An inequality for the constant heat flux to obtain a steady-state two-phase Stefan problem

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We consider a material $\Omega \subset \mathscr{R}^n$ with regular boundary $\Gamma = \Gamma_1 \cup \Gamma_2$ and we assume that the melting temperature is 0°C. We apply a temperature $b > 0$ on Γ_1 and a heat flux $q > 0$ on Γ_2. We prove that there exists a constant $q_1 > 0$ such that, for $q > q_1$, we have a steady-state two-phase Stefan problem. This result is verified numerically, by using Modulef, with two cases with analytical solutions.

1. INTRODUCTION

We consider a material Ω, a bounded domain of $\mathscr{R}^n$ ($n = 1, 2, 3$ for the applications), with a sufficiently regular boundary $\Gamma = \Gamma_1 \cup \Gamma_2$ (with measure $(\Gamma_1) > 0$) and we assume that the phase-change temperature is 0°C. We apply a constant temperature $b > 0$ on Γ_1 and a constant (outcoming) heat flux $q > 0$ on Γ_2. If we consider in Ω a steady-state heat conduction problem, from the physical point of view, we arrive at the following conclusions:

(i) If q is small, then the temperature in Ω will be positive, and so a change of phase in the material will not occur. In this case, the resulting problem will be one of conduction, only for the liquid phase.
(ii) If q is large, then the temperature in Ω will take positive and negative values, and so a change of phase in the material will occur.

In this paper, we shall find for q a sufficient condition for the occurrence of a change of phase in Ω, i.e., we shall prove that there exists $q_1 > 0$ so that for all $q > q_1$ we can have a steady-state two-phase Stefan problem in Ω. Moreover, in two examples where the sufficient conditions is also necessary[1], we shall compute numerically the constant q_1 through a simulation process by using the Modulef software (Finite Elements Modules).

2. MATHEMATICAL FORMULATION OF THE PROBLEM

Following[2] we study the temperature $\theta = \theta(x)$, defined for $x \in \Omega$. The set Ω can be expressed in this form

$$\Omega = \Omega_1 \cup \Omega_2 \cup L \tag{1}$$

where

$$\Omega_1 = \{x \in \Omega / \theta(x) < 0\} \quad \Omega_2 = \{x \in \Omega / \theta(x) > 0\}$$
$$L = \{x \in \Omega / \theta(x) = 0\} \tag{2}$$

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are the solid phase, the liquid phase and the free boundary that separates them respectively.

The temperature θ can be represented in Ω in the following way:

$$\theta(x) = \begin{array}{ll} \theta_1(x) < 0 & x \in \Omega_1 \\ 0 & x \in L \\ \theta_2(x) > 0 & x \in \Omega_2 \end{array} \tag{3}$$

and satisfies the following conditions

$$\begin{array}{ll} \Delta\theta_i = 0 & \text{in } \Omega_i \ (i = 1, 2) \\ \theta_1 = \theta_2 = 0 & k_1 \dfrac{\partial \theta_1}{\partial n} = k_2 \dfrac{\partial \theta_2}{\partial n} \text{ on } L \end{array} \tag{4}$$

$$\theta_2 / \Gamma_1 = b$$
$$-k_2 \left.\frac{\partial \theta_2}{\partial n}\right|_{\Gamma_2} = q \quad \text{si } \theta/_{\Gamma_2} > 0$$
$$-k_1 \left.\frac{\partial \theta_1}{\partial n}\right|_{\Gamma_2} = q \quad si\ \theta/_{\Gamma_2} < 0$$

where $k_i > 0$ is the thermal conductivity of phase i ($i = 1$: solid phase, $i = 2$: liquid phase), $b > 0$ is the constant temperature given on Γ_1, and $q > 0$ is the constant heat flux given on Γ_2.

If we define the function u in Ω as follows

$$u = k_2 \theta^+ - k_1 \theta^- \quad \text{in } \Omega \tag{5}$$

where θ^+ and θ^- represent the positive part and the negative part of the function θ respectively, then problem (4) is transformed into

$$\Delta u = 0 \quad \text{in } D'(\Omega)$$
$$u/_{\Gamma_1} = b_0 \equiv k_2 b$$
$$-\left.\frac{\partial u}{\partial n}\right|_{\Gamma_2} = q \tag{6}$$

whose variational formulation is given by

$$a(u, v-u) = -q \int_{\Gamma_2} (v-u)\,d\gamma \qquad v \in K$$
$$u \in K \tag{7}$$

where

$$V = H^1(\Omega) \qquad V_0 = \{v \in V / v/_{\Gamma_1} = 0\}$$
$$K = \{v \in V / v/_{\Gamma_1} = b_0\}$$
$$a(u,v) = \int_\Omega \nabla u \cdot \nabla v\,dx \tag{8}$$

