IFIP AICT 494

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Jean-Antoine Désidéri
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(Eds.)

System Modeling and Optimization

27th IFIP TC 7 Conference, CSMO 2015
Sophia Antipolis, France, June 29 – July 3, 2015
Revised Selected Papers

Springer

Double Convergence of a Family of Discrete Distributed Mixed Elliptic Optimal Control Problems with a Parameter

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Abstract. The convergence of a family of continuous distributed mixed elliptic optimal control problems (P_α), governed by elliptic variational equalities, when the parameter $\alpha \to \infty$ was studied in Gariboldi - Tarzia, Appl. Math. Optim., 47 (2003), 213-230 and it has been proved that it is convergent to a distributed mixed elliptic optimal control problem (P). We consider the discrete approximations $(P_{h\alpha})$ and (P_h) of the optimal control problems (P_α) and (P) respectively, for each $h > 0$ and $\alpha > 0$. We study the convergence of the discrete distributed optimal control problems $(P_{h\alpha})$ and (P_h) when $h \to 0$, $\alpha \to \infty$ and $(h, \alpha) \to (0, +\infty)$ obtaining a complete commutative diagram, including the diagonal convergence, which relates the continuous and discrete distributed mixed elliptic optimal control problems $(P_{h\alpha})$, (P_α), (P_h) and (P) by taking the corresponding limits. The convergent corresponds to the optimal control, and the system and adjoint system states in adequate functional spaces.

Keywords: Double convergence · Distributed optimal control problems · Elliptic variational equalities · Mixed boundary conditions · Numerical analysis · Finite element method · Fixed points · Optimality conditions · Error estimations

1 Introduction

The purpose of this paper is to do the numerical analysis, by using the finite element method, of the convergence of the continuous distributed mixed optimal control problems with respect to a parameter (the heat transfer coefficient) given in [10, 11] obtaining a double convergence when the parameter of the finite element method goes to zero and the heat transfer coefficient goes to infinity.

We consider a bounded domain $\Omega \subset \mathbb{R}^n$ whose regular boundary $\Gamma = \partial\Omega = \Gamma_1 \cup \Gamma_2$ consists of the union of two disjoint portions Γ_1 and Γ_2 with meas $(\Gamma_1) > 0$. We consider the following elliptic partial differential problems with mixed boundary conditions, given by:

$$-\Delta u = g \quad \text{in } \Omega; \quad u = b \quad \text{on } \Gamma_1; -\frac{\partial u}{\partial n} = q \quad \text{on } \Gamma_2, \tag{1}$$

L. Bociu et al. (Eds.): CSMO 2015, IFIP AICT 494, pp. 493–504, 2016.
DOI: 10.1007/978-3-319-55795-3_47

$$-\Delta u = g \quad \text{in } \Omega; -\frac{\partial u}{\partial n} = \alpha(u-b) \quad \text{on } \Gamma_1; -\frac{\partial u}{\partial n} = q \quad \text{on } \Gamma_2 \tag{2}$$

where g is the internal energy in Ω, $b = Const. > 0$ is the temperature on Γ_1 for the system (1) and the temperature of the external neighborhood on Γ_1 for the system (2) respectively, q is the heat flux on Γ_2 and $\alpha > 0$ is the heat transfer coefficient on Γ_1. The systems (1) and (2) can represent the steady-state two-phase Stefan problem for adequate data [21, 22]. We consider the following continuous distributed optimal control problem (P) and a family of continuous distributed optimal control problems (P_α) for each parameter $\alpha > 0$, defined in [10], where the control variable is the internal energy g in Ω, that is: Find the continuous distributed optimal controls $g_{op} \in H = L^2(\Omega)$ and $g_{\alpha_{op}} \in H$ (for each $\alpha > 0$) such that:

$$\text{Problem (P): } J(g_{op}) = \min_{g\in H} J(g), \quad \text{Problem } (P_\alpha): J_\alpha(g_{\alpha_{op}}) = \min_{g\in H} J_\alpha(g) \tag{3}$$

where the quadratic cost functional $J, J_\alpha : H \to \mathbb{R}_0^+$ are defined by [2, 18, 26]:

$$(a)\ J(g) = \frac{1}{2}\|u_g - z_d\|_H^2 + \frac{M}{2}\|g\|_H^2, \quad (b)\ J_\alpha(g) = \frac{1}{2}\|u_{\alpha g} - z_d\|_H^2 + \frac{M}{2}\|g\|_H^2 \tag{4}$$

with $M > 0$ and $z_d \in H$ given, $u_g \in K$ and $u_{\alpha g} \in V$ are the state of the systems defined by the mixed ellliptic differential problems (1) and (2) respectively whose elliptic variational equalities are given by [16]:

$$u_g \in K: \quad a(u_g, v) = (g, v) - \int_{\Gamma_2} qv d\gamma, \quad \forall v \in V_0 \tag{5}$$

$$u_{\alpha g} \in V: \quad a_\alpha(u_{\alpha g}, v) = (g, v) - \int_{\Gamma_2} qv d\gamma + \alpha \int_{\Gamma_1} bv d\gamma, \quad \forall v \in V \tag{6}$$

and their adjoint system states $p_g \in V$ and $p_{\alpha g} \in V$ are defined by the following elliptic variational equalities:

$$\begin{aligned} &(a)\ p_g \in V_o: \ a(p_g, v) = (u_g - z_d, v), \forall v \in V_0; \\ &(b)\ p_{\alpha g} \in V: a_\alpha(p_{\alpha g}, v) = (u_{\alpha g} - z_d, v), \forall v \in V \end{aligned} \tag{7}$$

with the spaces and bilinear forms defined by:

$$V = H^1(\Omega),\ V_0 = \{v \in V, v/\Gamma_1 = 0\},\ K = b + V_0,\ H = L^2(\Omega),\ Q = L^2(\Gamma_2) \tag{8}$$

$$a(u,v) = \int_\Omega \nabla u . \nabla v dx, \quad a_\alpha(u,v) = a(u,v) + \alpha \int_{\Gamma_1} uv d\gamma, \quad (u,v) = \int_\Omega uv\, dx \tag{9}$$

where the bilinear, continuous and symmetric forms a and a_α are coercive on V_0 and V respectively, that is [16]:

$$\exists \lambda > 0 \text{ such that } \lambda \|v\|_V^2 \leq a(v,v), \quad \forall v \in V_0 \tag{10}$$

$$\exists \lambda_\alpha = \lambda_1 \min(1,\alpha) > 0 \text{ such that } \lambda_\alpha \|v\|_V^2 \leq a_\alpha(v,v), \quad \forall v \in V \tag{11}$$

and $\lambda_1 > 0$ is the coercive constant for the bilinear form a_1[16, 21].

The unique continuous distributed optimal energies g_{op} and $g_{\alpha_{op}}$ have been characterized in [10] as a fixed point on H for a suitable operators W and W_α over their optimal adjoint system states $p_{g_{op}} \in V_0$ and $p_{\alpha g_{\alpha_{op}}} \in V$ defined by:

$$W, W_\alpha : H \to H \quad \text{such that} \quad (a)\ W(g) = -\frac{1}{M} p_g, \quad (b)\ W_\alpha(g) = -\frac{1}{M} p_{\alpha g}. \tag{12}$$

The limit of the optimal control problem (P_α) when $\alpha \to \infty$ was studied in [10] and it was proven that:

$$\lim_{\alpha\to\infty} \left\| u_{\alpha g_{\alpha_{op}}} - u_{g_{op}} \right\|_V = 0, \quad \lim_{\alpha\to\infty} \left\| p_{\alpha g_{\alpha_{op}}} - p_{g_{op}} \right\|_V = 0, \quad \lim_{\alpha\to\infty} \left\| g_{\alpha_{op}} - g_{op} \right\|_H = 0 \tag{13}$$

for a large constant $M > 0$ by using the characterization of the optimal controls as fixed points through operators (12a) and (12b); this restrictive hypothesis on data was eliminated in [11] by using the variational formulations. We can summary the conditions (13) saying that the distributed optimal control problems (P_α) converges to the distributed optimal control problem (P) when $\alpha \to +\infty$.

Now, we consider the finite element method and a polygonal domain $\Omega \subset \mathbb{R}^n$ with a regular triangulation with Lagrange triangles of type 1, constituted by affine-equivalent finite element of class C^0 being h the parameter of the finite element approximation which goes to zero [3, 7]. Then, we discretize the elliptic variational equalities for the system states (6) and (5), the adjoint system states (7a) and (7b), and the cost functional (4a, b) respectively. In general, the solution of a mixed elliptic boundary problem belongs to $H^r(\Omega)$ with $1 < r \leq 3/2 - \varepsilon$ $(\varepsilon > 0)$ but there exist some examples which solutions belong to $H^r(\Omega)$ with $2 \leq r$ [1, 17, 20]. Note that mixed boundary conditions play an important role in various applications, e.g. heat conduction and electric potential problems [12].

