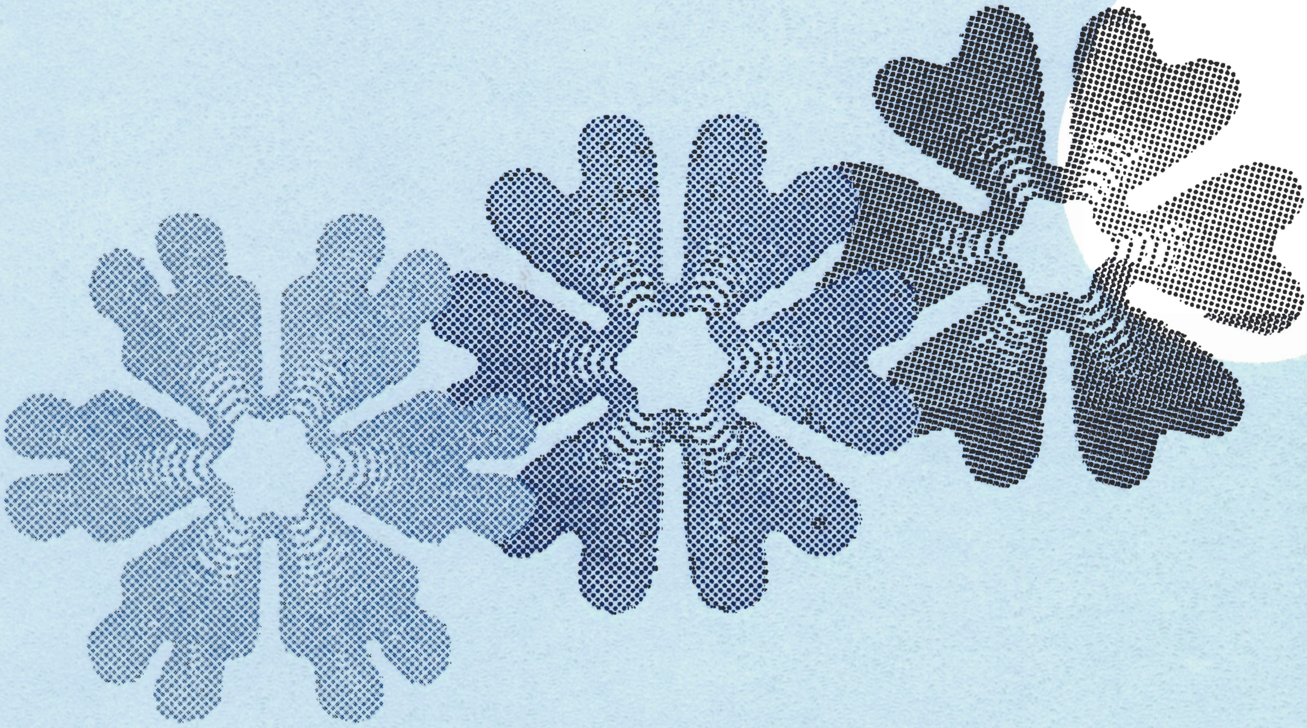
Computational Modelling of Free and Moving Boundary Problems III

Editors: L.C. Wrobel,
B.Šarler, C.A. Brebbia



Computational Mechanics Publications

L.C. Wrobel
Wessex Institute of Technology
Ashurst Lodge
Ashurst
Southampton
SO40 7AA, UK

B. Šarler
Laboratory for Fluid Dynamics
and Thermodynamics
Faculty of Mechanical Engineering
University of Ljubljana
Slovenia

C.A. Brebbia
Wessex Institute of Technology
Ashurst Lodge
Ashurst
Southampton
SO40 7AA, UK

Published by

Computational Mechanics Publications
Ashurst Lodge, Ashurst, Southampton, SO40 7AA, UK
Tel: 44 (0)1703 293223; Fax: 44 (0)1703 292853

For USA, Canada and Mexico

Computational Mechanics Inc
25 Bridge Street, Billerica, MA 01821, USA
Tel: 508 667 5841; Fax: 508 667 7582

British Library Cataloguing-in-Publication Data

A Catalogue record for this book is available
from the British Library

ISBN: 1 85312 322 6 Computational Mechanics Publications, Southampton
ISBN: 1 56252 246 9 Computational Mechanics Publications, Boston

Library of Congress Catalog Card Number 95 67482

The texts of the various papers in this volume were set individually by the authors or under their supervision

Printed and bound in Great Britain by Bell & Bain Ltd, Glasgow, UK.

