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A VARIANT OF THE HEAT BALANCE INTEGRAL METHOD AND A NEW PROOF OF THE EXPONENTIALLY FAST ASYMPTOTIC BEHAVIOR OF THE SOLUTIONS IN HEAT CONDUCTION PROBLEMS WITH ABSORPTION

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Abstract—We give a new and explicit estimate for the asymptotic behavior of the solutions of the problem $u_t - u_{xx} + u_+^p = 0$, $x > 0$, $t > 0$, with conditions $u(0, t) = 1$, $t > 0$ and $u(x, 0) = U_0(x) \geq 0$, $x > 0$, for a class of functions U_0 and parameter $0 < p < 1$. We use an approximate solution given by the heat balance integral method with the innovation property which fixes appropriately the asymptotic limit of the corresponding approximate free boundary.

I. INTRODUCTION

The purpose of this paper is to give a new and explicit estimate for the asymptotic behavior of the solutions of the problem:

$$\begin{aligned} &\text{(i)}\ L(u) = u_t - u_{xx} + \lambda^2 u_+^p = 0, \qquad x > 0, \qquad t > 0,\\ &\text{(ii)}\ u(0, t) = 1, \qquad t > 0,\\ &\text{(iii)}\ u(x, 0) = U_0(x) \geq 0, \qquad x > 0, \end{aligned} \tag{1}$$

for a class of functions $U_0 = U_0(x)$ corresponding to the initial condition (1iii), and parameters $p > 0$ and $\lambda > 0$.

We denote with x_+ the positive part of x, that is $x_+ = \text{Max}(0, x)$.

If $0 < p < 1$, it is well known [1, 3, 9] that equation (1i) has a stationary solution corresponding to datum (1ii), which has compact support in $[0, +\infty)$ and is given by

$$u_\infty(x) = \left(1 - \frac{\lambda}{I} x\right)_+^{2/(1-p)}, \qquad I = I(p) = \frac{\sqrt{2(1+p)}}{1-p}. \tag{2}$$

In the case $0 < p < 1$ and $U_0 \leq u_\infty$, the solution $u = u(x, t)$ of (1) satisfies

$$0 < u(x, t) < u_\infty(x, t), \qquad 0 < x < \frac{I}{\lambda}, \qquad t > 0, \tag{3}$$

because of the comparison principle for equation (1i) [2]. This means that $u(t) = u(.; t)$ has compact support in variable x for any $t > 0$ and

$$s(t) = \text{Sup}(x > 0 / u(x, t) > 0\}, \qquad t > 0, \tag{4}$$

is a free boundary which is moving with finite speed for $t > 0$.

We shall give an estimate of how fast the free boundary $s(t)$ tends to its limit I/λ as $t \to +\infty$. The estimate we get implies that this convergence is exponentially fast in time, in a similar form to the one given in [8]. The purpose of the present paper is to show how this result can be obtained in a different way to [8] by using the Goodman heat balance integral method [6, 7]. To prove that we use an approximate solution given and motivated by the heat balance integral method with the innovation property (8) which fixes appropriately the asymptotic limit of the corresponding approximate free boundary. This method was very useful to some phase-change problems, see for instance [4, 5, 10]. This approximate solution to (1), approaches exponentially fast the stationary solution $u_\infty = u_\infty(x)$ when $U_0 \leq u_\infty$ and $0 < p < 1$ for all $\lambda > 0$. This is a new and explicit proof (with respect to [8]) of the exponentially fast asymptotic behavior of the solutions in heat conduction problems with absorption.

