state two-phase Stefan problem.

Let $g : (\mathbf{R}^+)^3 \to \mathbf{R}$ be the real function defined by

(62) $$g(\alpha,q,B) = G_{\alpha q B}(u_{\alpha q B}) \,,\quad \alpha, q, B > 0 \;.$$

THEOREM 14. (i) Function g has partial derivatives with respect to variables α, q and B, and they are given by the following expressions for all α, q, $B > 0$:

(63) $$\frac{\partial g}{\partial \alpha}(\alpha,q,B) = \int_{\Gamma_1} \left(\frac{1}{2}\, u^2_{\alpha q B} - B\, u_{\alpha q B} \right) d\gamma \;,$$

(64) $$\frac{\partial g}{\partial q}(\alpha,q,B) = \int_{\Gamma_2} u_{\alpha q B}\, d\gamma \;,$$ (65) $$\frac{\partial g}{\partial B}(\alpha,q,B) = -\,\alpha \int_{\Gamma_1} u_{\alpha q B}\, d\gamma \;.$$

(ii) There exists a function $A = A(\alpha) > 0$, defined for $\alpha > 0$, such that

(66) $$g(\alpha,q,B) = -\,\frac{A(\alpha)}{2}\, q^2 + B\, q\,|\,\Gamma_2\,| - \frac{B^2\,\alpha}{2}\,|\,\Gamma_1\,| \,,$$

(67) $$\int_{\Gamma_2} u_{\alpha q B}\, d\gamma = B\,|\,\Gamma_2\,| - q\, A(\alpha) \;,\quad \forall q, B > 0 \,.$$

(iii) Function $A = A(\alpha)$ is a decreasing positive function of α which verifies

(68) $$A(\alpha) > \frac{|\,\Gamma_2\,|^2}{|\,\Gamma_1\,|}\,\frac{1}{\alpha} \;,\qquad \lim_{\alpha \to +\infty} A(\alpha) = C \;,$$
$$\lim_{\alpha \to +\infty} \alpha\, A'(\alpha) = 0 \,,\qquad (\alpha\, A(\alpha)\,)' = \frac{1}{q^2}\, a(u_{\alpha q B}, u_{\alpha q B}) \;,$$

where C > 0 is the constant defined in Theorem 4.

THEOREM 15. (i) Let $q_m = q_m(\alpha,B)$ and $q_M = q_M(\alpha,B)$ be real functions, defined for $\alpha,B>0$ by the following expressions

$$(69) \qquad q_m(\alpha,B) = \frac{B \mid \Gamma_2 \mid}{A(\alpha)} \quad , \quad q_M(\alpha,B) = \frac{B\ \alpha \mid \Gamma_1 \mid}{\mid \Gamma_2 \mid} \ .$$

They verifies the conditions

$$(70) \qquad \begin{array}{l} q_m(0^+,B) = q_M(0^+,B) = 0 \quad , \qquad q_m(\alpha,B) < q_M(\alpha,B) \ , \ \forall \alpha > 0, \ B > 0 \ , \\ \lim_{\alpha \to +\infty} q_m(\alpha,B) = q_0(B) \quad , \quad q_m \text{ is an increasing function of variable } \alpha \ . \end{array}$$

The set

$$(71) \qquad S^{(2)}(B) = \left\{ (\alpha,q) \in (\mathbb{R}^+)^2 \ / \ q_m(\alpha,B) < q < q_M(\alpha,B) \ , \ \alpha > 0 \right\}$$

is not empty, for all B > 0 .

(ii) We have the following equivalences :

$$(72) \qquad \text{i)} \int_{\Gamma_1} u_{\alpha qB}\, d\gamma > 0 \ \Leftrightarrow \ q < q_M(\alpha,B) \ , \quad \text{ii)} \int_{\Gamma_2} u_{\alpha qB}\, d\gamma < 0 \ \Leftrightarrow \ q > q_m(\alpha,B) \ .$$

COROLLARY 16. If $(\alpha,q) \in S^{(2)}(B)$, then (52) is a two-phase steady-state Stefan problem.

Remark 9. In the case where, due to symmetry, we find that $u_{\alpha qB}$ is constant on Γ_1 and Γ_2 respectively, then the sufficient condition, given by Corollary 16 is also necessary for problem (52) to be a two-phase Stefan problem.

