Fractional Dynamics in Natural Phenomena and Advanced Technologies

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CHAPTER 9

A SOLUTION TO A ONE-DIMENSIONAL TWO-PHASE FRACTIONAL STEFAN-LIKE PROBLEM WITH A CONVECTIVE BOUNDARY CONDITIONS AT THE FIXED FACE

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1. Introduction

In recent years, some works on the fractional Lamé-Clapeyron-Stefan problem (called in the literature as Stefan problem) were published (Atkinson, 2012; Ceretani, 2020; Ceretani and Tarzia, 2017; Falcini, Garra and Voller, 2013; Jinyi and Mingyu, 2009; Kholpanov, Zhalev and Fedotov, 2003; Roscani and Santillan Marcus, 2013 and 2014; Roscani, Bollati and Tarzia, 2018; Roscani, Caruso and Tarzia, 2020; Roscani and Tarzia, 2014, 2018a, 2018b, 2018c; Tarzia, 2015; Voller, 2010 and 2014).

In this paper, a generalized Neumann solution for the two-phase fractional Lamé-Clapeyron-Stefan problems for a semi-infinite material will be obtained with a constant initial temperature and a convective (Robin) boundary condition at the fixed face $x = 0$. Recently, a generalized Neumann solution for the two-phase fractional Lamé-Clapeyron-Stefan problem for a semi-infinite material with constant initial temperature, and a constant temperature condition at the fixed face $x = 0$ was given in (Roscani and Tarzia, 2014), and with a particular heat flux condition at the fixed face $x = 0$ was given in (Roscani and Tarzia, 2018a).

In these problems, the two governing heat equations and a governing condition for the free boundary include a fractional time derivative in the Caputo sense of order $0 < \alpha < 1$. The Caputo fractional derivative was defined in (Caputo, 1967) by:

$$D^{\alpha} f(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{f'(\tau)}{(t-\tau)^{\alpha}} \, d\tau \quad \text{for } 0 < \alpha < 1$$
$$= f'(t) \quad \text{for } \alpha = 1 \tag{1}$$

where $\Gamma = \Gamma(x)$ is the Gamma function defined by:

$$\Gamma(x) = \int_0^{+\infty} t^{x-1} \exp(-t) \, dt \, . \tag{2}$$

Now, we define two functions (Wright and Mainardi functions) which are very important in order to obtain explicit solutions in the following Sections.

The Wright function is defined in (Wright, 1933):

$$W(z;\alpha,\beta) = \sum_{n=0}^{+\infty} \frac{z^n}{n \, \Gamma(n\alpha+\beta)}, \quad z \in C, \quad \alpha > -1, \quad \beta \in R \, . \tag{3}$$

and the Mainardi function is defined in (Gorenflo, Luchko and Mainardi, 1999):

$$M_{\upsilon}(z) = W(-z;-\upsilon,1-\upsilon) = \sum_{n=0}^{+\infty} \frac{(-z)^n}{n \, \Gamma(-n\upsilon+1-\upsilon)}, \quad z \in C, \quad \upsilon < 1 \tag{4}$$

which is a particular case of the Wright function. Some basic properties are given by:

$$\frac{\partial W}{\partial z}(z;\alpha,\beta) = W(z;\alpha,\alpha+\beta) \tag{5}$$

$$W(-x;-\frac{1}{2},1) = \operatorname{erfc}\left(\frac{x}{2}\right), \quad 1 - W(-x;-\frac{1}{2},1) = \operatorname{erf}\left(\frac{x}{2}\right) \tag{6}$$

$$W(-\infty;-\frac{\alpha}{2},1) = 0, \quad W(0;-\frac{\alpha}{2},1) = 1 \tag{7}$$

$$D^{\alpha}\left(t^{\beta}\right) = \frac{\Gamma(1+\beta)}{\Gamma(1+\beta-\alpha)} t^{\beta-\alpha} \, . \tag{8}$$

In the last decades, fractional differential equations were developed by (Gorenflo, Luchko and Mainardi, 1999; Hristov, 2023; Kilbas, Srivastava and Trujillo, 2006; Luchko, 2010; Mainardi, 2010; Mainardi, Luchko and Pagnini, 2001; Mainardi, Mura and Pagnini, 2010; Podlubny, 1999).

