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Approximate and analytic methods to solve some parabolic free boundary problems

By using approximate and analytic methods we obtain an answer to some parabolic free boundary problems [Ta2] :

I. Generalized Lame'–Clapeyron solution for a one–phase source Stefan problem

Goal : "We give a generalized Lamé–Clapeyron solution for a one–phase Stefan problem with a particular type of sources. Necessary and sufficient conditions are given in order to characterize the source term which provides a unique solution".

We consider the following singular free boundary problem for the heat equation : Find the free boundary $x = s(t) > 0$, defined for $t > 0$ and $s(0) = 0$, and the temperature $\theta = \theta(x,t) > 0$, defined $0<x<s(t)$, $t> 0$, such that they satisfy the following conditions [MeTa]:

$$(1)\qquad \begin{aligned} &\rho\, c\, \theta_t - k\, \theta_{xx} = \frac{\rho\, h}{t}\, \beta(x/2a\sqrt{t}) \;,\; 0 < x < s(t) \;,\; t > 0 \;, \\ &\theta(0,t) = B > 0 \;,\; t > 0 \;, \qquad s(0) = 0 \;, \\ &\theta(s(t),t) = 0 \;,\; k\, \theta_x(s(t),t) = -\, \rho h\, \dot{s}(t) \;,\; t > 0 \;. \end{aligned}$$

THEOREM 1: An explicit solution of (1), as function of β, is given by

$$(2)\qquad \theta(x,t) = B \left\{ 1 - \frac{\sqrt{\pi}}{\mathrm{Ste}}\, \xi \exp(\xi^2)\, \mathrm{erf}(\eta) + \frac{4}{\mathrm{Ste}} \int_0^\eta \left(\int_r^\xi \beta(y) \exp(y^2)\, dy \right) \exp(-r^2)\, dr \right\},$$

$$s(t) = 2\, a\, \xi \sqrt{t} \;, \qquad \eta = x/(2\, a \sqrt{t}) \in (0\,,\xi)\,,\; a^2 = \frac{k}{\rho\, c} > 0\,,$$

where the number $\xi>0$ is a solution of the equation :

$$(3)\qquad F(x,\beta) = \frac{\mathrm{Ste}}{\sqrt{\pi}} \;,\quad x > 0 \;,\quad \mathrm{Ste} = \frac{B\, c}{h} > 0 \text{ (Stefan number)},$$

with an appropriate function $F = F(x,\beta)$.

THEOREM 2: Let β be a continuous real function on $\mathbb{R}^+$ such that $x\beta(x)$ is locally integrable on $\mathbb{R}^+$. Define the function Z by :

$$(4)\qquad \begin{aligned} & Z = Z_\beta(x) = \exp(x^2)\,\mathrm{erf}(x)\,[\,\psi_0(x) - \beta(x)\,]\,,\ x > 0\,, \\ & \psi_0(x) = \tfrac{1}{2} + x^2 + \tfrac{x}{\sqrt{\pi}}\,G(x)\,, \qquad G(x) = \frac{\exp(-x^2)}{\mathrm{erf}(x)}\,. \end{aligned}$$

If the function Z satisfies the following conditions

$$(5)\qquad Z(x) > 0\,,\ \forall\, x \in (\nu,+\infty) \quad \text{and} \quad \int_0^{+\infty} Z(t)\,dt = +\infty\,,$$

where $\nu = \nu_Z \geq 0$ is defined by

$$(6)\qquad \nu = \mathrm{Inf}\left\{ x \geq 0 \,/\, \int_0^x Z(t)\,dt > 0 \right\},$$

then for any Ste > 0, there exists a unique $\xi = \xi(\mathrm{Ste}) > 0$ which is the solution of the equation (3) for the given function β. Conversely, if for the given function β the equation (3) has a unique root $\xi = \xi(\mathrm{Ste}) > 0$ for any Ste>0 then there exists a continuous and locally integrable function Z on $\mathbb{R}^+$ satisfying (5) and (6) such that

$$(7)\qquad \beta(x) = \psi_0(x) - Z(x)\,G(x) \quad (= \beta_Z(x)\,)\,,\ x > 0\,.$$

Moreover in any case the root $\xi > \nu$.