The function $A = A(\alpha)$, defined for $\alpha > 0$, is not explicitly known but has properties (68) . Now, we shall consider a particular case for which we can obtain more information about the expression of $A(\alpha)$. We consider the <u>particular</u> <u>case</u> when $u_{\alpha qB}$ verifies the condition [TT]

$$(73) \qquad \frac{1}{q^2}\, a(u_{\alpha qB}, u_{\alpha qB}) = \text{Const.} \ (= \text{Const}(\alpha,q,B)\ ) \ , \ \forall \alpha, \ q, \ B > 0 \ ,$$

or in an equivalent way $(\alpha A(\alpha))' = A(\alpha) + \alpha A'(\alpha) = \text{Const.}$, $\forall \alpha > 0$, due to (68). In this case, we have necessarily that $\text{Const}(\alpha,q,B) = C > 0$, $\forall \alpha, q, B > 0$, where C is the constant defined in Theorem 4.

LEMMA 18. (i) We have the following equivalence

(74) $$u_{qB} - u_{\alpha qB} \text{ is constant in } \Omega \Leftrightarrow (\alpha A(\alpha))' = C .$$

(ii) For the particular case (73), we have the following properties :

(75) $$u_{qB} - u_{\alpha qB} = \frac{q \mid \Gamma_2 \mid}{\alpha \mid \Gamma_1 \mid} \text{ in } \Omega ,$$ (76) $$u_{\alpha qB} \mid_{\Gamma_1} = B - \frac{q \mid \Gamma_2 \mid}{\alpha \mid \Gamma_1 \mid} ,$$

(77) $$\frac{\partial u_{\alpha qB}}{\partial n} \mid_{\Gamma_1} = \frac{q \mid \Gamma_2 \mid}{\mid \Gamma_1 \mid} ,$$ (78) $$\frac{\partial u_{qB}}{\partial n} \mid_{\Gamma_1} = \text{Const.}$$

Moreover, the function $A(\alpha)$ is given by the expression

(79) $$A(\alpha) = C + \frac{1}{\alpha} \frac{\mid \Gamma_2 \mid^2}{\mid \Gamma_1 \mid} .$$

Remark 10. For the particular case (73), a complete description of the set $S^{(2)}(B)$ was obtained.

We shall give three examples in which the solution is explicitly known [Ta2] so that we can verify all the theoretical results obtained in this paper.

Example 1. We consider the following data

$$n = 2 \ , \ \Omega = (0,x_0) \times (0,y_0) \ , \ x_0 > 0 \ , \ y_0 > 0 \ ,$$

(80) $$\Gamma_1 = \{0\} \times [0,y_0] \ , \ \Gamma_2 = \{x_0\} \times [0,y_0] \ , \ \Gamma_3 = (0,x_0) \times \{0\} \cup (0,x_0) \times \{y_0\}.$$

Example 2. Next we consider

$n = 2 \ , \ 0 < r_1 < r_2 \ , \ \Gamma_3 = \phi$,

Ω : annulus of radius r_1 and r_2 , centered at (0,0) ,

(81) Γ_1 : circumference of radius r_1 and center (0,0) ,

Γ_2 : circumference of radius r_2 and center (0,0) .

Example 3. Finally, we take into account the same information of Example 2 but now for the case n=3.

Remark 11. The three examples verifies condition (73), that is, they are particular cases.

Remark 12. The two elliptic variational equalities (7) and (52) appear if we consider the asymptotic behavior when the time $t \to +\infty$ in four parabolic variational inequalities of type II, defined in [Ta1], for the evolution two-phase Stefan problem (See also [Da, Du2, Fre, Fr2]).

