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On Heat Flux in Materials With or Without Phase Change

1 Introduction

We consider a heat conducting material occupying Ω, a bounded domain of $\mathbb{R}^n$ ($n = 1, 2, 3$ in practice), with a sufficiently regular boundary $\Gamma = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3$ (with meas $(\Gamma_1) > 0$ and meas $(\Gamma_2) > 0$). We assume that the phase-change temperature of the material is 0°C. We impose a temperature $b = b(x) > 0$ on Γ_1 (or a Newton law with coefficient $\alpha > 0$) and an outcoming heat flux $q = q(x) > 0$ on Γ_2; we also suppose that the portion of the boundary Γ_3 (when it exists) is a wall impermeable to heat, i.e. the heat flux on Γ_3 is null.

If we consider in Ω a steady-state heat conduction problem, then we are interested in finding necessary and/or sufficient conditions for the heat flux q on Γ_2 (and/or the heat transfer coefficient α on Γ_1) to obtain a change of phase in Ω, that is, a steady-state two-phase Stefan problem in Ω.

We shall consider several cases.

2 Problem I

For constant data $b > 0$ and $q > 0$, find a constant $q_0 > 0$ such that for $q > q_0$ we have a two-phase Stefan problem.

The temperature $\theta = \theta(x)$ can be represented in $\Omega = \Omega_1 \cup \Omega_2 \cup \mathcal{L}$ in the following way:

$$\theta(x) = \begin{cases} \theta_1(x) < 0, & x \in \Omega_1 \quad \text{(solid phase)} \\ 0, & x \in \mathcal{L} \quad \text{(free boundary)} \\ \theta_2(x) > 0, & x \in \Omega_2 \quad \text{(liquid phase)} \end{cases} \tag{1}$$

Denoting by $k_i > 0$ the thermal conductivity of phase i ($i = 1$: solid phase, $i = 2$: liquid phase), we define the new unknown function u as follows [3,5,13]

$$u = k_2\theta^+ - k_1\theta^- \quad \text{in } \Omega \tag{2}$$

and we obtain the problem ($B = k_2 b > 0$):

$$\left.\begin{aligned} &\Delta u = 0, \quad \text{in } \Omega, \\ &u|_{\Gamma_1} = B, \quad -\frac{\partial u}{\partial n}|_{\Gamma_2} = q, \quad \frac{\partial u}{\partial n}|_{\Gamma_3} = 0, \end{aligned}\right\} \tag{3}$$

whose variational formulation is given by:

$$\left.\begin{aligned} &a(u, v - u) = L(v - u), \quad \forall v \in K, \\ &u \in K, \end{aligned}\right\} \tag{4}$$

where

$$\left.\begin{aligned} &V = H^1(\Omega), \quad K = \{v \in V \mid v|_{\Gamma_1} = B\}, \\ &a(u,v) = \int_\Omega \nabla u \cdot \nabla v \, dx, \quad L(v) = -\int_{\Gamma_2} qv \, d\gamma. \end{aligned}\right\} \tag{5}$$

Moreover, the element $u \in K$ is also characterized by the following minimum problem [10,15]

$$\left.\begin{aligned} &J(u) \leq J(v), \quad \forall v \in K, \\ &u \in K, \end{aligned}\right\} \tag{6}$$

where $J(v) = \frac{1}{2}a(v,v) - L(v)$.

Let $u = u_q$ be the unique solution of the above variational equality for $q > 0$ and let $f : \mathbb{R}^+ \to \mathbb{R}$ be the real function defined by

$$f(q) = J(u_q) = \frac{1}{2}a(u_q, u_q) + q \int_{\Gamma_2} u_q \, d\gamma. \tag{7}$$

Theorem I-1. We have the following properties [16]:

i) There exists a constant $C > 0$ such that

$$\left.\begin{aligned} &a(u_q, u_q) = C\, q^2, \\ &f(q) = -\frac{C}{2}q^2 + B\, q \text{ meas } (\Gamma_2). \end{aligned}\right\} \tag{8}$$

ii) If $q > q_0(B)$, then we obtain a two-phase Stefan problem in Ω, where

$$q_0(B) = \frac{B}{C} \text{ meas } (\Gamma_2). \tag{9}$$

We remark that the element q_0 verifies $f'(q_0) = \int_{\Gamma_2} u q_0 \, d\gamma = 0$.