On the determination of the unknown coefficients through phase-change processes

D.A. Tarzia
Depto. Matemática, FCE, Universidad Austral, Paraguay 1950, (2000) Rosario, Argentina

Abstract

We present two different approaches on the determination of thermal coefficients of a semi-infinite material through a phase-change process with an overspecified heat flux condition on the fixed face.

1) We use a simple mushy zone model in a two-phase solidification problem (Stefan problem) for the simultaneous determination of some unknown coefficients. We also find the necessary and sufficient conditions for the existence of a solution and the corresponding formulae for the unknown coefficients.

2) We use approximate methods (heat balance integral and variational methods) for a one-phase Stefan problem for the simultaneous determination of some unknown coefficients. We propose a convexe linear expansion (with a parameter to be also determined) of the first and second order approximation solutions for the temperature. We also find the necessary and sufficient conditions for the existence of a solution and the corresponding formulae for the unknown coefficients .

1 Introduction

Heat transfer problems with phase-change such as melting and freezing have been studied in the last century because of their wide scientific and technological applications (e.g. Alexiades & Solomon [1], Lunardini [10], Rubinstein [11]). For example, a review of a long bibliography on moving and free boundary problems for the heat equation, particularly concerning the Stefan problem, is presented in Tarzia [16] with a large bibliography (we are now preparing an updated one with approximately 4500 references).

First, we shall consider a semi-infinite material with mass densities $\rho > 0$ equal in both solid and liquid phases and we assume, without loss of generality, that the phase-change temperature is $0^{\underline{o}}$ C.

If the material is initially assumed to be liquid at the constant temperature $E > 0$ and a constant temperature $-D < 0$ is imposed on the fixed face $x=0$, then three distinct regions can be distinguished (for a mathematical and properties description of this simple model see Tarzia [20]; for the one-phase model see Solomon, Wilson & Alexiades [12]) :

(H_1) The liquid phase, at temperature $\theta_2=\theta_2(x,t) > 0$, occupying the region $x > r(t)$, $t > 0$.

(H_2) The solid phase, at temperature $\theta_1=\theta_1(x,t) < 0$, occupying the region $0 < x < s(t)$, $t > 0$.

(H_3) The mushy zone, at temperature 0, occupying the region $s(t) \leq x \leq r(t)$, $t > 0$. We make two assumptions on its structure :

(a) The material in the mushy zone contains a fixed fraction ϵh (with $0 < \epsilon < 1$) of the total latent heat $h > 0$, i.e.,

$$k_1 \, \theta_{1_x}(s(t),t) - k_2 \, \theta_{2_x}(r(t),t) = \rho \, h \Big(\epsilon \, \dot{s}(t) + (1 - \epsilon) \, \dot{r}(t)\Big), \quad t > 0. \tag{1}$$

(b) The width of the mushy zone is inversely proportional (with constant $\gamma > 0$) to the temperature gradient at the point $(s(t), t)$, i.e.,

$$\theta_{1_x}(s(t),t) \Big(r(t) - s(t)\Big) = \gamma \, , \quad t > 0. \tag{2}$$

We suppose that the temperature $\theta = \theta(x,t)$ of the material is defined by

$$\theta(x,t) = \left| \begin{array}{lll} \theta_1(x,t) < 0 & \text{if} & 0 < x < s(t), \, t > 0 \\ 0 & \text{if} & s(t) \leq x \leq r(t), \, t > 0 \\ \theta_2(x,t) > 0 & \text{if} & x > r(t), \, t > 0 . \end{array} \right. \tag{3}$$

The governing differential equations take the following forms for the solid and liquid phases :

$$\alpha_1 \, \theta_{1_{xx}}(x,t) = \theta_{1_t}(x,t) \quad , \; 0 < x < s(t), \, t > 0 \, , \tag{4}$$

$$\alpha_2\, \theta_{2_{xx}}(x,t) = \theta_{2_t}(x,t) \quad , \quad x > r(t),\ t > 0 \ , \tag{5}$$

where $c_i > 0$, $k_i > 0$ and $\alpha_i = a_i^{\,2} = k_i / \rho c_i > 0$ are the specific heat, the thermal conductivity and the diffusion coefficient for the phase i (i = 1 : solid phase; i = 2 : liquid phase) respectively.

