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NUMERICAL ANALYSIS FOR THE HEAT FLUX IN A MIXED ELLIPTIC PROBLEM WITH SOLUTION OF NON-CONSTANT SIGN

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ABSTRACT: A discretized mixed elliptic differential problem with solutions of non-constant sign, as functions of the Dirichlet and Neumann data, are studied in a convex polygonal bounded domain Ω of $\mathbf{R}^n$. An inequality for the heat flux is given in order to obtain a continuous and a discrete change of phase, that is, a continuous or discrete solution of non-constant sign in Ω (steady-state, two-phase, continuous or discretized Stefan problem). A convergence for the two inequalities, as function of the parameter h of the finite element method, is also obtained.

KEY WORDS: Steady-state Stefan problem, free boundary problems, phase-change problems, variational inequalities, Mixed elliptic problems, Numerical Analysis, Finite Element Method, Error bounds.

AMS SUBJECT CLASSIFICATION: 35R35, 35J85, 65N15, 65N30.

I. INTRODUCTION

The present talk can be considered as a review of the two papers [12, 13]. We consider a heat conducting material occuping Ω, a convex polygonal bounded domain of $\mathbb{R}^n$ ($n = 1, 2, 3$ in practice), with a sufficiently regular boundary $\Gamma = \Gamma_1 \cup \Gamma_2$ (with $meas(\Gamma_1) \equiv |\Gamma_1| > 0, |\Gamma_2| > 0$). We assume, without loss of generality, that the phase-change temperature is 0°C. We impose a temperature $b > 0$ on Γ_1 and an outcoming heat flux $q > 0$ on Γ_2. If we consider in Ω a steady-state heat conduction problem, then we are interested in finding sufficient and/or necessary conditions for the heat flux q on Γ_2 to obtain a change of phase in Ω, that is, a steady-state two-phase Stefan problem in Ω (i.e. the temperature is a function of non-constant sign in Ω) [10].

Following [9] we study the temperature $\theta = \theta(x)$, defined for $x \in \Omega$. The set Ω can be expressed in the form

$$(1) \qquad \Omega = \Omega_1 \cup \Omega_2 \cup \mathcal{L} .$$

where

$$(2) \qquad \left| \begin{aligned} \Omega_1 &= \left\{ x \in \Omega \ / \ \theta(x) < 0 \right\} , \\ \Omega_2 &= \left\{ x \in \Omega \ / \ \theta(x) > 0 \right\} , \\ \mathcal{L} &= \left\{ x \in \Omega \ / \ \theta(x) = 0 \right\} , \end{aligned} \right.$$

are, respectively, the solid phase, the liquid phase, and the free boundary (e.g. a surface in $\mathbb{R}^3$) that separates them . The temperature θ can be represented in Ω in the following way :

$$(3) \qquad \theta(x) = \left| \begin{array}{ll} \theta_1(x) < 0 , & x \in \Omega_1 , \\ 0 , & x \in \mathcal{L} , \\ \theta_2(x) > 0 , & x \in \Omega_2 , \end{array} \right.$$

and satisfies the conditions below :

$$
(4)\quad
\begin{cases}
\text{i) } \Delta\theta_i = 0 \quad \text{in } \Omega_i \ (i = 1, 2)\ , \\[2mm]
\text{ii) } \theta_1 = \theta_2 = 0\ , \quad k_1 \dfrac{\partial\theta_1}{\partial n} = k_2 \dfrac{\partial\theta_2}{\partial n} \quad \text{on } \mathcal{L}\ , \\[2mm]
\text{iii) } \theta_2 \mid_{\Gamma_1} = b\ , \\[2mm]
\text{iv) } \begin{cases} -k_2 \dfrac{\partial\theta_2}{\partial n} \mid_{\Gamma_2} = q & \text{if } \theta \mid_{\Gamma_2} > 0\ , \\[2mm] -k_1 \dfrac{\partial\theta_1}{\partial n} \mid_{\Gamma_2} = q & \text{if } \theta \mid_{\Gamma_2} < 0\ , \end{cases}
\end{cases}
$$

where $k_i > 0$ is the thermal conductivity of phase i (i = 1 : solid phase, i = 2 : liquid phase), $b > 0$ is the temperature given on Γ_1 , and $q > 0$ is the heat flux given on Γ_2.