II. MIXED PARABOLIC PROBLEMS WITH OR WITHOUT PHASE CHANGE.

We consider a semi-infinite material, represented by $\Omega = (0,+\infty)$, with an initial uniform temperature $\theta_0 > 0$. On the fixed face $x = 0$, the body may have a temperature $-D < 0$ (solidification problem) or an outward heat flux $q(t) > 0$ for all instant $t > 0$. We enlarge the problem by taking into account the effect of the density change during the phase change. Moroever, the material has constant thermal coefficient, e.g. :

$$(1)\qquad \begin{aligned} &k_i > 0 \text{ : thermal conductivity of the phase i , } c_i > 0 \text{ : specific heat of the phase i ,}\\ &\rho_i > 0 \text{ : mass density of the phase i , } \quad h > 0 \text{ : latent heat of fusion,}\\ &\alpha_i = a_i^2 = \frac{k_i}{\rho_i c_i} > 0 \text{ : thermal diffusivity of the phase i ,}\end{aligned}$$

where $i = 1$ and $i = 2$ represent the solid and liquid phase respectively. Without loss of generality, we take null phase-change temperature (i.e. we consider the case : ice-water).

The problem consists in finding the function $x = s(t) > 0$ (free boundary), defined for $t > 0$ with $s(0) = 0$, and the temperature

$$(2)\qquad \theta(x,t) = \begin{cases} \theta_1(x,t) < 0 & \text{if } 0 < x < s(t)\ ,\ t > 0\ ,\\ 0 & \text{if } x = s(t)\ ,\ t > 0\ ,\\ \theta_2(x,t) > 0 & \text{if } x > s(t)\ ,\ t > 0\ ,\end{cases}$$

defined for $x > 0$ and $t > 0$, such that they satisfy the following conditions [CJ,Ru] :

$$\begin{aligned}
&\text{i) } \alpha_1\, \theta_{1_{xx}} = \theta_{1_t}\ ,\quad 0 < x < s(t)\ ,\ t > 0\ ,\\
&\text{ii) } \alpha_2\, \theta_{2_{xx}} + \frac{\rho_1 - \rho_2}{\rho_2}\, \dot{s}(t)\, \theta_{2_x} = \theta_{2_t}\ ,\quad x > s(t)\ ,\ t > 0\ ,\\
&\text{iii) } s(0) = 0\ ,\\
(3)\qquad &\text{iv) } \theta_1(s(t),t) = \theta_2(s(t),t) = 0\ ,\ t > 0\\
(3bis)\qquad &\text{v) } k_1\, \theta_{1_x}(s(t),t) - k_2\, \theta_{2_x}(s(t),t) = \rho_1\, h\, \dot{s}(t)\ ,\quad t > 0\ ,\\
&\text{vi) } \theta_2(x,0) = \theta_2(+\infty,t) = \theta_0 > 0\ ,\quad x > 0\ ,\quad t > 0\ ,\\
&\text{vii) } \theta_1(0,t) = -D < 0\ ,\quad (\text{ (viibis) } k_1\, \theta_{1_x}(0,t) = q(t) > 0\ )\ ,\quad t > 0.
\end{aligned}$$

We shall give the explicit Neumann solution to problem (3) [BT,CJ,Ru].

LEMMA 1. The solution to the problem (3) (known as Neumann solution) is given by

$$(4)\qquad \begin{aligned} &\theta_1(x,t) = A_1 + B_1\, f\Big(\frac{x}{2\, a_1 \sqrt{t}}\Big)\ , \qquad s(t) = 2\,\gamma\,\sqrt{t}\quad (\gamma > 0)\ ,\\ &\theta_2(x,t) = A_2 + B_2\, f\Big(\delta_1 + \frac{x}{2\, a_2 \sqrt{t}}\Big)\ ,\end{aligned}$$

where

$$f(x) = \frac{2}{\sqrt{\pi}} \int_0^x \exp(-u^2)\,du \quad (= \mathrm{erf}(x)), \quad A_1(\gamma) = -D \quad , \quad B_1(\gamma) = \frac{D}{f(\frac{\gamma}{a_1})} ,$$

(5) $$A_2(\gamma) = -\frac{\theta_0 f(\frac{\gamma}{a_0})}{1 - f(\frac{\gamma}{a_0})} \quad , \quad B_2(\gamma) = \frac{\theta_0}{1 - f(\frac{\gamma}{a_0})} ,$$

$$\epsilon = \frac{\rho_1 - \rho_2}{\rho_2} \quad , \quad \delta_1 = \frac{\gamma}{a_2} \, | \epsilon | \quad , \quad a_0 = \frac{a_2}{1 + | \epsilon |} ,$$

and γ is the unique solution of the equation

(6) $$F(x) = x \; , \quad x > 0 \; ,$$

with

(7) $$F(x) = \frac{k_1}{h \, \rho_1 \, a_1 \sqrt{\pi}} B_1(x) \exp\left(-\frac{x^2}{a_1^2} \right) - \frac{k_2}{h \, \rho_1 \, a_2 \sqrt{\pi}} B_2(x) \exp\left(-\frac{x^2}{a_0^2} \right),$$

which satisfies the following properties

(8) $$F(0^+) = +\infty \quad , F(+\infty) = -\infty \quad , F' < 0 \; .$$

Now, we shall analyse the solution of problem (3bis) for different heat fluxes $q = q(t)$.

