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NEUMANN-LIKE SOLUTION FOR THE TWO-PHASE STEFAN PROBLEM WITH A SIMPLE MUSHY ZONE MODEL

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ABSTRACT: *We obtain an exact solution of the Neumann type for a simple mushy zone model with parameters ($\gamma > 0$ and $0 < \varepsilon < 1$) for the two-phase Stefan problem for a semi-infinite material with mass densities equal in both solid and liquid phases, and constant initial ($\theta_0 > 0$) and boundary ($-D < 0$) temperature. We suppose that, without loss of generality, the phase-change temperature is $0°C$. We generalize the Solomon-Wilson-Alexiades model given for the one-phase Lamé-Clapeyron (Stefan) problem to the two-phase case.*

KEY WORDS: Stefan problem, similarity variable, Neumann solution, phase-change problem, free boundary problems, exact solutions, mushy zone.

RESUMO: *SOLUÇÃO DO TIPO DE NEUMANN PARA O PROBLEMA BIFÁSICO DE STEFAN COM UM MODELO DE ZONA PASTOSA SIMPLES. Obtemos uma solução exata do tipo de Neumann para um modelo de zona pastosa simples com parâmetro ($\gamma > 0$ e $0 < \varepsilon < 1$) para o problema bifásico de Stefan para um material semi-infinito com densidade igual em ambas as fases líquidas e sólidas, e constante inicial ($\theta_0 > 0$) e temperatura de fronteira ($-D < 0$). Supomos que, sem perda de generalidade, a temperatura de mudança de fase é de $0°C$. Generalizamos o modelo de Solomon-Wilson-Alexiades referente ao problema de uma fase de Lamé-Clapeyron (Stefan) para o caso bifásico.*

PALAVRAS-CHAVE: problema de Stefan, similaridade variável, solução de Neumann, problema de mudança de fase, problemas de fronteira livre, soluções exatas, zonas pastosas.

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1. INTRODUCTION

We consider a semi-infinite material with mass density equal in both solid and liquid phases and the phase-change temperature at 0°C. We generalize the mushy zone model given for the one-phase Lamé-Clapeyron (Stefan) problem in [6] (see also [10]) to the two-phase case. Three distinct regions can be distinguished, as follows:

(H_1) The liquid phase, at temperature $\theta_2 = \theta_2(x,t) > 0$, occupying the region $x > r(t)$, $t > 0$.

(H_2) The solid phase, at temperature $\theta_1 = \theta_1(x,t) < 0$, occupying the region $0 < x < s(t)$, $t > 0$.

(H_3) The mushy zone, at temperature 0, occupying the region $s(t) < x < r(t)$, $t > 0$.

We make the following two assumptions on its structure following the paraffin case [6, 11 (with experimental results)] (the parameter ε and γ are characteristics of the phase-change material):

(a) The material in the mushy zone contains a fixed fraction εh (with constant $0 < \varepsilon < 1$) of the total latent heat h.

(b) The width of the mushy zone is inversely proportional (with constant $\gamma > 0$) to the temperature gradient at the point $(s^-(t), t)$.

If the phase-change semi-infinite material is initially in liquid phase at the constant temperature $\theta_0 > 0$ and a constant temperature $-D < 0$ is imposed on the fixed face $x = 0$, then we obtain the following results:

(i) An exact solution of the Neumann type for $\theta_1(x,t)$, $\theta_2(x,t)$, $s(t)$ and $r(t)$ as functions of the initial and boundary temperature θ_0 and D, mushy zone parameters ε and γ, and thermal coefficients of the material.

(ii) An analogous property of (i) if we replace in the hypothesis (H_3b) the temperature gradient at the point $(s^-(t), t)$ (i.e. $\theta_{1_x}(s(t), t)$) by the temperature gradient at the point $(s^+(t), t)$ (i.e. $\theta_{2_x}(r(t), t)$).