II. THE HEAT BALANCE INTEGRAL METHOD APPLIED TO PROBLEM (1) AND A NEW PROOF OF THE ASYMPTOTIC BEHAVIOR IN HEAT CONDUCTION PROBLEMS WITH ABSORPTION

We consider a related problem to (1) which consists in finding the function $C = C(x, t)$ and the free boundary $s = s(t)$ such that they satisfy the following conditions:

$$\begin{aligned} &\text{(i)}\ C_t - C_{xx} + \lambda^2 C_+^p = 0, \qquad 0 < x < s(t), \qquad t > 0,\\ &\text{(ii)}\ C(0, t) = 1, \qquad t > 0,\\ &\text{(iii)}\ s(0) = 0,\\ &\text{(iv)}\ C(s(t), t) = 0, \qquad t > 0,\\ &\text{(v)}\ C_x(s(t), t) = 0, \qquad t > 0. \end{aligned} \tag{5}$$

Taking into account the heat balance integal method we replace equation (5i) by its integral in the variable x from 0 to $s(t)$, that is

$$\begin{aligned} -\lambda^2 \int_0^{s(t)} C^p(x, t)\,\mathrm{d}x &= \int_0^{s(t)} C_t(x, t)\,\mathrm{d}x - \int_0^{s(t)} C_{xx}(x, t)\,\mathrm{d}x \\ &= \int_0^{s(t)} C_t(x, t)\,\mathrm{d}x + C_x(0, t), \qquad t > 0, \end{aligned} \tag{6}$$

and then we propose for the approximate problem (6)–(5ii–v) the following expression for C, namely [5–7]:

$$C(x, t) = \left(1 - \frac{x}{s(t)}\right)_+^{\alpha} \tag{7}$$

where $s = s(t)$ is a function to be determined and $\alpha > 1$ is a parameter to be chosen so that

$$\lim_{t\to\infty} s(t) = \frac{I(p)}{\lambda}. \tag{8}$$

This is the present innovation of the heat balance integral method applied to free boundary problems [6, 7].

Function C satisfies conditions (5ii, iv, v). If we put expression (7) into (6), after some manipulations, we obtain for $s = s(t)$ an ordinary differential equation, i.e. the following Cauchy problem:

$$\begin{aligned} &\dot{y}(t) = \alpha(\alpha + 1)\left[\frac{1}{y(t)} - \frac{\lambda^2}{\alpha(1 + p\alpha)} y(t)\right], \qquad t > 0,\\ &y(0) = 0, \end{aligned} \tag{9}$$

whose solution is given by

$$s(t) = \frac{1}{\beta}[1 - \exp(-2\alpha(\alpha + 1)\beta^2 t)]^{1/2}, \qquad t \ge 0, \tag{10}$$

with

$$\beta^2 = \frac{\lambda^2}{\alpha(1 + p\alpha)}. \tag{11}$$

If we choose $\alpha > 1$ by imposing the limit condition (8), we obtain that

$$\beta = \frac{\lambda}{I(p)}. \tag{12}$$

Condition (12) is an equation for $\alpha > 1$, and its solution is given by

$$\alpha = \alpha(p) = \frac{2}{1 - p} > 2, \tag{13}$$

which is the same exponent of the stationary solution (2) and independent of the parameter λ. Therefore, we have obtained the following result:

THEOREM 1. Let $p \in (0, 1)$ and $\lambda > 0$ be. If we apply Goodman heat balance integral method to problem (5), that is the problem defined by (6) and (5ii–v), with the innovation property (8), we obtain the solutions $C_B = C_B(x, t)$ and $s_B = s_B(t)$ which are given respectively by (7) with (13) and

$$s_B(t) = \frac{I}{\lambda}\left[1 - \exp\left(-\frac{2\lambda^2(3-p)t}{1+p}\right)\right]^{1/2}, \qquad t \geq 0. \tag{14}$$

We can define the following functions:

$$u_1(x, t) = \left[1 - \frac{x}{s_1(t)}\right]_+^{2/(1-p)}, \qquad x \geq 0, \qquad t > 0, \tag{15}$$

$$s_1(t) = \frac{I}{\lambda}\left[1 - \exp\left(-\frac{2\lambda t}{I}\right)\right], \qquad t \geq 0. \tag{16}$$

If we consider the heat conduction problem with absorption (1), we obtain:

THEOREM 2. Let $0 < p < 1$, $\lambda > 0$ and $0 \leq U_0 \leq u_\infty$ in $\mathbb{R}^+$ be. If $u = u(x, t)$ is a solution of (1) and $s = s(t)$ is defined by (4), we have the following comparison properties:

$$u_1(x, t) \leq u(x, t) \leq u_\infty(x), \qquad 0 \leq x \leq \frac{I}{\lambda}, \qquad t > 0, \tag{17}$$

$$s_1(t) \leq s(t) \leq \frac{I}{\lambda}, \qquad t \geq 0, \tag{18}$$

and the following estimates

$$0 < \frac{I}{\lambda} - s(t) \leq \frac{I}{\lambda} - s_1(t) \leq \frac{I}{\lambda}\exp\left(-\frac{2\lambda t}{I}\right), \qquad t \geq 0, \tag{19}$$

$$0 \leq u_\infty^{(1-p)/2}(x) - u^{(1-p)/2}(x, t) \leq u_\infty^{(1-p)/2}(x) - u_1^{(1-p)/2}(x, t) \leq \frac{\exp\left(-\frac{2\lambda t}{I}\right)}{1 - \exp\left(-\frac{2\lambda t}{I}\right)},$$
$$x \in \left[0, \frac{I}{\lambda}\right], \qquad t > 0. \tag{20}$$

PROOF. To prove (17) it is sufficient to verify that $L(u_1) \leq 0$ because of the comparison principle for the operator L [2].

Taking into account the properties

$$\left(1 - \frac{x}{s}\right)\frac{x}{s} \leq \frac{1}{4}, \qquad \forall x \in [0, s], \tag{21}$$

$$\frac{\dot{s}_1(t)}{s_1(t)} = 2\left(\frac{1}{s_1(t)} - \frac{\lambda}{I}\right), \qquad t > 0, \tag{22}$$

we obtain that

$$L(u_1) \leq \left[1 - \frac{x}{s_1(t)}\right]_+^{(2p)/1-p}\left(\frac{I}{s_1(t)} - \lambda\right)\left(\frac{1}{\sqrt{2(1+p)}} - \lambda\right) \leq 0, \tag{23}$$

for

$$\lambda \geq \frac{1}{\sqrt{2(1+p)}}. \tag{24}$$

Owing to the fact that

$$\frac{1}{2} < \frac{1}{\sqrt{2(1+p)}} < \frac{1}{\sqrt{2}} < 1, \qquad 0 < p < 1, \tag{25}$$

and the following scalling property:

$$v_\lambda(x, t) = v_1(\lambda x, \lambda^2 t) \tag{26}$$

where v_λ and v_1 denote the solutions of (1) for $\lambda > 0$ and $\lambda = 1$ respectively, we deduce that $L(u_1) \leq 0$ for all $\lambda > 0$. Then u_1 is a subsolution of (1) and we obtain (17) and (18). After that, the estimates (19) and (20) follow by the expressions of u_1 and s_1.

COROLLARY 3. The Theorem 2 implies that the solution u to (1) converges uniformly and exponentially fast to the stationary solution u_∞ when t goes to infinity. Moreover, the rate of convergence is faster than $2\lambda/I$.

REMARK 1. We shall prove later that function $s_B = s_B(t)$ is better than $s_1 = s_1(t)$ to approximate the free boundary $s = s(t)$.

From now on, without loss of generality, we consider the case $\lambda = 1$, $0 < p < 1$ and $0 \leq U_0 \leq u_\infty$ in $\mathbb{R}^+$ in problem (1). The results obtained in [8] are given by

$$s_0(t) \leq s(t) \leq I, \qquad t \geq 0, \qquad \left(I = I(p) = \frac{\sqrt{2(1+p)}}{1-p}\right) \tag{27}$$

$$u_0(x, t) \leq u(x, t) \leq u_\infty(x), \qquad 0 \leq x \leq I, \qquad t \geq 0, \tag{28}$$

where functions s_0 and u_0 are defined by (take $L_0 = 0$ and $m = 1$ in [8])

$$s_0(t) = I[1 - \exp(-c_0 t)]^{1/2}, \qquad t \geq 0, \tag{29}$$

$$u_0(x, t) = \left[1 - \frac{x}{s_0(t)}\right]_+^{2/(1-p)}, \qquad 0 \leq x \leq I, \qquad t > 0, \tag{30}$$

