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EXPLICIT SOLUTIONS TO SOME FREE BOUNDARY PROBLEMS FOR THE HEAT EQUATION

DOMINGO ALBERTO TARZIA

Abstract. A review on explicit solutions of a similarity type to some free boundary problems for the heat equation for a semi-infinite material is presented. In general, a heat flux condition of the type $q_0 t^{\frac{-1}{2}}$ on the fixed face x=0 is imposed which depends on a positive parameter q_0. That coefficient $q_0$must verify an inequality in order to obtain an explicit similarity solution for the following free boundary problems:
1) a mathematical model for the two-phase thawing in a saturated porous medium is considered when change of phase induces a density jump and considering the influence of pressure on the melting temperature. It has been analysed a heat flux and a temperature condition on the fixed face. The mathematical analysis is made for the different cases depending on the sign of three physical parameters .
2) a two-phase Stefan problem for a nonlinear heat conduction equation for materials of Storm-type with both heat flux and temperature boundary conditions on the fixed face is analysed.
3) drying with coupled phase change in a porous medium with a heat flux condition on the fixed face is considered. Moreover, for large Luikov number the temperature of the humid porous medium reaches a minimum value which is smaller than its initial temperature.

1. Introduction.

We present explicit solutions of a similarity type to some free boundary problems for the heat-diffusion equation for a semi-infinite material [$Ta2$]. In general, we impose a heat flux condition of the type $q_0 t^{\frac{-1}{2}}$ [$Ta1$] on the fixed face x=0 which depends on a positive parameter q_0 which must verify an inequality in order to obtain an explicit similarity solution for the following free boundary problems:

1) a mathematical model for the two-phase thawing in a saturated porous medium is considered when change of phase induces a density jump and considering the influence of pressure on the melting temperature [$FaGuPrRu$]. It has been analysed a heat flux condition on the fixed face in [$LoTa$] and a temperature condition on the fixed face in [$FaPrTa$]. The mathematical analysis is made for the different cases depending on the sign of three physical parameters;

2) a two-phase Stefan problem for a nonlinear heat conduction equation for materials of Storm-Type [Ro] with both heat flux and temperature boundary conditions on the fixed face is analysed in [$NaTa$];

3) drying with coupled phase change in a porous medium [Ch, Gu, Lu] with a heat flux condition on the fixed face is considered in [$SaTa$]. Moreover, for large Luikov number the temperature of the humid porous medium reaches a minimum value which is smaller than its initial temperature.

2. Two-phase thawing in a saturated porous medium.

2.1. Introduction.

In this part we consider the problem of thawing of partially frozen porous media, saturated with an incompressible liquid, with the aim of constructing similarity solutions. More specifically we deal with the following situations (for a detailed exposition of the physical background we refer to [$FaGuPrRu$]):

(i) a sharp interface between the frozen part and the unfrozen part of the domain exists (sharp, in the macroscopic sense);

(ii) the frozen phase is at rest with respect to the porous skeleton, which will be considered to be undeformable;

(iii) due to the density jump between the liquid and solid phase, thawing can induce either desaturation or water movement in the unfrozen region. We will consider the latter situation, assuming that liquid is continuously supplied to keep the medium saturated.

The unknowns of the problem are a function x=s(t) representing the free boundary separating $Q_1 \equiv \{(x, t) : 0 < x < s(t), t > 0\}$ and $Q_2 \equiv \{(x, t) : s(t) < x, t > 0\}$ and two functions $u(x,t)$ and $v(x,t)$ defined in Q_1 and in Q_2 respectively, representing the temperature of the unfrozen and of the frozen zone. Besides standard regularity requirements, $s(t)$, $u(x,t)$ and $v(x,t)$ fulfill the following conditions [$FaPrTa$]:

$$u_t = a_1 u_{xx} - b\rho\, \dot{s}(t)\, u_x \quad , in\ Q_1 \tag{1}$$

$$v_t = a_2 v_{xx} \ , in\ Q_2 \tag{2}$$

$$u(s(t),t) = v(s(t),t) = d\rho s(t)\, \dot{s}(t) \ , \ t > 0 \tag{3}$$

$$k_F v_x(s(t), t) - k_U u_x(s(t), t) = \alpha\, \dot{s}(t) + \beta\rho\, s(t) \left(\dot{s}(t)\right)^2 , \ t > 0 , \tag{4}$$

$$u(0, t) = B > 0 \ , t > 0 \ , \tag{5}$$

$$v(x, 0) = v(+\infty, t) = -A < 0,\ x > 0 ,\ t > 0 \tag{6}$$

$$s(0) = 0 \tag{7}$$

with

$$a_1 = \alpha_1^2 = \frac{k_U}{\rho_U c_U}, a_2 = \alpha_2^2 = \frac{k_F}{\rho_F c_F}, b = \frac{\epsilon \rho_W c_W}{\rho_U c_U},$$

$$d = \frac{\epsilon\gamma\mu}{K}, \rho = \frac{\rho_W - \rho_I}{\rho_W}, \alpha = \epsilon\rho_I\lambda,$$

$$\beta = \frac{\epsilon^2 \rho_I (c_W - c_I)\gamma\mu}{K} = \epsilon d \rho_I (c_W - c_I),$$

where $\epsilon > 0$: porosity; ρ_I and $\rho_W > 0$: density of water and ice; $c > 0$: specific heat at constant density; k_U and $k_F > 0$: conductivity of the unfrozen and frozen zone; u : temperature of the unfrozen zone; v : temperature of the frozen zone; $u = v = 0$: melting point at atmospheric pressure; $\lambda > 0$: latent heat at $u = 0$; $\mu > 0$: viscosity of liquid; γ : coefficient in the Clausius-Clapeyron law; $K > 0$: hydraulic permeability; $-A < 0$ $(A > 0)$: initial temperature; $B > 0$: boundary temperature at the fixed face $x = 0$.

