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Edited by:

Sergio R. Idelsohn (Chairman)
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Centro de Investigaciones de Métodos Computacionales (CIMEC), Santa Fe, Argentina

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Caracas, Venezuela

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NEUMANN SOLUTIONS TO FRACTIONAL LAMÉ-CLAPEYRON-STEFAN PROBLEMS WITH HEAT FLUX OR CONVECTIVE BOUNDARY CONDITIONS

DOMINGO A. TARZIA[1]

[1]CONICET – Depto. Matemática, Univ. Austral
Paraguay 1950, S2000FZF Rosario, Argentina
E-mail: DTarzia@austral.edu.ar

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Abstract. In this paper, generalized Neumann solutions for the two-phase fractional Lamé-Clapeyron-Stefan problems for a semi-infinite material are obtained with constant initial condition, and a boundary condition at the fixed face $x = 0$ given by a heat flux or a convective (Robin) condition. In these problems, the two governing diffusion equations and a governing condition for the free boundary include a fractional time derivative in the Caputo sense of order $0 < \alpha < 1$. When $\alpha \to 1^-$ we recover the classical Neumann solutions for the two-phase Lamé-Clapeyron-Stefan problem through the error function, given in:

(i) Tarzia, Quart. Appl.Math., 39 (1981), 491-497, for a heat flux boundary condition at the fixed face $x = 0$ when an inequality for the coefficient which characterizes the heat flux boundary condition is satisfied;
(ii) Tarzia, MAT – Serie A, 8 (2004), 21-27, for a convective boundary condition at the fixed face $x = 0$ when an inequality for the coefficient which characterizes the convective boundary condition is satisfied.

1 INTRODUCTION

In the last decades the fractional differential equations were developed [10, 14, 16, 18-21] and in the recent years some works on the fractional Lamé-Clapeyron-Stefan problem were published [1, 8, 12, 13, 22-24, 32, 33].

In this paper, generalized Neumann solutions for the two-phase fractional Lamé-Clapeyron-Stefan problems for a semi-infinite material are obtained with constant initial temperature, and a boundary condition at the fixed face $x = 0$ given by a heat flux or a convective (Robin) condition. Recently, a generalized Neumann solution for the two-phase fractional Lamé-Clapeyron-Stefan problem for a semi-infinite material with constant initial temperature, and a constant temperature condition at the fixed face $x = 0$ was given in [24].

In these problems, the two governing diffusion equations and a governing condition for the free boundary include a fractional time derivative in the Caputo sense of order

$0 < \alpha < 1$ which is defined in [4]:

$$D^{\alpha} f(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{f'(\tau)}{(t-\tau)^{\alpha}} \, d\tau \quad \text{for } 0 < \alpha < 1$$
$$= f'(t) \quad \text{for } \alpha = 1 \tag{1}$$

where Γ is the Gamma function defined by:

$$\Gamma(x) = \int_0^{+\infty} t^{x-1} \exp(-t) \, dt \,. \tag{2}$$

Now, we define two functions (Wright and Mainardi functions) which are very important in order to obtain the explicit solutions in the following Sections.

The Wright function is defined in [34]:

$$W(z;\alpha,\beta) = \sum_{n=0}^{+\infty} \frac{z^n}{n\,\Gamma(n\alpha+\beta)}, \quad z \in \mathbb{C}, \quad \alpha > -1, \quad \beta \in \mathbb{R} \,. \tag{3}$$

and the Mainardi function is defined in [10]:

$$M_{\upsilon}(z) = W(-z;-\upsilon,1-\upsilon) = \sum_{n=0}^{+\infty} \frac{(-z)^n}{n\,\Gamma(-n\upsilon+1-\upsilon)}, \quad z \in \mathbb{C}, \quad \upsilon < 1 \tag{4}$$

which is a particular case de the Wright function. Some basic properties are given by:

$$\frac{\partial W}{\partial z}(z;\alpha,\beta) = W(z;\alpha,\alpha+\beta) \tag{5}$$

$$W(-x;-\frac{1}{2},1) = \operatorname{erfc}\left(\frac{x}{2}\right), \quad 1 - W(-x;-\frac{1}{2},1) = \operatorname{erf}\left(\frac{x}{2}\right) \tag{6}$$

$$D^{\alpha}\left(t^{\beta}\right) = \frac{\Gamma(1+\beta)}{\Gamma(1+\beta-\alpha)} t^{\beta-\alpha} \quad . \tag{7}$$

Moreover, for the classical Lamé-Clapeyron-Stefan problem there exist thousands of papers on the subject, for example the first published papers [15, 26], the books [2,3,5-7, 9, 11, 17, 25, 31] and a large bibliography given in [28]. A review on explicit solutions with moving boundaries was given in [30].