The conditions at the solid-mushy interface $x = s(t)$ and the mushy-liquid interface $x = r(t)$ are given by (1), (2) and the requirement of the continuity of the temperature, i.e.,

$$\theta_1(s(t),t) = \theta_2(r(t),t) = 0 \quad , \quad t > 0. \tag{6}$$

The initial and boundary conditions are given by

$$\theta_1(0,t) = -D < 0 \quad , \quad t > 0 \ , \tag{7}$$

$$\theta_2(x,0) = \theta_2(+\infty, t) = E > 0 \quad , \quad x > 0 \ , t > 0 \ , \tag{8}$$

$$s(0) = r(0) = 0. \tag{9}$$

We consider an overspecified heat flux condition (e.g. Cannon [4], Tarzia [15]) on the fixed face $x = 0$ which is given by (e.g. Stampella & Tarzia [14], Tarzia [17 – 19], Solomon, Wilson & Alexiades [13])

$$k_1\, \theta_{1_x}(0,t) = \frac{q_o}{\sqrt{t}} \quad , \quad t > 0 \quad , \quad \text{with } q_o > 0. \tag{10}$$

Moreover, the coefficients $q_o > 0$ (which characterizes the heat flux at the fixed face x=0) and $T_o > 0$ (which is the temperature at the fixed face x = 0) must be found through an experimental phase-change process (e.g. Arderius, Lara & Tarzia [1]). If by means of a phase-change experiment we are able to measure these quantities, then we shall find formulae for the determination of the unknown coefficients (ϵ, γ : parameters of the mushy zone; h, ρ, c_1, c_2, k_1, k_2 : thermal coefficients of the material). Moreover, it does exist iff some complementary conditions for the corresponding data are verified. We generalize some of the results obtained in Stampella & Tarzia [14] for the particular case $\epsilon = 1$ and $\gamma = 0$ (i.e., for a sharp interphase, without mushy region) and those obtained in Tarzia [19] for the one-phase case. In Tarzia [18] several references on the determination of physical coefficients were given. We shall only consider here the respective properties for the determination of the thermal conductivity k_2 of the liquid phase (initial phase) or the parameter $\epsilon \in (0,1)$ which characterizes the free boundary condition (1). Other cases will be considered in Gonzalez & Tarzia [8].

Secondly, we shall consider the determination of the unknown coefficients through approximate methods (heat balance integral and variational methods) corresponding to the one-phase Stefan problem. We consider the following melting problem for a semi-infinite material with an overspecified condition on the fixed face :

$$\rho\, c\, T_t = k\, T_{xx} \;, \qquad 0 < x < s(t) \;, t > 0, \tag{11}$$

$$s(0) = 0 \;, \tag{12}$$

$$T(0,t) = T_o > 0 \;, t > 0 \;, \tag{13}$$

$$T(s(t),t) = 0 \;, t > 0 \;, \tag{14}$$

$$k\, T_x(s(t),t) = -\, \rho\, h\, \dot{s}(t) \;, t > 0 \;, \tag{15}$$

$$k\, T_x(0,t) = -\, \frac{q_o}{\sqrt{t}} \;, t > 0 \;, \tag{16}$$

where $T = T(x,t)$ is the temperature of the liquid phase.

In Tarzia [17,18] one or two thermal coefficients were determined, and formulae for the unknown coefficients were given by using the exact Lamé-Clapeyron solution. In Garguichevich, Sanziel & Tarzia[7] the approximate solution is given by using the quasi-stationnary, heat balance integral and variational methods. Now, we shall consider the heat balance integral and variational methods (e.g. Goodman [9], Biot [3]) for the determination of one thermal coefficient. We propose a different approach to the one proposed before by considering a convexe linear expansion (with a parameter to be also determinated) of the first and second order approximation solutions for the temperature. We also find the necessary and sufficient conditions for the existence of a solution and the corresponding formulae for the unknown coefficients. In Castellini & Tarzia [6] comparison results are obtained for the quasi-stationnary, heat balance integral and variational methods.