Let $q_c > 0$ be the critical heat flux which characterizes a two-phase Stefan problem in Ω, that is

$$\left.\begin{aligned} &q > q_c \;\Leftrightarrow\; \exists \text{ 2-phases}, \\ &q \leq q_c \;\Leftrightarrow\; \exists \text{ 1-phase}. \end{aligned}\right\} \tag{10}$$

Theorem I-2. We obtain some estimates for q_c [2]:

i) If ω is the solution to

$$\left.\begin{aligned} &\Delta\omega = 0, \quad \text{in } \Omega \\ &\omega|_{\Gamma_1} = B, \quad \omega|_{\Gamma_2} = 0, \quad \frac{\partial \omega}{\partial n}|_{\Gamma_3} = 0 \end{aligned}\right\} \tag{11}$$

then we have

$$q_i \equiv \min_{\Gamma_2}(-\frac{\partial \omega}{\partial n}|_{\Gamma_3}) \leq q_c. \tag{12}$$

ii) Let $d = \text{Sup}_{x \in \Gamma_2}$ dist $(x, \overline{\Gamma}_1) > 0$ and let $P_1 \in \overline{\Gamma}_1$, $P_2 \in \Gamma_2$ be such that $d =$ dist (P_1, P_2). Now let π be an affine function such that

$$\left.\begin{array}{ll} \pi(P_1) = B, & \pi|_{\Gamma_1} \geq B \\ \pi(P_2) = 0, & \pi|_{\Gamma_2} \geq 0 \\ \dfrac{\partial \pi}{\partial n}|_{\Gamma_3} \geq 0 & \end{array}\right\} \tag{13}$$

then we have

$$q_s \equiv \max_{\Gamma_2}(-\frac{\partial \pi}{\partial n}|_{\Gamma_2}) \geq q_c. \tag{14}$$

iii) If $\omega \neq \pi$ (i.e. $\omega < \pi$) in Ω then $q_i < q_s$.

iv) $q_c(\Omega)$ is a non-increasing function of Ω in the following sense:

$$\Omega_1 \subset \Omega_2 \text{ with a common } \Gamma_2 \text{ and } \Gamma_3) \Rightarrow q_c(\Omega_2) \leq q_c(\Omega_1). \tag{15}$$

3 Problem II

For the general case $b = b(x) > 0$ on Γ_1 and $q = q(x)$ on Γ_2, we consider the following optimization problem: Find $q \in Q^+$ that produces the maximum total heat flux on Γ_2 without change of phase within Ω, i.e. find

$$\max_{q \in Q^+} F(q) \tag{16}$$

where

$$\left.\begin{array}{l} Q = H^{1/2}(\Gamma_2), \quad S^+ = \{u \in S \mid u > 0 \text{ in } \Omega\}, \\ S = \{u \in K \mid \Delta u = 0 \text{ in } \Omega, \dfrac{\partial u}{\partial n}|_{\Gamma_3} = 0\}, \\ Q^+ = T^{-1}(S^+) = \{q \in Q \mid u_q \geq 0 \text{ in } \Omega\}, \\ f : Q \to \mathbb{R} \mid F(q) = \displaystyle\int_{\Gamma_2} q \, d\gamma, \end{array}\right\} \tag{17}$$

where the application $T : Q \to S$ is defined by $T(q) = u$ where $u = u_q$ is the unique solution of the variational equality (4). We suppose that $b \in H^{3/2}(\Gamma_1)$ and Ω is sufficiently regular to have the reglarity property $u \in C^0(\overline{\Omega})$ (See, for instance, our three examples).