Remark 1. The free boundary problem (1)−(7) reduces to the usual Stefan problem when $d\rho = 0$, since in that case we have the classical Stefan conditions on $x = s(t)$:

$$u\,(s(t),\,t) = v\,(s(t),\,t) = 0, t > 0, \tag{8}$$

$$k_F\, v_x(s(t),\,t) - k_U\, u_x(s(t),\,t) = \alpha\, \dot{s}\,(t), t > 0\,. \tag{9}$$

From now on we suppose that $d\rho \neq 0$. In Section 2.2. we study problem (1)−(7) with a temperature boundary condition on $x = 0$ [$FaPrTa$] and in Section 2.3. we study the problem (1)−(4), (21), (6), (7) with a heat flux condition on $x = 0$ [$LoTa$]. The existence of a similarity solution depends on the value of three parameters in both cases and an inequality for q_0 must be verified for the second case.

2.2 Similarity solutions with temperature boundary condition.

Now, we will look for similarity solutions to problem (1)−(7) by considering different cases as a function of the sign of the parameters ρ, β and d (that is γ). Our results include the cases considered in [$ChRu, FaGuPrRu$].We consider [$FaPrTa$] where the following results were obtained.

First of all, we note that the function

$$u\,(x,\,t) \;=\; \Phi\,(\eta) \quad , \quad with \quad \eta \;=\; \frac{x}{2\,\alpha_1\sqrt{t}} \tag{10}$$

is a solution of equation (1) if and only if Φ satisfies the following equation :

$$\frac{1}{2}\,\Phi''(\eta) + \left(\eta - \frac{b\rho}{\alpha_1}\,\dot{s}(t)\,\sqrt{t}\right))\Phi'(\eta) = 0\,. \tag{11}$$

Theorem 1. The free boundary problem (1)−(7) has the similarity solution

$$s(t) \;=\; 2\,\xi\,\alpha_1\,\sqrt{t}\;, \tag{12}$$

$$u\,(x,\,t) = B + \frac{m\xi^2 - B}{g(p,\xi)} \quad \int_0^{\frac{x}{2\alpha_1\sqrt{t}}} exp\,(\,-r^2\, +\, p\,\xi\, r\,)dr\;, \tag{13}$$

$$v(x,\,t) = \quad \frac{m\xi^2 \mathrm{erfc}\,(\frac{x}{2\,\alpha_2\sqrt{t}}) + A(\mathrm{erf}(\gamma_0\xi) - \mathrm{erf}(\frac{x}{2\,\alpha_2\sqrt{t}}))}{\mathrm{erfc}(\gamma_o\xi)} \tag{14}$$

if and only if the coefficient $\xi > 0$ satisfies the following equation

$$K_1\,(B\, -\, m\,y^2)H(p,\,y)\;-\;K_2\;F(m,y) = \delta\,y + \nu\,y^3, \quad y > 0, \tag{15}$$

where

$$erf(x) = \frac{2}{\sqrt{\pi}} \int_0^x exp(-r^2)dr , \quad erfc(x) = 1 - erf(x), \tag{16}$$

$$g(p,y) = \int_0^y exp(pyr - r^2)dr , \ H(p, y) = \frac{\exp((p-1)y^2)}{\frac{2}{\sqrt{\pi}} g(p,y)} \tag{17}$$

$$F(m, y) = (A + my^2) \frac{\exp(-\gamma_o^2 y^2)}{\text{erfc}(\gamma_o y)} \tag{18}$$

and the constants K_1 , K_2 , γ_o , δ, p, m, ν are defined as follows:

$$K_1 = \frac{k_U}{\alpha_1\sqrt{\pi}} > 0, K_2 = \frac{k_F}{\alpha_2\sqrt{\pi}} > 0, \gamma_o = \frac{\alpha_1}{\alpha_2} > 0, \delta = \alpha\,\alpha_1 > 0,$$

$$p = 2b\rho , \ m = 2d\rho\alpha_1^2 , \ \nu = 2\beta\rho\alpha_1^3 .$$

Now we are in position to discuss the solvability of (15), which we rewrite as follows

$$K_1 I(y) - K_2 F(y) = \delta y + \nu y^3, \qquad y > 0. \tag{19}$$

It will be convenient to denote by q_i, $i = 1, 2, \dots , N$ the zeroes of the l.h.s. of (19), whenever they exist, i.e.

$$K_1 I(q_i) - K_2 F(q_i) = 0, \ \ i = 1, 2, \dots N. \tag{20}$$

Then, we have:

Theorem 2. (i) If $m > 0$ and $\nu \geq 0$, then (19) always admits solutions.