II Determination of one unknown thermal coefficient through a mushy zone model for the two-phase Stefan problem

Taking into account the hypothese $(H_1) - (H_3)$ we can formulate the following

Problem (P_1) : Find the free boundaries $x=s(t)$ and $x=r(t)$, defined for $t > 0$ with $0 < s(t) < r(t)$ and $s(0) = r(0) = 0$, the temperature $\theta = \theta(x,t)$, defined by (3) for $x > 0$ and $t > 0$, and k_2 or ϵ such that they satisfy the conditions (1), (2), (4) – (10) where $D > 0$, $E > 0$ and $q_o > 0$ are data and they must be known or determined by an experience of phase-change.

The solution of this problem is given by (e.g. Carslaw & Jaeger [5], Rubinstein [11], Solomon, Wilson & Alexiades [12], Tarzia [15, 20])

$$\theta_1(x,t) = -D + \frac{D}{f\left(\frac{\sigma}{a_1}\right)} f\left(\frac{x}{2\, a_1 \sqrt{t}}\right), \tag{17}$$

$$\theta_2(x,t) = \frac{-E\, f\left(\frac{\omega}{a_2}\right)}{1 - f\left(\frac{\omega}{a_2}\right)} + \frac{E}{1 - f\left(\frac{\omega}{a_2}\right)} f\left(\frac{x}{2\, a_2 \sqrt{t}}\right), \tag{18}$$

$$s(t) = 2\,\sigma \sqrt{t}\ ,\ \sigma > 0\ , \tag{19}$$

$$r(t) = 2\,\omega \sqrt{t}\ ,\ \omega > \sigma\ , \tag{20}$$

the coefficient ω is given by

$$\omega = \omega(\sigma) = a_1\, W\left(\frac{\sigma}{a_1}\right), \tag{21}$$

and, the coefficient σ and the unknown thermal coefficient k_2 or ϵ are obtained by solving the following system of equations

$$\frac{q_o}{h\,\rho\, a_1} \exp\left(-\frac{\sigma^2}{{a_1}^2}\right) - \frac{E\, k_2}{h\,\rho\, a_1\, a_2 \sqrt{\pi}} F_1\left(\frac{\omega(\sigma)}{a_2}\right) = G\left(\frac{\sigma}{a_1}\right)\ , \tag{22}$$

$$\frac{a_1}{k_1} f\left(\frac{\sigma}{a_1}\right) = \frac{D}{q_o \sqrt{\pi}} \tag{23}$$

where

$$f(x) = \operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x \exp(-t^2)\, dt\ , \qquad F_1(x) = \frac{\exp(-x^2)}{1 - f(x)}\ , \tag{24}$$

$$W(x) = x + \frac{\gamma \sqrt{\pi}}{2\,D} f(x) \exp(x^2)\ , \qquad G(x) = x + \frac{(1-\epsilon)\,\gamma \sqrt{\pi}}{2\,D} f(x) \exp(x^2)\,. \tag{25}$$

Theorem 1. – (i) (Determination of k_2) The necessary and sufficient condition for Problem (P_1), with σ and k_2 unknown, to have a unique solution is that data $D > 0$, $E > 0$, $q_o > 0$, mushy zone coefficients $0 < \epsilon < 1$ and $\gamma > 0$, and thermal coefficients of the phase-change material h, ρ, c_1, c_2, $k_1 > 0$ do verify the condition

$$\frac{D\, k_1}{q_o\, a_1 \sqrt{\pi}} < f(p)\ , \tag{26}$$

where p is the unique positive zero of function P. In such case, the solution is given by (17) – (20) with

$$k_2 = \frac{k_1\, c_2}{c_1}\, \frac{W^2(\xi_1)}{B^2} \;, \qquad \sigma = a_1\, \xi_1 \;, \qquad \omega = a_1\, W(\xi_1) \,, \tag{27}$$