Problem (4) represents a free boundary elliptic problem (when $\mathcal{L} \neq \emptyset$) where the free boundary $\mathcal{L}$ (unknown a priori) is characterized by the three conditions (4ii). Following the idea of [1, 3, 4, 9] we shall transform (4) into a new elliptic problem but now without a free boundary. If we define the function u in Ω as follows

$$
(5)\qquad u = k_2\,\theta^+ - k_1\,\theta^- \quad \left(\theta = \frac{1}{k_2}\,u^+ - \frac{1}{k_1}\,u^- \right) \quad \text{in } \Omega\ ,
$$

where θ^+ and θ^- represent the positive and the negative parts of the function θ respectively, then problem (4) is transformed into

$$
(6)\quad
\begin{cases}
\text{i) } \Delta u = 0 \quad \text{in } D'(\Omega), \\[2mm]
\text{ii) } u \mid_{\Gamma_1} = B\ , \quad B = k_2\, b > 0\ , \\[2mm]
\text{iii) } -\dfrac{\partial u}{\partial n} \mid_{\Gamma_2} = q\ ,
\end{cases}
$$

whose variational formulation is given by

$$(7) \qquad u \in K \quad , \qquad a(u, v-u) = L(v-u) \quad , \qquad \forall\, v \in K \quad ,$$

where

$$(8) \qquad \left| \begin{array}{l} V = H^1(\Omega) \quad , \quad V_0 = \left\{ v \in V \,/\, v\,|_{\Gamma_1} = 0 \right\} \, , \\ \\ K = K_B = \left\{ v \in V \,/\, v\,|_{\Gamma_1} = B \right\} \, , \\ \\ a(u, v) = \int_\Omega \nabla u \,.\, \nabla u \;\, dx \;\; , \;\; L(v) = L_q(v) = - \int_{\Gamma_2} q\, v\, d\gamma \,. \end{array} \right.$$

Under the hypotheses $L \in V_0'$ (e.g. $q \in L^2(\Gamma_2)$) and $B \in H^{1/2}(\Gamma_1)$, there exists a unique solution of (7) which is characterized by the following minimization problem [1,6]

$$(9) \qquad \left| \begin{array}{l} J(u) \leq J(v) \quad , \qquad \forall\, v \in K \quad , \\ \\ u \in K \quad , \end{array} \right.$$

where

$$(10) \qquad J(v) = J_q(v) = \tfrac{1}{2}\, a(v,v) - L(v) = \tfrac{1}{2}\, a(v,v) + \int_{\Gamma_2} q\, v\, d\gamma \quad .$$

<u>LEMMA</u> <u>1</u>.– If $u = u_{qB}$ is the unique solution of problem (7) for data q on Γ_2 and $B > 0$ on Γ_1 , then we have the monotony property :

$$(11) \qquad B_1 \leq B_2 \text{ on } \Gamma_1 \text{ and } q_2 \leq q_1 \text{ on } \Gamma_2 \;\Rightarrow\; u_{q_1B_1} \leq u_{q_2B_2} \text{ in } \bar{\Omega} \,.$$

Moreover,

$$(12) \qquad q > 0 \text{ on } \Gamma_2 \;\Rightarrow\; u_{qB} \leq \underset{\Gamma_1}{\text{Max}}\; B \;\text{ in } \bar{\Omega} \;,$$

and function $u = u_{qB}$ satisfies the equality

(13) $$a(u^-, u^-) = \int_{\Gamma_2} q\, u^- \, d\gamma \quad .$$

COROLLARY 2.— From (13), we deduce

(14) $$u^- \neq 0 \text{ in } \bar{\Omega} \iff u^- \neq 0 \text{ on } \Gamma_2 ,$$

where $q > 0$ and $B > 0$.

NOTE 1.— We shall denote by (N−n) the formula (n) of Section N and we shall omit N in the same paragraph. Idem for theorems, lemmas, corollaries, remarks and notes. We shall also omit the space variable $x \in \Omega$ for every function defined in Ω.