Moreover

(a) if p$\leq$ 2, there exists a unique ξ satisfying (19) and $\xi < q_1 < y_o = \sqrt{\frac{B}{m}}$.

(b) if $p > 2$, there is one and only one solution ξ in $(0, q_1)$

(ii) If $m > 0$, $\nu < 0$, a solution to (19) exists whenever $\frac{B}{m} < \frac{\delta}{|\nu|}$.

(iii) If $m < 0$, $\nu \geq 0$, sufficient conditions for the existence of solutions are the existence of a root of (20) or

$$\nu > \left(K_1\left(\frac{p}{2}-1\right)^+ + K_2\gamma_0\right) |m| \sqrt{\pi},$$

where $(\ )^+$ denotes the positive part of the quantity in bracket.

(iv) If $m < 0$, $\nu < 0$, (19) has at least two solutions if (20) has roots and $q_1 < \sqrt{\frac{\delta}{|\nu|}}$.

Remark 2. When $c_W \neq c_I$ the temperature $u^* = \dfrac{\lambda}{c_I - c_W}$ is the intersection of the two lines representing the energy of the solid and the liquid as a function of temperature. We note that the similarity solution is such that

$$u(s(t), t) = v(s(t), t) = m\xi^2.$$

Therefore it seems appropriate to say that a similarity solution is physically acceptable if

$$m\xi^2 < \frac{\lambda}{c_I - c_W} \quad \text{when} \quad c_I > c_W \ , m\xi^2 > \frac{\lambda}{c_I - c_W} \quad \text{when} \quad c_I < c_W$$

that is

$$2d\rho\alpha_1^2(c_I - c_W)\xi^2 < \lambda \, .$$

2.3 Similarity solutions with heat flux boundary condition.

If we replace the condition (5) by the following one

$$k_U u_x(0,t) = -\frac{q_0}{\sqrt{t}} \ , t > 0 \tag{21}$$

we can consider the problem II given by (1)–(4), (21), (6) and (7) [$LoTa$] where the following results were obtained.

Theorem 3. The free boundary problem II has the following similarity solution

$$s(t) = 2\xi^* \alpha_1 \sqrt{t} \ , \tag{22}$$

$$u(x,t) = m(\xi^*)^2 + \frac{2q_o\alpha_1}{K_U} g(p,\xi^*) - \frac{2q_o\alpha_1}{K_U} \int_0^{\frac{x}{2\alpha_1\sqrt{t}}} exp(-r^2 + p\xi^* r)dr, \tag{23}$$

$$v(x,\,t) = \frac{m(\xi^*)^2 + A \,\mathrm{erf}(\gamma_0\xi^*)}{\mathrm{erfc}(\gamma_0\xi^*)} - \frac{m\,(\xi^*)^2}{\mathrm{erfc}(\gamma_0\xi^*)} erf\left(\frac{x}{2\,\alpha_1\sqrt{t}}\right) \tag{24}$$

if and only if the coefficient $\xi^* > 0$ satisfies the following equation

$$q_o \, exp\left((p-1)y^2\right) - K_2 F(m,y) = \delta\, y + \nu\, y^3, y > 0 \tag{25}$$

or its equivalent

$$Q_o(y) = q_o \, , \; y > 0$$

where Q_o is defined by the following expression

$$Q_o(y) = \frac{K_2 F(m,y) + \delta y + \nu y^3}{exp((p-1)y^2)} \ , \, y > 0 \, . \tag{26}$$

Theorem 4. Let $m > 0$, $\nu > 0$ be. Then:

(i) If $p \leq 1$, then the problem II has a unique similarity solution if and only if q_o verifies the following inequality

$$q_0 > \frac{K_F A}{\alpha_2\sqrt{\pi}} \, . \tag{27}$$

(ii) If $p > 1$, then the problem II has a unique similarity solution if and only if q_o verifies the following inequality

$$0 < q_o < \max_{y \geq 0} Q_o(y) \, . \tag{28}$$

Remark 3. For $\rho = 0$ (then $p = 0$ without jump of density) the inequality (27) has yet been found in $[Ta1]$.

We have now the following relationship between problems I and II. Let (s, u, v) be the solution of the problem II given by (22)–(24) where ξ^* is the unique root of the equation (25). Then $u(0,t)$ is a constant in time which is given by

$$u(0,t) = m(\xi^*)^2 + \frac{2q_o\alpha_1}{K_U} g(p,\xi^*) > 0 \,. \tag{29}$$

Then, we can consider the problem I by impossing this new temperature on the fixed face $x = 0$.

Theorem 5. Let $m > 0$, $\nu > 0$, $p \le 1$ be. We assume that q_o verifies the inequality (27). If (s, u, v) is the unique similarity solution of the problem II then (s, u, v) is the unique solution of the problem I where the constant B in the condition (5) is given by $B = T_o(q_o, \xi^*(q_o))$ where $\xi^*(q_o)$ is the unique solution of the equation (25) and $T_o(q, y) = my^2 + \frac{2q_0\,\alpha_1}{K_U} g(p, y)$. Moreover the two solutions are coincidental.

3. Two-phase Stefan problem for Storm's materials.

3.1. Introduction.

We consider a two-phase Stefan problem for a semi-infinite region $x > 0$ with phase change temperature T_f. It is required to determine the evolution of the moving phase separation boundary $x = X(t)$ and temperature distributions for each phase.