If we replace the constant temperature $-D < 0$ by a heat flux of type $q_0 t^{-\frac{1}{2}}$ (with $q_0 > 0$) on the fixed face $x = 0$, then we obtain the following results:

(iii) There exists an exact solution $\theta_1^*(x,t), \theta_2^*(x,t), s^*(t)$ and $r^*(t)$ of the Neumann type of the mushy zone model, as functions of $\theta_0, q_0, \varepsilon, \gamma$ and the thermal coefficients of the material, if and only if the coefficient q_0 satisfies the inequality

$$q_0 > \frac{\gamma\, k_1}{2\, a_2\, \eta_0}, \tag{1}$$

where $\eta_0 = \eta_0(\varepsilon, \gamma, \theta_0, h, k_1, k_2, c_2) = \eta_0 \left(\frac{\theta_0\, c_2}{h(1-\varepsilon)}, \frac{\gamma\, k_1\, c_2}{h\, k_2 (1-\varepsilon)} \right) > 0$ is the unique positive zero of a given function G (see definition in eq. (64)). Moreover, for the solution given in (i), the inequality for q_0 turns into

$$\operatorname{erf}\left(\frac{\sigma}{a_1}\right) < \frac{2\, D\, \eta_0}{\gamma} \left(\frac{k_2\, c_1}{\pi\, k_1\, c_2}\right)^{1/2}, \tag{2}$$

where $\sigma > 0$ is the coefficient that characterizes the first free boundary $s(t) = 2\sigma\sqrt{t}$ of the two-phase mushy zone model.

(iv) If $q_0 = \frac{\gamma\, k_1}{2\, a_2\, \eta_0}$, then there exists an exact solution for $\theta_2^*(x,t), r^*(t)$ and $s^*(t) = 0$ for the corresponding one-phase mushy zone model (the solid phase there does not exist). If $0 < q_0 < \frac{\gamma\, k_1}{2\, a_2\, \eta_0}$, then there does not exist an exact solution of the Neumann type for the corresponding mushy zone model. Moreover, for the particular case $\gamma = 0$, that is the mushy zone model is identical to the classical Neumann model [2, 5, 9], we find the inequality $q_0 > \frac{k_2\, \theta_0}{a_2\, \sqrt{\pi}}$ [8] to obtain a phase-change problem.

A review of the mushy zone model for the unidimensional Stefan problem can be found in [3].

2. NEUMANN-LIKE SOLUTION FOR THE TWO-PHASE MUSHY ZONE MODEL WITH TEMPERATURE CONDITION ON THE FIXED CASE $x = 0$

Taking into account the hypothesis (H$_1$)-(H$_3$) we can formulate the following Problem (P$_1$): Find the free boundaries $x = s(t)$ and $x = r(t)$, defined for $t > 0$ with $s(t) < r(t)$ and $s(0) = r(0) = 0$, and the temperature $\theta = \theta(x,t)$, defined for $x > 0$ and $t > 0$ by

$$\theta(x,t) = \begin{cases} \theta_1(x,t) < 0 & \text{if} \quad 0 < x < s(t),\ t > 0, \\ 0 & \text{if} \quad s(t) \le x \le r(t),\ t > 0, \\ \theta_2(x,t) > 0 & \text{if} \quad r(t) < x,\ t > 0, \end{cases} \tag{3}$$

such that they satisfy the following conditions.

$$\alpha_1\, \theta_{1_{xx}} = \theta_{1_t}, \quad 0 < x < s(t),\ t > 0, \tag{4}$$

$$\alpha_2\, \theta_{2_{xx}} = \theta_{2_t}, \quad r(t) < x,\ t > 0, \tag{5}$$

$$s(0) = r(0) = 0, \tag{6}$$

$$\theta_1(s(t),t) = \theta_2(r(t),t) = 0,\ t > 0, \tag{7}$$

$$k_1\, \theta_{1_x}(s(t),t) - k_2\, \theta_{2_x}(r(t),t) = \rho h[(1-\varepsilon)\dot{r}(t) + \varepsilon\, \dot{s}(t)],\ t > 0, \tag{8}$$