The goal of this paper is to study the numerical analysis, by using the finite element method, of the convergence results (13) corresponding to the continuous distributed elliptic optimal control problems (P_α) and (P) when $\alpha \to +\infty$. The main result of this paper can be characterized by the following result:

Theorem 1. We have the following complete commutative diagram which relates the continuous distributed mixed optimal control problems (P_α) and (P), with the discrete distributed mixed optimal control problems $(P_{h\alpha})$ and (P_h) and it is obtained by taking the limits $h \to 0, \alpha \to +\infty$ and $(h,\alpha) \to (0,+\infty)$, as in Fig. 1, where $g_{h\alpha_{op}}$, $u_{h\alpha g_{h\alpha_{op}}}$ and $p_{h\alpha g_{h\alpha_{op}}}$ are respectively the optimal control, the system and the adjoint system

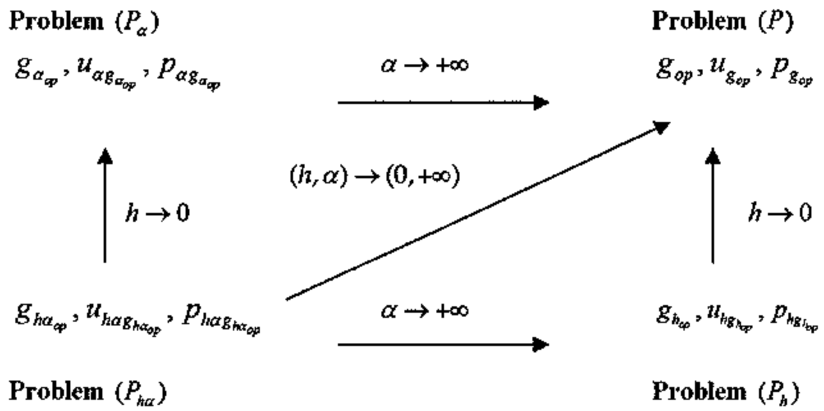

Fig. 1. Relationship among optimal control problems $(P_{h\alpha})$, (P_α), (P_h) and (P) by taking the limits $h \to 0$, $\alpha \to +\infty$ and $(h,\alpha) \to (0,+\infty)$.

states of the discrete distributed mixed optimal control problem $(P_{h\alpha})$ for each $h > 0$ and $\alpha > 0$, and the double convergence is the diagonal one.

The study of the limit $h \to 0$ of the discrete solutions of optimal control problems can be considered as a classical limit, see [4–6, 8, 9, 13–15, 19, 23, 24, 27, 28] but the limit $\alpha \to +\infty$, for each $h > 0$, and the double limit $(h,\alpha) \to (0,+\infty)$ can be considered as a new ones.

The paper is organized as follows. In Sect. 2 we define the discrete elliptic variational equalities for the state systems u_{hg} and $u_{h\alpha g}$, we define the discrete distributed cost functional J_h and $J_{h\alpha}$, we define the discrete distributed optimal control problems (P_h) and $(P_{h\alpha})$, and the discrete elliptic variational equalities for the adjoint state systems p_{hg} and $p_{h\alpha g}$ for each $h > 0$ and $\alpha > 0$, and we obtain properties for the discrete optimal control problems (P_h) and $(P_{h\alpha})$. In Sect. 3 we study the classical convergences of the discrete distributed optimal control problems (P_h) to (P), and $(P_{h\alpha})$ to (P_α) when $h \to 0$ (for each $\alpha > 0$) and the estimations for the discrete cost functional J_h and $J_{h\alpha}$. In Sect. 4 we study the new convergence of the discrete distributed optimal control problems $(P_{h\alpha})$ to (P_h) when $\alpha \to +\infty$ for each $h > 0$ and we obtain a commutative diagram which relates the continuous and discrete distributed mixed optimal control problems $(P_{h\alpha})$, (P_α), (P_h) and (P) by taking the limits $h \to 0$ and $\alpha \to +\infty$. In Sect. 5 we study the new double convergence of the discrete distributed optimal control problems $(P_{h\alpha})$ to (P) when $(h,\alpha) \to (0,+\infty)$ and we obtain the diagonal convergence in the previous commutative diagram.

