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A steady-state two-phase Stefan–Signorini problem with mixed boundary data

Abstract. We consider a steady-state heat conduction problem in a multidimensional bounded domain Ω which has a regular boundary Γ composed by the union of two parts Γ_1 and Γ_2. We assume, without loss of generality, that the melting temperature is zero degree centigrade. We consider a source term g in the domain Ω. On the boundary Γ_2 we have a positive heat flux q and on the boundary Γ_1 we have a Signorini type condition with a positive external temperature b.

We obtain sufficient conditions on data q, g, b to obtain a change of phase (steady-state, two-phase, Stefan-Signorini problem) in Ω, that is a temperature of non-constant sign in Ω. We use the elliptic variational inequalities theory. We also find that the solution of the corresponding elliptic variational inequality is differentiable with respect to the Neumann datum q on Γ_2. Several properties already obtained for variational equalities can also be generalized for variational inequalities.

Moreover, by using the finite element method, we also obtain sufficient conditions on data to obtain a steady-state, two-phase, discretized Stefan-Signorini problem in the corresponding discretized domain, that is a discrete temperature of non-constant sign in Ω.

1. Introduction.

We consider a bounded domain $\Omega \subset \mathbf{R}^n$ with regular boundary $\Gamma = \Gamma_1 \cup \Gamma_2$ with $|\Gamma_2| = \text{meas}(\Gamma_2) > 0$ and $|\Gamma_1| > 0$. We suppose that $\Gamma_1 = \Gamma_{1_t} \cup \Gamma_{1_s}$ with $|\Gamma_{1_i}| > 0$ for $i = t, s$.

We consider a steady-state heat conduction problem in Ω. We assume, without loss of generality, that the melting temperature is zero degree centigrade. We consider a source term g in the domain Ω. On the boundary Γ_2 we have a positive heat flux q and on the boundary Γ_{1_t} we impose a positive temperature b. On the boundary Γ_{1_s} we have a Signorini type condition with a positive external temperature b. If θ is the temperature of the material we can consider the new unkown function in Ω defined by[Du, Ta1]

$$u = k_2\theta^+ - k_1\theta^- \tag{1}$$

where $k_i > 0$ is the thermal conductivity of the phase i ($i = 1$: solid phase, $i = 2$: liquid phase). Let $B = k_2 b > 0$ where $b > 0$ is the temperature imposed on Γ_{1_t}.

We consider the following steady-state Stefan-Signorini free boundary problem

$$-\Delta u = g \quad \text{in} \quad \Omega \tag{2}$$

$$-\frac{\partial u}{\partial n}/_{\Gamma_2} = q \quad \text{on} \quad \Gamma_2 \tag{3}$$

$$u/\Gamma_{1_t} = B \quad \text{on} \quad \Gamma_{1t} \tag{4}$$

$$u \geq B\,,\ \frac{\partial u}{\partial n} \geq 0\,,\ (u-B)\frac{\partial u}{\partial n} = 0 \quad \text{on} \quad \Gamma_{1_s}\,. \tag{5}$$

The goal of this paper is to find sufficient conditions on $q = \text{Const.} > 0$ on Γ_2 to obtain a temperature u of non-constant sign in Ω , that is a steady-state, two-phase, Stefan-Signorini problem. When $\Gamma_1 = \Gamma_{1_t}$ (i.e. $\Gamma_{1_s} = \emptyset$) the corresponding free boundary problem without Signorini boundary conditions was studied in [GaTa]. We follow a method similar to the one developed in [BoShTa, GaTa, GoTa, Sa, Ta1, Ta2, Ta3].

We shall present some theoretical and numerical (by finite element approximation) results through variational inequalities and the corresponding related estimates in terms of the finite element approximation parameter h.

2. Continuous analysis.

The variational formulation of the problem (2)-(5) is given by

$$\begin{cases} a(u, v-u) \geq L(v-u), \quad \forall v \in K_B \\ u \in K_B \end{cases} \tag{6}$$

where

$$\begin{cases} V = H^1(\Omega)\,, \\ W_0 = \{v \in V/\ v/\Gamma_{1_t} = v/\Gamma_{1_s} = 0\} \subset V_0 = \{v \in V/\ v/\Gamma_{1_t} = 0\} \\ K_B = \{v \in V/\ v/\Gamma_{1_t} = B\,,\ v/\Gamma_{1_s} \geq B\} = B + K_0 \\ K_0 = \{v \in V/\ v/\Gamma_{1_t} = 0\,,\ v/\Gamma_{1_s} \geq 0\} \supset W_0 \end{cases} \tag{7}$$

and

$$\begin{cases} a(u,v) = \int_\Omega \nabla u . \nabla v \, dx \\ L(v) = L_{qg}(v) = \int_\Omega gv\, dx - \int_{\Gamma_2} qv\, d\gamma. \end{cases} \tag{8}$$

For $g \in L^2(\Omega)$, we have a unique solution $u = u_{qgB}$ (it will be denoted by u_q) of the variational inequality (6) [KiSt, Ta1].