where ξ_1 is the unique solution of the equation

$$f(x) = \frac{D\, k_1}{q_o\, a_1\, \sqrt{\pi}} \;, \quad x > 0 \,. \tag{28}$$

and B is the only solution of the equation

$$\frac{1}{Q(x)} = \frac{h}{E\, c_2}\, \frac{H(\xi_1)}{W(\xi_1)} \;, \quad x > 0 \,. \tag{29}$$

where

$$H(x) = \frac{q_o}{h\, \rho\, a_1} \exp(-x^2) - G(x) \,, \qquad Q(x) = \sqrt{\pi}\, x \exp(x^2) \Big(1 - f(x)\Big) \,, \tag{30}$$

$$P(x) = \frac{q_o}{E\, \rho\, a_1\, c_2} \exp(-x^2) - \Big(1 + \frac{(1-\epsilon)\, h}{E\, c_2}\Big) \frac{\gamma \sqrt{\pi}}{2\, D}\, f(x) \exp(x^2) - \Big(1 + \frac{h}{E\, c_2}\Big)\, x \,. \tag{31}$$

(ii) (Determination of ϵ) The necessary and sufficient condition for Problem (P_1), with σ and ϵ unknown, to have a unique solution is that data $D > 0$, $E > 0$, $q_o > 0$, mushy zone coefficient $\gamma > 0$ and thermal coefficients of the phase-change material h, ρ, c_1, c_2, k_1, $k_2 > 0$ do verify the conditions

$$q_o \;>\; \frac{E\, k_2}{a_2\, \sqrt{\pi}} \qquad , \qquad f(v) \;<\; \frac{D\, k_1}{q_o\, a_1\, \sqrt{\pi}} \;<\; f(u) \;, \tag{32}$$

where u and v are the unique positive zeros of functions U and V respectively, which are defined by

$$U(x) = \frac{q_o}{h\, \rho\, a_1} \exp(-x^2) - x - \frac{E\, k_2}{h\, \rho\, a_1\, a_2\, \sqrt{\pi}}\, F_1\Big(\frac{a_1}{a_2}\, W(x)\Big) \,, \tag{33}$$

$$V(x) = \frac{\gamma\sqrt{\pi}}{2\, D}\, f(x) \exp(x^2) - U(x) \;, \qquad F_2(x) = \frac{\exp(-x_2)}{f(x)} \,. \tag{34}$$

In such case, the solution is given by (17) – (20) with

$$\epsilon = \frac{2\, D}{\gamma\, \sqrt{\pi}}\, F_2(\xi_1)\, V(\xi_1) \,, \qquad \sigma = a_1\, \xi_1 \,, \qquad \omega = a_1\, W(\xi_1) \,, \tag{35}$$

where ξ_1 is the unique solution of the eqn (28).

III Determination of one unknown thermal coefficient through the heat balance integral method in a one-phase Stefan problem

By using the heat balance integral method (e.g. Goodman [9]) we replace eqn (11) by

$$\frac{d}{dt}\left(\int_0^{s(t)} \theta(x,t)\ dx\right) = -\frac{k}{\rho c}\left(\frac{\rho h}{k}\ \dot{s}(t) + \theta_x(0,t)\right),\ 0 < x < s(t)\ ,\ t > 0\ , \quad (36)$$

and condition (15) by

$$\theta_{xx}\,(s(t),t) = \frac{c}{h}\ \theta_x^2(s(t),t)\quad ,\ t > 0\ . \quad (37)$$

We can formulate the following

Problem (P_2) : Find the free boundary $x=s(t)$, defined for $t>0$, the temperature $\theta=\theta(x,t)$, defined for $0<x<s(t)$, $t>0$, and one of the four thermal coefficient k, ρ, h and c such that they satisfy the conditions (12) – (14), (16), (36) and (37).