Moreover, the solution of (7) is characterized by the following minimum problem[3-5]:

$$\begin{cases} J(u) \leq J(v) & v \in K \\ u \in K \end{cases} \tag{9}$$

where

$$J(v) = \frac{1}{2} a(v,v) + q \int_{\Gamma_2} v\,d\gamma \tag{10}$$

Remark 1

The inverse transformation of (5) is given by

$$\theta = \frac{1}{k_2} u^+ - \frac{1}{k_1} u^- \qquad \text{in } \Omega \tag{5 bis}$$

3. PROPERTIES

Let u_q be the unique solution of the variational equation (7) for $q > 0$ (Ref. 2).

Property 1

We have the following expression

$$a(u_q^-, u_q^-) = q \int_{\Gamma_2} u_q^-\,d\gamma \tag{11}$$

Proof. It is enough to choose $v = u_q^+ \in K$ in (7) to obtain (11).

Remark 2

From (11) and from the fact that $u_q^- \in V_0$, we deduce the equivalence

$$u_q^- \not\equiv 0 \quad \text{en } \Omega \Leftrightarrow u_q^- \not\equiv 0 \quad \text{sobre } \Gamma_2 \tag{12}$$

from which, for a given value of q, we have that there will be a change of phase in Ω (u_q or θ_q take positive and negative values in Ω) iff the function u_q takes negative values on the boundary Γ_2. In other words, the function u_q will begin to take negative values on Γ_2. (This fact will be taken into account when we carry out the numerical simulation for the computation of the coefficient q_1.)

Property 2

If $u_i \equiv u_{q_i}$ is the solution of (7) for q_i ($i = 1, 2$), then we have the following equalities:

$$\text{(i)} \quad a(u_2 - u_1, u_2 - u_1) = (q_1 - q_2) \int_{\Gamma_2} (u_2 - u_1)\,d\gamma$$
$$\text{(ii)} \quad a(u_2, u_2) - a(u_1, u_1) = a(u_2 + u_1, u_2 - u_1) = (q_1 + q_2) \int_{\Gamma_2} (u_1 - u_2)\,d\gamma \tag{13}$$

Proof. If we take $v = u_2 \in K$ in the variational equality corresponding to u_1, and $v = u_1 \in K$ in the one corresponding to u_2 and we add up and subtract both equalities, then we obtain (13i) and (13ii) respectively.

Property 3

If $u_i = u_{q_i}$ is the unique solution of (7) for q_i ($i = 1, 2$), then we have the following properties:

(i) If $q_2 \leq q_1$ then

$$\text{(a) } u_1 \leq u_2 \quad \text{in } \Omega \qquad \text{(b) } \int_{\Gamma_2} u_1\,d\gamma \leq \int_{\Gamma_2} u_2\,d\gamma \tag{14}$$

(ii) The applications $q \to u_q$ and $q \to \int_{\Gamma_2} u_q\, d_\gamma$ are strictly decreasing functions. i.e.,

$$q_2 < q_1 \Rightarrow \begin{array}{l} \text{(a) } u_1 \leq u_2,\ u_1 \not\equiv u_2 \quad \text{in } \Omega \\ \text{(b) } \int_{\Gamma_2} u_1\,d\gamma < \int_{\Gamma_2} u_2\,d\gamma \end{array} \tag{15}$$

Proof. (i) Condition (14b) follows directly from (13i). To prove (14a) we shall take into account the following equivalence:

$$\begin{cases} u_1 \leq u_2 \quad \text{in } \Omega \Leftrightarrow W = 0 \quad \text{in } \Omega \\ \text{where } W = (u_2 - u_1)^- \end{cases} \tag{16}$$

Since $W \in V_0$, then, if we use $v = u_2 + W \in K$ in the variational equality corresponding to u_1, and $v = u_1 + W \in K$ in the one corresponding to u_2 and we later add them up, we have

$$0 \leq (q_1 - q_2) \int_{\Gamma_2} W\,d\gamma = a(u_2 - u_1, W) = -a(W, W) \leq 0 \tag{17}$$

that is, $W = 0$ in Ω.