Moreover, for the classical Lamé-Clapeyron-Stefan problem there exist thousands of papers on the subject, for example, the first published papers (Lamé and Clapeyron, 1831; Stefan, 1889), the books (Alexiades and Solomon, 1996; Cannon, 1984; Carslaw and Jaeger, 1959; Crank, 1984; Elliot and Ockendon, 1982; Fasano, 2005; Gupta, 2017; Lunardini, 1991; Rubinstein, 1971; Tayler, 1986), and a large bibliography given in (Tarzia, 2000). A review of explicit solutions with moving or free boundaries was given in (Tarzia, 2011).

The further text of this chapter is organized as follows: In Section 2, we will obtain the necessary condition for the coefficient which characterizes the convective boundary condition at $x = 0$ to have an instantaneous two-phase fractional phase-change process. In Section 3, we will obtain a generalized Neumann solution for the two-phase fractional Lamé-Clapeyron-Stefan problem for a semi-infinite material with a constant initial condition, and a convective (Robin) boundary condition at the fixed face $x = 0$ when the necessary inequality obtained in Section 2 is satisfied. When $\alpha \to 1^-$, we also recover the Neumann solution for the classical two-phase Lamé-Clapeyron-Stefan problem through the error function when an inequality is satisfied for the coefficient that characterizes the convective boundary condition, which was previously obtained in (Tarzia, 2017) which is of the type given in (Tarzia 1981).

2. A necessary condition to obtain an instantaneous two--phase fractional Stefan problem with a convective boundary condition at the fixed face

To obtain the necessary condition for data to have an instantaneous phase-change process to the problem (20) - (27), we will first consider the following fractional heat conduction problem of order $0 < \alpha < 1$ for the liquid phase in the first quadrant with an initial constant temperature and a convective (Robin) boundary condition at the fixed face $x = 0$:

$$D^{\alpha} T_{\ell} - \lambda_{\ell}^{2} T_{\ell_{xx}} = 0, \qquad x > 0, \quad t > 0 \tag{9}$$

$$T_{\ell}(x,0) = T_{\ell}(+\infty, t) = T_i < T_f, \quad x > 0, \quad t > 0, \tag{10}$$

$$kT_{\ell_x}(0,t)=\frac{h_0}{t^{\alpha/2}}\left(T_\ell(0,t)-T_\infty\right),\quad t>0\,, \tag{11}$$

where $\lambda_\ell^2=\dfrac{k_\ell}{\rho c_\ell}$ is the diffusion coefficient of the liquid phase material.

Lemma 1.

The solution to the problem (9) - (11) is given by:

$$T_\ell(x,t)=T_\infty+(T_i-T_\infty)\left[\frac{\dfrac{k_\ell}{h_0\lambda_\ell\Gamma(1-\alpha/2)}+1-W\left(-\dfrac{x}{\lambda_\ell t^{\alpha/2}};-\dfrac{\alpha}{2},1\right)}{1+\dfrac{k_\ell}{h_0\lambda_\ell\Gamma(1-\alpha/2)}}\right], \tag{12}$$

$$x>0,\ t>0$$

Proof:

The solution to problem (9)-(11) can be obtained by:

$$T_\ell(x,t)=A+B\left(1-W\left(-\frac{x}{\lambda_\ell t^{\alpha/2}};-\frac{\alpha}{2},1\right)\right),\ x>0,t>0 \tag{13}$$

where the coefficients A and B must be determined by using conditions (10) and (11). First, we can obtain that the gradient of the fractional temperature (13), given by:

$$T_{\ell_x}(x,t)=\frac{B}{\lambda_l t^{\alpha/2}}W\left(-\frac{x}{\lambda_\ell t^{\alpha/2}};-\frac{\alpha}{2},1-\frac{\alpha}{2}\right),\ x>0,t>0\,. \tag{14}$$