Problem 1. For which heat fluxes $q = q(t)$ do (3bis) have a solution of the Neumann type, i.e. when does problem (3bis) represent an evolution two-phase Stefan problem for that fluxes ?

We shall prove that there is not always solution of the Neumann type for the problem (3bis), i.e., problem (3bis) does not always represent an evolution two-phase Stefan problem; the cases considered will be [BT,SWA,Ta4]

(9) $$q(t) = q_0 \, t^{n/2} \quad (q_0 > 0) \; , \; t > 0 \; , \quad n = -1, 0, 1, \ldots$$

For the case $n = -1$ we instantaneously have a two-phase Stefan problem (evolution case) if and only if the coefficient q_0 verifies the following inequality [BT (for $\rho_1 \neq \rho_2$), Ta4(for $\rho_1 = \rho_2$)]

(10) $$q_0 > \frac{k_2 \, \theta_0}{a_2 \sqrt{\pi}} = \theta_0 \sqrt{\frac{k_2 \, \rho_2 \, c_2}{\pi}} \; .$$

For the cases $n = 0, 1, \ldots$ solidification does not immediately begin at $t = 0$ because the material temperature in $x = 0$ must be raised from θ_0 to 0 before solidification begins and a waiting time t_n is necessary, where [SWA]

(11) $$t_n = \left(\frac{k_2 \, \Gamma(\frac{3}{2} + \frac{n}{2})}{a_2 \, q_0 \, \Gamma(1 + \frac{n}{2})} \, \theta_0 \right)^{\frac{2}{n+1}} .$$

THEOREM 2. (i) When the heat flux is $q(t) = q_0\, t^{-1/2}$ $(t > 0)$, then there exists a solution of the Neumann type for the problem (3bis) if and only if q_0 verifies the inequaliuty (10). In this case, the salution of (3bis) is given by

$$(12)\qquad \begin{aligned} &\theta_1(x,t) = C_1 + D_1\, f\Big(\frac{x}{2\, a_1 \sqrt{t}}\Big), \quad s(t) = 2\,\omega \sqrt{t} \quad (\omega > 0)\ , \\ &\theta_2(x,t) = C_2 + D_2\, f\Big(\delta_2 + \frac{x}{2\, a_2 \sqrt{t}}\Big), \end{aligned}$$

where

$$(13)\qquad \begin{aligned} &C_1(\omega) = -\frac{a_1\, q_0 \sqrt{\pi}}{k_1} f\big(\tfrac{\omega}{a_1}\big)\ , \quad D_1(\omega) = \frac{a_1\, q_0 \sqrt{\pi}}{k_1}\ , \\ &C_2(\omega) = -\frac{\theta_0\, f\big(\frac{\omega}{a_0}\big)}{1 - f\big(\frac{\omega}{a_0}\big)}\ , \quad D_2(\omega) = \frac{\theta_0}{1 - f\big(\frac{\omega}{a_0}\big)}\ , \qquad \delta_2 = \frac{\omega\, |\, \epsilon\, |}{a_2}\ , \end{aligned}$$

and ω is the unique solution of the equation

$$(14)\qquad F_0(x) = x\ , \quad x > 0\ ,$$

with

$$(15)\qquad F_0(x) = \frac{q_0}{h\, \rho_1} \exp\big(-\tfrac{x^2}{a_1^2}\big) - \frac{k_2\, \theta_0}{h\, \rho_1\, a_2 \sqrt{\pi}}\, \frac{\exp\big(-\frac{x^2}{a_0^2}\big)}{1 - f\big(\frac{x}{a_0}\big)}\ .$$

(ii) If $q_0 \le \frac{k_2\, \theta_0}{a_2 \sqrt{\pi}}$ there is no solution for the solidification problem (3bis), we just have a problem of the heat conduction in the initial liquid phase.