(ii) To prove (15a,b) we use the following results:

$$\text{(A) } u_1 = u_2 \quad \text{in } \Omega \Rightarrow q_1 = q_2 \quad \text{or} \quad \int_{\Gamma_2} (u_2 - u_1)\,d\gamma = 0 \tag{18}$$

$$\text{(B) } \int_{\Gamma_2} (u_2 - u_1)\,d\gamma = 0 \Rightarrow \begin{cases} \text{(Bi)} & u_2 = u_1 \quad \text{in } \Omega \\ \text{(Bii)} & q_1 = q_2 \end{cases} \tag{19}$$

Condition (A) results directly from (13i) and condition (Bi) is deduced from (13i) and from the fact that $u_2 - u_1 \in V_0$. Taking into account (B)'s hypothesis, the result (Bi) and the variational equalities corresponding to u_2 and u_1, we obtain

$$-q_1 \int_{\Gamma_2} (v-u_1)\,d\gamma = a(u_1, v-u_1) = a(u_2, v-u_2)$$

$$= -q_2 \int_{\Gamma_2} (v-u_2)\,d\gamma = -q_2 \int_{\Gamma_2} (v-u_1)\,d\gamma \quad v \in K$$

i.e.,

$$(q_1 - q_2) \int_{\Gamma_2} (v-u_1)\,d\gamma = 0 \qquad v \in K \tag{20}$$

Taking one element $v_0 \in V_0$ so that $\int_{\Gamma_2} v_0 \, d\gamma \neq 0$ and choosing $v = u_1 + v_0 \in K$, from (20) we deduce (Bii).

Let $f: \mathbb{R}^+ \to \mathbb{R}$ be the real function, defined in the following way:

$$f(q) = J(u_q) = \frac{1}{2} a(u_q, u_q) + q \int_{\Gamma_2} u_q \, d\gamma \tag{21}$$

Remark 3
To find the element q_1 (see Introduction), and taking into account (12), (15b) and function f, it will be enough to find a value $q > 0$ for which we have $f(q) < 0$. We shall further see that this technique can still be improved.

Property 4
For all $q > 0$ and h such that $q + h > 0$, we have the following estimations:

$$\text{(i)} \quad \left\| \frac{1}{h}(u_{q+h} - u_q) \right\|_V \leqslant C_1 \equiv \frac{\|\gamma_0\|}{\alpha_0} [\text{meas}(\Gamma_2)]^{1/2} \tag{22}$$

$$\text{(ii)} \quad \left\| \frac{1}{h}(u_q - u_{q+h}) \right\|_{L^2(\Gamma_2)} \leqslant C_2 \equiv C_1 \|\gamma_0\| \tag{23}$$

where γ_0 is the trace operator (linear and continuous, defined on V), and $\alpha_0 > 0$ is the coercivity constant on V_0 of the bilinear form a, i.e.,

$$a(v, v) \geqslant \alpha_0 \|v\|_V^2 \qquad v \in V_0 \tag{24}$$

Proof. (i) Taking into account (24), (13i) with $q_1 = q + h$ and $q_2 = q$, the Cauchy-Schwarz inequality and the continuity of γ_0 we obtain (22). (ii) Taking into account (22) and the continuity of γ_0, we deduce (23).

From (15b) and (23) we deduce the following

Corrollary 5
From all $q > 0$ and $h > 0$ we have

$$0 < \int_{\Gamma_2} u_q \, d\gamma - \int_{\Gamma_2} u_{q+h} \, d\gamma < C_2 h \tag{25}$$

and therefore the function $q \to \int_{\Gamma_2} u_q \, d\gamma$ is continuous.

Property 6
Function f is derivable. Moreover, f' is continuous and strictly decreasing function, and it is given by the following expression

$$f'(q) = \int_{\Gamma_2} u_q \, d\gamma \tag{26}$$

Proof. From (13ii) we obtain

$$\frac{f(q+h) - f(q)}{h} = \frac{1}{2} \int_{\Gamma_2} u_q \, d\gamma + \frac{1}{2} \int_{\Gamma_2} u_{q+h} \, d\gamma \tag{27}$$

and the expression (26) is deduced from (25) and (27).

Property 7
For all $q > 0$ we have the following expressions

$$\text{(i)} \quad a(u_q, u_q) = k_2 b \int_{\Gamma_1} \frac{\partial u_q}{\partial n} \, d\gamma - q \int_{\Gamma_2} u_q \, d\gamma \tag{28}$$

$$\text{(ii)} \quad \int_{\Gamma_1} \frac{\partial u_q}{\partial n} \, d\gamma = q \, \text{meas}(\Gamma_2) \tag{29}$$

$$\text{(iii)} \quad f(q) = k_2 b \, \text{meas}(\Gamma_2) q - \frac{1}{2} a(u_q, u_q) \tag{30}$$

$$\text{(iv)} \quad \frac{d}{dq}[a(u_q, u_q)] = 2\left[k_2 b \, \text{meas}(\Gamma_2) - \int_{\Gamma_2} u_q \, d\gamma \right] = \frac{2}{q} a(u_q, u_q) \tag{31}$$

Proof. Expressions (28) and (29) are obtained by multiplying the differential equation of (6) by u_q and 1 respectively, by integrating on Ω and by using Green's formula. Expression (30) is deduced from (21), (28) and (29). Expression (31) is obtained by deriving (30) with respect to q and by using (26).