Then, the initial condition (10) is transformed by:

$$T_i=T_\ell(x,0)=A+B\left[1-W(-\infty,-\frac{\alpha}{2},1\right]=A+B\ , \tag{15}$$

and taking into account that $T_l(0,t) = A$, the boundary condition (11) is transformed by:

$$A = T_\infty + \frac{Bk_\ell}{h_0 \lambda_\ell \Gamma(1-\frac{\alpha}{2})}, \tag{16}$$

Therefore, from the previous two equations (15) and (16), we obtain:

$$B = \frac{T_i - T_\infty}{1 + \frac{k_l}{h_0 \lambda_\ell \Gamma(1-\alpha/2)}}, \quad A = T_\infty + \frac{k_\ell}{h_0 \lambda_\ell \Gamma(1-\frac{\alpha}{2})} \frac{T_i - T_\infty}{1 + \frac{k_\ell}{h_0 \lambda_\ell \Gamma(1-\alpha/2)}}, \tag{17}$$

and then, the temperature of the problem (9) – (11) is given by (12), and its temperature at the fixed face $x = 0$ is given by:

$$T_\ell(0,t) = T_\infty + \frac{T_i - T_\infty}{1 + \frac{h_0 \lambda_\ell \Gamma(1-\alpha/2)}{k_\ell}}, \quad t > 0, \tag{18}$$

which is constant in time.

Corollary 2.

We will have an instantaneous fractional change of phase if and only if the fractional temperature (12) at the fixed face $x = 0$ is less than the phase-change temperature T_f, that is, $T_l(0,t) < T_f$ which is equivalent to the following inequality for the coefficient h_0, given by:

$$h_0 > \frac{k_\ell (T_i - T_f)}{\lambda_\ell (T_f - T_\infty)} \frac{1}{\Gamma(1-\alpha/2)}, \tag{19}$$

which implies that the coefficient h_0 must be sufficiently larger to obtain an instantaneous fractional change of phase.

3. The Two-Phase Fractional Lamé-Clapeyron-Stefan Problem (Solidification Process) with a Convective Boundary Condition at The Fixed Face

We consider the following solidification process:

Problem (FCP_α **)** Find the free boundary $x = s(t)$ (the phase-change interface), and the temperature $T = T(x,t)$ such that the following equations and conditions must be satisfied $(0 < \alpha < 1)$:

$$D^\alpha T_\ell - \lambda_\ell^2 T_{\ell_{xx}} = 0, \qquad x > s(t), \quad t > 0 , \tag{20}$$

$$D^\alpha T_s - \lambda_s^2 T_{s_{xx}} = 0, \qquad 0 < x < s(t), \quad t > 0 , \tag{21}$$

$$s(0) = 0 , \tag{22}$$

$$T_s(x,0) = T_s(+\infty, t) = T_i > T_f , \qquad x > 0, \quad t > 0 , \tag{23}$$

$$T_s(s(t),t) = T_f , \qquad t > 0 , \tag{24}$$

$$T_l(s(t),t) = T_f , \qquad t > 0 , \tag{25}$$

$$k_s T_{s_x}\left(s(t),t\right) - k_\ell T_{\ell_x}\left(s(t),t\right) = \rho\ell\, D^\alpha s(t), \quad t > 0 , \tag{26}$$

$$k_s T_{s_x}(0,t) = \frac{h_0}{t^{\alpha/2}}\left(T_s(0,t) - T_\infty\right), \qquad t > 0 , \tag{27}$$

where $\lambda_s^2 = \dfrac{k_s}{\rho c_s}$, $\lambda_\ell^2 = \dfrac{k_\ell}{\rho c_\ell}$ are the diffusion coefficients for the solid and liquid phases.