(iii) The case $q_0 \le \frac{k_2\, \theta_0}{a_2 \sqrt{\pi}}$ corresponds to the limit case of problem (3) when the latent heat of fusion $h \to +\infty$.

Since the temperature θ_1 , defined in (12), verifies that $\theta_1(0,t) = C_1(\omega) < 0$, then we can consider the problem (3) for $D = -C_1(\omega)$ and so we obtain the following

LEMMA 3. If the condition (10) is valid and we take $D = -C_1(\omega)$ in problem (3), we have :

i) $\gamma = \omega$,

ii) the coefficient γ, which characterizes the free boundary of the Neumann solution for the problem (3), verifies the following inequality

$$(16)\qquad f\big(\tfrac{\gamma}{a_1}\big) < \frac{D}{\theta_0} \Big[\frac{\rho_1\, c_1\, k_1}{\rho_2\, c_2\, k_2}\Big]^{\frac{1}{2}}\ .$$

Now, we consider a slab, represented by the interval $0 < x < x_o$, at the initial temperature $\theta_o = \theta_o(x) \geq 0$, having a heat flux $q = q(t) > 0$ on the left face $x = 0$ and a temperature condition $b(t) > 0$ on the right face $x = x_o$ (x_o could be also $+\infty$, i.e., a semi-infinite material). We consider the corresponding heat conduction problem $(0 < x_o \leq +\infty)$:

$$
(17)\qquad
\begin{aligned}
&\text{i)}\quad \rho\, c\, \theta_t - k\, \theta_{xx} = 0, \quad 0 < x < x_o\,, \quad t > 0,\\
&\text{ii)}\quad \theta(x,0) = \theta_o(x) > 0\,, \quad 0 \leq x \leq x_o\,,\\
&\text{iii)}\quad k\, \theta_x(0,t) = q(t), \quad t > 0\,,\\
&\text{iv)}\quad \theta(x_o,t) = b(t), \quad t > 0.
\end{aligned}
$$

We replace the condition (17iv) by $\theta(+\infty,t) = \theta_o(+\infty) > 0$, $t > 0$ for the case $x_o = +\infty$. We assume that the data satisfy the hypothese that ensure the existence and uniqueness property of the solution of (17).

We consider the following posibilities:

(a) The heat conduction problem is defined for all $t > 0$ (waiting- time $t^* = +\infty$);

(b) there exist a time $t^* < +\infty$ such that another phase (i.e. the solid phase) appears for $t \geq t^*$ (waiting-time $0 \leq t^* < +\infty$) and then we have a two-phase Stefan problem for $t > t^*$. In this case, there exist a free boundary $x = s(t)$ which separates the liquid and solid phases with $s(t^*) = 0$.

We will separate the cases waiting-time $t^* = 0$ (i.e. there exists an <u>instantaneus change of phase</u>) and $0 < t^* \leq +\infty$. These possibilities depend on the data θ_o, q, b.

Problem 2 : Clarify this dependence by finding necessary or sufficient conditions on data θ_o, q, b to have the different possibilities, i.e. an instantaneous change of phase ($t^* = 0$) or a waiting-time $t^* > 0$ [TTu].

Remark 1. The term waiting-time was used for free boundary problems corresponding to the porous medium equation (See, for instance [Ar]).

THEOREM 4. If the data $q = q(t)$, $\theta_o = \theta_o(x)$ and $b = b(t)$ verify the conditions

$$\text{(18)} \quad \begin{aligned} &\text{i)} \quad 0 < q(t) \le q_o,\ 0 < t \le t_1 \quad \text{with } t_1 > 0, \\ &\text{ii)} \quad \theta'_o(x) \ge 0 \text{ and } \beta_1 \ge \theta_o(x) \ge \beta_o > 0,\ 0 \le x \le x_o \quad \text{with } \beta_o \le \beta_1, \\ &\text{iii)} \quad b(t) \ge \beta_1 \quad \text{and} \quad \dot{b}(t) \ge 0,\ t > 0\,, \end{aligned}$$

then there exists a waiting-time $t^*>0$ for problem (33), (i.e. another phase could appear at $t \ge t^*$), where t^* verifies the following inequality

$$\text{(19)} \qquad t^* \ge \text{Min}(t_1, t_o^*), \quad t_o^* = \pi k \rho c \beta_o^2 / 4 q_o^2 \ .$$

Remark 2. When the data verify conditions (18), problem (17) represents a heat conduction problem for the initial phase (in our case, the liquid phase) for $t \le t^*$.