$$\theta_{1_x}(s(t),t)(r(t) - s(t)) = \gamma,\ t > 0, \tag{9}$$

$$\theta_2(x,0) = \theta_2(+\infty,t) = \theta_0 > 0,\ x > 0,\ t > 0, \tag{10}$$

$$\theta_1(0,t) = -D < 0,\ t > 0, \tag{11}$$

where $h > 0$ is the latent heat of fusion, $\rho > 0$ is the mass density equal in both solid and liquid phases, and $k_i > 0$, $c_i > 0$, $\alpha_i = a_i^2 = \frac{k_i}{\rho\, c_i} > 0$ are the thermal conductivity, the specific heat and the diffusion coefficient for the phase i ($i = 1$: solid phase, $i = 2$: liquid phase) respectively. The condition (8) is the Stefan condition (energy conservation) corresponding to the hypothesis (H_3a) and the condition (9) concerns to the hypothesis (H_3b) [6, 10, 11].

Following the Neumann method [2, 5, 9] we propose for the Problem (P_1) the functions

$$\theta_1(x,t) = A_1 + B_1 f\left(\frac{x}{2\, a_1 \sqrt{t}}\right), \tag{12}$$

$$\theta_2(x,t) = A_2 + B_2 f\left(\frac{x}{2\, a_2 \sqrt{t}}\right), \tag{13}$$

$$s(t) = 2\, \sigma \sqrt{t}, \tag{14}$$

$$r(t) = 2\, \omega \sqrt{t}, \tag{15}$$

which satisfy condition (4)-(6), where

$$f(x) = \text{erf}\,(x) = \frac{2}{\sqrt{\pi}} \int_0^x \exp\,(-t^2) dt \tag{16}$$

is the error function. If we impose the other six conditions (7)-(11), we obtain that the 5 coefficients A_1, A_2, B_1, B_2 and ω are given, as functions of the coefficient σ, as follows.

$$A_1 = -D, \qquad B_1 = \frac{D}{f\left(\frac{\sigma}{a_1}\right)}, \tag{17}$$

$$A_2 = -\frac{\theta_0 f\left(\frac{\omega}{a_2}\right)}{1 - f\left(\frac{\omega}{a_2}\right)}, \qquad B_2 = \frac{\theta_0}{1 - f\left(\frac{\omega}{a_2}\right)}, \tag{18}$$

$$\omega = \omega(\sigma) = \sigma + \frac{\gamma\, a_1 \sqrt{\pi}}{2D}\, f\left(\frac{\sigma}{a_1}\right)\, \exp\left(\frac{\sigma^2}{a_1^2}\right) = a_1 W\left(\frac{\sigma}{a_1}\right), \tag{19}$$

where function W is defined by

$$W(x) = x + \frac{\gamma \sqrt{\pi}}{2D} f(x) \exp(x^2), \quad x > 0. \tag{20}$$

The coefficient

$$\varepsilon_1 = \frac{\sigma}{a_1} > 0 \tag{21}$$

must satisfy the following dimensionless equation.

$$F_0(x) = G_1(x), \quad x > 0, \tag{22}$$

where

$$F_1(x) = \frac{\exp(-x^2)}{1-f(x)}, \qquad F_2(x) = \frac{\exp(-x^2)}{f(x)}, \tag{23}$$

$$F_0(x) = \frac{D\,c_1}{h\sqrt{\pi}}\,F_2(x) - \frac{\theta_0}{h}\sqrt{\frac{k_2\,c_1\,c_2}{\pi\,k_1}}\,F_1\left(\frac{a_1}{a_2}W(x)\right), \tag{24}$$

$$G_1(x) = x + \frac{(1-\varepsilon)\gamma\sqrt{\pi}}{2D}f(x)\exp(x^2) = x + (1-\varepsilon)[W(x)-x]. \tag{25}$$

Taking into account that those functions F_1, F_2, G_1, W and F are such that

$$F_1(0^+) = 1, \qquad F_1(+\infty) = +\infty, \qquad F_1' > 0, \tag{26}$$

$$F_2(0^+) = +\infty, \qquad F_2(+\infty) = 0, \qquad F_2' < 0, \tag{27}$$

$$G_1(0^+) = 0, \qquad G_1(+\infty) = +\infty, \qquad G_1' > 0, \tag{28}$$

$$W(0^+) = 0, \qquad W(+\infty) = +\infty, \qquad W' > 0, \tag{29}$$

$$F_0(0^+) = +\infty, \qquad F_0(+\infty) = -\infty, \qquad F_0' < 0, \tag{30}$$

we deduce that equation (22) has a unique solution $\varepsilon_1 > 0$ (that is a unique $\sigma > 0$), and therefore we obtain the following result.