2 Discretization by Finite Element Method and Properties

We consider the finite element method and a polygonal domain $\Omega \subset \mathbb{R}^n$ with a regular triangulation with Lagrange triangles of type 1, constituted by affine-equivalent finite element of class C^0 being h the parameter of the finite element approximation which goes to zero [3, 7]. We can take h equal to the longest side of the triangles $T \in \tau_h$ and we can approximate the sets V, V_0 and K by:

$$V_h = \left\{v_h \in C^0(\overline{\Omega})/v_h/T \in P_1(T),\ \forall T \in \tau_h\right\}, V_{0h} = \{v_h \in V_h/v_h/\Gamma_1 = 0\}; K_h = b + V_{0h} \quad (14)$$

where P_1 is the set of the polymonials of degree less than or equal to 1. Let $\pi_h : C^0(\bar{\Omega}) \to V_h$ be the corresponding linear interpolation operator. Then there exists a constant $c_0 > 0$ (independent of the parameter h) such that [3]:

$$(a)\ \|v - \pi_h(v)\|_H \le c_0 h^r \|v\|_r; (b)\ \|v - \pi_h(v)\|_V \le c_0 h^{r-1} \|v\|_r; \forall v \in H^r(\Omega), 1 < r \le 2. \quad (15)$$

We define the discrete cost functional $J_h, J_{h\alpha} : H \to \mathbb{R}_0^+$ by the following expressions:

$$(a)\ J_h(g) = \frac{1}{2}\left\|u_{hg} - z_d\right\|_H^2 + \frac{M}{2}\|g\|_H^2, \quad (b)\ J_{h\alpha}(g) = \frac{1}{2}\left\|u_{h\alpha g} - z_d\right\|_H^2 + \frac{M}{2}\|g\|_H^2 \quad (16)$$

where u_{hg} and $u_{h\alpha g}$ are the discrete system states defined as the solution of the following discrete elliptic variational equalities [16, 24]:

$$u_{hg} \in K_h : \quad a\left(u_{hg}, v_h\right) = (g, v_h) - \int_{\Gamma_2} q v_h d\gamma, \quad \forall v_h \in V_{0h}, \quad (17)$$

$$u_{h\alpha g} \in V_h : \quad a_\alpha\left(u_{h\alpha g}, v_h\right) = (g, v_h) - \int_{\Gamma_2} q v_h d\gamma + \alpha \int_{\Gamma_1} b v_h d\gamma, \quad \forall v_h \in V_h. \quad (18)$$

The corresponding discrete distributed optimal control problems consists in finding $g_{h_{op}}, g_{h\alpha_{op}} \in H$ such that:

$$\begin{aligned} &(a)\ \text{Problem } (P_h) : J_h\left(g_{h_{op}}\right) = \underset{g \in H}{Min}\ J_h(g), \\ &(b)\ \text{Problem } (P_{h\alpha}) : J_{h\alpha}\left(g_{h\alpha_{op}}\right) = \underset{g \in H}{Min}\ J_{h\alpha}(g) \end{aligned} \quad (19)$$

and their corresponding discrete adjoint states p_{hg} and $p_{h\alpha g}$ are defined respectively as the solution of the following discrete elliptic variational equalities:

$$p_{hg} \in V_{0h} : \quad a\left(p_{hg}, v_h\right) = \left(u_{hg} - z_d, v_h\right), \quad \forall v_h \in V_{0h} \quad (20)$$

$$p_{h\alpha g} \in V_h : \quad a_\alpha\left(p_{h\alpha g}, v_h\right) = \left(u_{h\alpha g} - z_d, v_h\right), \quad \forall v_h \in V_h \quad (21)$$

Remark 1. We note that the discrete (in the n-dimensional space) distributed optimal control problem (P_h) and $(P_{h\alpha})$ are still infinite dimensional optimal control problems since the control space is not discretized.

Lemma 2.

(i) *There exist unique solutions* $u_{hg} \in K_h$ *and* $p_{hg} \in V_{0h}$, *and* $u_{h\alpha g} \in V_h$ *and* $p_{h\alpha g} \in V_h$ *of the elliptic variational equalities* (17) *and* (20), (18), *and* (21) *respectively* $\forall g \in H, \forall q \in Q, b > 0 \, on \, \Gamma_1$.

(ii) *The operators* $g \in H \to u_{hg} \in V$, and $g \in H \to u_{h\alpha g} \in V$ are *Lipschitzians. The operators* $g \in H \to p_{hg} \in V_{0g}$, and $g \in H \to p_{h\alpha g} \in V_h$ are *Lipschitzians and strictly monotone operators.*

Proof. We use the Lax-Milgram Theorem, the variational equalities (17), (18), (20) and (21), the coerciveness (10) and (11) and following [10, 18, 25]. □

Theorem 3.