Property 8
For all $q > 0$, we have the following expressions

$$\text{(i)} \quad f'(q) = k_2 b \, \text{meas}(\Gamma_2) - \frac{1}{q} a(u_q, u_q) \tag{32}$$

$$\text{(ii)} \quad f''(q) = -\frac{1}{q^2} a(u_q, u_q) < 0 \tag{33}$$

Proof. Expression (32) is deduced from (26), (28) and (29), and expression (33) is obtained by using (31) and by deriving (32) with respect to q.

Property 9
There exists a constant $C > 0$ such that

$$a(u_q, u_q) = Cq^2 \tag{34}$$

Proof. Let the real function be

$$Y(q) = \frac{1}{q} a(u_q, u_q) \tag{35}$$

defined for $q > 0$. Function Y satisfies the following Cauchy problem:

$$Y'(q) = -f''(q) = \frac{1}{q^2} a(u_q, u_q) = \frac{1}{q} Y(q)$$

$$Y(0^+) = \lim_{q \to 0} 2\left[k_2 b \, \text{meas}(\Gamma_2) - \int_{\Gamma_2} u_q \, d\gamma \right] = 0 \tag{36}$$

The solution of (36) is given by

$$Y(q)=Cq \qquad \text{with } C>0 \text{ (constant)} \tag{37}$$

and therefore we obtain (34).

Remark 4
Constant $C>0$ has the following physical dimension

$$[C]=(cm)^n \tag{38}$$

where n is the dimension of the space $\mathbb{R}^n$ in question.

From (30) *y* Property 9, it follows:

Corollary 10
Function f, defined by (21), is given by

$$f(q)=-\frac{C}{2}q^2+k_2b\,\text{meas}(\Gamma_2)q \tag{39}$$

Theorem 11
For all $q>q_1$ problem (7) is a two-phase one, where

$$q_1=\frac{k_2b}{C}\text{meas}(\Gamma_2) \tag{40}$$

Proof. Since $f'(q_1)=0$, the result follows from (12) and (26).

Property 12
In the case where, because of symmetry, we find that function u_q is constant on Γ_2, the sufficient condition, given by Theorem 11, is also necessary for problem (7) to be a two-phase one.
Proof. Since $u_q/\Gamma_2=\text{constant}$, the property follows from the following equivalence

$$\int_{\Gamma_2} u_q\,d\gamma=0 \Leftrightarrow u_q/\Gamma_2=0 \tag{41}$$

Remark 5
Every thing we proved in this paper is still valid if the boundary Γ of the bounded domain Ω is represented by the union of the three portions $(\Gamma=\Gamma_1\cup\Gamma_2\cup\Gamma_3)$ such that they have the following characteristics:

(i) Γ_1 and Γ_2 have the same conditions as the ones previously described in (4).
(ii) Γ_3 is a wall impermeable to heat, i.e., we have $\partial\theta/\partial n|\Gamma_3=0$ in (4) and therefore $\partial u/\partial n|\Gamma_3=0$ in (6).

Remark 6
An analogous problem to the one posed in this paper but for the evolution case has been solved in Ref. 6 for a semi-infinite material which is initially in solid phase, at constant temperature, and which receives a heat flux of the form $-h_0/\sqrt{t}$ $(h_0>0)$ on its fixed face $x=0$.

4. NUMERICAL RESULTS

We shall next see the numerical results obtained by using Modulef[7,8] in two cases for which Property 12 is valid and the solution is explicitly known[1].

Example 1
We consider the following data

$$\begin{aligned}&n=2,\ \Omega=(0,x_0)x(0,y_0)\\&\Gamma_1=\{0\}x[0,y_0],\ \Gamma_2=\{x_0\}x[0,y_0]\\&\Gamma_3=[0,x_0]x\{0\}\cup[0,x_0]x\{y_0\}\end{aligned} \tag{42}$$

The solution of (6) or (7) is given by

$$u_q(x,y)=k_2b-qx \tag{43}$$

and then we obtain

$$C=x_0y_0 \qquad q_1=\frac{k_2b}{x_0} \tag{44}$$

The numerical results which are below exposed were obtained by doing a simulation process of problem (6) or (7), with the following data:

$$\begin{aligned}&x_0=1\,[\text{cm}] \quad y_0=1\,[\text{cm}] \quad b=5\,[^\circ C]\\&k_2=0.0014\left[\frac{\text{cal}}{\text{cm seg }^\circ\text{C}}\right] \text{ (thermal conductivity of water)}\end{aligned} \tag{45}$$

and by using the following triangulation[9,10]: 100 2-rectangles of type 1 and 121 vertexes.