The following free boundary (fusion process) problem is considered $[NaTa, Ro, TrBr]$:

$$\rho c_{p_1}(T_1)\frac{\partial T_1}{\partial t} = \frac{\partial}{\partial x}\left(k_1(T_1)\frac{\partial T_1}{\partial x}\right) \,,\ X(t) < x < \infty \,,\ t > 0 \tag{30}$$

$$k_1(T_1)\frac{\partial T_1}{\partial x} - k_2(T_2)\frac{\partial T_2}{\partial x} = L\rho\,\dot{X} \,,\ x = X(t) \tag{31}$$

$$T_1 = T_2 = T_f \quad,\ x = X(t) \tag{32}$$

$$\rho c_{p_2}(T_2)\frac{\partial T_2}{\partial t} = \frac{\partial}{\partial x}\left(k_2(T_2)\frac{\partial T_2}{\partial x}\right) \,,\ 0 < x < X(t) \,,\ t > 0 \tag{33}$$

$$T_1(x,0) = T_0 < T_f \tag{34}$$

$$X(0) = 0 \tag{35}$$

$$k_2\left(T_2(0,t)\right)\frac{\partial T_2}{\partial x}(0,t) = -\frac{q_0}{\sqrt{t}} \,,\ q_0 > 0 \,,\ t > 0 \tag{36}$$

In the above, $T_i(x,t)$, $c_{p_i}(T_i)$, $k_i(T_i)$, $i = 1,2$ represent in turn the temperature distribution, specific heat and thermal conductivity in the two phases, solid and liquid respectively. The density ρ of the medium is assumed to be constant and L denotes the latent heat of fusion of the medium. Here $-q_0/\sqrt{t}$ denotes the prescribed flux on the boundary $x = 0$ while T_0 represents the initial temperature of the medium. It is noted that the two-phase Stefan problem in linear heat conduction with constant thermal coefficients and a heat flux of the type (36) was investigated by $[Ta1]$. It was proved in $[NaTa]$ that a necessary and sufficient condition in order to have an instantaneous phase-change process is that an inequality for the coefficient q_0 should be verified. Our investigation is confined to materials for which

$$K_i \frac{\Phi_i'}{\Phi_i^2} = k_i(T_i) \quad , \quad i = 1,2 \quad , \quad K_i > 0 \tag{37}$$

where

$$\Phi_i(T_i) = \int_{T_{0i}}^{T_i} S_i(\sigma)\, d\sigma \quad , \quad S_i(T_i) = \rho c_{p_i}(T_i) \quad i = 1,2 \ . \tag{38}$$

We remark that if (37) is true then $k_i(T_i)$ and $S_i(T_i)$ verify the Storm's relation

$$\frac{1}{\sqrt{k_i(T_i) S_i(T_i)}} \frac{d}{dT_i} \left(log \sqrt{\frac{S_i(T_i)}{k_i(T_i)}} \right) = \frac{1}{\sqrt{K_i}} \quad , \quad i = 1,2\, . \tag{39}$$

The above condition was originally obtained by Storm $[St]$ in an investigation of heat conduction in simple monatomic metals There the validity of the approximation (39) was examined for aluminum, silver, sodium, cadmium, zinc, copper and lead.

We follow now $[NaTa]$ which can be considered as a complement to $[Ro]$. We prove in Section 3.2. the existence and uniqueness of solution of problem (30) – (36) if and only if the positive constant q_0 is large enough, i.e.

$$q_0 > \sqrt{K_2} G^{-1} \left(\sqrt{\frac{K_2}{K_1}} \frac{1}{\left(\frac{\Phi_1(T_f)}{\Phi_1(T_0)} - 1 \right)} \right) \tag{40}$$

where $G^{-1} : (1, +\infty) \longmapsto (0, +\infty)$ is the inverse function of G with

$$G(x) = \operatorname{erf}(x) + \frac{1}{\sqrt{\pi}} \frac{\exp(-x^2)}{x} \quad , \quad x > 0. \tag{41}$$

which was defined in $[BrNaTa]$ and it was proved that

$$G(0^+) = +\infty \quad , \quad G(+\infty) = 1 \quad \text{and} \quad G'(x) < 0 \quad \forall x > 0. \tag{42}$$

The inequality (40) for the coefficient q_0 generalizes the corresponding inequality which has been obtained for phase-change materials with constant thermal coefficients $[Ta1]$

In Section 3.3. we consider the problem (30) – (35), and the flux condition (36) will be replace for the following temperature condition

$$T_2(0,t) = T_m > T_f \ . \tag{43}$$

on the fixed face. We can remark that there exists a relationship between both condition (36) and (43) on the fixed face $x = 0$ which is given by (55). We prove the existence and uniqueness of solution of problem (30) – (35) and (43) for all thermal coefficients.