(i) *The discrete cost functional* J_h *and* $J_{h\alpha}$ *are* H *- elliptic and strictly convexe applications, that is* $(\forall g_1, g_2 \in H, \forall t \in [0,1])$:

$$(1-t)J_h(g_2) + tJ_h(g_1) - J_h(tg_1 + (1-t)g_2) \geq M\frac{t(1-t)}{2}\|g_2 - g_1\|_H^2 \quad (22)$$

$$(1-t)J_{h\alpha}(g_2) + tJ_{h\alpha}(g_1) - J_{h\alpha}(tg_1 + (1-t)g_2) \geq M\frac{t(1-t)}{2}\|g_2 - g_1\|_H^2 \quad (23)$$

(ii) *There exist a unique optimal controls* $g_{h_{op}} \in H$ *and* $g_{h\alpha_{op}} \in H$ *that satisfy the optimization problems* (19*a*) *and* (19*b*) *respectively.*

(iii) J_h *and* $J_{h\alpha}$ *are Gâteaux differentiable applications and their derivatives are given by the following expressions:*

$$(a)\; J_h'(g) = Mg + p_{hg}, \quad (b)\; J_{h\alpha}'(g) = Mg + p_{h\alpha g}, \quad \forall g \in H, \quad \forall h > 0 \quad (24)$$

(iv) *The optimality condition for the optimization problems* (19*a*) *and* (19*b*) *are given by:*

$$\text{(a)}\; J_h'(g_{h_{op}}) = 0 \Leftrightarrow g_{h_{op}} = -\frac{1}{M}p_{hg_{h_{op}}};\; \text{(b)}\; J_{h\alpha}'(g_{h\alpha_{op}}) = 0 \Leftrightarrow g_{h\alpha_{op}} = -\frac{1}{M}p_{h\alpha g_{h\alpha_{op}}} \quad (25)$$

(v) J_h' *and* $J_{h\alpha}'$ are *Lipschitzians and strictly monotone operators.*

Proof. We use the definitions (16a, b), the elliptic variational equalities (17) and (18) and the coerciveness (10) and (11), following [10, 18, 25]. □

We define the operators:

$$W_h, W_{h\alpha} : H \to H \text{ such that (a) } W_h(g) = -\frac{1}{M} p_{hg}, \quad (b)\ W_{h\alpha}(g) = -\frac{1}{M} p_{h\alpha g}. \quad (26)$$

Theorem 4. *We have that:*

(i) *W_h and $W_{h\alpha}$ are Lipschitzian operators, and W_h ($W_{h\alpha}$) is a contraction operator if and only if M is large, that is:*

$$\text{(a) M} > \frac{1}{\lambda^2}, \quad (b)\ \text{M} > \frac{1}{\lambda_\alpha^2}. \quad (27)$$

(ii) *If M verifies the inequalities (27a, b) then the discrete distributional optimal control $g_{h_{op}} \in H$ ($g_{h\alpha_{op}} \in H$) is obtained as the unique fixed point of W_h $(W_{h\alpha})$, i.e.:*

$$\begin{aligned} g_{h_{op}} &= -\frac{1}{M} p_{hg_{h_{op}}} \Leftrightarrow W_h\left(g_{h_{op}}\right) = g_{h_{op}}, \\ g_{h\alpha_{op}} &= -\frac{1}{M} p_{h\alpha g_{h\alpha_{op}}} \Leftrightarrow W_{h\alpha}\left(g_{h\alpha_{op}}\right) = g_{h\alpha_{op}}. \end{aligned} \quad (28)$$

Proof. We use the definitions (25a, b), and the properties (25a, b) and Lemma 2. □

3 Convergence of the Discrete Distributed Optimal Control Problems (P_h) to (P) and $(P_{h\alpha})$ to (P_α) When $h \to 0$

We obtain the following error estimations between the continuous and discrete solutions:

Theorem 6. *We suppose the continuous system states and adjoint system states have the regularities $u_g, u_{\alpha g_{\alpha_{op}}} \in H^r(\Omega)$ and $p_g, p_{\alpha g_{\alpha_{op}}} \in H^r(\Omega)$ $(1 < r \le 2)$. If M verifies the inequalities (27a, b) then we have the following error bonds:*

$$\left\|g_{h_{op}} - g_{op}\right\|_H \le ch^{r-1}, \quad \left\|u_{hg_{h_{op}}} - u_{g_{op}}\right\|_V \le ch^{r-1}, \quad \left\|p_{hg_{h_{op}}} - p_{g_{op}}\right\|_V \le ch^{r-1} \quad (29)$$

$$\begin{aligned} &\left\|g_{h\alpha_{op}} - g_{\alpha_{op}}\right\|_H \le ch^{r-1}, \quad \left\|u_{h\alpha g_{h\alpha_{op}}} - u_{\alpha g_{\alpha_{op}}}\right\|_V \le ch^{r-1}, \\ &\left\|p_{h\alpha g_{h\alpha_{op}}} - p_{\alpha g_{\alpha_{op}}}\right\|_V \le ch^{r-1} \end{aligned} \quad (30)$$

where c's are constants independents of h.