$q\left[\frac{\text{cal}}{\text{cm}^2\text{ seg}}\right]$	$u/\Gamma_2(\text{const})\left[\frac{\text{cal}}{\text{cm seg}}\right]$
0.0071	−0.0188679
0.00705	−0.00943398
0.007001	−0.000188704
0.00700001	−0.00000191009
0.007	−0.0000000234828
0.0069999998	−0.0000000234828
0.0069999995	−0.0000000234828
0.0069999994	+0.000000627643
0.0069999993	+0.000000627643
0.006999999	+0.000000669955
0.00699999	+0.00000707206
0.0069999	+0.0000713957
0.006999	+0.000714244
0.006998	+0.00142851
0.0699	+0.00714280

We take for q_1 the following approximate value

$$q_{1\,\text{approx.}}=0.00699999945\pm0.00000000005 \tag{46}$$

Since the exact value for q_1 is given by

$$q_{1\,\text{exact}}=0.007 \tag{47}$$

the error made, by defect, is bounded by

$$0<q_{1\,\text{exact}}-q_{1\,\text{approx.}}<6\cdot10^{-10}\left[\frac{\text{cal}}{\text{cm}^2\text{ seg}}\right] \tag{48}$$

Example 2
We consider the following data:

$$n=2 \qquad 0<r_1<r_2$$

Ω : annulus of radius r_1 and r_2, centred in $(0,0)$

Γ_1: circumference of radius r_1 and centre $(0,0)$

Γ_2: circumference of radius r_2 and centre $(0,0)$

(49)

The solution of (6) or (7) is given by

$$u_q(x,y)=k_2b-qr_2\log\frac{r}{r_1} \quad r=(x^2+y^2)^{1/2} \tag{50}$$

and then we obtain

$$C=2\pi r_2^2\log\frac{r_2}{r_1} \quad \text{meas}(\Gamma_2)=2\pi r_2$$

$$q_1=\frac{k_2b}{r_2\log\frac{r_2}{r_1}} \tag{51}$$

The numerical results which are exposed below were obtained by doing a simulation process of problems (6) or (7), with the following data:

$$r_1=1\,[\text{cm}] \quad r_2=2\,[\text{cm}] \quad b=5\,[°\text{C}]$$

$$k_2=0.0014\left[\frac{\text{cal}}{\text{cm seg °C}}\right] \text{(thermal conductivity of water)} \tag{52}$$

Owing to the symmetry of the problem, it was solved for a quarter of the annulus (the one corresponding to the first quadrant), bearing in mind that in this case a new portion of the boundary Γ_3 appears, which is given by

$$\Gamma_3=\{0\}x[1,2]\cup[1,2]x\{0\} \tag{53}$$

Therefore the values for meas(Γ_2) and C are modified in a 1/4 factor, but the expression of q_1, which is the value of our interest, does not vary.

We have used in the new domain the following triangulation[9,10]: 100 2-quadrilateral (two of its sides are segments of lines and the other two are portions of circumferences) of type 1 and 122 vertexes, and we have obtained:

$q\left[\frac{\text{cal}}{\text{cm}^2\text{ seg}}\right]$	u/Γ_2 (const)$\left[\frac{\text{cal}}{\text{cm seg}}\right]$
0.004	+0.00147440
0.005	+0.0000930041
0.00505	+0.0000239342
0.00506	+0.0000101201
0.0050673	+0.00000597593
0.0050674	−0.000000102207
0.0050675	−0.000000240374
0.0050677	−0.000000516654
0.005068	−0.000000931046
0.00507	−0.00000369387
0.0051	−0.0000451358
0.0052	−0.000183276
0.00535	−0.000390486
0.0055	−0.000597696
0.006	−0.00128840

which gives us for q_1 the following value

$$q_{1\,\text{approx.}}=0.00506735\pm0.00000005 \tag{54}$$

Since the exact value for q_1 is given by

$$q_{1\,\text{exact}}=\frac{0.007}{2\log 2}\cong 0.00504943 \tag{55}$$

the error made, by excess, is bounded by

$$0<q_{1\,\text{approx.}}-q_{1\,\text{exact}}<2\cdot 10^{-5}\left[\frac{\text{cal}}{\text{cm}^2\text{ seg}}\right] \tag{56}$$

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