3.2. Existence and uniqueness of solution of the free boundary problem with flux condition on the fixed face.

We obtain that the temperature distributions T_1 and T_2, and the free boundary X(t), for problem (30) – (36), are given parametrically by

$$T_1 = \Phi_1^{-1}\left\{A_1 \operatorname{erf}\left[\left(\frac{\gamma}{2K_1}\right)^{\frac{1}{2}}\xi_1^*\right] + B_1\right\}^{-1}, \; \xi_1 = 1 + \int_{\lambda_1}^{\xi_1^*}\left\{A_1 \operatorname{erf}\left[\left(\frac{\gamma}{2K_1}\right)^{\frac{1}{2}}\sigma\right] + B_1\right\} d\sigma \tag{44}$$

$$T_2 = \Phi_2^{-1}\left\{A_2 \operatorname{erf}\left[\left(\frac{\gamma}{2K_2}\right)^{\frac{1}{2}}\xi_2^*\right] + B_2\right\}^{-1}, \; \xi_2 = \int_{-\sqrt{2/\gamma q_0}}^{\xi_2^*}\left\{A_2 \operatorname{erf}\left[\left(\frac{\gamma}{2K_2}\right)^{\frac{1}{2}}\sigma\right] + B_2\right\} d\sigma \tag{45}$$

$$X(t) = \sqrt{2\gamma t}\ . \tag{46}$$

where the unknowns γ, A_i, B_i, λ_i $(i = 1, 2)$ are given by

$$\lambda_1 = L\rho + \Phi_1(T_f) - \Phi_2(T_f) + \lambda_2 \tag{47}$$

$$A_1 = \frac{1}{1-\operatorname{erf}\left(\sqrt{\frac{\gamma}{2K_1}}\lambda_1\right)}\left(\frac{1}{\Phi_1(T_0)} - \frac{1}{\Phi_1(T_f)}\right), B_1 = \frac{1}{1-\operatorname{erf}\left(\sqrt{\frac{\gamma}{2K_1}}\lambda_1\right)}\left(\frac{1}{\Phi_1(T_f)} - \frac{\operatorname{erf}\left(\sqrt{\frac{\gamma}{2K_1}}\lambda_1\right)}{\Phi_1(T_0)}\right) \tag{48}$$

$$A_2 = \frac{1}{\Phi_2(T_f)\left(G\left(\frac{q_0}{\sqrt{K_2}}\right) + \operatorname{erf}\left(\sqrt{\frac{\gamma}{2K_2}}\lambda_2\right)\right)}, B_2 = \frac{G\left(\frac{q_0}{\sqrt{K_2}}\right)}{\Phi_2(T_f)\left(G\left(\frac{q_0}{\sqrt{K_2}}\right) + \operatorname{erf}\left(\sqrt{\frac{\gamma}{2K_2}}\lambda_2\right)\right)} \tag{49}$$

and $\lambda_2 = \lambda_2(\gamma)$ is given implicitely by the equation

$$F(\gamma, \lambda_2) = 0 \tag{50}$$

where $F = F(\gamma, \lambda_2)$ is defined by

$$F(\gamma, \lambda_2) = -1 + \frac{\lambda_2}{\Phi_2(T_f)}\,\frac{G(u(\gamma,\lambda_2)) + m}{m + \operatorname{erf}(u(\gamma,\lambda_2))}, \; \gamma > 0, \; \lambda_2 > 0 \tag{51}$$

with

$$m = G\left(\frac{q_0}{\sqrt{K_2}}\right) > 1 \quad \text{and} \quad u(\gamma, \lambda_2) = \sqrt{\frac{\gamma}{2K_2}}\lambda_2 \tag{52}$$

and γ must satisfies the equation

$$\Psi(\gamma) = L\rho\sqrt{\gamma} \quad , \quad \gamma > 0 \tag{53}$$

where function Ψ is defined by

$$\Psi(\gamma) = -\sqrt{\frac{2K_1}{\pi}}\,\frac{\Phi_1(T_f) - \Phi_1(T_0)}{\Phi_1(T_0)}\,\frac{\exp(-\frac{\gamma}{2K_1}\lambda_1^2(\gamma))}{1 - \operatorname{erf}\left(\sqrt{\frac{\gamma}{2K_1}}\lambda_1(\gamma)\right)} + \sqrt{\frac{2K_2}{\pi}}\,\frac{\exp(-\frac{\gamma}{2K_2}\lambda_2^2(\gamma))}{m + \operatorname{erf}\left(\sqrt{\frac{\gamma}{2K_2}}\lambda_2(\gamma)\right)}. \tag{54}$$

Theorem 6. (i) The equation (50) defines implicitly an increasing function $\lambda_2 = \lambda_2(\gamma)$ such as $F(\gamma, \lambda_2(\gamma)) = 0$. Moreover, we have $\lambda_2(0^+) = 0$ and $\lambda_2(+\infty) = \Phi_2(T_f)$.

(ii) The free boundary problem (30) – (36) has a Neumann type unique solution if and only if the coefficient q_0 verifies the inequality (40). In this case the solution is given by (46), (44), (45), (48), (49), $\lambda_2 = \lambda_2(\gamma)$ is given by previous part (i), $\lambda_1 = \lambda_1(\gamma)$ is given by (47) and γ is the unique solution of the equation (53).

Theorem 6 shows us that when the thermal heat flux input coefficient q_0 has a lower bound of the type (40) we obtain an instantaneous phase-change process. On the contrary, if q_0 does not verify (40) then we only have a heat conduction problem for the initial solid phase.