II. MIXED ELLIPTIC PROBLEMS WITH OR WITHOUT PHASE CHANGE.

We shall give a problem which are related to the mixed elliptic partial differential equations (I−6) or (I−7).

Problem P : For the constant case $B > 0$ and $q > 0$, find a constant $q_0 = q_0(B) > 0$ such that for $q > q_0(B)$ we have a steady-state, two-phase Stefan problem in Ω, that is the solution u of (I−7) is a function of non-constant sign in Ω.

REMARK 1.— From (I−14) we deduce that an answer to problem P is the element q for which u takes negative values on the boundary Γ_2 .

LEMMA 1.— Let $u = u_q$ be the unique solution of the variational equality (I−7) for $q > 0$ (for a given $B > 0$). Then

(i) The mappings

$$(1) \qquad q > 0 \to u_q \in V \qquad \text{and} \qquad q > 0 \to \int_{\Gamma_2} u_q \, d\gamma \in \mathbb{R}$$

are strictly decreasing functions.

(ii) For all $q > 0$ and $\delta > 0$ we have the following estimates :

$$(2) \qquad \| \frac{1}{\delta} (u_{q+\delta} - u_q) \|_V \leq C_1 = \frac{\| \gamma_0 \|}{\alpha_0} \mid \Gamma_2 \mid^{1/2},$$

$$(3) \qquad \| \frac{1}{\delta} (u_q - u_{q+\delta}) \|_{L^2(\Gamma_2)} \leq C_2 = C_1 \| \gamma_0 \| \quad ,$$

where γ_0 is the trace operator (linear and continuous, defined on V), and $\alpha > 0$ is the coercivity constant on V_0 of the bilinear a , i.e. :

$$(4) \qquad \exists \, \alpha > 0 \,/\, a(v, v) = \| v \|^2_{V_0} \geq \alpha \| v \|^2_V \quad , \forall v \in V_0 \ .$$

(iii) For all $q > 0$ and $\delta > 0$ we have

$$(5) \qquad 0 < \int_{\Gamma_2} u_q \, d\gamma - \int_{\Gamma_2} u_{q+\delta} \, d\gamma \leq C_3 \, \delta \ \ (C_3 = C_2 \mid \Gamma_2 \mid^{1/2} > 0)$$

and therefore the function $q > 0 \to \int_{\Gamma_2} u_q \, d\gamma$ is continuous.

<u>PROOF</u>.— If $u_i = u_{q_i}$ is the solution of (I–7) for $q_i > 0$ $(i = 1, 2)$, then we have the following equalities :

$$(6) \quad a(u_2 - u_1, u_2 - u_1) = (q_1 - q_2) \int_{\Gamma_2} (u_2 - u_1) \, d\gamma \ ,$$

$$(7) \quad a(u_2, u_2) - a(u_1, u_1) = a(u_2 + u_1, u_2 - u_1) = (q_1 + q_2) \int_{\Gamma_2} (u_1 - u_2) \, d\gamma \ ,$$

because we take $v = u_2 \in K$ in the variational equality corresponding to u_1, and $v = u_1 \in K$ in the one corresponding to u_2, and we add up and subtract both equalities. From (6) and (7) we obtain (2) and (3) [12].

Let $f : \mathbb{R}^+ \to \mathbb{R}$ be the real function defined by

$$(8) \qquad f(q) = J(u_q) = \frac{1}{2}\, a(u_q, u_q) + q \int_{\Gamma_2} u_q \, d\gamma \, .$$

REMARK 2.— To solve Problem P it is sufficient to find a value $q > 0$ for which we have $f(q) < 0$. We shall further see that this technique can still be improved.