The problem is solved by using optimization techniques of convex functionals with restriction [1,4].

Theorem II-1. [8] There exists a unique $\overline{q} \in Q^+$ such that

$$F(\overline{q}) = \max_{q \in Q^+} F(q). \tag{18}$$

Moreover, the element $\overline{q}$ is defined by

$$\overline{q} = -\frac{\partial \omega}{\partial n}|_{\Gamma_2} \quad (\omega \text{ is given by (11)}). \tag{19}$$

We have other problems related to problem II [8]:

II-i) Find the maximum upper bound for q such that there is no change of phase within Ω, that is:

$$\text{Find } q_M^0 > 0 \mid u_q \geq 0 \text{ in } \Omega, \quad \forall q = q(x) \leq q_M^0 \text{ on } \Gamma_2. \tag{20}$$

II-ii) For $q(x) = Q\, q_1(x)$ on Γ_2 (Q = const. > 0, $q_1 > 0$ on Γ_2 given) find the maximum upper bound for Q such that there is no change of phase within Ω, that is:

$$\text{Find } Q_M > 0 \mid u_q \geq 0 \text{ in } \Omega, \quad \forall Q \leq Q_M. \tag{21}$$

4 Problem III

For the constant case $b > 0$ and $q > 0$ we change the condition $u|_{\Gamma_1} = B$ into the following

$$-\frac{\partial u}{\partial n}|_{\Gamma_1} = \alpha(u - B) \tag{22}$$

and then we can pose the same problem I, now with the variables α, q, as characterizing also a two-phase Stefan problem.

The variational equality for $u = u_{\alpha q B}$ is given by [13]

$$\left.\begin{array}{l} a_\alpha(u, v) = L_{\alpha q B}(v), \quad \forall v \in V \\ u \in V \end{array}\right\} \tag{23}$$

where

$$\left.\begin{array}{l} a_\alpha(u, v) = a(u, v) + \alpha \displaystyle\int_{\Gamma_1} uv\, d\gamma, \\ L_{\alpha q B}(v) = -q \displaystyle\int_{\Gamma_2} v\, d\gamma + \alpha B \int_{\Gamma_1} v\, d\gamma. \end{array}\right\} \tag{24}$$

Theorem III-1. [12] If $q > q_0$ (q_0 is given by (9)), then we have a two-phase Stefan problem in Ω, $\forall \alpha > \alpha_0$, where

$$\alpha_0 \equiv \alpha_0(q, B) = \frac{\text{meas}(\Gamma_2)}{\text{meas}(\Gamma_1)} \frac{q}{B}. \tag{25}$$

Let $g : (\mathbb{R}^+)^3 \to \mathbb{R}$ be the real function defined by

$$g(\alpha, q, B) = \frac{1}{2} a_\alpha(u_{\alpha q B}, u_{\alpha q B}) - L_{\alpha q B}(u_{\alpha q B}) \leq 0. \tag{26}$$

Theorem III-2. [12] There exists a function $A = A(\alpha) > 0$, defined for $\alpha > 0$, such that:

$$g(\alpha, q, B) = -\frac{A(\alpha)}{2} q^2 + Bq \operatorname{meas}(\Gamma_2) - \frac{B^2\alpha}{2} \operatorname{meas}(\Gamma_1). \tag{27}$$

Moreover, A is a decreasing function which verifies

$$\left.\begin{array}{l} A(\alpha) > \dfrac{[\operatorname{meas}(\Gamma_2)]^2}{\operatorname{meas}(\Gamma_1)} \dfrac{1}{\alpha}, \quad \lim\limits_{\alpha\to+\infty} A(\alpha) = C, \\ \lim\limits_{\alpha\to+\infty} \alpha A'(\alpha) = 0, \quad (\alpha A(\alpha))' = \dfrac{1}{q^2} a(u_{\alpha q B}, u_{\alpha q B}). \end{array}\right\} \tag{28}$$