II. Neumann-like solution for the two-phase Stefan problem with a simple mushy zone model

Goal : "We give a generalized Neumann solution for a simple mushy zone model with two parameters for the two-phase Stefan problem for a semi-infinite material with equal mass densities in both solid and liquid phases, and constant thermal coefficients".

We consider a semi-infinite material with mass density equal in both solid and liquid phases and the phase-change temperature at 0°C. We generalize the mushy zone model given for the one-phase Lamé-Clapeyron (Stefan) problem in [SoWiAl] (See also [Ta1]) to the two-phase case [Ta5]. Three distinct regions can be distinguished, as follows :

H_1)The liquid phase, at temperature $\theta_2 = \theta_2(x,t) > 0$, occupying the region $x > r(t)$, $t > 0$.

H_2)The solid phase, at temperature $\theta_1 = \theta_1(x,t) < 0$, occupying the region $0 < x < s(t)$, $t > 0$.

H_3) The mushy zone, at temperature 0 , occupying the region $s(t) < x < r(t)$, $t > 0$. We make two assumptions on its structure following the paraffin case [SoWiAl] (the parameter ϵ and γ are characteristics of the phase-change material) :

a) The material in the mushy zone contains a fixed fraction ϵh (with constant $0 < \epsilon < 1$) of the total latent heat h .

b) The width of the mushy zone is inversely proportional (with constant $\gamma > 0$) to the temperature gradient at the point $(s^-(t),t)$.

THEOREM 1: If the phase-change semi-infinite material is initially in liquid phase at the constant temperature $\theta_0 > 0$ and a constant temerature $-D < 0$ is imposed on the fixed face $x = 0$, then we obtain the following results :

(i) We obtain an exact solution of the Neumann type for $\theta_1(x,t)$, $\theta_2(x,t)$, s(t) and r(t) as functions of θ_0 , D, ϵ , γ and the thermal coefficients of the material.

Moreover , If we replace the constant temperature $-D<0$ by a heat flux of the type $q_0/\sqrt{t}$ (with $q_0 > 0$) on the fixed face $x = 0$, then we obtain the results :
(ii) There exists an exact solution $\theta_1^*(x,t)$, $\theta_2^*(x,t)$, $s^*(t)$ and $r^*(t)$ of the Neumann type of the mushy zone model if and only if the coefficient q_0 satisfies the inequality

$$(1) \qquad q_0 > \frac{\gamma\, k_1}{2\, a_2\, \eta_0} \quad ,$$

where $\eta_0 = \eta_0(\epsilon,\gamma,\theta_0,h,k_1,k_2,c_2)$ is the unique positive zero of a given function G.

Moreover, for the solution given in (i), the inequality for q_0 turns into an inequality for σ, where $\sigma > 0$ is the coefficient that characterizes the first free boundary $s(t) = 2\,\sigma\,\sqrt{t}$ of the two-phase mushy zone model.

III. A new proof of the exponentially fast asymptotic behavior of the solutions in heat conduction problems with absorption

Goal : "We give a new proof of the exponentially fast asymptotic behavior of the solutions in heat conduction problems with absorption by using a variant of the heat balance integral method".

We consider the following heat conduction problem with absorption [St] :

$$(1) \qquad \begin{aligned} &\text{i) } u_t - u_{xx} + \lambda^2 u_+^p = 0 \ , \ x > 0 \ , \ t > 0 \ , \\ &\text{ii) } u(0,t) = 1 \ , \ t > 0 \ , \qquad \text{iii) } u(x,0) = U_0(x) \geq 0 \ , \ x > 0 \ , \end{aligned}$$

for a class of functions $U_0 = U_0(x)$ and parameters $p>0$ and $\lambda>0$. If $0<p<1$, equation (1i) has a stationary solution corresponding to datum (1ii), which has compact support in $[0,+\infty)$ and is given by:

$$(2) \qquad u_\infty(x) = \left(1 - \frac{\lambda}{I}\, x\right)_+^{\frac{2}{1-p}} \ , \ I = I(p) = \frac{\sqrt{2(1+p)}}{1-p} \ .$$