THEOREM 2.— (i) The function f is differentiable. Moreover, f' is a continuous and strictly decreasing function, and it is given by the following expression

$$(9) \qquad f'(q) = \int_{\Gamma_2} u_q \, d\gamma \, .$$

(ii) There exists a geometric constant $C > 0$ such that

$$(10) \qquad a(u_q, u_q) = C\, q^2 \, ,$$

$$(11) \qquad f(q) = -\frac{C}{2}\, q^2 + B\, |\, \Gamma_2 \,|\, q \, .$$

(iii) If

$$(12) \qquad q > q_0(B) \, ,$$

then we obtain a two-phase, steady-state Stefan problem in Ω (i.e. u_q is a function of non-constant sign in Ω), where

(13) $$q_0(B) = \frac{B \mid \Gamma_2 \mid}{C} .$$

(iv) Constant $C = C(\Omega, \Gamma_1, \Gamma_2) > 0$ is given by

(14) $$C = a(u_3, u_3) = \int_{\Gamma_2} u_3 \, d\gamma ,$$

where u_3 is the solution of the variational equality

(15) $$u_3 \in V_0 , \quad a(u_3, v) = \int_{\Gamma_2} v \, d\gamma , \quad \forall v \in V_0 ,$$

PROOF.– See [12]

REMARK 3.– The sufficient condition $f(q) < 0$, to solve Problem P, was improved by the condition $f'(q) < 0$, which is optimal (see examples more later). In the case where, because of symmetry, we find that the function u_q is constant on Γ_2, the sufficient condition, given by (12), is also necessary to have a steady-state, two-phase Stefan problem.

III. NUMERICAL ANALYSIS OF MIXED ELLIPTIC PROBLEMS WITH OR WITHOUT PHASE CHANGE

Now, we consider τ_h, a regular triangulation of the polygonal domain Ω with Lagrange triangles of type 1, constituted by affine-equivalent finite element of class C^0, where $h > 0$ is a parameter which goes to zero. We can take h equal to the longest side of the triangles $T \in \tau_h$ and we can approximate V_0 by [2] :

(1) $$V_h = \left\{ v_h \in C^0(\bar{\Omega}) \,/\, v_h \mid_T \in P_1(T) , \forall T \in \tau_h , v_h \mid_{\Gamma_1} = 0 \right\},$$

where P_1 is the set of the polynomials of degree less than or equal to 1. Let Π_h be the corresponding linear interpolatation operator. Then, we can consider that there exists a constant $C_o > 0$ (independent of the parameter h) such that

$$(2) \qquad \| v - \Pi_h v \|_V \leq C_o h^{r-1} \| v \|_{r,\Omega} , \quad \forall v \in H^r(\Omega) \ , \text{with } 1 < r \leq 2 .$$

We consider the following finite dimensional approximate variational problem, corresponding to the continuous variational problem (I—7), given by :

$$(3) \qquad \left| \begin{array}{l} a(u_h, v_h) = L(v_h) , \quad \forall v_h \in V_h , \\ \\ u_h \in K_h = B + V_h , \end{array} \right.$$

and we can obtain the following results.

LEMMA 1 .— We have

$$(4) \qquad \lim_{h \to 0^+} \| u_h - u \|_V = 0 ,$$

where u is the unique solution of the variational equality (I—7).

PROOF .— Since meas$(\Gamma_1) > 0$, we have that the bilinear form a is coercive over V_o and therefore $\| \cdot \|_{V_o}$ and $\| \cdot \|_V$ are two equivalents norms in V_0 . We conclude the proof by following a method similar to the one developed in [2].

COROLLARY 2 .— If we define

$$(5) \qquad \theta_h = \frac{1}{k_2} u_h^+ - \frac{1}{k_1} u_h^- \in V \ , \ \theta = \frac{1}{k_2} u^+ - \frac{1}{k_1} u^- \in V ,$$

then, we have

(6) $$\lim_{h \to 0^+} \| \theta_h - \theta \|_H = 0 \ ,$$

where $H = L^2(\Omega)$.

PROOF .— See [13]

The goal of this part is to consider the discrete equivalent of the inequality (II—12). We study sufficient (and/or necessary) conditions on the constant heat flux q on Γ_2 to obtain a change of phase (steady-state, two-phase, discretized Stefan problem) in the corresponding discretized domain, that is a discrete temperature of non-constant sign in Ω. We obtain that (similarly to the continuous problem) :

(i) there exists a constant $C_h > 0$ (which depends only of the geometry of the domain Ω for each $h > 0$, and which is characterized by a variational problem) such that, if $q > q_{0h}(B) = B|\Gamma_2|/C_h$ then the steady-state discretized problem presents two phases.