Let $q_m = q_m(\alpha, B)$ and $q_M = q_M(\alpha, B)$ be the real functions defined by:

$$q_m(\alpha, B) = \frac{B \operatorname{meas}(\Gamma_2)}{A(\alpha)}, \quad q_M(\alpha, B) = \frac{B\alpha \operatorname{meas}(\Gamma_1)}{\operatorname{meas}(\Gamma_2)}. \tag{29}$$

Theorem III-3. [12] If $(\alpha, q) \in S^{(2)}(B)$, then we have a two-phase Stefan problem in Ω, where

$$S^{(2)}(B) = \{(\alpha, q) \in (\mathbb{R}^+)^2 \mid q_m(\alpha, B) < q < q_M(\alpha, B)\} \quad (B > 0). \tag{30}$$

We consider now the *particular case* when $u_{\alpha q B}$ verifies the condition

$$\frac{1}{q^2} a(u_{\alpha q B}, u_{\alpha q B}) = \text{Const.} \tag{31}$$

In this case, necessarily we have that

$$\text{Const.} = C > 0. \tag{32}$$

Theorem III-4. [12] We have the following properties for this particular case

$$\left.\begin{array}{l} u_{\alpha q B}\big|_{\Gamma_1} = B - \dfrac{q \operatorname{meas}(\Gamma_2)}{\operatorname{meas}(\Gamma_1)}, \\ A(\alpha) = C + \dfrac{[\operatorname{meas}(\Gamma_2)]^2}{\operatorname{meas}(\Gamma_1)} \dfrac{1}{\alpha}, \end{array}\right\} \tag{33}$$

then we obtain a complete description of the set $S^{(2)}(B)$.

Remarks. i) When, because of symmetry, u_q is constant on Γ_2 [$u_{\alpha q B}$ is constant on $\Gamma_1 \cup \Gamma_2$] the sufficient condition in Theorem (I-1ii) [Theorem II-3] becomes also necessary. ii) We can verify all theoretical results obtained in this paper in three different examples [14]:

a)

$$\left.\begin{array}{l} n = 2, \quad \Omega = (0, x_0) \times (0, y_0), \quad x_0 > 0, \quad y_0 > 0, \\ \Gamma_1 = \{0\} \times [0, y_0], \quad \Gamma_2 = \{x_0\} \times [0, y_0], \\ \Gamma_3 = (0, x_0) \times \{0\} \cup (0, x_0) \times \{y_0\}. \end{array}\right\} \tag{34}$$

For this case, we obtain

$$\left.\begin{array}{l} c = x_0 y_0, \quad q_c = q_i = q_s = q_0 = \dfrac{B}{x_0}, \quad \alpha_0 = \dfrac{q}{B}, \\ A(\alpha) = y_0(x_0 + \dfrac{1}{\alpha}), \quad q_m = \dfrac{B}{x_0 + \frac{1}{\alpha}}, \quad q_M = B\alpha. \end{array}\right\} \tag{35}$$

b)

$$\left.\begin{array}{l} n = 2, \quad 0 < r_1 < r_2, \quad \Gamma_3 = \emptyset, \\ \Omega: \text{ annulus of radius } r_1 \text{ and } r_2, \text{ centred in } (0,0), \\ \Gamma_1(\Gamma_2): \text{ circumference of radius } r_1(r_2) \text{ and centre } (0,0). \end{array}\right\} \tag{36}$$

c) We take into account the same information of (b) but now for the case $n = 3$.

iii) The two elliptic variational equalities (5) and (25) appear if we consider the asymptotic behaviour when the time $t \to +\infty$ in four parabolic variational inequalities of type II, defined in [13] for the evolution two-phase Stefan problem. The asymptotic behaviour for the weak formulation was given in [6].

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