with

$$c_0 = c_0(p) = 4(1-p). \tag{31}$$

If we consider the functions C_B and s_B given by (Theorem 1):

$$C_B(x, t) = \left[1 - \frac{x}{s_B(t)}\right]_+^{2/(1-p)}, \qquad 0 \leq x \leq I, \qquad t \geq 0, \tag{32}$$

$$s_B(t) = I[1 - \exp(-c_B t)]^{1/2}, \qquad t > 0, \tag{33}$$

with

$$c_B = c_B(p) = \frac{2(3-p)}{1+p}, \tag{34}$$

and functions u_1 and s_1 given respectively by (15) and

$$s_1(t) = I[1 - \exp(-c_1 t)], \qquad t \geq 0, \tag{35}$$

with

$$c_1 = c_1(p) = \frac{2}{I} = (1-p)\sqrt{\frac{2}{1+p}}, \tag{36}$$

then we obtain the following comparison and relationship properties among them.

THEOREM 4. Under the hypotheses and definitions given above we have:

(A) Relationship between u_0, s_0 and C_B, s_B:

$$s_0(t) < s_B(t) < I, \qquad t > 0, \tag{37}$$

$$u_0(x, t) \leq C_B(x, t) \leq u_\infty(x), \qquad 0 \leq x \leq I, \qquad t > 0. \tag{38}$$

(B) Relationship between u_1, s_1 and C_B, s_B:

$$s_1(t) < s_B(t), \qquad t > 0, \tag{39}$$

$$u_1(x, t) \leq C_B(x, t), \qquad 0 \leq x \leq I, \qquad t > 0. \tag{40}$$

(C) Relationship between u_0, s_0 and u_1, s_1:

$$s_1(t) < s_0(t), \qquad t > 0, \tag{41}$$

$$u_1(x, t) \leq u_0(x, t), \qquad 0 \leq x \leq I, \qquad t > 0. \tag{42}$$

PROOF. (A) Owing to the expressions (29) and (33), we deduce

$$\frac{s_0^2(t) - s_B^2(t)}{l^2(p)} = -\exp(-c_0 t)[1 - \exp(-(c_B - c_0)t)] < 0, \qquad t > 0, \tag{43}$$

because the function

$$g(x) = \frac{c_0(x)}{c_B(x)} = 2\,\frac{(1-x)^2}{3-x}, \qquad 0 \le x \le 1, \tag{44}$$

verifies the following properties:

$$\begin{aligned} &g(0) = \frac{2}{3}, \qquad g(1) = 0,\\ &g'(x) = \frac{2(x^2 - 6x + 1)}{(3-x)^2}, \qquad g'(x) = 0 \Leftrightarrow x = 3 - 2\sqrt{2} \in (0, 1),\\ &g(x) \le g(3 - 2\sqrt{2}) = 12 - 8\sqrt{2} \approx 0.686 < 1, \qquad 0 \le x \le 1. \end{aligned} \tag{45}$$

Therefore, we have

$$u_0^{(1-p)/2}(x, t) - C_B^{(1-p)/2}(x, t) = x\left(\frac{1}{s_B(t)} - \frac{1}{s_0(t)}\right) < 0, \qquad 0 < x < s_0(t), \qquad t > 0, \tag{46}$$

that is (38).

(B) Owing to the expressions (33) and (35), we define the function $G = G(p, t)$, given by (p is a parameter in s_B and s_1):

$$G(p, t) = \frac{s_B(t)}{s_1(t)} = \frac{\sqrt{1 - \exp(-c_B(p)t)}}{1 - \exp(-c_1(p)t)}, \qquad 0 < p < 1, \qquad t > 0. \tag{47}$$