If we choose a convexe linear expansion for the temperature profile given by

$$\theta(x,t) = \theta_0\left[\lambda\left(1-\frac{x}{s(t)}\right)^2 + \left(1-\lambda\right)\left(1-\frac{x}{s(t)}\right)\right]\ ,\quad 0<\lambda<1\ , \quad (38)$$

the free boundary of problem (P_2) is given by

$$s(t) = 2\,\sigma\,\sqrt{t} = 2\,\mu\,\sqrt{\alpha\, t}\ ,\qquad \alpha = \frac{k}{\rho c}\ , \quad (39)$$

where $\sigma>0$ is the coefficient which caracterizes the free boundary and the unknown coefficients λ, σ (or μ) and one of the four elements $\{k, \rho, h, c\}$ must satisfy the following system of equation

$$\mu^2 = \frac{\frac{Ste}{2}\,(1+\lambda)}{1+\frac{Ste}{2}\,(1-\frac{\lambda}{3})}\ ,\qquad Ste = \frac{c\,T_o}{h}\ , \quad (40)$$

$$\lambda = \beta - \sqrt{\beta^2-1}\ ,\qquad \beta = 1+\frac{1}{Ste}\ , \quad (41)$$

$$T_o^2\, k\rho c = \frac{4\,\mu^2}{(1+\lambda)^2}\, q_o^2\ , \quad (42)$$

Theorem 2. – The solution of the four cases is summarized in Table 1.

Unknown Coefficients	Restriction	Solution
$k\,,\,\sigma$	–	$k=\dfrac{4\,q_o^2}{\rho c \theta_o^2}\,\dfrac{\mu^2}{(1+\lambda)^2}\;,\quad \sigma=\dfrac{2\,q_o}{\rho c \theta_o}\,\dfrac{\mu^2}{(1+\lambda)^2}$ where μ, λ are given by (40) and (41) respectively.
$\rho\,,\,\sigma$	–	$\rho=\dfrac{4\,q_o^2}{k c \theta_o^2}\,\dfrac{\mu^2}{(1+\lambda)^2}\;,\quad \sigma=\dfrac{k\,\theta_o\,(1+\lambda)}{2\,q_o}$ where μ, λ are given by (40) and (41) respectively.
$c\,,\,\sigma$	$\dfrac{k\rho h T_o}{2\,q_o^2}<1$	$\sigma=\sqrt{\dfrac{3\,k\theta_o}{2\,\rho\,h}}\,\sqrt{\dfrac{\lambda+1}{2\lambda^2-3\lambda+3}}\left(1-\lambda\right),$ $c=\dfrac{h}{\theta_o}\,\dfrac{2\lambda}{(1-\lambda)^2}\,,$ where λ is the unique solution of the equation $M(\lambda)=\dfrac{k\rho h\theta_o}{6q_o^2}\;,\;0<\lambda<1\,.$
$h,\,\sigma$	$\dfrac{k\rho c T_o^2}{2\,q_o^2}<1$	$h=c\,\theta_o\,\dfrac{(1-\lambda)^2}{2\lambda}\;,\;\sigma=\sqrt{\dfrac{k}{\rho c}}\,\sqrt{\dfrac{3\lambda(\lambda+1)}{2\lambda^2-3\lambda+3}}$ where λ is the unique solution of the equation $N(\lambda)=\dfrac{1}{12}\,\dfrac{\rho c k\theta_o^2}{q_o^2}\;,\quad 0<\lambda<1\,.$

Table 1. Restrictions and formulae for the unknown thermal coefficients through the heat balance integral method, where M and N are defined by

$$M(\lambda)=\frac{(1-\lambda)^2}{(1+\lambda)(2\lambda^2-3\lambda+3)}\;,\qquad N(\lambda)=\frac{\lambda}{(1+\lambda)(2\lambda^2-3\lambda+3)}$$

and verify the conditions

$$M(0)=\tfrac{1}{3}\,,\;M(1)=0,\;M'(\lambda)<0 \text{ in } (0,1)\quad;\quad N(0)=0\,,\;N(1)=\tfrac{1}{4}\,,\;N'(\lambda)>0 \text{ in } (0,1).$$

IV Determination of one unknown thermal coefficient through the Biot's variational method in a one-phase Stefan problem