In the case where q_0 verifies the inequality (40), we can compute the temperature on the fixed face $x = 0$ which is given by

$$T_2(0,t) = T_m = \Phi_2^{-1}\left(\frac{q_0\sqrt{\pi}\Phi_2(T_f)}{\sqrt{K_2}}\frac{\operatorname{erf}\left(\sqrt{\frac{\gamma}{2K_2}}\lambda_2(\gamma)\right) + G\left(\frac{q_0}{\sqrt{K_2}}\right)}{\exp\left(-\frac{q_0^2}{K_2}\right)}\right) \tag{55}$$

which satisfies the condition $T_2(0,t) > T_f$, $\forall t > 0$. Therefore, we can consider the problem (30) – (35) and (43).

3.3. Existence and uniqueness of solution of the free boundary problem with temperature condition on the fixed face.

The temperature distributions T_1 and T_2, and the free boundary $X(t)$ of the problem (30) – (35), (43) are given parametrically by

$$T_1 = \Phi_1^{-1}\left\{A_1 \operatorname{erf}\left[\left(\frac{\gamma}{2K_1}\right)^{\frac{1}{2}}\xi_1^*\right] + B_1\right\}^{-1}, \xi_1 = \int_{\lambda_1}^{\xi_1^*}\left\{A_1 \operatorname{erf}\left[\left(\frac{\gamma}{2K_1}\right)^{\frac{1}{2}}\sigma\right] + B_1\right\} d\sigma + 1 \tag{56}$$

and

$$T_2 = \Phi_2^{-1}\left\{A_2 \operatorname{erf}\left[\left(\frac{\gamma}{2K_2}\right)^{\frac{1}{2}}\xi_2^*\right] + B_2\right\}^{-1}, \xi_2 = \int_{0}^{\xi_2^*}\left\{A_2 \operatorname{erf}\left[\left(\frac{\gamma}{2K_2}\right)^{\frac{1}{2}}\sigma\right] + B_2\right\} d\sigma \tag{57}$$

$$X(t) = \sqrt{2\gamma t} \tag{58}$$

where

$$\lambda_1 = \lambda_2 + L\rho + \Phi_1(T_f) - \Phi_2(T_f) \tag{59}$$

$$A_1 = \frac{1}{1-\operatorname{erf}\left(\sqrt{\frac{\gamma}{2K_1}}\lambda_1\right)}\left(\frac{1}{\Phi_1(T_0)} - \frac{1}{\Phi_1(T_f)}\right), A_2 = \frac{1}{\left(\frac{1}{\Phi_2(T_f)} - \frac{1}{\Phi_2(T_m)}\right)\operatorname{erf}\left(\sqrt{\frac{\gamma}{2K_2}}\lambda_2\right)} \tag{60}$$

$$B_1 = \frac{1}{1-\operatorname{erf}\left(\sqrt{\frac{\gamma}{2K_1}}\lambda_1\right)}\left(\frac{1}{\Phi_1(T_f)} - \frac{\operatorname{erf}\left(\sqrt{\frac{\gamma}{2K_1}}\lambda_1\right)}{\Phi_1(T_0)}\right), B_2 = \frac{1}{\Phi_2(T_m)} \tag{61}$$

and $\lambda_2 = \lambda_2(\gamma)$ is given implicitely by the following equation

$$H(\gamma, \lambda_2) = 0 \tag{62}$$

where $H = H(\gamma, \lambda_2)$ is defined by

$$H(\gamma, \lambda_2) = -1 + A_2(\gamma, \lambda_2) \int_0^{\lambda_2} \operatorname{erf}\left(\sqrt{\frac{\gamma}{2K_2}}\sigma\right) d\sigma + B_2 \lambda_2 \tag{63}$$

and γ must be a solution of the equation

$$\Lambda(\gamma) = L\rho\sqrt{\gamma} \quad , \quad \gamma > 0 \tag{64}$$

where function Λ is defined by

$$\begin{aligned} \Lambda(\gamma) \;=\; & -\sqrt{\frac{2K_1}{\pi}} \left(\frac{\Phi_1(T_f) - \Phi_1(T_0)}{\Phi_1(T_0)}\right) \frac{\exp\left(-\frac{\gamma}{2K_1}\lambda_1^2(\gamma)\right)}{\left(1 - \operatorname{erf}\left(\sqrt{\frac{\gamma}{2K_1}}\lambda_1(\gamma)\right)\right)} + \\ & +\sqrt{\frac{2K_2}{\pi}} \left(\frac{\Phi_2(T_m) - \Phi_2(T_f)}{\Phi_2(T_m)}\right) \frac{\exp\left(-\frac{\gamma}{2K_2}\lambda_2^2(\gamma)\right)}{\operatorname{erf}\left(\sqrt{\frac{\gamma}{2K_2}}\lambda_2(\gamma)\right)} \, , \; \gamma > 0 \, . \end{aligned} \tag{65}$$

Theorem 7. (i) There exist an increasing function $\lambda_2 = \lambda_2(\gamma)$ such as $H(\gamma, \lambda_2(\gamma)) = 0$ for all γ and the equation (64) has a unique solution $\gamma > 0$.
(ii) The free boundary problem (30) – (35) and (43) has a unique solution for all data which is given by (56), (57), (58), (60), (61), $\lambda_2 = \lambda_2(\gamma)$ is given by previous part (i), $\lambda_1 = \lambda_1(\gamma)$ is given by (59) and γ is the unique solution of the equation (64).