Theorem 1. *For any data $\theta_0, D > 0$, for any mushy zone coefficients $0 < \varepsilon < 1$ and $\gamma > 0$, and for any thermal coefficients of the phase change material $\rho, h, k_1, k_2, c_1, c_2 > 0$, Problem* (P$_1$) *has a unique solution of the Neumann type* (12)-(15) *and* (17)-(19)*, where the coefficient σ is given by $\sigma = a_1\varepsilon_1$ and $\varepsilon_1 > 0$ is the unique solution of equation* (22).

If we replace in the hypothesis (H$_3$b) the temperature gradient at the point $(s^-(t), t)$ (i.e. $\theta_{1_x}(s(t), t)$) by the temperature gradient at the point $(r^+(t), t)$ (i.e. $\theta_{2_x}(r(t), t)$), that is condition (9) is replaced by

$$\theta_{2_x}(r(t), t)(r(t) - s(t)) = \gamma, \qquad t > 0, \tag{31}$$

we can formulate the following Problem (P$_2$): Find the free boundaries $x = s(t)$ and $x = r(t)$, defined for $t > 0$ with $0 < s(t) < r(t)$ and $s(0) = r(0) = 0$, and the temperature $\theta = \theta(x, t)$, defined for $x > 0$ and $t > 0$ by (3), such that they satisfy the conditions (4)-(8), (10), (11) and (31).

Following the method, given before, we propose for the Problem (P$_2$) the functions (12)-(15), where the coefficients A_1, B_1 and A_2, B_2 are given, as functions of ω, by (17) and (18) respectively, the coefficient σ is given by

$$\sigma = \sigma(\omega) = \omega - \frac{\gamma\,a_2\sqrt{\pi}}{2\,\theta_0}g\left(\frac{\omega}{a_2}\right) = a_2\,g_2\left(\frac{\omega}{a_2}\right), \tag{32}$$

with the condition $\sigma(\omega) > 0$, where

$$g(x) = \exp(x^2)(1 - f(x)) = \frac{1}{F_1(x)}, \tag{33}$$

$$g_2(x) = x - \frac{\gamma\sqrt{\pi}}{2\,\theta_0}g(x). \tag{34}$$

The equation for the coefficient

$$\varepsilon_2 = \frac{\omega}{a_2} > 0 \tag{35}$$

is given, through condition (8), as follows.

$$H_0(x) = G_2(x), \qquad g_2(x) > 0,\ x > 0, \tag{36}$$

where

$$G_2(x) = x - \frac{\varepsilon\,\gamma\sqrt{\pi}}{2\,\theta_0}\,g(x), \tag{37}$$

$$H_0(x) = \frac{D}{h}\sqrt{\frac{k_1\,c_1\,c_2}{\pi\,k_2}}\,F_2\left(\frac{a_2}{a_1}\,g_2(x)\right) - \frac{c_2\,\theta_0}{h\sqrt{\pi}}\,F_1(x). \tag{38}$$

Moreover, for the trivial condition $\sigma = \sigma(\omega)$ (i.e. $s(t) > 0$), we have the following equivalence:

$$\sigma(\omega) > 0 \qquad \Longleftrightarrow \qquad g_2(\varepsilon_2) > 0 \qquad \Longleftrightarrow \qquad \varepsilon_2 > x_2, \tag{39}$$

where $x_2 > 0$ is the unique positive zero of function g_2, which is such that

$$g_2(0^+) = -\frac{\gamma\,\sqrt{\pi}}{2\,\theta_0} < 0, \qquad g_2(+\infty) = +\infty,\ g_2' > 0, \tag{40}$$

because

$$g(0^+) = 1, \qquad g(+\infty) = 0,\ g' < 0. \tag{41}$$

Function G_2 has the following properties.