Remark 3. We can see that t_0^* does not depend on the length of the slab $x_0 > 0$.

COROLLARY 5. Under the hypothese (18ii,iii), then a necessary condition to have (17) an instantaneus change of phase (i.e. $t^* = 0$) is $q(0^+) = +\infty$.

Remark 4. If we consider the following case

$$\text{(20)} \quad \begin{aligned} &x_o = +\infty\,, \qquad \theta_o(x) \ge \beta_o > 0\,, \forall\ x \ge 0\,, \\ &q(t) \le q_o(t) = \frac{\beta_o\, k}{a\sqrt{\pi\, t}} \qquad , \forall\ t > 0, \end{aligned}$$

then problem (17) is a heat conduction problem for the liquid phase for all $t>0$, i.e. there is not a phase-change process for any $t > 0$. Moreover, the particular case

$$\text{(21)} \qquad q(t) = \frac{\beta_o\, k}{a\sqrt{\pi\, t}} \ (= q_o(t)) \quad , \ t > 0\ ,$$

show us that condition $q(0^+) = +\infty$ is not sufficient.

Remark 5. If $x_o = +\infty$ and $\theta_o(x) \ge \beta_o > 0$ for $x \ge 0$, then a necessary condition to have (17) an instantaneous change of phase (i.e. the wainting-time is $t^* = 0$) is to exist an instant $t_o > 0$ such that $q(t_o) > \dfrac{\beta_o\, k}{a\sqrt{\pi\, t_o}}$.

THEOREM 6. If the data verify the conditions

$$x_o = +\infty \ ; \ 0 \le \theta_o(x) \le \beta_1 \quad \forall\, x \ge 0 \,,$$

$$(22) \qquad q(t) \ge \frac{q_o}{t^\beta} \ , \ 0 < t < 1 \ , \text{ with } q_o > 0 \text{ and } \tfrac{1}{2} < \beta < 1 \ ,$$

then an instantaneous phase- change occurs, that is the waiting-time is $t^* = 0$.

We consider the case of constant temperature $b(t) = b > 0$, $t > 0$ on $x = x_o$ and constant heat flux $q(t) = q > 0$, $t > 0$ on $x = 0$. The steady- state solution is given by $\theta_\infty(x) = \frac{q}{k}(x - x_o) + b$ and a necessary and sufficient condition to have a steady-state two-phase Stefan problem is [Ta2] $q > q_0(B) = \frac{B}{x_o}$, $B = k\,b > 0$, where k is the thermal conductivity of the liquid phase.

Using the fact that $\theta = \theta(x,t)$, solution of (17) with data $q > 0$ and $b > 0$, converges to $\theta_\infty = \theta_\infty(x)$ when t goes to $+\infty$ [Fr1], for any initial temperature $\theta_0 = \theta_0(x)$, we can formulate the following.

Problem 3 : find the relation between the heat flux $q > 0$ on $x = 0$ and a time t_1 such that another phase appears for $t \ge t_1$, and we can reformulate problem (1) in a two-phase Stefan problem for $t \ge t_1$.

THEOREM 7. The initial temperature verifies the conditions $\theta_o \ge 0$, $\theta_o' \ge 0$, $\theta_o'' \le 0$ in $[0,x_o]$ and $\theta_o(x_o) = b$. If the time $t_1 > 0$ and the constant heat flux $q > 0$ verify the inequality

$$(23) \qquad q > \frac{b\,k}{x_o\left(1 - \exp\left(-\dfrac{\alpha\,\pi^2\,t_1}{4\,x_0^2}\right)\right)} \quad , \ \alpha = \frac{k}{\rho\,c}$$

then another phase (the solid phase) there exists for $t \ge t_1$. Moreover, $\theta(0,t) < 0$ for all $t \ge t_1$ and the free boundary $x = s(t)$ begins at a point $(0,t')$ with $0 \le t' < t_1$.