Theorem 3

Let $T_\infty < T_f < T_i$ *be.*

a) *If the coefficient* h_0 *satisfies the inequality (19) then there exists an instantaneous phase-change (solidification) process and the problem (* FCP_α *) has the generalized Neumann explicit solution given by:*

$$T_s(x,t) = T_f - (T_f - T_\infty)\left[1 - \frac{\dfrac{k_s}{h_0\lambda_s\Gamma(1-\alpha/2)} + 1 - W\left(-\dfrac{x}{\lambda_s t^{\alpha/2}};-\dfrac{\alpha}{2},1\right)}{\dfrac{k_s}{h_0\lambda_s\Gamma(1-\alpha/2)} + 1 - W\left(-\dfrac{\xi_{C\alpha}}{\lambda};-\dfrac{\alpha}{2},1\right)}\right] \quad (28)$$

$$= T_\infty + (T_f - T_\infty)\left[\frac{\dfrac{k_s}{h_0\lambda_s\Gamma(1-\alpha/2)} + 1 - W\left(-\dfrac{x}{\lambda_s t^{\alpha/2}};-\dfrac{\alpha}{2},1\right)}{\dfrac{k_s}{h_0\lambda_s\Gamma(1-\alpha/2)} + 1 - W\left(-\dfrac{\xi_{C\alpha}}{\lambda};-\dfrac{\alpha}{2},1\right)}\right],$$

$$T_\ell(x,t) = T_i - (T_i - T_f)\frac{W\left(-\dfrac{x}{\lambda_\ell t^{\alpha/2}};-\dfrac{\alpha}{2},1\right)}{W\left(-\xi_{C\alpha};-\dfrac{\alpha}{2},1\right)} \quad (29)$$

$$= T_f + (T_i - T_f)\left[1 - \frac{W\left(-\dfrac{x}{\lambda_\ell t^{\alpha/2}};-\dfrac{\alpha}{2},1\right)}{W\left(-\xi_{C\alpha};-\dfrac{\alpha}{2},1\right)}\right],$$

$$s(t) = \xi\,\lambda_\ell\, t^{\alpha/2}, \quad (30)$$

where the coefficient $\xi > 0$ *is the solution of the following equation:*

$$F_{C_\alpha}(x) = \rho \ell \lambda_\ell \frac{\Gamma\left(1+\alpha/2\right)}{\Gamma\left(1-\alpha/2\right)} x, \quad x > 0 \tag{31}$$

with

$$F_{C_\alpha}(x) = \frac{k_s(T_f - T_\infty)}{\lambda_s} F_{4\alpha}(x/\lambda) - \frac{k_\ell(T_i - T_f)}{\lambda_\ell} F_{2\alpha}(x) \tag{32}$$

where

$$F_{4\alpha}(x) = \frac{M_{\alpha/2}(x)}{\frac{k_s}{h_0 \lambda_s \Gamma(1-\alpha/2)} + 1 - W\left(-x; -\frac{\alpha}{2}, 1\right)}. \tag{33}$$

$$F_{2\alpha}(x) = \frac{M_{\alpha/2}(x)}{W\left(-x; -\frac{\alpha}{2}, 1\right)}. \tag{34}$$

b) *If the coefficient* h_0 *satisfies the inequalities*

$$0 < h_0 \le \frac{k_\ell(T_i - T_f)}{\lambda_\ell(T_f - T_\infty)} \frac{1}{\Gamma(1-\alpha/2)}, \tag{35}$$

then the problem (FCP_α *) is a fractional heat transfer problem for the initial liquid phase whose solution is given by (12).*

Proof.