By using the variational method (e.g. Biot [3]) we replace condition (15) by (37) and eqn (11) by

$$\frac{\partial V_o}{\partial s} + \frac{\partial D_o}{\partial \dot{s}} = Q \ , 0 < x < s(t) \ , t > 0 \ , \tag{43}$$

where

$$V_o = \frac{1}{2} \int_0^{s(t)} \rho c \ \theta^2(x,t) \ dx \ : \text{thermal potential} \ , \tag{44}$$

$$D_o = \frac{1}{2} \int_0^{s(t)} \frac{1}{k} \ (\dot{H}_o(x,t))^2 \ dx \ : \text{dissipation function} \ , \tag{45}$$

$$Q_o = \theta_o \frac{\partial H_o}{\partial s}(0,t) \ : \ \text{thermal forces} \ , \tag{46}$$

where H_o is the heat displacement defined as the solution of the problem

$$\frac{\partial H_o}{\partial x} = - \rho c \ \theta \ , \quad H_o(s(t),t) = \rho L \ s(t) \ . \tag{47}$$

We can then formulate the following

Problem (P$_3$) : Find the free boundary $x=s(t)$, defined for $t>0$, the temperature $\theta=\theta(x,t)$, defined for $0<x<s(t)$, $t>0$, and one of the four thermal coefficient k, ρ, h and c such that they satisfy the conditions (12) – (14), (16), (37) and (43).

If we choose a convexe linear expansion for the temperature profile given by (38), then the free boundary of problem (P$_3$) is given by (39) and the unknown coefficients λ, σ (or μ) and one of the four elements $\{k, \rho, h, c\}$ must satisfy the following system of equation

$$\mu^2 = \frac{21}{2} \lambda R(\lambda) \ , \qquad T_o^2 \ k\rho c = \frac{4\ \mu^2}{(1+\lambda)^2} q_o^2 \ , \qquad \frac{c\ T_o}{h} = \frac{2\ \lambda}{(1-\lambda)^2} \tag{48}$$

where

$$R(\lambda) = \frac{30 - 40\lambda + 25\lambda^2 - \lambda^3}{315 - 840\lambda + 10008\lambda^2 - 574\lambda^3 + 143\ \lambda^4} \tag{49}$$

$$R(0) = \frac{2}{21} \ , \ R(1) = \frac{7}{26} \ , \ R'(1) = 0 \ , \ R' > 0 \ \text{in} \ (0,1) \ . \tag{50}$$

Theorem 3. – The solution of the four cases is summarized in Table 2.

Unknown Coefficients	Restriction	Solution
$k\ ,\ \sigma$	–	$k=\dfrac{42\,q_o^2}{\rho c\theta_o^2}\,\dfrac{\lambda R(\lambda)}{(1+\lambda)^2}\ ,\quad \sigma=\dfrac{21\,q_o}{\rho c\theta_o}\,\dfrac{\lambda R(\lambda)}{1+\lambda}$ where λ is given by (41).
$\rho\ ,\ \sigma$	–	$\rho=\dfrac{42\,q_o{}^2}{k c\theta_o{}^2}\,\dfrac{\lambda R(\lambda)}{(1+\lambda)^2}\ ,\quad \sigma=\dfrac{k\theta_o(1+\lambda)}{2\,q_o}$ where λ is given by (41).
$c\ ,\ \sigma$	$\dfrac{k\rho h T_o}{2\,q_o^2}<1$	$c=\dfrac{2h}{\theta_o}\,\dfrac{\lambda}{(1-\lambda)^2}\ ,\quad \sigma=\dfrac{k\theta_o}{2\,q_o}\,(1+\lambda)$ where λ is the unique solution of the equation $\dfrac{(1-\lambda)^2 R(\lambda)}{(1+\lambda)^2}=\dfrac{k\rho h\ \theta_o}{21\,q_o^2}\ ,\quad 0<\lambda<1\ .$
$h,\ \sigma$	$\dfrac{52\ k\rho c T_o^2}{147\ q_o^2}\ <\ 1$	$h=\dfrac{c\theta_o}{2}\,\dfrac{(1-\lambda)^2}{\lambda}\ ,\quad \sigma=\dfrac{k\theta_o}{2\,q_o}\,(1+\lambda)$ where λ is the unique solution of the equation $\dfrac{\lambda R(\lambda)}{(1+\lambda)^2}=\dfrac{k\rho c\theta_o^2}{42\,q_o^2}\ ,\quad 0<\lambda<1\ .$

Table 2. Restrictions and formulae for the unknown thermal coefficients through the variational method.