COROLLARY 8. If we consider the q,t plane and we define the following set

$$(24) \qquad Q = \{ (t,q) \ / \ q > f(t) = \frac{b\,k}{x_o[1 - \exp(-\frac{\alpha\,\pi^2\,t}{4\,x_0^2})]} \ , \ t > 0 \}$$

then we obtain that for all $(q,t) \in Q$ we have a two-phase problem.

NOTE. Many others free boundary problems for elliptic or parabolic partial differential equations (of Stefan type) can be found in [BC, Ca, CJ, Cr, Di, Du2, EO, Fa, Fr1, Fr3, Li, Ma1, Ma2, Pr, Ro, Ru, Ta3, Ta5, Ta7].

ACKNOWLEDGMENT. This paper has been partially sponsored by the Project "Problemas de Frontera Libre de la Física-Matemática", from CONICET-UNR, Rosario, Argentina.

I wish to express my acknowledgements to Profs. C. Baiocchi, P Colli, E. Magenes, M. Paolini, C. Verdi for many stimulating conversations we had during my stay 13-15 July 1988 in Pavia Numerical Analysis Institute.

REFERENCES

[Ar] D.G. ARONSON, "The porous medium equation", in Nonlinear Diffusion Problems, A. Fasano- M. Primicerio (Eds.), Lecture Notes in Math. N. 1224, Springer Verlag, Berlin (1986), 1-46.

[Ba] C. BAIOCCHI, "Sur un problème à frontière libre traduisant le filtrage de liquides à travers des milieux poreux", C.R. Acad. Sc. Paris, 273A(1971), 1215-1217.

[BC] C. BAIOCCHI - A. CAPELO, "Disequazioni variazionali e quasivariazionali. Applicazioni a problemi di frontiera libera", Vol. 1: Problemi variazionali, Vol. 2: Problemi quasivariazionali, Quaderni dell'Unione Matematica Italiana, N 4, 7, Pitagora Editrice, Boyna (1978).

[BT] A.B. BANCORA- D.A. TARZIA, "On the Neumann solution for the two-phase Stefan problem including the density jump at the free boundary", Lat. Am. J. Heat Mass Transfer, 9(1985), 215-222.

[BST] J.E. BOUILLET - M. SHILLOR - D.A. TARZIA, "Critical outflow for a steady-state Stefan problem", Applicable Analysis, 32(1989), 31-51.

[Ca] J.R. CANNON, "The one-dimensional heat equation", Addison-Wesley, Menlo Park, California (1984).

[CJ] H.S. CARSLAW - J.C. JAEGER, "Conduction of heat in solids", Clarendon Press, Oxford (1959).

[Cr] J. CRANK, "Free and moving boundary problems", Clarendon Press, Oxford (1984).

[Da] A. DAMLAMIAN, "Résolution de certaines inéquations variationnelles stationnaries et d'évolution", Thèse d'Etat, Univ. Paris VI, Paris (1976).

[Di] J.I. DIAZ, "Nonlinear partial differential equations and free boundaries", Vol. I : Elliptic equations, Research Notes in Math. N_0 106, Pitman, London (1985).

[Du1] G. DUVAUT, "Résolution d'un problème de Stefan (Fusion d'un bloc de glace à zéro degré)", C.R. Acad. Sc. Paris, 276A(1973), 1461-1463.

[Du2] G. DUVAUT, "Problèmes à frontière libre en théorie des milieux continus", Rapport de Recherche N^0 185, LABORIA - IRIA, Rocquencourt (1976).

[EO] C.M. ELLIOTT - J.R. OCKENDON, "Weak and variational methods for moving boundary problems", Research Notes in Math., N_0 59, Pitman, London (1982).

[Fa] A. FASANO, "Las zonas pastosas en el problema de Stefan", CUADERNOS del Instituto de Matemática "Beppo Levi", N^0 13, Rosario (1987).

[Fr] M. FREMOND, "Diffusion problems with free boundaries", in Autumn Course on Applications of Analysis to Mechanics, ICTP, Trieste (1976).

[Fr1] A. FRIEDMAN, "Partial differential equations of parabolic type", Prentice-Hall, Englewood Cliffs (1964).

[Fr2] A. FRIEDMAN, "The Stefan problem in several space variables", Trans. Amer. Math. Soc., 132(1968), 51-87.

