ADVANCES IN APPLIED MATHEMATICS 3, 74–82 (1982)

Determination of the Unknown Coefficients in the Lamé–Clapeyron Problem (or One-Phase Stefan Problem)

DOMINGO ALBERTO TARZIA

PROMAR(CONICET–UNR), Instituto de Matemática "Beppo Levi," Universidad Nacional de Rosario, Avenida Pellegrini 250, 2000 Rosario, Argentina

We study the Lamé–Clapeyron problem (or one-phase Stefan problem) with unknown coefficients. We give an overspecified condition $k(\partial\theta/\partial x)(0,t) = -h_0/\sqrt{t}$ in the fixed face $x = 0$ of the phase-change material to solve it. We obtain:

(i) If the thermal conductivity or mass density coefficient of the material is unknown, the corresponding free boundary problem always has a solution of Lamé–Clapeyron–Neumann type. (ii) If the latent heat of fusion or specific heat coefficient of the material is unknown, the corresponding free boundary problem has a solution of Lamé–Clapeyron–Neumann type if and only if a supplementary condition is verified. Moreover, if we consider the original problem without unknown coefficients, we deduce the inequality $\operatorname{erf}(\xi) < \sqrt{2\theta_0 c/\pi l}$ for the coefficient ξ of the free boundary $s(t) = 2a\xi\sqrt{t}$ of the Lamé–Clapeyron solution of the one-phase Stefan problem.

I. INTRODUCTION

We shall consider the Lamé–Clapeyron problem (or one-phase Stefan problem). That is, we shall find the functions $s = s(t) > 0$ (free boundary), defined for $t > 0$ with $s(0) = 0$, and the temperature $\theta = \theta(x, t)$, defined for $0 < x < s(t)$ and $t > 0$, such that they satisfy the following conditions:

$$\begin{aligned}
&\frac{\partial\theta}{\partial t} = a^2\frac{\partial^2\theta}{\partial x^2}, \qquad 0 < x < s(t), \quad t > 0,\\
&\theta(s(t), t) = 0, \qquad t > 0,\\
&-k\frac{\partial\theta}{\partial x}(s(t), t) = Ls'(t), \qquad t > 0,\\
&s(0) = 0,\\
&\theta(0, t) = \theta_0, \qquad t > 0,
\end{aligned} \tag{1}$$

0196-8858/82/010074-09$05.00/0

where

$$\begin{array}{ll} k > 0: & \text{thermal conductivity,} \\ c > 0: & \text{specific heat,} \\ \rho > 0: & \text{mass density,} \\ l > 0: & \text{latent heat of fusion,} \\ L = \rho l, & C = \rho c, \\ a^2 = \dfrac{k}{C}: & \text{thermal diffusivity,} \\ \theta_0 > 0: & \text{temperature in the fixed face } x = 0. \end{array} \tag{2}$$

The solution of the problem (1) is given in [17]. But, if one of the coefficients k, l, c, ρ of the phase change material (we suppose that its melting temperature is null) is unknown, then it is necessary to give an overspecified condition in the fixed face $x = 0$ to solve it. This idea was suggested in papers [2–12, 14–16]. The overspecified condition is given by the knowledge of the heat flux that the phase change material receives in its fixed face $x = 0$, that is,

$$k\frac{\partial \theta}{\partial x}(0, t) = -\frac{h_0}{\sqrt{t}}, \qquad t > 0, \text{ with } h_0 > 0, \tag{3}$$

which was suggested in [20].

We shall consider in Section II four different cases with the following unknown coefficient:

$$\begin{array}{llll} \text{(i)} & k, & \text{(ii)} & l, \\ \text{(iii)} & c, & \text{(iv)} & \rho. \end{array} \tag{4}$$

We shall prove that, under condition (3), there is not always a solution of Lamé–Clapeyron–Neumann type [1, 13, 17, 18, 20, 21] for the problem (1) and (3), in the four different cases (4). Moreover, the explicit solution exists for the cases (ii) and (iii) if a complementary condition is satisfied and the explicit solution always exists for the cases (i) and (iv). This fact has been already observed in [19, 20] for the other problems of Stefan type. Moreover, in Section III, for the problem (1) with solution (5), we obtain the inequality (29) for the coefficient ξ of the free boundary $s(t) = 2a\xi\sqrt{t}$ using the inequality (22) obtained in Section II.