We obtain for u_q the following properties:

Lemma 1. We have

(i)(9) $$\alpha \left\| u_{q_2} - u_{q_1} \right\|_V^2 \leq a\left(u_{q_2} - u_{q_1}, u_{q_2} - u_{q_1}\right) \leq (q_1 - q_2) \int_{\Gamma_2} \left(u_{q_2} - u_{q_1}\right) d\gamma$$

where $\alpha > 0$ is the coercive constant of the bilinear form a.

(ii)(10)
$$\begin{cases} \left\|u_{q_2}-u_{q_1}\right\|_V \le \dfrac{|\Gamma_2|^{\frac{1}{2}}\,\|\gamma_0\|}{\alpha}\,|q_2-q_1| \\ \\ \left\|u_{q_2}-u_{q_1}\right\|_{L^2(\Gamma_2)} \le \dfrac{|\Gamma_2|^{\frac{1}{2}}\,\|\gamma_0\|}{\alpha}\,|q_2-q_1| \end{cases}$$

where γ_0 is the trace operator.

(iii) The function $\mathbf{R}^+ \to \mathbf{R}$,

(11)
$$q \to \int_{\Gamma_2} u_q d\gamma$$

is a continuous and strictly decreasing function. Moreover, we have

$$q_1 \le q_2 \Longrightarrow u_{q_2} \le u_{q_1} \quad \text{in} \quad \overline{\Omega} \quad \text{and} \quad \Gamma_2 \int_{\Gamma_2} u_{q_2}\, d\gamma \le \int_{\Gamma_2} u_{q_1}\, d\gamma \, .$$

(iv) There exists $u'_q \in V_0$ such that:

(12)
$$\begin{cases} (i)\ \dfrac{u_{q+\delta}-u_q}{\delta} \rightharpoonup u'_q \text{ in } V-\text{weak} \ , \text{ when } \delta \to 0 \\ (ii)\ \dfrac{u_{q+\delta}-u_q}{\delta} \rightharpoonup u'_q \text{ in } L^2(\Gamma_2)-\text{weak} \ , \text{ when } \delta \to 0 \end{cases}$$

and

(13)
$$a(u_q, u'_q) = L(u'_q) \left(= \int_\Omega g u'_q \, dx - q \int_{\Gamma_2} u'_q \, d\gamma \right) .$$

The element u, unique solution of (6), is also characterized by the following minimization problem:

(14)
$$\begin{cases} J(u) \le J(v) \, , \quad \forall v \in K_B \\ u \in K_B \end{cases}$$

where

(15)
$$J(v) = J_{qg}(v) = \tfrac{1}{2} a(v.v) - L_{qg}(v).$$

We can define the real function $f : \mathbf{R}^+ \to \mathbf{R}$ in the following way [GaTa, Ta2]

(16)
$$f(q) = J(u_q) = \tfrac{1}{2} a(u_q, u_q) - \int_\Omega g u_q \, dx + q \int_{\Gamma_2} u_q \, d\gamma$$

where u_q is the unique solution of the variational inequality (6) for each heat flux q> 0.

Theorem 2. The function f is differentiable. Moreover, f' is a continuous and strictly decreasing function, and it is given by the following expression

(17)
$$f'(q) = \int_{\Gamma_2} u_q \, d\gamma.$$

Proof.- We use (6), (13) and the definition of f'.

Corollary 3. We have the following properties:

$$(18)\qquad \frac{d}{dq}\left[\int_\Omega g u_q \, d\gamma\right] = \int_\Omega g u_q' \, dx$$

$$(19)\qquad \frac{d}{dq}\left[a(u_q, u_q)\right] = 2a(u_q, u_q')$$

$$(20)\qquad f''(q) = \int_{\Gamma_2} u_q' \, d\gamma \, .$$

Theorem 4. The element u_q' does not depend on q, that is $u_q' = \eta \in K_0$ where η is the unique solution of the following elliptic variational inequality:

$$(21)\qquad \begin{cases} a(\eta, v-\eta) \geq -\int_{\Gamma_2} (v-\eta)\, d\gamma, & \forall v \in K_0 \\ \eta \in K_0 \, . \end{cases}$$