Remark 4. The two boundary conditions on the fixed face (36), with datum q_0, and (43), with datum T_m, are related through the relationship given by (55).

Moreover, all results obtained for the fusion process for the Storm's type materials can be also got for the solidification process with the corresponding analogous initial and boundary conditions.

4. Drying with coupled phase change in a porous medium.
4.1 Introduction.

A semi-infinite porous medium is dried by maintaining a heat flux condition at $x = 0$ of the type $-q_0/\sqrt{t}$, with $q_0 > 0$ [Tal]. Initially, the whole body is at uniform temperature t_0 and uniform moisture potential u_0. The moisture is assumed to evaporate completely at a constant temperature, evaporation point t_v. It is also assumed that the moisture potential in the first region, $0 < x < s(\tau)$, is constant at u_v, where $x = s(\tau)$ locates the evaporation front at time $\tau > 0$. It is further assumed that the moisture in vapor form does not take away any appreciable amount of heat from the system. Neglecting mass diffusion due to temperature variation, the problem can be expressed as $[Ch, Gu, Lu, SaTa]$:

$$\frac{\partial t_1}{\partial \tau}(x,\tau) = a_1 \frac{\partial^2 t_1}{\partial x^2}(x,\tau), \qquad 0 < x < s(\tau), \tau > 0 \tag{66}$$

$$u_1 = u_v, \qquad 0 < x < s(\tau), \tau > 0 \tag{67}$$

$$\frac{\partial t_2}{\partial \tau}(x,\tau) = a_1 \frac{\partial^2 t_2}{\partial x^2} + \frac{\varepsilon L c_m}{c_q}\frac{\partial u_2}{\partial \tau}, \qquad x > s(\tau), \tau > 0 \tag{68}$$

$$\frac{\partial u_2}{\partial \tau}(x,\tau) = a_m \frac{\partial^2 u_2}{\partial x^2}(x,\tau), \qquad x > s(\tau), \tau > 0 \tag{69}$$

$$k_1 \frac{\partial t_1}{\partial x} = -\frac{q_0}{\sqrt{\tau}} \quad at \quad x = 0, \tau > 0 \tag{70}$$

$$t_2 = t_0 \quad in \quad x > 0, \tau = 0 \tag{71}$$

$$u_2 = u_0 \quad in \quad x > 0, \tau = 0 \tag{72}$$

$$t_1(s(\tau),\tau) = t_2(s(\tau),\tau) = t_v > t_0 \quad at \quad x = s(\tau) \tag{73}$$

$$u_1(s(\tau),\tau) = u_2(s(\tau),\tau) = u_v < u_0 \quad at \quad x = s(\tau) \tag{74}$$

$$-k_1 \frac{\partial t_1}{\partial x}(s(\tau),\tau) + k_2 \frac{\partial t_2}{\partial x}(s(\tau),\tau) = (1-\varepsilon)\rho_m L \tfrac{ds}{dt} \quad at \quad x = s(\tau) \tag{75}$$

where t_1 : temperature of the dried porous medium; t_2 : temperature of the humid porous medium; u_2 : mass-transfer potential of the humid porous medium; a_i $(i = 1, 2)$: thermal diffusivity of the phase-i; a_{12} : ratio of thermal diffusivities from phase 1 to phase 2; a_m : moisture diffusivity; c_m : specific mass capacity; c_q : specific heat capacity; k_i $(i = 1, 2)$: thermal conductivity of the phase-i; k_{21} : ratio of thermal conductivity from phase 2 to phase 1; $K_0 = Lc_m(u_0 - u_v)/c_q(t_v - t_0)$: Kossovitch number; L : latent heat of evaporation of liquid per unit mass; t_0 : initial temperature; t_v : temperature at the phase-change state; u_0 : initial mass-transfer potential; ε : coefficient of internal evaporation; ρ_m : density of moisture; $L_u = a_m/a_1$: Luikov number and $\nu = (1-\varepsilon)L\rho_m a_1/k_1(t_v - t_0) > 0$.

We follow $[SaTa]$. In Section 4.2 we find a solution of this problem, depending on the value of the Luikov number L_u and in Section 4.3 we give a sufficient condition for the Luikov number L_u in order to obtain when the temperature distribution has a minimum value less than its initial temperature.

4.2 Similarity solution.

We have

Theorem 8. If the Luikov number is equals to one, and the coefficient q_0 verifies the condition

$$q_0 > \frac{k_2(t_v - t_0)}{2\sqrt{\pi a_1}}[\varepsilon K_0 + 2], \tag{76}$$

then there exists one and only one solution $\lambda > 0$ of the following equation

$$\frac{k_{21}}{\sqrt{\pi}}\frac{e^{-x^2}}{1-\operatorname{erf}(x)}\left[-\frac{2\varepsilon K_0}{\sqrt{\pi}}x\frac{e^{-x^2}}{1-\operatorname{erf}(x)}+2\varepsilon K_0 x^2-\varepsilon K_0-2\right]+ \\ +\frac{2\sqrt{a_1}q_0}{k_1\left(t_v-t_0\right)}e^{-x^2}=2\nu x, \qquad x>0. \tag{77}$$