II. Solution of the Different Cases

Case (i). Determination of the Coefficient k

The solution of the problem (1), (3) is given by

$$\theta(x,t) = \theta_0 - \frac{\theta_0}{f(\sigma/a)} f\big(x/2a\sqrt{t}\big),$$
$$s(t) = 2\sigma\sqrt{t}, \qquad \sigma > 0, \text{ with } f(x) = \frac{2}{\sqrt{\pi}} \int_0^x \exp(-u^2)\, du = \operatorname{erf}(x), \tag{5}$$

where the two unknown coefficients $\sigma > 0$, $k > 0$ must satisfy the following system of equations:

$$\begin{aligned} \frac{k\theta_0}{La\sqrt{\pi}} \frac{\exp(-\sigma^2/a^2)}{f(\sigma/a)} &= \sigma, \\ \frac{k\theta_0}{a\sqrt{\pi}} &= h_0 f(\sigma/a), \end{aligned} \tag{6}$$

or equivalently

$$\begin{aligned} \sigma \exp(\sigma^2/a^2) &= \frac{h_0}{L}, \\ \frac{a}{f(\sigma/a)} &= \frac{h_0\sqrt{\pi}}{C\theta_0}, \qquad a = \sqrt{\frac{k}{C}}. \end{aligned} \tag{7}$$

If we define

$$\xi = \frac{\sigma}{a} \tag{8}$$

we have

$$G_1(\xi) = \frac{C\theta_0}{L\sqrt{\pi}} = \frac{c\theta_0}{l\sqrt{\pi}}, \qquad \xi > 0, \tag{9}$$

where the function G_1 verifies

$$\begin{aligned} &G_1(x) = x\exp(x^2)f(x), \\ &G_1(0) = 0, \qquad G_1(+\infty) = +\infty, \\ &G_1'(x) > 0, \qquad \forall x > 0. \end{aligned} \tag{10}$$

With the properties (10), Eq. (9) has a unique solution $\xi > 0$. Moreover, the coefficients σ, k are given, as functions of element ξ, by:

$$\begin{aligned} \sigma &= \frac{h_0}{L}\exp(-\xi^2), \\ k &= \frac{\pi h_0^2}{C\theta_0^2} f^2(\xi), \end{aligned} \tag{11}$$

and then we obtain

LEMMA 1. *For any data* $h_0 > 0$, $\theta_0 > 0$ *and for any coefficients of the phase change material* $\rho > 0$, $L = \rho l > 0$, $C = \rho c > 0$, *the problem* (1), (3) *has the solution* (5), *where the coefficients* σ, k *are given by* (11), *and* ξ *is the solution of Eq.* (9).

Case (*ii*). *Determination of the Coefficient* l

The solution of the problem (1), (3) is given by (5), where the two unknown coefficients $\sigma > 0$, $l > 0$ must satisfy the following system of equations:

$$\begin{aligned} \rho l &= \frac{h_0}{\sigma}\exp(-\sigma^2/a^2), \\ f(\sigma/a) &= \frac{k\theta_0}{a h_0\sqrt{\pi}}. \end{aligned} \tag{12}$$

If we define ξ by (8), we obtain

$$\begin{aligned} \sigma &= \xi\sqrt{\frac{k}{C}}, \\ l &= \frac{h_0}{\rho}\sqrt{\frac{C}{k}}\,\frac{\exp(-\xi^2)}{\xi}, \end{aligned} \tag{13}$$

where ξ must satisfy the equation:

$$f(\xi) = \frac{k\theta_0}{a h_0\sqrt{\pi}}, \qquad \xi > 0. \tag{14}$$

As $0 < f(x) < 1$, $\forall x > 0$, the solution $\xi > 0$ of Eq. (14) exists if and only if the condition

$$\frac{k\theta_0}{ah_0\sqrt{\pi}} = \frac{\theta_0}{h_0}\sqrt{\frac{k\rho c}{\pi}} < 1 \tag{15}$$

is verified. Then we obtain

LEMMA 2. *If the data* $h_0 > 0$, $\theta_0 > 0$ *and the coefficients of the phase change material* $\rho > 0$, $C = \rho c > 0$, $k > 0$ *verify the condition* (15), *the problem* (1), (3) *has the solution* (5) *where the coefficients* σ, l *are given by* (13) *and* ξ *is the solution of Eq.* (14).