Moreover, $\eta/_{\Gamma_2} \leq 0$ with

$$(22)\qquad -\int_{\Gamma_2} \eta \, d\gamma \geq a(\eta, \eta) \geq \alpha \|\eta\|_V^2 > 0.$$

Corollary 5. We have the following properties :
(i) The element u_q can be written by

$$(23)\qquad u_q = u_{qgB} = B + U_g + q\eta$$

where U_g is the unique solution of the following elliptic variational equality

$$(24)\qquad \begin{cases} a(U_g, v) = \int_\Omega g v \, dx, & \forall v \in W_0 \\ U_g \in W_0 \, . \end{cases}$$

(ii) We have

$$(25)\qquad f'(q) = (B\,|\Gamma_2| + C_g) - Dq, \qquad f''(q) = \int_{\Gamma_2} \eta \, d\gamma$$

where

$$(26)\qquad C_g = \int_{\Gamma_2} U_g \, d\gamma, \qquad D = -\int_{\Gamma_2} \eta \, d\gamma > 0.$$

We can define the real function $R = R(B, g)$ in the following way

$$(27)\qquad R(B,g) = \frac{B\,|\Gamma_2| + C_g}{D}.$$

Theorem 6. For $B > 0$ and $g \in L^2(\Omega)$, we have:

$$(28)\qquad q > R(B,g) \Rightarrow u \text{ is of non-constant sign in } \Omega,$$

i.e. there exists a steady-state two-phase Stefan-Signorini problem.

Proof. The result (28) is obtained by considering the following equivalence

$$(29)\qquad q > R(B,g) \Longleftrightarrow f'(q) = \int_{\Gamma_2} u_q \, d\gamma < 0.$$

3. Numerical analysis.

We suppose that $\Omega \subset \mathbf{R}^n$ is a convexe polygonal bounded domain. We consider τ_h, a regular triangulation of the polygonal domain Ω with Lagrange triangles of type 1, constituted by affine-equivalent finite element of class C^0, where $h > 0$ is a parameter which goes to zero. We can take h equal to the longest side of the triangles $T \in \tau_h$ [BrSc, Ci, GlLiTr]. We follow a method similar to the one developed in [Ta3] to obtain the discrete equivalent of the continuous result (28).

The variational formulation of the continuous problem (6) is given by

$$(30) \qquad \begin{cases} a(u_h, v_h - u_h) \geq L(v_h - u_h)\ , \quad \forall v_h \in K_{B_h} \\ u_h \in K_{B_h} \end{cases}$$

where

$$(31) \qquad \begin{cases} K_{B_h} = B + K_{0_h} \subset K_B\ ,\ P_1 = \text{set of the polynomials of degree} \leq 1 \\ K_{0_h} = \left\{ v_h \in C^0(\overline{\Omega}) /\ v_h/_T \in P_1(T), \forall T \in \tau_h, v_h/_{\Gamma_{1t}} = 0, v_h/_{\Gamma_{1s}} \geq 0 \right\} \\ V_{0_h} = \left\{ v_h \in C^0(\overline{\Omega}) /\ v_h/_T \in P_1(T), \forall T \in \tau_h, v_h/_{\Gamma_{1t}} = 0 \right\} \\ W_{0_h} = \left\{ v_h \in C^0(\overline{\Omega}) /\ v_h/_T \in P_1(T), \forall T \in \tau_h, v_h/_{\Gamma_{1t}}, v_h/_{\Gamma_{1s}} = 0 \right\} \end{cases}$$

with

$$(32) \qquad \begin{cases} W_{0_h} \subset K_{0_h} \subset V_{0_h} \\ W_{0_h} \subset W_0\ ,\ K_{0_h} \subset K_0\ ,\ V_{0_h} \subset V_0 . \end{cases}$$

For $g \in L^2(\Omega)$, the unique solution of the variational inequality (30) will be denoted by $u_h = u_{h_q}$.The element u_h is also characterized by the minimization problem:

$$(33) \qquad \begin{cases} J(u_h) \leq J(v_h)\ , \quad \forall v_h \in K_{B_h} \\ u_h \in K_{B_h} . \end{cases}$$

For each $h > 0$, we define the real function $f_h : \mathbf{R}^+ \to \mathbf{R}$ in the following way

$$(34) \qquad f_h(q) = J(u_{h_q}) = \tfrac{1}{2} a(u_{h_q}, u_{h_q}) - L_q(u_{h_q}).$$

We obtain the following properties for the discrete solution u_{h_q} of the elliptic variational inequality (30).