(ii) we have the estimations $C_h < C$ and $q_0(B) < q_{0h}(B)$, where C and $q_0(B)$ are given for the continuous problem respectively by (II—14) and (II—13) .

(iii) we deduce error bounds for $C - C_h$ and $q_{0h}(B) - q_0(B)$ as functions of the parameter h.

In other words, we obtain for the mixed elliptic discretized problem, defined by u_h, analogous conditions to the ones obtained for the corresponding continuous problem [12], defined by u .

For each $q > 0$ we consider the functions $u(q) \in K$ and $u_h(q) \in K_h$, respectively, as the unique solution of the variational equalities (I—7) (continuous problem) and (3) (discrete problem). For each $h > 0$, we define the real function f_h : $\mathbf{R}^+ \to \mathbf{R}$, in the following way

(7) $$f_h(q) = J_q(u_h(q)) = \frac{1}{2} a(u_h(q), u_h(q)) + q \int_{\Gamma_2} u_h(q) \, d\gamma \ , \ q > 0 \ .$$

We obtain the following properties :

THEOREM 3.– (i) If $u_i = u_h(q_i)$ is the solution of (3) for $q_i > 0$ (i = 1, 2), then we have the following relations:

$$(8) \qquad a(u_2 - u_1 , u_2 - u_1) = (q_1 - q_2) \int_{\Gamma_2} (u_2 - u_1)\, d\gamma ,$$

$$(9) \qquad a(u_2, u_2) - a(u_1, u_1) = a(u_2 + u_1, u_2 - u_1) = (q_1 + q_2) \int_{\Gamma_2} (u_1 - u_2)\, d\gamma .$$

(ii) For all real numbers $q > 0$ and δ such that $(q + \delta) > 0$, we obtain the following estimations :

$$(10) \qquad \left\| \frac{1}{\delta}[u_h(q) - u_h(q + \delta)] \right\|_V \leq D_1 = \frac{\|\gamma_o\|}{\alpha} \, |\Gamma_2|^{1/2} ,$$

$$(11) \qquad \left\| \frac{1}{\delta}[u_h(q) - u_h(q + \delta)] \right\|_{L^2(\Gamma_2)} \leq D_2 = D_1 \|\gamma_o\| ,$$

where γ_o is the linear and continuous trace operator defined over V. Moreover, the function $\mathbf{R}^+ \to \mathbf{R}$

$$(12) \qquad q \to \int_{\Gamma_2} u_h(q)\, d\gamma \in \mathbf{R} ,$$

is a continuous and strictly decreasing function.

(iii) The function $f_h = f_h(q)$ is differentiable. Moreover, $f_h{}'$ is a continuous and strictly decreasing function given by the following expression

$$(13) \qquad f_h{}'(q) = \int_{\Gamma_2} u_h(q)\, d\gamma .$$

PROOF .— (i) If we take $v = u_2 - u_1 \in V_h$ in the variational equality corresponding to u_1 and $v = u_1 - u_2 \in V_h$ in the one corresponding to u_2, add and subtract the resulting relations, then we obtain, respectively (8) and (9) .

(ii) Taking into account (II−4), the Cauchy-Schwarz inequality, and the continuity of the operator γ_o, we deduce (14). From (14) and the continuity of γ_o we have (11). Therefore, we have (12) because

$$(14) \qquad \left| \int_{\Gamma_2} [u_h(q) - u_h(q+\delta)] \, d\gamma \right| \le D_2 \, |\Gamma_2|^{1/2} \, \delta \, .$$

Moreover, the monotony property is a consequence of (8).

(iii) From (7) and elementary computations, we deduce

$$(15) \qquad \frac{1}{\delta} [f_h (q+\delta) - f_h(q)] = \frac{1}{2} \int_{\Gamma_2} [u_h(q) + u_h(q+\delta)] \, d\gamma \, ,$$

that is (13), by using (12).