Case (iii). Determination of the Coefficient c

The solution of the problem (1), (3) is given by (5) where the two unknown coefficients $\sigma > 0$, $c > 0$ are given, as functions of the element ξ, defined in (8), by:

$$\sigma = \frac{h_0}{L}\exp(-\xi^2),$$
$$c = \frac{\pi h_0^2}{\rho k \theta_0^2} f^2(\xi), \tag{16}$$

where ξ is the solution of the equation:

$$G_2(\xi) = \frac{kL\theta_0}{h_0^2\sqrt{\pi}}\exp(\xi^2), \qquad \xi > 0, \tag{17}$$

with

$$G_2(x) = \frac{f(x)}{x}. \tag{18}$$

The function G_2 has the following properties:

$$G_2(0^+) = \frac{2}{\sqrt{\pi}}, \qquad G_2(+\infty) = 0,$$
$$G_2'(x) = \frac{\frac{2}{\sqrt{\pi}}x\exp(-x^2) - f(x)}{x^2} < 0, \qquad \forall x > 0. \tag{19}$$

This is true, since we have

$$f(x) = \frac{2}{\sqrt{\pi}} \int_0^x \exp(-u^2)\, du > \frac{2}{\sqrt{\pi}} x \exp(-x^2), \qquad \forall x > 0. \tag{20}$$

With the properties (19), Eq. (17) has a unique solution $\xi > 0$ if and only if the condition

$$\frac{kL\theta_0}{h_0^2\sqrt{\pi}} < G_2(0^+) = \frac{2}{\sqrt{\pi}} \tag{21}$$

or its equivalent

$$\frac{kL\theta_0}{2h_0^2} < 1 \tag{22}$$

is verified. Then, we obtain

LEMMA 3. *If the data $h_0 > 0$, $\theta_0 > 0$ and the coefficients of the phase change material $\rho > 0$, $k > 0$, $L = \rho l > 0$ verify the condition* (22), *the problem* (1), (3) *has the solution* (5) *where the coefficients σ, c are given by* (16) *and ξ is the solution of Eq.* (17).

Case (*iv*). *Determination of the Coefficient* ρ

The solution of the problem (1), (3) is given by (5), where the two unknown coefficients $\sigma > 0$, $\rho > 0$ must satisfy the following system of equations:

$$\begin{aligned} &\frac{\sigma}{a}\exp(\sigma^2/a^2) f(\sigma/a) = \frac{k\theta_0}{La^2\sqrt{\pi}} = \frac{c\theta_0}{l\sqrt{\pi}}, \\ &f(\sigma/a) = \frac{\theta_0}{h_0}\sqrt{\frac{k\rho c}{\pi}}. \end{aligned} \tag{23}$$

If we define

$$\xi = \frac{\sigma}{a} = \sigma\sqrt{\frac{\rho c}{k}}, \qquad \eta = \sqrt{\rho}, \tag{24}$$

then the coefficients σ, ρ are given, as functions of the element ξ, by

$$\begin{aligned} &\sigma = \frac{k\theta_0}{h_0\sqrt{\pi}} \frac{\xi}{f(\xi)}, \\ &\sqrt{\rho} = \eta = \frac{h_0}{\theta_0}\sqrt{\frac{\pi}{kc}}\, f(\xi), \end{aligned} \tag{25}$$

where ξ must satisfy the equation:

$$G_1(\xi) = \frac{c\theta_0}{l\sqrt{\pi}}, \qquad \xi > 0. \tag{26}$$

As the function G_1 verifies (10), Eq. (26) has a unique solution $\xi > 0$ and we obtain

LEMMA 4. *For any data $h_0 > 0$, $\theta_0 > 0$ and for any coefficients of the phase change material $k > 0$, $l > 0$, $c > 0$, the problem* (1), (3) *has the solution* (5), *where the coefficients σ, ρ are given by* (25) *and ξ is the solution of Eq.* (26).