The solution to the problem (20) - (27) can be obtained by the following expressions:

$$T_s(x,t) = E + F\left(1 - W\left(-\frac{x}{\lambda_s t^{\alpha/2}}; -\frac{\alpha}{2}, 1\right)\right), 0 < x < s(t), t > 0 \qquad (36)$$

$$T_\ell(x,t) = C + D\left(1 - W\left(-\frac{x}{\lambda_\ell t^{\alpha/2}}; -\frac{\alpha}{2}, 1\right)\right), x > s(t), t > 0 \qquad (37)$$

$$s(t) = \xi\, \lambda_\ell\, t^{\frac{\alpha}{2}}, \quad t > 0, \qquad (38)$$

where the coefficients C, D, E, F and $\xi > 0$ must be determined by using the boundary conditions (23) - (27).

From conditions (23) - (25) and (27), we can obtain a system of 4 equations for unknowns C, D, E and F as a function of the coefficient ξ. Defining the parameter

$$\lambda = \frac{\lambda_\ell}{\lambda_s} > 0, \qquad (39)$$

we can solve that system by obtaining:

$$F = \frac{T_f - T_\infty}{1 - W(-\xi\lambda, -\alpha/2, 1) + \dfrac{k_s}{h_0 \lambda_s \Gamma(1 - \alpha/2)}} \qquad (40)$$

$$E = T_\infty + \frac{k_s}{h_0 \lambda_s \Gamma(1 - \alpha/2)} \frac{T_f - T_\infty}{1 - W(-\xi\lambda, -\alpha/2, 1) + \dfrac{k_s}{h_0 \lambda_s \Gamma(1 - \alpha/2)}}, \qquad (41)$$

$$D = \frac{T_i - T_f}{W(-\xi, -\alpha/2, 1)} > 0 \qquad (42)$$

$$C = T_i - \frac{T_i - T_f}{W(-\xi, -\alpha/2, 1)} \quad . \qquad (43)$$

From expressions (40) - (43), we can obtain the fractional temperatures for the solid and liquid phases (28) and (29) respectively. By using now (28) and (29) and the fractional Stefan condition (26) we get for ξ the equation (31) which has a solution by using the properties of the real functions given in (Roscani and Tarzia, 2014 and 2018a).

Corollary 4.

By considering a similar method to the one developed by Roscani and Tarzia (2018a), the inequality (19) for the coefficient h_0 can be transformed into an inequality for the coefficient that characterizes the fractional free boundary for the fractional two-phase Stefan problem when a temperature boundary condition ($T_s(0,t)=T_0$) is assumed at the fixed face $x=0$. Moreover, by equivalence, this inequality is also transformed for the coefficient $\xi>0$ of the free boundary (38), given by the following inequality ($T_0<T_f<T_i$):

$$1-W(-\xi\lambda,-\alpha/2,1)<\frac{T_f-T_0}{T_i-T_f}\frac{k_s\lambda_\ell}{k_\ell\lambda_s}. \tag{44}$$

Theorem 5

Let $T_\infty<T_f<T_i$ be. If the coefficient h_0 satisfies the inequality (19) then the solution of the problem (FCP_α*) converges to the classical solution of the problem (*FCP_1*) when $\alpha\to 1^-$, and then we recover the classical Neumann explicit solution and the inequality for the coefficient h_0 which characterized the convective (Robin) boundary condition at $x=0$ obtained for $\alpha=1$ in (Tarzia, 2017), that is:*

$$h_0>\frac{k_\ell}{\lambda_\ell\sqrt{\pi}}\frac{T_i-T_f}{T_f-T_\infty}. \tag{35}$$

Conclusions

We have obtained generalized Neumann solutions for two-phase fractional Lamé-Clapeyron-Stefan problems for a semi-infinite material with a constant initial condition when a convective (Robin) boundary condition is imposed on the fixed face $x = 0$. When $\alpha \to 1^-$, we recover the two classical Neumann solutions (which are equivalents among them) for the corresponding classical two-phase Lamé-Clapeyron-Stefan problem given through the error function, and also the inequalities for the corresponding coefficients which characterizes the convective boundary condition at $x = 0$.

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