THEOREM 4 .— (i) The element $u_h = u_h(q) \in V_h$ can be written as

$$(16) \qquad u_h(q) = B - q \, u_{3h}$$

where u_{3h} is the unique solution of the variational equality

$$(17) \qquad \left| \begin{array}{l} a(u_{3h}, v_h) = \int_{\Gamma_2} v_h \, d\gamma \, , \quad \forall \, v_h \in V_h \, , \\ u_{3h} \in V_h \, . \end{array} \right.$$

(ii) There exists a constant $C_h > 0$ such that

$$(18) \qquad f_h(q) = q \, B \, | \Gamma_2 | - \frac{1}{2} C_h \, q^2 \, , \; \forall \, q > 0 \, ,$$

$$(19) \qquad a(u_h(q), u_h(q)) = C_h \, q^2 \, , \; \forall \, q > 0 \, .$$

where the constant C_h is given by

$$C_h = a(u_{3_h}, u_{3_h}) = \int_{\Gamma_2} u_{3_h} \, d\gamma \,. \tag{20}$$

(iii) If

$$q > q_{0_h}(B) \,, \tag{21}$$

then problem (3) represents a discretized steady-state two-phase Stefan problem (i.e. $u_h(q)$ is a function of non-constant sign in Ω), where

$$q_{0_h}(B) = \frac{B \, |\Gamma_2|}{C_h} \,. \tag{22}$$

PROOF.— (i) It follows from (3), (7) and (16) by uniqueness of the variational equalities (3) and (17);

(ii) It follows from (7) and (16);

(iii) It follows taking into account

$$f_h'(q_{0_h}(B) = 0 \,, \tag{23}$$

and the monotony property of the function f_h'.

THEOREM 5 .— (i) We have the following equality :

$$a(u(q), u_h(q)) = C_h \, q^2 \,, \ \forall \, q > 0 \,. \tag{24}$$

(ii) Also, we have the following inequalities :

$$\text{(a) } C_h < C \,, \qquad \text{(b) } q_0(B) < q_{0_h}(B) \,. \tag{25}$$

PROOF.— (i) If we take $v = u_h(q) \in K_h = B + V_h \subset B + V_0 = K$ in the variational equality (I−7), and we take into account the expressions (II−10) and (18), then we obtain (24).

(ii) On the other hand, from (II−4) and (24) we have

$$(26)\quad \alpha \, \| u(q) - u_h(q) \|_V^2 \leq a(u(q) - u_h(q), u(q) - u_h(q)) = (C - C_h)\, q^2 ,$$

that is (25a). Moreover, (25b) follows from (II−13), (22) and (25a).

Now we shall use the interpolation result (2) for the function $u_3 \in H^r(\Omega)$, as a hypothesis of regularity of the continuous problem (I − 7) (in general, $1 < r < \frac{3}{2}$ [5, 7, 8]). In [11], we present three examples with explicit solution were presented. In those cases, $u(q)$, $u_3 \in C^\infty(\Omega)$.

THEOREM 6 .— We have the following relations and estimations :

$$(27)\qquad a(u(q) - u_h(q), v_h) = 0 \ , \ \forall \, v_h \in V_h ,$$

$$(28)\qquad (C - C_h)\, q^2 = a(u(q) - u_h(q), u(q) - u_h(q)) \leq$$

$$\leq \inf_{v_h \in V_h} a(u(q) - v_h \, , u(q) - v_h) ,$$

$$(29)\qquad 0 < C - C_h \leq C_o^2 \, h^{2(r-1)} \, | u_3 |^2_{r,\Omega} ,$$

$$(30)\qquad 0 < q_{0_h}(B) - q_0(B) \leq \frac{C_o^2 \, h^{2(r-1)}}{C} \, | u_3 |^2_{r,\Omega} \, q_{0_h}(B) .$$

PROOF.— See [13].

REMARK 1 .— If we only have $u(q) \in V$ (i.e. $u_3 \in V$), we can obtain

$$(31)\quad 0 < C - C_h \le \frac{1}{q^2} \| u(q) - \Pi_h(u(q)) \|_V^2 = \| u_3 - \Pi_h(u_3) \|_V^2 ,$$

where the second term converges to zero when $h \to 0^+$ [2] , but we cannot give an order of convergence.

REMARK 2.– If the constant heat flux on Γ_2 verifies the inequality $q > q_{0_h}(B)$, then both the discrete and continuous problems represent steady-state, two-phase, Stefan problems, that is, their temperatures are of non-constant sign in Ω.