In Section 2, we will obtain a generalized Neumann solution for the two-phase fractional Lamé-Clapeyron-Stefan problem for a semi-infinite material with constant initial condition, and a heat flux boundary condition at the fixed face $x = 0$. When $\alpha \to 1^-$ we recover the Neumann solution for the classical two-phase Lamé-Clapeyron-Stefan problem through the error function, given in [27], when an inequality for the coefficient which characterizes the heat flux boundary condition is satisfied.

In Section 3, we will obtain a generalized Neumann solution for the two-phase fractional Lamé-Clapeyron-Stefan problem for a semi-infinite material with constant initial condition, and a convective (Robin) boundary condition at the fixed face $x = 0$. When $\alpha \to 1^-$ we also recover the Neumann solution for the classical two-phase Lamé-Clapeyron-Stefan problem through the error function, given in [29], when an inequality for the coefficient which characterizes the convective boundary condition is satisfied.

2 THE TWO-PHASE FRACTIONAL LAMÉ-CLAPEYRON-STEFAN PROBLEM (MELTING PROCESS) WITH A HEAT FLUX BOUNDARY CONDITION AT THE FIXED FACE

We consider the following melting process:

Problem (FFP_α **)** Find the free boundary $x = s(t)$, and the temperature $T = T(x,t)$ such that the following equations and conditions are satisfied $(0 < \alpha < 1)$:

$$D^\alpha T_s - \lambda_s^2 T_{s_{xx}} = 0, \qquad x > s(t), \quad t > 0, \tag{8}$$

$$D^\alpha T_\ell - \lambda_\ell^2 T_{\ell_{xx}} = 0, \qquad 0 < x < s(t), \quad t > 0, \tag{9}$$

$$s(0) = 0, \tag{10}$$

$$T_s(x,0) = T_s(+\infty,t) = T_i < T_f, \qquad x > 0, \quad t > 0, \tag{11}$$

$$T_s(s(t),t) = T_f, \qquad t > 0, \tag{12}$$

$$T_l(s(t),t) = T_f, \qquad t > 0, \tag{13}$$

$$k_s T_{s_x}\left(s(t),t\right) - k_\ell T_{\ell_x}\left(s(t),t\right) = \rho\ell\, D^\alpha s(t), \quad t > 0, \tag{14}$$

$$k_\ell T_{\ell_x}(0,t) = -\frac{q_0}{t^{\alpha/2}}, \qquad t > 0, \tag{15}$$

where $\lambda_s^2 = \dfrac{k_s}{\rho c_s}$, $\lambda_\ell^2 = \dfrac{k_\ell}{\rho c_\ell}$.

Theorem 1 *Let* $T_i < T_f$ *be.*

a) *If the coefficient* q_0 *satisfies the inequality:*

$$q_0 > \frac{k_s(T_f - T_i)}{\lambda_s \Gamma(1-\alpha/2)}, \tag{16}$$

then there exists an instantaneous phase-change (melting) process and the problem (FFP_α*) has the generalized Neumann explicit solution given by:*

$$T_\ell(x,t) = T_f + \frac{q_0 \lambda_\ell \Gamma(1-\alpha/2)}{k_\ell}\left[W\left(-\frac{x}{\lambda_\ell t^{\alpha/2}};-\frac{\alpha}{2},1\right) - W\left(-\lambda \xi_{F\alpha};-\frac{\alpha}{2},1\right)\right] \tag{17}$$

$$T_s(x,t) = T_i + (T_f - T_i)\frac{W\left(-\frac{x}{\lambda_s t^{\alpha/2}};-\frac{\alpha}{2},1\right)}{W\left(-\xi_{F\alpha};-\frac{\alpha}{2},1\right)}, \tag{18}$$

$$s(t) = \xi_{F\alpha}\lambda_s t^{\alpha/2}, \tag{19}$$

where the coefficient $\xi = \xi_{F\alpha} > 0$ is the solution of the following equation:

$$F_{F_\alpha}(x) = \frac{\Gamma\left(1+\alpha/2\right)}{\Gamma\left(1-\alpha/2\right)}x, \quad x > 0 \tag{20}$$