[Fr3] A. FRIEDMAN, "Variational principles and free boundary problems", J. Wiley, New York (1982).

[GT] R.L.V. GONZALEZ - D.A. TARZIA, "Optimization of heat flux in domain with temperature constraints", J. Optimiz. Th. Appl., To appear.

[Ke] O.D. KELLOGG, "Foundations of potential theory", Springer Verlag, Berlin (1929.

[KS] D. KINDERLEHRER – G. STAMPACCHIA, "An introduction to variational inequalities and their applications", Academic Press, New York (1980).

[Li] J.L. LIONS, "Problèmes aux limites dans les équations aux derivées partielles", Presses de l'Univ. de Montréal, Montréal (1962).

[Ma1] E. MAGENES, "Topics in parabolic equations : some typical free boundary problems", in Boundary Value Problems for Linear Evolution Partial Differential Equations, H.G. Garnier (Ed.), D. Reidel Publ. Comp., Dordrecht (1976), 239-312.

[Ma2] E. MAGENES, "Problemi di Stefan bifase in piu variabili spaziali", Le Matematiche, 36(1981), 65-108.

[Pr] M. PRIMICERIO, "Problemi di diffusione a frontiera libera", Bollettino Un. Mat. Italiana, 18A(1981), 11-68.

[Ro] J.F. RODRIGUES, "Obstacle problems in mathematical physics", North-Holland Mathematics Studies N_0 134, North-Holland, Amsterdam (1987).

[Ru] L.I. RUBINSTEIN, "The Stefan problem", Translations of Mathematical Monographs, Vol. 27, Amer. Math. Soc., Providence, R.I. (1971).

[SWA] A.D. SOLOMON-D.G. WILSON-V. ALEXIADES, "Explicit solutions to change problems", Quart. Appl. Math., 41(1983), 237-243

[TT] E.D. TABACMAN - D.A. TARZIA, "Sufficient and/or necessary conditions for the heat transfer coefficient on Γ_1 and the heat flux on Γ_2 to obtain a steady-state two-phase Stefan problem", J. Diff. Eq., 77(1989), 16-37.

[Ta1] D.A. TARZIA, "Sur le problème de Stefan à deux phases", Thèse 3ème Cycle, Univ. Paris VI (Mars 1979). See also C.R. Acad. Sc. Paris, 288A(1979), 941-944; Math. Notae, 27(1979/80), 145-156 and 157-165.

[Ta2] D.A. TARZIA, "Sobre el caso estacionario del problema de Stefan a dos fases", Math. Notae, 28(1980/81), 73-89.

[Ta3] D.A. TARZIA, "Introducción a las inecuaciones variacionales elípticas y sus aplicaciones a problemas de frontera libre", Centro Latinoamericano de Matemática e Informática, CLAMI - CONICET, No. 5, Buenos Aires (1981).

[Ta4] D.A. TARZIA, "An inequality for the coefficient σ of the free boundary $s(t)=2\sigma\sqrt{t}$ of the Neumann solution for the two-phase Stefan problem", Quart. Appl. Math., 39(1981-82), 491-497.

[Ta5] D.A. TARZIA, "Una revisión sobre problemas de frontera móvil y libre para la ecuación del calor. El problema de Stefan", Math. Notae, 29(1981), 147-241. See also "A bibliography on moving-free boundary problems for the heat-diffusion equation. The Stefan problem", (with 2528 references), Progetto Nacionale M.P.I. "Equazioni di evoluzione e applicazioni fisico-matematiche", Firenze (1988).

[Ta6] D.A. TARZIA, "An inequality for the constant heat flux to obtain a steady-state two-phase Stefan problem", Engineering Analysis, 5(1988), 177-181.

[Ta7] D.A. TARZIA, "The two-phase Stefan problem and some related conduction problems", Reuniões em Matemática Aplicada e Computacão Científica, Vol. 5, SBMAC, Gramado (1987). See also "On heat flux in material with or without phase change", in Int. Colloqium on Free Boundary Problems : Theory and Aplications, Irsee/Bavaria, 11-20 June 1987, Research Notes in Math., Pitman, To appear.

[TTu] D.A. TARZIA - C.V. TURNER, "A note on the existence of a waiting time for a two-phase Stefan problem", To appear.