Acknowledgment

This paper has been partially sponsored by the Project No. 221 "Aplicaciones de Problemas de Frontera Libre" from CONICET – UA, Rosario (Argentina).

References

1. Alexiades, V. & Solomon, A.D. *Mathematical modeling of melting and freezing processes*, Hemisphere Publishing Corp., Washington, 1993.

2. Arderius, J.C., Lara, M.A. & Tarzia, D.A. *Experimental-numerical determination of thermal coefficients through a phase-change process*, To appear.

3. Biot, M.A. *Variational principle in heat transfer*, Clarendon Press, Oxford, 1970.

4. Cannon, J.R. *The one-dimensional heat equation*, Addison-Wesley, Menlo Park, California, 1984.

5. Carslaw, H.S. & Jaeger, J.C. *Conduction of heat in solids*, Oxford University Press, London, 1959.

6. Castellini, H. & Tarzia, D.A. , In preparation.

7. Garguichevich, G.G., Sanziel, M.C. & Tarzia, D.A. Comparison of approximate methods for the determination of thermal coefficients through a phase-change problems, *Int. Comm. Heat Mass Transfer*, 1985, **12**, 451–464.

8. Gonzalez, A.M. & Tarzia, D.A. , In preparation.

9. Goodman, T.R. The heat balance integral and its applications to problems to involving a change of phase, *Trans. of the ASME*, 1958, **80**, 335 – 342.

10. Lunardini, V.J. *Heat transfer with freezing and thawing*, Elsevier, Amsterdam, 1991.

11 Rubinstein, L.I. *The Stefan problem*, Translations of Mathematical Monographs, Vol. 27, Amer. Math. Soc., Providence, 1971.

12 Solomon, A.D., Wilson, D.G. & Alexiades, V. A mushy zone model with an exact solution, *Letters Heat Mass Transfer*, 1982,**9**, 319–324.

13 Solomon, A.D., Wilson, D.G. & Alexiades, V. Explicit solutions to phase change problems, *Quart. Appl. Math.*, 1983, **41**, 237 – 243.

14 Stampella, M.B. & Tarzia, D.A. Determination of one or two unknown thermal coefficients of a semi-infinite material through a two-phase Stefan problem", *Int. J. Engng. Sci.*, 1989, **27**, 1407–1419.

15 Tarzia, D.A. An inequality for the coefficient σ of the free boundary $s(t) = 2\sigma\sqrt{t}$ of the Neumann solution for two-phase Stefan problem, *Quart. Appl. Math.*, 1981 – 82, **39**, 491 – 497.

16. Tarzia, D.A. "Una revisión sobre problemas de frontera móvil y libre para la ecuación del calor. El problema de Stefan", *Math. Notae*, 1981, **29**, 147 – 241. See also *"A bibliography on moving–free boundary problems for the heat diffusion equation. The Stefan problem"* (with 2528 titles), Progetto Nazionale M.P.I. : "Equazioni di Evoluzione e Applicazioni Fisico– Matematiche", Firenze, 1988.

17. Tarzia, D.A. "Determination of the unknown coefficients in the Lamé-Clapeyron problem (or one-phase Stefan problem)", *Advances in Applied Mathematics*, 1982, **3**, 74 – 82.

18. Tarzia, D.A. "Simultaneous determination of two unknown thermal coeficients through an inverse one-phase Lamé-Clapeyron (Stefan) problem with an overspecified condition on the fixed face", *Int. J. Heat Mass Transfer*, 1983, **26**, 1151 – 1157.

19 Tarzia, D.A. Determination of unknown thermal coefficients of a semi-infinite material for the one-phase Lamé-Clapeyron (Stefan) problem through the Solomon-Wilson-Alexiades' mushy zone model, *Int. Comm. Heat Mass Transfer*, 1987, **17**, 219–228.

20 Tarzia, D.A. Neumann-like solution for the two-phase Stefan problem with a simple mushy zone model, *Comp. Appl. Math.*, 1990, **9**, 201–211.