REMARK 3.– When the function $u_h(q)$ is constant on Γ_2 (as a function of $x \in \Gamma_2$), then the sufficient condition given by (21) is also necessary in order to have a two-phase discrete problem, because

$$(32)\qquad \int_{\Gamma_2} u_h(q)\, d\gamma < 0 \quad \Leftrightarrow \quad u_h(q) < 0 \ \text{ on } \ \Gamma_2 .$$

THEOREM 7.– If we let h, $B > 0$, and $0 < \epsilon_0 < 1$ (ϵ_0 is a parameter to be chosen arbitrarily), then we have the following estimations :

$$(33)\qquad q_0(B) < q_{0_h}(B) \le \frac{q_0(B)}{\epsilon_0} \quad \text{and} \quad C_h \ge C\ \epsilon_0 , \quad \forall\ h \le h_r(\epsilon_0),$$

$$(34)\quad 0 < q_{0_h}(B) - q_0(B) \le \frac{C_0^2 \ | u_3 |^2_{r,\Omega}}{C\ \epsilon_0}\ q_0(B)\ h^{2(r-1)} , \quad \forall\ h \le h_r(\epsilon_0),$$

where

$$(35)\qquad h_r(\epsilon_0) = \left(\frac{C\,(1 - \epsilon_0)}{C_0^2\ | u_3 |^2_{r,\Omega}} \right)^{\frac{1}{2(r-1)}} .$$

PROOF .– From (30) we deduce

(36) $$A(h)\, q_{0_h}(B) \le q_0(B)\,,$$

where

(37) $$A(h) = 1 - \frac{C_0^2 \ | u_3 |^2_{r,\Omega}}{C} \ h^{2(r-1)} < 1\ .$$

If we consider, for each ϵ_0, $0 < \epsilon_0 < 1$, the following equivalence :

(38) $$0 < \epsilon_0 < A(h) < 1 \quad \Leftrightarrow \quad 0 < h < h_r(\epsilon_0)\,,$$

then we deduce the inequalities (33) and (34).

<u>COROLLARY 8</u> .— If $B > 0$, then we have the following limit

(39) $$\lim_{h \to 0^+} q_{0_h}(B) = q_0(B)\ .$$

<u>REMARK 4</u>.— Everything we proved in this paper is still valid if the boundary Γ of the bounded domain Ω is represented by the union of three portions ($\Gamma = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3$) having the following characteristics :

(i) Γ_1 and Γ_2 have the same conditions as the ones previously described in (I−4),

(ii) Γ_3 is a wall impermeable to heat, i.e. we have $\frac{\partial \theta}{\partial n} \mid_{\Gamma_3} = 0$ in (I−4) and therefore $\frac{\partial u}{\partial n} \mid_{\Gamma_3} = 0$ in (I−6).

Moreover, the first example considered (see below) verifies this condition.

We shall give three examples in which the solution is explicitly known [11] so that we can verify all the theoretical results obtained in this work.

<u>Example 1</u>.— We consider the following data

$$
(40) \quad \left|
\begin{array}{l}
n = 2 \quad , \ \Omega = (0, x_0) \times (0, y_0) \quad , \ x_0 > 0 \ , \ y_0 > 0 \quad , \\
\Gamma_1 = \{0\} \times [0, y_0] \quad , \quad \Gamma_2 = \{x_0\} \times [0, y_0] \quad , \\
\Gamma_3 = (0, x_0) \times \{0\} \ \cup \ (0, x_0) \times \{y_0\} \ .
\end{array}
\right.
$$

Example 2.— Next we consider

$$
(41) \quad \left|
\begin{array}{l}
n = 2 \quad , \quad 0 < r_1 < r_2 \quad , \ \Gamma_3 = \phi \ , \\
\Omega : \text{annulus of radius } r_1 \text{ and } r_2 \text{ , centered at } (0, 0) \quad , \\
\Gamma_1 : \text{circumference of radius } r_1 \text{ and center } (0, 0) \quad , \\
\Gamma_2 : \text{circumference of radius } r_2 \text{ and center } (0,0) \quad .
\end{array}
\right.
$$

Example 3.— Finally, we take into account the same information of Example 2 but now for the case $n = 3$.

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