In the case $0 < p < 1$ and $U_0 \leq u_\infty$, the solution $u = u(x,t)$ of (1) satisfies

$$(3) \qquad 0 < u(x,t) < u_\infty(x) \ , \ 0 < x < \frac{I}{\lambda} \ , \ t > 0 \ ,$$

and it means that $u(t) = u(.;t)$ has compact support in variable x for any $t > 0$ and

$$(4) \qquad s(t) = \text{Sup}\,\{\, x > 0 \,/\, u(x,t) > 0 \,\} \ , \ t > 0 \ ,$$

is a free boundary which is moving with finite speed for $t > 0$.

We give an estimate of how fast the free boundary s(t) tends to its limit I/λ as $t \to +\infty$ [Ta3,Ta4]. The estimate we get implies that this convergence is exponentially fast in time, in a similar form to the one given in [RiTa]. Our purpose is to show how this result can be obtained in a different way to [RiTa] by using the Goodman heat balance integral method [Go]. To prove that we use the innovation property (7) which fixes appropriately the asymptotic limit of the corresponding approximate free boundary. We consider a related problem to (1) which consists in finding the function

C=C(x,t) and the free boundary s=s(t) such that they satisfy the conditions:

$$(5)\qquad \begin{aligned} &\text{i)}\quad C_t - C_{xx} + \lambda^2 C_+^p = 0 \ , \ 0 < x < s(t) \ , \ t > 0 \ , \\ &\text{ii)}\quad C(0,t)=1 \ , \ t > 0 \ , \qquad \text{iii)}\ s(0)=0 \ , \\ &\text{iv)}\quad C(s(t),t) = 0 \ , \ t > 0 \ , \qquad \text{v)}\ C_x(s(t),t) = 0 \ , \ t > 0 \ . \end{aligned}$$

Taking into account the heat balance integral method we replace equation (5i) by its integral in the variable x from 0 to s(t), we propose

$$(6)\qquad C_B(x,t) = \left(1 - \frac{x}{s_B(t)}\right)_+^{\alpha}$$

where $s_B = s_B(t)$ is a function to be determined and $\alpha > 1$ is a parameter to be chosen so that

$$(7)\qquad \lim_{t\to\infty} s(t) = \frac{I(p)}{\lambda} \ ,$$

then we obtain

$$(8)\quad s_B(t) = \frac{I}{\lambda}\left[1 - \exp\left(- \frac{2\lambda^2(3-p)t}{1+p}\right)\right]^{1/2} , \ t \geq 0, \quad \alpha=\alpha(p)= \frac{2}{1-p}>2.$$

THEOREM 1. Let $0 < p < 1$, $\lambda > 0$ and $0 \leq U_0 \leq u_\infty$ in $\mathbb{R}^+$ be. If u=u(x,t) is a solution of (1) and s=s(t) is defined by (4), we have the comparison properties:

$$(9)\qquad u_1(x,t) \leq u(x,t) \leq u_\infty(x) \ , \ 0 \leq x \leq \frac{I}{\lambda} \ , \ t > 0 \ ,$$

$$(10)\qquad s_1(t) \leq s(t) \leq \frac{I}{\lambda} \ , \ 0 < \frac{I}{\lambda} - s(t) \leq \frac{I}{\lambda} - s_1(t) \leq \frac{I}{\lambda}\exp\left(- \frac{2\lambda t}{I}\right) \ , \ t \geq 0 \ ,$$

where $u_1(x,t)$ and $s_1(t)$ are appropriate functions.

THEOREM 2. For the case $\lambda = 1$, $0<p<1$ and $0 \leq U_0 \leq u_\infty$ in $\mathbb{R}^+$ in problem (1), we obtain the following estimates:

$$(11)\qquad s_1(t) < s_0(t) \leq s(t) \leq I \ , \quad s_1(t) < s_0(t) < s_B(t) < I \ , \ t > 0 \ ,$$

$$(12)\qquad u_1(x,t) \leq u_0(x,t) \leq u(x,t) \leq u_\infty(x), \ u_1(x,t) \leq u_0(x,t) \leq C_B(x,t) \leq u_\infty(x), 0 \leq x \leq I, \ t>0,$$

where functions s_0 and u_0 are defined by taking $L_0 = 0$ and m=1 in [RiTa].