$$G_2(0^+) = -\frac{\varepsilon\,\gamma\sqrt{\pi}}{2\,\theta_0} < 0, \qquad G_2(+\infty) = +\infty,\ G_2' > 0, \tag{42}$$

and therefore it has a unique positive zero $y_2 > 0$. Owing to $g_2(x_2) = 0$, that is $g(x_2) = \frac{2\,\theta_0\,x_2}{\gamma\sqrt{\pi}}$, we deduce that $G_2(x_2) = (1-\varepsilon)\,x_2 > 0$, i.e.

$$x_2 > y_2. \tag{43}$$

Function H_0 is such that

$$H_0(x_2^+) = +\infty, \quad H_0(+\infty) = -\infty, \quad H_0' < 0 \ \text{ in } \ (x_2, +\infty), \tag{44}$$

and therefore equation (36) which is equivalent to

$$H_0(x) = G_2(x), \quad x > x_2, \tag{45}$$

has a unique solution $\varepsilon_2 > x_2 > 0$. Then, we have obtained the following result:

Theorem 2. *For any data $\theta_0, D > 0$, for any mushy zone coefficients $0 < \varepsilon < 1$ and $\gamma > 0$, and for any thermal coefficients of the phase change material $\rho, h, k_1, k_2, c_1, c_2 > 0$, Problem* (P$_2$) *has a unique solution of the Neumann type* (12)-(15) *and* (17),(18),(32), *where the coefficient ω is given by $\omega = a_2 \varepsilon_2$ and $\varepsilon_2 > 0$ is the unique solution of equation* (36) *or its equivalent* (45).

Remark 1. We obtain the same results, given by Theorem 1 (Theorem 2), if we replace the temperature gradient by the heat flux at the point $(s^-(t), t)((r^+(t), t)$.

Remark 2. It is an open problem to connect the exact solution, given by Theorem 1 or 2, with the mathematical theory developed in [1, 4] through similarity solutions.

4. EXACT SOLUTION FOR THE TWO-PHASE MUSHY ZONE MODEL WITH HEAT FLUX CONDITION ON THE FIXED FACE $x = 0$

Following the idea developed in [8], we replace condition (11) by a heat flux of the type $q_0 t^{-\frac{1}{2}}$ (with $q_0 > 0$) on the fixed $x = 0$, that is

$$k_1\, \theta_{1_x}(0,t) = \frac{q_0}{\sqrt{t}}, \qquad t > 0. \tag{46}$$

Then we can formulate the following Problem (P$_3$): Find the free boundaries $x = s^*(t)$ and $x = r^*(t)$, defined for $t > 0$ with $0 < s^*(t) < r^*(t)$ and $s^*(0) = r^*(0) = 0$, and the temperature $\theta^* = \theta^*(x,t)$, defined for $x > 0$ and $t > 0$ by

$$\theta^*(x,t) = \begin{cases} \theta_1^*(x,t) < 0 & \text{if} \quad 0 < x < s^*(t),\ t > 0, \\ 0 & \text{if} \quad s^*(t) \le x \le r^*(t),\ t > 0, \\ \theta_2^*(x,t) > 0 & \text{if} \quad r^*(t) < x,\ t > 0, \end{cases} \tag{47}$$

such that they satisfy the conditions (4)-(10) and (46).

Theorem 3. *There exists an exact solution of the Neumann type for the Problem* (P$_3$), *determined by the data $\theta_0, q_0, \varepsilon, \gamma$ and thermal coefficients of the phase change material, if and only if the coefficient q_0 satisfies the inequality* (1). *In this case, the solution is given by*

$$\theta_1^*(x,t) = A_1^* + B_1^* f\left(\frac{x}{2\, a_1\, \sqrt{t}}\right), \tag{48}$$

$$\theta_2^*(x,t) = A_2^* + B_2^* f\left(\frac{x}{2\, a_2\, \sqrt{t}}\right), \tag{49}$$

$$s^*(t) = 2\, \sigma^* \sqrt{t}, \tag{50}$$

$$r^*(t) = 2\, \omega^* \sqrt{t}, \tag{51}$$

where the five coefficients $A_1^*, A_2^*, B_1^*, B_2^*$ and ω^* are given, as functions of the coefficients σ^*, by