Proof. It is useful to use the restriction $\alpha > 1$ by splitting a_α by [21, 24, 25].

$$a_\alpha(u,v) = a_1(u,v) + (\alpha - 1)\int_{\Gamma_1} uv d\gamma \tag{31}$$

but then it can be replaced by $\alpha \geq \alpha_0$ for any $\alpha_0 > 0$. We follow a similar method to the one developed in [25] for Neumann boundary optimal control problems by using the elliptic variational equalities (17), (18), (20) and (21), the thesis holds. □

Remark 2. If M verifies the inequalities (27a, b) we can obtain the convergence in Theorem 6 by using the characterization of the fixed point (28a, b), and the uniqueness of the optimal controls $g_{op} \in H$ and $g_{\alpha_{op}} \in H$.

Now, we give some estimations for the discrete cost functional $J_{h\alpha}$ and J_h.

Lemma 7. *If M verifies the inequality (27a, b) and the continuous system states and adjoint system states have the regularities $u_g, u_{\alpha g} \in H^r(\Omega)$ $p_g, p_{\alpha g} \in H^r(\Omega)(1 < r \leq 2)$ then we have the following error bonds:*

$$\begin{aligned} &\frac{M}{2}\|g_{h_{op}} - g_{op}\|_H^2 \leq J(g_{h_{op}}) - J(g_{op}) \leq Ch^{2(r-1)}, \\ &\frac{M}{2}\left\|g_{h\alpha_{op}} - g_{\alpha_{op}}\right\|_H^2 \leq J_\alpha(g_{h\alpha_{op}}) - J_\alpha(g_{\alpha_{op}}) \leq Ch^{2(r-1)} \end{aligned} \tag{32}$$

$$\begin{aligned} &\frac{M}{2}\|g_{h_{op}} - g_{op}\|_H^2 \leq J_h(g_{op}) - J_h(g_{h_{op}}) \leq Ch^{2(r-1)}; \\ &\frac{M}{2}\|g_{h\alpha_{op}} - g_{\alpha_{op}}\|_H^2 \leq J_{h\alpha}(g_{\alpha_{op}}) - J_{h\alpha}(g_{h\alpha_{op}}) \leq Ch^{2(r-1)} \end{aligned} \tag{33}$$

$$|J_h(g_{op}) - J(g_{op})| \leq Ch^{r-1}, \quad |J_h(g_{h_{op}}) - J(g_{op})| \leq Ch^{r-1} \tag{34}$$

$$|J_{h\alpha}(g_{op}) - J_\alpha(g_{op})| \leq Ch^{r-1}, \quad |J_{h\alpha}(g_{h\alpha_{op}}) - J_\alpha(g_{\alpha_{op}})| \leq Ch^{r-1} \tag{35}$$

where C's are constants independents of h and α.

Proof. Estimations (32) and (33) follow from the estimations (29), and the equalities (similar relationship for J and J_α):

$$J_\alpha(g_{h\alpha_{op}}) - J_\alpha(g_{\alpha_{op}}) = \frac{1}{2}\left\|u_{h\alpha g_{h\alpha_{op}}} - u_{\alpha g_{op}}\right\|_H^2 + \frac{M}{2}\|g_{h\alpha_{op}} - g_{\alpha_{op}}\|_H^2 \tag{36}$$

$$J_{h\alpha}(g_{\alpha_{op}}) - J_{h\alpha}(g_{h\alpha_{op}}) = \frac{1}{2}\left\|u_{h\alpha g_{h_{op}}} - u_{h\alpha g_{h\alpha_{op}}}\right\|_H^2 + \frac{M}{2}\|g_{h\alpha_{op}} - g_{\alpha_{op}}\|_H^2 \tag{37}$$

$$|J_{h\alpha}(g) - J_\alpha(g)| \leq \left(\frac{1}{2}\|u_{h\alpha g} - u_{\alpha g}\|_H + \|u_{\alpha g} - z_d\|_H\right)\|u_{h\alpha g} - u_{\alpha g}\|_H, \quad \forall g \in H. \tag{38}$$

□

4 Convergence of the Discrete Optimal Control Problems $(P_{h\alpha})$ to (P_h) When $\alpha \to +\infty$

Theorem 9. We have the following limits:

$$\lim_{\alpha\to+\infty}\left\|u_{h\alpha g_{h\alpha op}} - u_{hg_{hop}}\right\|_V = \lim_{\alpha\to+\infty}\left\|p_{h\alpha g_{h\alpha op}} - p_{hg_{hop}}\right\|_V = \lim_{\alpha\to+\infty}\left\|g_{h\alpha_{op}} - g_{h_{op}}\right\|_H = 0, \forall h > 0. \tag{39}$$

Proof. We omit this proof because we prefer to prove the next one with more details.