with

$$F_{F_\alpha}(x) = \frac{q_0\,\Gamma(1-\alpha/2)}{\rho\ell\lambda_s} M_{\alpha/2}(\lambda x) - \frac{k_s(T_f - T_i)}{\rho\ell\lambda_s^2} F_{2\alpha}(x). \tag{21}$$

$$F_{2\alpha}(x) = \frac{M_{\alpha/2}(x)}{W\left(-x;-\frac{\alpha}{2},1\right)}. \tag{22}$$

b) *If the coefficient* q_0 *satisfies the inequalities*

$$0 < q_0 \le \frac{k_s(T_f - T_i)}{\lambda_s \Gamma(1-\alpha/2)}, \tag{23}$$

then the problem (FFP_α *) is a fractional diffusion problem for the initial solid phase whose solution is given by:*

$$T_s(x,t) = T_i + \frac{q_0 \lambda_s \Gamma(1-\alpha/2)}{k_s} W\left(-\frac{x}{\lambda_s t^{\alpha/2}}; -\frac{\alpha}{2}, 1\right), \quad x>0, \quad t>0. \tag{24}$$

Theorem 2 *Let* $T_i < T_f$ *be. If the coefficient* q_0 *satisfies the inequality (16) then the solution of the problem (* FFP_α *) converges to the solution of the classical Lamé-Clapeyron-Stefan problem (* FFP_1 *) when* $\alpha \to 1^-$*, and then we recover the classical Neumann explicit solution and the inequality for the coefficient which characterized the heat flux at* $\mathrm{x}=0$ *obtained for* $\alpha = 1$ *in [27], that is:*

$$q_0 > \frac{k_s (T_f - T_i)}{\sqrt{\pi \alpha_s}}. \tag{25}$$

3 THE TWO-PHASE FRACTIONAL LAMÉ-CLAPEYRON-STEFAN PROBLEM (SOLIDIFICATION PROCESS) WITH A CONVECTIVE BOUNDARY CONDITION AT THE FIXED FACE

We consider the following solidification process:

Problem (FCP_α **)** Find the free boundary $x = s(t)$, and the temperature $T = T(x,t)$ such that the following equations and conditions are satisfied $(0<\alpha<1)$:

$$D^\alpha T_\ell - \lambda_\ell^2 T_{\ell_{xx}} = 0, \qquad s(t) < x, \quad t > 0, \tag{26}$$

$$D^\alpha T_s - \lambda_s^2 T_{s_{xx}} = 0, \qquad 0 < x < s(t), \quad t>0, \tag{27}$$

$$s(0) = 0, \tag{28}$$

$$T_s(x,0) = T_s(+\infty, t) = T_i > T_f, \qquad x>0, \quad t>0, \tag{29}$$

$$T_s(s(t),t) = T_f, \qquad t>0, \tag{30}$$

$$T_l(s(t),t) = T_f, \qquad t>0, \tag{31}$$

$$k_s T_{s_x}\left(s(t),t\right) - k_\ell T_{\ell_x}\left(s(t),t\right) = \rho \ell \, D^\alpha s(t), \quad t>0, \tag{32}$$

$$k_s T_{s_x}(0,t) = \frac{h_0}{t^{\alpha/2}}\left(T_s(0,t) - T_\infty\right), \qquad t>0, \tag{33}$$

where $\lambda_s^2 = \dfrac{k_s}{\rho c_s}$, $\lambda_\ell^2 = \dfrac{k_\ell}{\rho c_\ell}$.

Theorem 3 *Let $T_\infty < T_f < T_i$ be.*

a) *If the coefficient h_0 satisfies the inequality:*

$$h_0 > \frac{k_\ell (T_i - T_f)}{\lambda_\ell (T_f - T_\infty)} \frac{1}{\Gamma(1-\alpha/2)}, \tag{34}$$

then there exists an instantaneous phase-change (solidification) process and the problem (FCP_α*) has the generalized Neumann explicit solution given by:*

$$T_s(x,t) = T_f - (T_f - T_\infty)\left[1 - \frac{\dfrac{k_s}{h_0\lambda_s\Gamma(1-\alpha/2)} + 1 - W\left(-\dfrac{x}{\lambda_s t^{\alpha/2}}; -\dfrac{\alpha}{2}, 1\right)}{\dfrac{k_s}{h_0\lambda_s\Gamma(1-\alpha/2)} + 1 - W\left(-\dfrac{\xi_{C\alpha}}{\lambda}; -\dfrac{\alpha}{2}, 1\right)}\right] \tag{35}$$