$$A_1^* = -\frac{a_1\, q_0 \sqrt{\pi}}{k_1} f\left(\frac{\sigma^*}{a_1}\right), \qquad B_1^* = \frac{a_1\, q_0 \sqrt{\pi}}{k_1}, \tag{52}$$

$$A_2^* = -\frac{\theta_0 f\left(\frac{\omega^*}{a_2}\right)}{1 - f\left(\frac{\omega^*}{a_2}\right)}, \qquad B_2^* = \frac{\theta_0}{1 - f\left(\frac{\omega^*}{a_2}\right)}, \tag{53}$$

$$\omega^* = \omega^*(\sigma^*) = \sigma^* + \frac{\gamma\, k_1}{2\, q_0} \exp\left(\frac{\sigma^{*2}}{a_1^2}\right) = a_1 W_1\left(\frac{\sigma^*}{a_1}\right), \tag{54}$$

where

$$W_1(x) = x + \frac{\gamma\, k_1}{2\, q_0\, a_1} \cancel{D(x)} \exp(x^2). \tag{55}$$

The coefficient $\sigma^* > 0$ is given by $\sigma^* = a_1 \varepsilon_1^*$ and $\varepsilon_1^* > 0$ is the unique solution of the equation

$$H(x) = H_1(x), \quad x > 0, \tag{56}$$

where

$$H_1(x) = x + \frac{(1-\varepsilon)\gamma\, k_1}{2\, q_0\, a_1} \exp(x^2), \tag{57}$$

$$H(x) = \frac{q_0}{\rho\, h\, a_1} \exp(-x^2) - \frac{\theta_0}{h} \sqrt{\frac{k_2\, c_1\, c_2}{\pi\, k_1}}\, F_1\left(\frac{a_1}{a_2} W_1(x)\right). \tag{58}$$

Proof. Owing to the method given in [8] and Section II, it is enough to prove that equation (56) has a unique solution $\varepsilon_1^* > 0$. The function W_1, H_1 and H are such that

$$W_1(0^+) = \frac{\gamma\, k_1}{2\, q_0\, a_1} > 0, \qquad W_1(+\infty) = +\infty,\ W_1' > 0, \tag{59}$$

$$H_1(0^+) = \frac{(1-\varepsilon)\gamma\, k_1}{2\, q_0\, a_1} > 0, \qquad H_1(+\infty) = +\infty,\ H_1' > 0, \tag{60}$$

$$\begin{cases} H(0^+) = \frac{q_0}{\rho\, h\, a_1} - \frac{\theta_0}{h}\sqrt{\frac{k_2\, c_1\, c_2}{\pi\, k_1}}\, F_1\left(\frac{\gamma\, k_1}{2\, q_0\, a_2}\right), \\ H(+\infty) = -\infty, \qquad H' < 0, \end{cases} \tag{61}$$

If we define the dimensionless auxiliary variable

$$\eta = \frac{\gamma\, k_1}{2\, q_0\, a_2}, \tag{62}$$

then equation (56) has a unique solution $\varepsilon_1^* > 0$ if and only if

$$H(0^+) > H_1(0^+) \iff G(\eta) < 0 \iff 0 < \eta < \eta_0 \iff q_0 > \frac{\gamma\, k_1}{2\, a_2\, \eta_0} \iff \text{(i)} \tag{63}$$

where $\eta_0 = \eta_0(\varepsilon, \gamma, \theta_0, h, k_1, k_2, c_2) = \eta_0\left(\frac{\theta_0\, c_2}{h(1-\varepsilon)}, \frac{\gamma\, k_1\, c_2}{h\, k_2(1-\varepsilon)}\right) > 0$ is the unique positive zero of the function G, defined by

$$G(x) = x + \frac{\theta_0\, c_2}{h(1-\varepsilon)\sqrt{\pi}}\, F_1(x) - \frac{\gamma\, k_1\, c_2}{2\, h\, k_2(1-\varepsilon)}\, \frac{1}{x}, \tag{64}$$

which satisfies the following properties.