5 Double Convergence of the Discrete Distributed Optimal Control Problem $(P_{h\alpha})$ to (P) When $(h,\alpha) \to (0,+\infty)$

For the discrete distributed optimal control problem $(P_{h\alpha})$ we will now consider the double limit $(h,\alpha) \to (0,+\infty)$.

Theorem 10. We have the following limits:

$$\lim_{(h,\alpha)\to(0,+\infty)}\left\|u_{h\alpha g_{h\alpha op}} - u_{g_{op}}\right\|_V = \lim_{(h,\alpha)\to(0,+\infty)}\left\|p_{h\alpha g_{h\alpha op}} - p_{g_{op}}\right\|_V = \lim_{(h,\alpha)\to(0,+\infty)}\left\|g_{h\alpha_{op}} - g_{op}\right\|_H = 0 \tag{40}$$

Proof. From now on we consider that c's represent positive constants independents simultaneously of $h > 0$ and $\alpha > 0$ (see (31)). We show a sketch of the proof by obtaining the following estimations (for $\forall h > 0$ and $\forall \alpha > 1$):

$$\|u_{h0}\|_V \le c_1, \quad \|u_{h\alpha 0}\|_V \le c_2, \quad (\alpha - 1)\int_{\Gamma_1}(u_{h\alpha 0} - b)^2 d\gamma \le c_3 \tag{41}$$

$$\|g_{h\alpha_{op}}\|_H \le c_4, \quad \left\|u_{h\alpha g_{h\alpha op}}\right\|_H \le c_5, \quad \|g_{h_{op}}\|_H \le c_6 \tag{42}$$

$$\left\|u_{hg_{hop}}\right\|_V \le c_7, \quad \left\|u_{h\alpha g_{h\alpha op}}\right\|_V \le c_8, \quad (\alpha - 1)\int_{\Gamma_1}\left(u_{h\alpha g_{h\alpha op}} - b\right)^2 d\gamma \le c_9 \tag{43}$$

$$\left\|p_{hg_{hop}}\right\|_V \le c_{10}, \quad \left\|p_{h\alpha g_{h\alpha op}}\right\|_V \le c_{11}, \quad (\alpha - 1)\int_{\Gamma_1} p^2_{h\alpha g_{h\alpha op}} d\gamma \le c_{12}. \tag{44}$$

For example, the constant c_{11} is a positive constant independent simultaneously of $h > 0$ and $\alpha > 0$, and it is given by the following expression:

$$
\begin{aligned}
c_{11} = \|z_d\|_H &\left[\frac{1}{\lambda_1}\left(1+\frac{1}{\sqrt{M}}\left(\frac{1}{\lambda_1}+\frac{1}{\lambda}+\frac{1}{\lambda\lambda_1}\right)\right)+\frac{1}{\lambda}\left(1+\frac{1}{\lambda_1}\right)\left(1+\frac{1}{\lambda\sqrt{M}}\right)\right] \\
&+ b\left[\frac{1}{\lambda_1}\left(1+\frac{1}{\lambda_1}\right)\left(1+\frac{1}{\lambda\sqrt{M}}+\frac{1}{\lambda_1\sqrt{M}}\right)+\frac{1}{\lambda}\left(1+\frac{1}{\lambda_1}\right)\left(1+\frac{1}{\lambda\sqrt{M}}\right)\right] \\
&+ \|q\|_Q\|y_0\|\left[\frac{1}{\lambda_1}\left[\frac{1}{\lambda_1}+\frac{1}{\lambda}\left(1+\frac{1}{\lambda_1}\right)+\frac{1}{\sqrt{M}}\left(\frac{1}{\lambda\lambda_1}+\frac{1}{\lambda^2}\left(1+\frac{1}{\lambda_1}\right)+\frac{1}{\lambda_1^2}\left(1+\frac{1}{\lambda}\right)\right)\right]\right. \\
&\left.+\frac{1}{\lambda^2}\left(1+\frac{1}{\lambda_1}\right)\left(1+\frac{1}{\lambda\sqrt{M}}\right)\right]
\end{aligned}
\tag{45}
$$