We have $c_1 < c_B$, because

$$c_1^2(x) - c_B^2(x) = \frac{2}{(1+x)^2}\,Q(x) < 0, \qquad 0 < x < 1, \tag{48}$$

where

$$Q(x) = x^3 - 3x^2 + 11x - 17 < 0, \qquad 0 \le x \le 1. \tag{49}$$

Function G verifies the following properties:

$$\begin{aligned} &G(p, 0^+) = +\infty, \qquad G(p, +\infty) = 1, \qquad 0 < p < 1,\\ &G_t(p, t) = \frac{1}{2} c_B c_1 \frac{\exp(-(c_B + c_1)t)}{[1 - \exp(-c_1 t)]^2 \sqrt{1 - \exp(-c_B t)}} [h(c_1, t) - h(c_2, t)] < 0,\\ &\qquad 0 < p < 1, \qquad t > 0 \end{aligned} \tag{50}$$

where function $h = h(c, t)$ is defined by:

$$h(c, t) = \frac{\exp(ct) - 1}{c}, \qquad c > 0, \qquad t > 0 \tag{51}$$

and satisfies

$$\begin{aligned} &h(0^+, t) = t, \qquad h(+\infty, t) = +\infty, \qquad t > 0,\\ &h_c(c, t) = \frac{1 + W(c, t)}{c^2} > 0, \qquad c > 0, \qquad t > 0, \end{aligned} \tag{52}$$

with

$$W(c, t) = (ct - 1)\exp(ct) > -1, \qquad c > 0, \qquad t > 0. \tag{53}$$

From (50) we obtain that $G(p, t) > 1$, $0 < p < 1$, $t > 0$ and therefore (39).
Moreover, we have

$$u_1^{(1-p)/2}(x, t) - C_B^{(1-p)/2}(x, t) = x\left(\frac{1}{s_B(t)} - \frac{1}{s_1(t)}\right) < 0, \qquad 0 < x < s_1(t), \qquad t > 0, \tag{54}$$

that is (40).

(C) Following a similar method developed before, we obtain that

$$\frac{s_0(t)}{s_1(t)}=\frac{\sqrt{1-\exp(-c_0(p)t)}}{1-\exp(-c_1(p)t)}>1, \qquad 0<p<1, \qquad t>0, \tag{55}$$

because

$$c_0^2(x)-c_1^2(x)=2(1-x)^2\frac{7+8x}{1+x}>0, \qquad 0<x<1, \tag{56}$$

that is (41) and (42).

Corollary 5. For any $0<p<1$, and taking into account Theorem 4 we obtain the following estimates:

$$s_1(t)<s_0(t)\le s(t)\le I, \qquad t>0, \tag{57}$$

$$s_1(t)<s_0(t)<s_B(t)<I, \qquad t>0, \tag{58}$$

and therefore

$$|s(t)-s_B(t)|\le I-s_0(t)\le I-s_1(t)\le I\exp\left(-\frac{2t}{I}\right), \qquad t>9. \tag{59}$$

Remark 2. The expression s_0 was obtained in [8] by constructing a sub-solution of the problem (1) ($\lambda=1$). Instead s_B was obtained by calculating the solution of an approximate problem (6) and (5ii–v)–(1) through the heat balance integral method with the innovation property (8). Both expressions, s_0 and s_B, give us a fast asymptotic behavior in heat conduction problems with absorption (1), but at present we cannot say which is the better. For t large both expressions are equivalent because

$$\lim_{t\to\infty}\frac{s_B(t)}{s_1(t)}=1, \qquad 0<p<1. \tag{60}$$

Corollary 6. We also obtain

$$u_1(x,t)\le u_0(x,t)\le u(x,t)\le u_\infty(x), \qquad 0\le x\le I, \qquad t>0, \tag{61}$$

$$u_1(x,t)\le u_0(x,t)\le C_B(x,t)\le u_\infty(x), \qquad 0\le x\le I, \qquad t>0, \tag{62}$$

and therefore:

$$|u^{(1-p)/2}(x,t)-C_B^{(1-p)/2}(x,t)|\le u_\infty^{(1-p)/2}(x,t)-u_1^{(1-p)/2}(x,t)\le u_\infty^{(1-p)/2}(x,t)-u_1^{(1-p)/2}(x,t)$$
$$\le\exp\left(-\frac{2t}{I}\right), \qquad 0<x<s_1(t), \qquad t>0, \tag{63}$$

which completes Corollary 3.

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