Furthermore, the solution of the problem (66) - (75) is given by :

$$u_1(x,\tau)=u_v, \qquad 0<x<s(\tau),\ \tau>0 \tag{78}$$

$$t_1(x,\tau)=1+\frac{q_0\sqrt{\pi a_1}}{k_1\left(t_v-t_0\right)}\left(\operatorname{erf}\lambda-\operatorname{erf}\left(\frac{x}{2\sqrt{a_1\tau}}\right)\right), \qquad 0<x<s(\tau),\ \tau>0 \tag{79}$$

$$u_2(x,\tau)=\frac{1-\operatorname{erf}\left(\frac{x}{2\sqrt{a_m\tau}}\right)}{1-\operatorname{erf}(\lambda)}, \qquad x>s(\tau),\ \tau>0 \tag{80}$$

$$t_2(\eta)=\frac{\varepsilon K_0}{\sqrt{\pi}\left(1-\operatorname{erf}(\lambda)\right)}\left[\lambda\, e^{-\lambda^2}\frac{1-\operatorname{erf}\left(\frac{x}{2\sqrt{a_1\tau}}\right)}{1-\operatorname{erf}(\lambda)}-\frac{x}{2\sqrt{a_1\tau}}e^{-\frac{x^2}{4a_1\tau}}\right]+ \\ +\frac{1-\operatorname{erf}\left(\frac{x}{2\sqrt{a_1\tau}}\right)}{1-\operatorname{erf}\lambda}, \qquad x>s(\tau),\ \tau>0 \tag{81}$$

$$s(\tau)=2\lambda\sqrt{a_1\tau}. \tag{82}$$

For the case $L_u \neq 1$, that is to say, $a_m \neq a_1$ we define the following functions:

$$\phi(x)=\frac{\sqrt{\pi a_1}q_0}{\left(t_v-t_0\right)}e^{-x^2}+P(x) \tag{83}$$

$$\varphi(x)=k_2F_1(x)+\sqrt{\pi}k_1\nu x. \tag{84}$$

where

$$P(x)=\frac{L_u\,\varepsilon K_0}{L_u-1}k_2\left(\frac{1}{\sqrt{L_u}}F_1\left(\frac{x}{\sqrt{L_u}}\right)-F_1(x)\right), \qquad x>0. \tag{85}$$

Theorem 9. If the Luikov number is different than one, and the coefficient q_0 verifies the condition

$$q_0>k_2\left(1+\frac{\sqrt{L_u}\varepsilon K_0}{1+\sqrt{L_u}}\right)\frac{t_v-t_0}{\sqrt{\pi a_1}}, \tag{86}$$

then there exists one and only one solution $\lambda>0$ of the equation

$$\phi(x)=\varphi(x), \qquad x>0. \tag{87}$$

Furthermore, the solution of the problem (66)–(75) is given by (78), (79), (80), (82) and

$$\eta = \frac{x}{2\sqrt{a_1\tau}}, \qquad t_2(\eta) = \frac{\varepsilon K_o L_u}{L_u - 1}\left[-\frac{1-\text{erf}\left(\frac{x}{2\sqrt{a_m\tau}}\right)}{1-\text{erf}\left(\frac{\lambda}{\sqrt{L_u}}\right)} + \frac{1-\text{erf}\left(\frac{x}{2\sqrt{a_1\tau}}\right)}{1-\text{erf}(\lambda)}\right] + \\ + \frac{1-\text{erf}\left(\frac{x}{2\sqrt{a_1\tau}}\right)}{1-\text{erf}\,\lambda}, \qquad x > s(\tau),\ \tau > 0. \tag{88}$$

Remark 5. The right side member of the inequality (86) goes to the right side member of the inequality (76) when L_u tends to 1, that is to say, we can study the case $L_u = 1$ considering the limit $L_u \to 1$ in the case $L_u \neq 1$.

4.3 Sufficient condition for the Luikov number in order to obtain the minimum value of the temperature distribution.

Some results of sample calculations are shown in [SaTa] with $\varepsilon K_0 = 2$, $a_1 = 1$, $k_2 = 1$, and $(t_v - t_0) = 1$ where we can see that the temperature distribution t_2 reaches to a minimum value which is smaller than the limit value t_0 that the function reaches at $+\infty$, i.e. the initial temperature. We shall find the values of the coefficient L_u for which the function t_2 has a minimum value which is smaller than its limit value when $\eta \to +\infty$.

Theorem 10. If the Luikov number L_u verifies the condition

$$L_u > \frac{1}{\varepsilon K_0 + 1} \tag{89}$$

then the temperature distribution t_2 reaches to a minimum value which is smaller than the initial temperature or its limit value at $+\infty$. The minimum value is attained when the dimensionless variable $\eta = x/2\sqrt{a_1\tau}$ takes the value

$$\eta = \sqrt{\left(\frac{L_u}{L_u - 1}\right)\log\left(\frac{((\varepsilon K_0 + 1)L_u - 1)}{\varepsilon K_o\sqrt{L_u}}\frac{1-\text{erf}\left(\frac{\lambda}{\sqrt{L_u}}\right)}{1-\text{erf}\,\lambda}\right)}\ . \tag{90}$$

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