IV. On the free boundary problem in the Wen–Langmuir shrinking core model for noncatalytic gas–solid reactions

Goal : "We give a local result in time for the existence and uniqueness of the solution of the free boundary problem in the shrinking core model for noncatalytic gas–solid reactions. We impose free boundary conditions which generalize Wen and Langmuir conditions".

We analyze a mathematical model of an isothermal noncatalytic diffusion—reaction process of a gas A with a solid slab S. The solid has a very low permeability and semi—thickness R along the gas diffusion direction[TaVi]. We assume the solid is chemically attacked from the surface $y = R$ with a quick and irreversible reaction of order $\nu > 0$ with respect to the gas A and zero order with respect to the solid S. We also assume that the solid has uniform and constant composition. As a result of the chemical reaction an inert layer is formed which is permeable to the gas and the process will exhibit a free boundary (the reaction front) as described in [We]. The corresponding mathematical scheme (Wen's model) is formulated as follows (in a dimensionless form) :

$$\text{(1)}\quad \begin{array}{lll} \text{i)} & u_{xx} - u_t = 0 \quad \text{in } D_T = \{(x,t) / 0 < x < s(t),\ 0 < t \leq T\}, & \\ \text{ii)} & u(0,t) = v_0 > 0,\quad 0 < t \leq T, & \\ \text{iii)} & u_x(s(t),t) = -u^{\nu}(s(t),t), & \text{iv) } u_x(s(t),t) = -\dot{s}(t),\quad 0 < t \leq T, \\ \text{v)} & s(0) = b > 0, & \text{vi) } u(x,0) = \Psi(x),\quad 0 \leq x \leq b. \end{array}$$

We can consider the following generalized free boundary conditions :

$$\text{(2)}\quad \text{i) } u_x(s(t),t) = g(u(s(t),t)),\quad \text{ii) } \dot{s}(t) = f(u(s(t),t)),\quad 0 < t \leq T,$$

where f and g are real functions which satisfy

$$\text{(3)}\quad \text{i) } f>0,\ f'>0 \text{ in } \mathbb{R}^+ \text{ and } f(0)=0,\quad \text{ii) } g<0,\ g'<0 \text{ in } \mathbb{R}^+ \text{ and } g(0)=0.$$

We remark here that functions f and g , defined by [We]: $g(x) = -x^{\nu} (= -f(x))$ or by a Langmuir type condition [Do] : $g(x) = -a\, x^n/(b+cx^n)$ $(=-f(x))$, verify conditions (3i,ii) for all constants a, b, c, n, $\nu > 0$.

Firstly, we study an auxiliary moving boundary problem. We generalize the results obtained in [FaPr1,FaPr2] changing the nonlinear condition on the fixed face $x = 0$ by other one on the moving boundary $x = s(t)$, given by (2i). Secondly, we study the Wen—Langmuir free boundary model for noncatalytic gas—solid reactions that consists in finding $T > 0$, $x = s(t)$ and $u = u(x,t)$ such that they satisfy conditions (2). We prove that there exists a unique solution for a sufficiently small $T > 0$. Moreover, the solution is given through the unique fixed point, in an adequate Banach space, of the following contraction operator F_2 : For $s = s(t) \in C^0([0,T])$ we define

$$\text{(4)}\quad F_2(s)(t) = \int_0^t f(v(s(\tau),\tau))\, d\tau$$

where v is the solution of problem (1i—ii—iv) and (2i). Here we exploit some techniques used in [CoRi] for sorption of swelling solvents in polymers.

V. <u>On the free boundary problem for the Michaelis—Menten absorption model for root growth</u>

<u>Goal</u> : "We give a growth absorption model for the surface of a root of a plant through an absorption mechanism. For low concentrations the resultant equations have been analytically solved by the quasi—stationary method. This solution is used to compute growth of the radius of the root".