$$G(0^+) = -\infty, \qquad G(+\infty) = +\infty, \quad G' > 0. \quad \square \tag{65}$$

Remark 3. If q_0 verifies the equality

$$q_0 = \frac{\gamma\, k_1}{2\, a_2\, \eta_0}, \tag{66}$$

then we have

$$\sigma^* = 0, \qquad \omega^* = \frac{\gamma\, k_1}{2\, q_0} = a_2\, \eta_0, \tag{67}$$

that is,

$$s^*(t) = 0, \qquad r^*(t) = 2\, a_2\, \eta_0 \sqrt{t}. \tag{68}$$

Therefore, there exists an exact solution for $\theta_2^*(x,t), r^*(t)$ and $s^*(t) = 0$ (the solid phase there does not exist) for the corresponding one-phase mushy zone model.

Remark 4. If q_0 satisfies the inequalities

$$0 < q_0 < \frac{\gamma\, k_1}{2\, a_2\, \eta_0}, \tag{69}$$

then there does not exist an exact solution of the Neumann type for the corresponding mushy zone model. It is only a conduction heat problem for the initial liquid phase.

Remark 5. In the particular case $\gamma = 0$ (no mushy region), we have

$$H_1(0^+) = 0, \qquad H(0^+) = \frac{1}{\rho\, h\, a_1}\left(q_0 - \frac{k_2\, \theta_0}{a_2\sqrt{\pi}}\right) \tag{70}$$

that is, the inequality (1) is given now by

$$q_0 > \frac{k_2\, \theta_0}{a_2\sqrt{\pi}}, \tag{71}$$

which is the necessary and sufficient condition for q_0 to obtain the classical phase-change problem [8].

Remark 6. The inequality (1) always implies inequality (71), which has a physical meaning.

Remark 7. The method applied in [7] to obtain the inequality (71) for the classical two-phase Stefan problem is not applicable for the present mushy zone model because this method (through an auxiliary heat conduction problem) can not take into account any mushy zone model.

4. ANOTHER PROPERTY FOR THE SOLUTION OF PROBLEM (P_1)

If the coefficient $q_0 > 0$ in the condition (46) for Problem (P$_3$) satisfies the inequality (1) and then we consider the boundary temperature (11) on the fixed face $x = 0$ where D is given, as a function of q_0, by the expression

$$D = \frac{a_1\, q_0 \sqrt{\pi}}{k_1}\, f(\varepsilon_1^*) > 0, \tag{72}$$

we obtain the following result.

Theorem 4. *If the coefficient $q_0 > 0$ satisfies the inequality* (1) *then Problem* (P$_1$) *is equivalent to Problem* (P$_3$). *Moreover, the coefficient σ of the free boundary $s(t)$, given by* (14), *corresponding to the Problem* (P$_1$) *satisfies the inequality* (2).

Proof. Following the method developed in [8], it is enough to prove that, after some manipulations, we have

(i) $W(\varepsilon_1^*) = W_1(\varepsilon_1^*)$,

(ii) $F_0(\varepsilon_1^*) = H(\varepsilon_1^*)$,

(iii) $G_1(\varepsilon_1^*) = H_1(\varepsilon_1^*)$,

(iv) $\varepsilon_1 = \varepsilon_1^*$,

(v) $\sigma = \sigma^*$,

(vi) $\omega = \omega^*$,

(vii) $\theta_1 = \theta_1^*, \ \ \theta_2 = \theta_2^*, \ \ s = s^*, \ \ r = r^*$.

Therefore, under the hypothesis (46) for the coefficient q_0, Problem (P$_1$) is equivalent to Problem (P$_3$). Then, inequality (2) is obtained by using (21), (46), (72)and (iv). □

Remark 8. (Continuation of Remark 5). In the particular case $\gamma = 0$, the inequality (2) is given by

$$\operatorname{erf}\left(\frac{\sigma}{a_1}\right) < \frac{D}{\theta_0}\sqrt{\frac{k_1\, c_1}{k_2\, c_2}} \tag{73}$$

which is obviously a nontrivial inequality when the right hand side is less than 1 [8].

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