$$= T_\infty + (T_f - T_\infty)\left[\frac{\dfrac{k_s}{h_0\lambda_s\Gamma(1-\alpha/2)} + 1 - W\left(-\dfrac{x}{\lambda_s t^{\alpha/2}}; -\dfrac{\alpha}{2}, 1\right)}{\dfrac{k_s}{h_0\lambda_s\Gamma(1-\alpha/2)} + 1 - W\left(-\dfrac{\xi_{C\alpha}}{\lambda}; -\dfrac{\alpha}{2}, 1\right)}\right],$$

$$T_\ell(x,t) = T_i - (T_i - T_f)\frac{W\left(-\dfrac{x}{\lambda_\ell t^{\alpha/2}}; -\dfrac{\alpha}{2}, 1\right)}{W\left(-\xi_{C\alpha}; -\dfrac{\alpha}{2}, 1\right)} \tag{36}$$

$$= T_f + (T_i - T_f)\left[1 - \frac{W\left(-\dfrac{x}{\lambda_\ell t^{\alpha/2}}; -\dfrac{\alpha}{2}, 1\right)}{W\left(-\xi_{C\alpha}; -\dfrac{\alpha}{2}, 1\right)}\right],$$

$$s(t) = \xi_{C\alpha}\lambda_\ell t^{\alpha/2}, \tag{37}$$

where the coefficient $\xi = \xi_{C\alpha} > 0$ is the solution of the following equation:

$$F_{C_\alpha}(x) = \rho \ell \lambda_\ell \frac{\Gamma\left(1+\alpha/2\right)}{\Gamma\left(1-\alpha/2\right)} x, \quad x>0 \tag{38}$$

with

$$F_{C_\alpha}(x) = \frac{k_s(T_f - T_\infty)}{\lambda_s} F_{4\alpha}(x/\lambda) - \frac{k_\ell(T_i - T_f)}{\lambda_\ell} F_{2\alpha}(x) \tag{39}$$

where

$$F_{4\alpha}(x) = \frac{M_{\alpha/2}(x)}{\dfrac{k_s}{h_0 \lambda_s \Gamma(1-\alpha/2)} + 1 - W\left(-x; -\dfrac{\alpha}{2}, 1\right)}. \tag{40}$$

b) *If the coefficient* h_0 *satisfies the inequalities*

$$0 < h_0 \le \frac{k_\ell(T_i - T_f)}{\lambda_\ell(T_f - T_\infty)} \frac{1}{\Gamma(1-\alpha/2)}, \tag{41}$$

then the problem (FCP_α*) is a fractional diffusion problem for the initial liquid phase whose solution is given by:*

$$T_\ell(x,t) = T_\infty + \frac{T_i - T_\infty}{1 + \dfrac{k_\ell}{h_0 \lambda_\ell \Gamma(1-\alpha/2)}} \left[\frac{k_\ell}{h_0 \lambda_\ell \Gamma(1-\alpha/2)} + 1 - W\left(-\frac{x}{\lambda_\ell t^{\alpha/2}}; -\frac{\alpha}{2}, 1\right) \right], \quad x>0, \quad t>0 \tag{42}$$

Theorem 4 *Let* $T_\infty < T_f < T_i$ *be. If the coefficient* h_0 *satisfies the inequality (34) then the solution of the problem (*FCP_α*) converges to the classical solution of the problem (*FCP_1*) when* $\alpha \to 1^-$*, and then we recover the classical Neumann explicit solution and the inequality for the coefficient* h_0 *which characterized the convective (Robin) boundary condition at* $x = 0$ *obtained for* $\alpha = 1$ *in [29], that is:*

$$h_0 > \frac{k_\ell}{\sqrt{\pi \alpha_\ell}} \frac{T_i - T_f}{T_f - T_\infty}. \tag{43}$$

4 CONCLUSIONS

- We have obtained generalized Neumann solutions for two two-phase fractional Lamé-Clapeyron-Stefan problems for a semi-infinite material with constant initial condition, when a heat flux or a convective (Robin) boundary condition is imposed on the fixed face $x = 0$.
- The explicit solutions are given through the Wright and Mainardi functions.
- When $\alpha \to 1^-$, we recover the two classical Neumann solutions (which are equivalents among them) for the corresponding classical two-phase Lamé-Clapeyron-Stefan problem given through the error function, and also the inequalities for the corresponding coefficients which characterized the heat flux or the convective boundary condition at $x = 0$.

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