Therefore, from the above estimations we have that:

$$\exists f \in H / g_{h\alpha_{op}} \longrightarrow f \text{ in } H \text{ weak as } (h,\alpha) \to (0,+\infty) \tag{46}$$

$$\exists\, \eta \in V / u_{h\alpha g_{h\alpha op}} \longrightarrow \eta \text{ in } V \text{ weak } (H \text{ strong}) \text{ as } (h,\alpha) \to (0,+\infty) \text{ with } \eta/\Gamma_1 = b \tag{47}$$

$$\exists\, \xi \in V / p_{h\alpha g_{h\alpha op}} \longrightarrow \xi \text{ in } V \text{ weak } (H \text{ strong}) \text{ as } (h,\alpha) \to (0,+\infty) \text{ with } \xi/\Gamma_1 = 0 \tag{48}$$

$$\exists f_h \in H / g_{h\alpha_{op}} \longrightarrow f_h \text{ in } H \text{ weak as } \alpha \to +\infty \tag{49}$$

$$\exists\, \eta_h \in V / u_{h\alpha g_{h\alpha op}} \longrightarrow \eta_h \quad \text{in } V \text{ weak (in } H \text{ strong) as } \alpha \to +\infty \text{ with } \eta_h/\Gamma_1 = b \tag{50}$$

$$\exists\, \xi_h \in V / p_{h\alpha g_{h\alpha op}} \longrightarrow \xi_h \quad \text{in } V \text{ weak (in } H \text{ strong) as } \alpha \to +\infty \text{ with } \xi_h/\Gamma_1 = 0 \tag{51}$$

$$\exists f_\alpha \in H / g_{h\alpha_{op}} \longrightarrow f_\alpha \text{ in } H \text{ weak as } h \to 0 \tag{52}$$

$$\exists\, \eta_\alpha \in V / u_{h\alpha g_{h\alpha op}} \longrightarrow \eta_\alpha \quad \text{in } V \text{ weak (in } H \text{ strong) as } h \to 0 \text{ with } \eta_\alpha/\Gamma_1 = b \tag{53}$$

$$\exists\, \xi_\alpha \in V / p_{h\alpha g_{h\alpha op}} \longrightarrow \xi_\alpha \quad \text{in } V \text{ weak (in } H \text{ strong) as } h \to 0 \text{ with } \xi_\alpha/\Gamma_1 = 0 \tag{54}$$

$$\exists f^* \in H / g_{h_{op}} \longrightarrow f^* \text{ in } H \text{ weak as } h \to 0 \tag{55}$$

$$\exists\, \eta^* \in V / u_{h g_{h_{op}}} \longrightarrow \eta^* \text{ in } V \text{ weak } (H \text{ strong}) \text{ as } h \to 0 \text{ with } \eta^*/\Gamma_1 = b \tag{56}$$

$$\exists\, \xi^* \in V / p_{h g_{h_{op}}} \longrightarrow \xi^* \text{ in } V \text{ weak } (H \text{ strong}) \text{ as } h \to 0 \text{ with } \xi^*/\Gamma_1 = 0 \tag{57}$$

Taking into account the uniqueness of the distributed optimal control problems $(P_{h\alpha})$, (P_α), (P_h) and (P), and the uniqueness of the elliptic variational equalities corresponding to their state systems we get

$$\eta_h = u_{hf_h} = u_{hg_{hop}}, \quad \xi_h = p_{hf_h} = p_{hg_{hop}}, \quad f_h = g_{hop} \tag{58}$$

$$\eta_\alpha = u_{\alpha f_\alpha} = u_{\alpha g_{\alpha op}}, \quad \xi_\alpha = p_{\alpha f_\alpha} = p_{\alpha g_{\alpha op}}, \quad f_\alpha = g_{\alpha op} \tag{59}$$

$$\eta = \eta^* = u_f = u_{g_{op}}, \quad \xi = \xi^* = p_f = p_{g_{op}}, \quad f = f^* = g_{op}. \tag{60}$$

Now, by using [11] we obtain

$$\lim_{\alpha\to+\infty} \|f_\alpha - g_{op}\|_H = 0, \quad \lim_{\alpha\to+\infty} \left\|\eta_\alpha - u_{g_{op}}\right\|_V = 0, \quad \lim_{\alpha\to+\infty} \left\|\xi_\alpha - p_{g_{op}}\right\|_V = 0 \tag{61}$$

and therefore the three double limits (40) hold when $(h, \alpha) \to (0, +\infty)$.

Proof of Theorem 1. It is a consequence of the properties (29), (30), (39), (40) and [10, 11].

Remark 3. We note that this double convergence is a novelty with respect to the recent results obtained for a family of discrete Neumann boundary optimal control problems [25].

Acknowledgements. The present work has been partially sponsored by the Projects PIP No 0534 from CONICET - Univ. Austral, Rosario, Argentina, and AFOSR-SOARD Grant FA9550-14-1-0122.

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