Theorem 7. We have the following properties:
(i) There exists an element $u'_{h_q} \in V_0$ such that

$$(35) \qquad \frac{u_{h_{q+\delta}} - u_{h_q}}{\delta} \rightharpoonup u'_{h_q} \ \text{ in } V\text{−weak}\ ,\ \text{when } \delta \to 0$$

$$(36) \qquad \frac{u_{h_{q+\delta}} - u_{h_q}}{\delta} \rightharpoonup u'_{h_q} \ \text{ in } L^2(\Gamma_2)\text{−weak}\ ,\ \text{when } \delta \to 0$$

(37) $$a(u_{h_q}, u'_{h_q}) = \int_\Omega g u'_{h_q}\, dx - q \int_{\Gamma_2} u'_{h_q}\, d\gamma.$$

(ii) The function $\mathbf{R}^+ \to \mathbf{R}$, $q \to \int_{\Gamma_2} u_{h_q} d\gamma$ is a continuous and strictly decreasing function.

(iii) The function f_h is differentiable. Moreover, we have the following expressions:

(38) $$f'_h(q) = \int_{\Gamma_2} u_{h_q}\, d\gamma \quad , \quad f''_h(q) = \int_{\Gamma_2} u'_{h_q}\, d\gamma.$$

(iv) The element u_{h_q} can be written as

(39) $$u_{h_q} = B + U_{h_g} + q\eta_h \quad , \quad \eta_h = u'_{h_q} \in K_{0_h}$$

where U_{h_g} and η_h are respectively the unique solutions of the variational equality (40) and inequality (41), that is:

(40) $$\begin{cases} a(U_{h_g}, v_h) = \int_\Omega g v_h\, dx\ , \quad \forall v_h \in W_{0_h} \\ U_{h_g} \in W_{0_h} \end{cases}$$

(41) $$\begin{cases} a(\eta_h, v_h - \eta_h) \geq - \int_{\Gamma_2} (v_h - \eta_h)\, d\gamma, \quad \forall v_h \in K_{0_h} \\ \eta_h \in K_{0_h}\ . \end{cases}$$

(v) We have that $\eta_{h/\Gamma_2} < 0$ and

(42) $$- \int_{\Gamma_2} {}_h\eta_h\, d\gamma \geq a(\eta_h, \eta_h) \geq \alpha \left\|\eta_h\right\|_V^2 > 0.$$

(vi) Also, we have

(43) $$f'_h(q) = \left(B\left|\Gamma_2\right| + C_{h_g}\right) - D_h q, \qquad f''_h(q) = \int_{\Gamma_2} \eta_h\, d\gamma < 0 \quad ,$$

where

(44) $$C_{h_g} = \int_{\Gamma_2} U_{h_g} d\gamma \quad , \quad D_h = - \int_{\Gamma_2} \eta_h\, d\gamma > 0.$$

(vii) If, for each $h > 0$, we define the real function

(45) $$R_h(B, g) = \frac{B\left|\Gamma_2\right| + C_{h_g}}{D_h}$$

then we obtain that

(46) $$q > R_h(B, g) \Rightarrow u_h \text{ is of non-constant sign in } \Omega,$$

i.e. there exists a discrete steady-state two-phase Stefan-Signorini problem.

Proof. We use a method similar to the one developed in [Ta3].

4. Error bounds.

Let Π_h be the corresponding linear interpolation operator for the finite element approximation. There is a constant $C_0 > 0$ (independent of h) such that [BrSc, Ci]

(47) $$\left\|v - \Pi_h v\right\|_V \leq C_0 h^{r-1} \left\|v\right\|_{r,\Omega} \quad , \quad \forall v \in H^r(\Omega) \quad , \quad r > 1.$$

If we suppose the regularity property:

(48) $$u_1 \in H^r(\Omega) \quad , \quad \eta \in H^r(\Omega)$$

we obtain the following error estimates.

Theorem 8. We have

(49) $$\|u_1 - u_{1_h}\|_V \leq O\left(h^{r-1}\right)$$

(50) $$0 < C_1 - C_{1_h} = O\left(h^{2r-2}\right) \quad , \quad 0 < q_{0_h}(B) - q_0(B) = O\left(h^{2r-2}\right)$$

(51) $$\left|C_{h_g} - C_g\right| = O\left(h^{r-1}\right)$$

(52) $$\|\eta - \eta_h\|_V \leq O\left(h^{\frac{r-1}{2}}\right) \quad , \quad |R_h(B,g) - R(B,g)| = O\left(h^{\frac{r-1}{2}}\right).$$

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