Many methods exist for studying the mechanism involved in nutrient uptake. One of the most promising methods is the mathematical model, which can be a satisfactory method of modelling the plant–root system by use of the partial differential equation for convective and diffusive flow to a root [Cu1,Cu2]. In general, these models have not considered computing root growth, but rather they have assumed young roots to be growing at exponential rates. We compute the free boundary (the root–soil interface) through the quasi–stationary method. We obtain an analytical solution for the nutrient interface concentration and the interface position. Taking into account the idea of the model used for the shrinking core problem for noncatalytic gas–solid reactions [TaVi], we propose the following free boundary problem for root growth assuming low concentrations (in cylindrical coordinates) [ReTaCa] :

$$\text{i)}\quad D\,C_{rr} + D\alpha_0 \frac{C_r}{r} = 0, \qquad s(t)<r<R,\ t>0,\ \alpha_0 = 1+\epsilon,\ \epsilon = v_0 s_0/Db > 0,$$

$$\text{(1)}\quad \text{ii)}\quad C(r,0) = \Phi(r)\,, \qquad s_0 \le r \le R\,, \qquad \text{iii)}\quad s(0) = s_0\,,\ 0<s_0<R\,,$$

$$\text{iv)}\quad C(R,t) = C_\infty > 0\,,\ t>0,$$

$$\text{v)}\quad Db\,C_r(s(t),t) + v_0\,C(s(t),t) = k\,C(s(t),t) - E = a\,C(s(t),t)\,\dot{s}(t)\,,\ t>0.$$

The solution of the problem is given by :

$$\text{(2)}\qquad C(r,t) = \beta(t) - \frac{\alpha(t)}{r^{\epsilon}}\,, \quad s(t)<r<R\,,\ t>0\,,$$

where:

$$\text{(3)}\qquad \alpha(t) = \left[\frac{1}{D\,b}\right] \frac{\left[(k-v_0)C_\infty - E\right]}{\dfrac{\epsilon}{s(t)^{1+\epsilon}} + \dfrac{(k-v_0)}{D\,b}\left[\dfrac{1}{s(t)^{\epsilon}} - \dfrac{1}{R^{\epsilon}}\right]}\,, \qquad \beta(t) = C_\infty + \frac{\alpha(t)}{R^{\epsilon}}\,,$$

$$\text{(4)}\qquad \Phi(r) = C_\infty - \frac{\left[(k-v_0)\,C_\infty - E\right]}{\dfrac{v_0}{s_0^{\,\epsilon}} + (k-v_0)\left[\dfrac{1}{s_0^{\,\epsilon}} - \dfrac{1}{R^{\epsilon}}\right]}\left[\frac{1}{r^{\epsilon}} - \frac{1}{R^{\epsilon}}\right]$$

and s(t) is the unique solution of the following Cauchy problem :

$$\text{(5)}\qquad \dot{s}(t) = F(s(t))\,,\ t>0, \qquad s(0) = s_0 \in (0,R)\,,$$

with:

$$\text{(6)}\qquad F(s) = \frac{k}{a}\left[1 - \alpha_3\,H(s)\right]\,, \quad H(s) = \frac{\left[1 + \alpha_2\,G(s)\right]}{\left[1 + \alpha_1\,G(s)\right]}\,, \quad G(s) = s\left[1 - \left(\frac{s}{R}\right)^{\epsilon}\right],$$

$$\text{(7)}\qquad \alpha_1 = \frac{E}{v_0\,s_0\,C_\infty} > 0\,, \qquad \alpha_2 = \frac{(k-v_0)}{v_0\,s_0} > 0\,, \qquad \alpha_3 = \frac{E}{k\,C_\infty} > 0\,.$$

The solution of the problem (5) is computed numerically and the results are plotted for the interface concentration $C(s(t),t)$ vs. s and the interface position $s(t)$ vs. t respectively as a function of the dimensionless parameter k/v_0 [ReTaCa] .

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