Remark 1. In the cases (i) and (iv), we have the same element ξ, defined by (9) or (26). Moreover, if we define σ_{I}, k_{I} by (11) and, σ_{IV}, ρ_{IV} by (25) corresponding to the cases (i) and (iv), respectively, we have

(a) If we put $k = k_{\mathrm{I}}$ in the definition of σ_{IV} we obtain σ_{I}.

(b) If we put $\rho = \rho_{\mathrm{IV}}$ in the definition of σ_{I} we obtain σ_{IV}.

III. AN INEQUALITY FOR THE COEFFICIENT σ OF THE FREE BOUNDARY $s(t) = 2\sigma\sqrt{t}$ OF THE LAMÉ–CLAPEYRON SOLUTION

If we consider the Lamé–Clapeyron problem without unknown coefficients, defined by (1), the solution of which is given by (5) where the coefficient σ (or $\xi = \sigma/a$) is the unique solution of the equation.

$$G_1(\xi) = \alpha_0, \qquad \xi > 0,$$
$$\text{with} \quad \alpha_0 = \frac{k\theta_0}{La^2\sqrt{\pi}} = \frac{c\theta_0}{l\sqrt{\pi}}. \tag{27}$$

In this case, the corresponding coefficient h_0 in (3) is given by

$$h_0 = \frac{k\theta_0}{af(\sigma/a)\sqrt{\pi}}. \tag{28}$$

Then, by Section II, the inequalities (15) and (22) are verified. Using (28), the condition (22) gives us the following inequality for the coefficient σ (or $\xi = \sigma/a$):

$$f(\xi) < \sqrt{\frac{2\theta_0 c}{\pi l}}. \tag{29}$$

On the other hand, the condition (15) does not give us anything since it is always verified by definition of the error function f. Then, we obtain

LEMMA 5. *The coefficient* ξ *of the free boundary* $s(t) = 2a\xi\sqrt{t}$ *of the Lamé–Clapeyron solution of the one-phase Stefan problem satisfies the inequality* (29).

Remark 2. The inequality (29) is of physical interest when the data θ_0, l, c verify the condition

$$\frac{c\theta_0}{l} < \frac{\pi}{2}. \tag{30}$$

In this case, we obtain an upper bound for the displacement $s(t) = 2\sigma\sqrt{t}$ and the velocity $s'(t) = \sigma/\sqrt{t}$ of the free boundary of the Lamé–Clapeyron solution of the one-phase Stefan problem.

An inequality for the Neumann solution of the two-phase Stefan problem is given in [20].

Remark 3. Using a development similar to the one described in this paper in an inverse one-phase Stefan problem, it is possible to obtain simultaneously two unknown coefficients of the phase-change material.

ACKNOWLEDGMENT

Part of this paper has been sponsored by the SUBCYT-CONICET (Argentina). This financial support was granted to the project "Problemas de frontera libre de la Física-Matemática".

REFERENCES

1. M. BRILLOUIN, Sur quelques problèmes non résolus de physique mathématique classique:propagation de la fusion, *Ann. Inst. H. Poincaré* **1** (1930–1931), 285–308.
2. B. M. BUDAK AND A. D. ISKENDEROV, On a class of boundary value problems with unknown coefficients, *Soviet Math. Dokl.* **8** (1967), 786–789.
3. B. M. BUDAK AND A. D. ISKENDEROV, A class of inverse boundary value problems with unknown coefficients, *Soviet Math. Dokl.* **8** (1967), 1026–1030.
4. J. R. CANNON, Determination of an unknown coefficient in a parabolic differential equation, *Duke Math. J.* **30** (1963), 313–323.
5. J. R. CANNON, Determination of certain parameters in heat conduction problems, *J. Math. Anal. Appl.* **8** (1964), 188–201.
6. J. R. CANNON, Determination of the unknown coefficient $k(u)$ in the equation $\nabla \cdot k(u)\nabla u = 0$ from overspecified boundary data, *J. Math. Anal. Appl.* **18** (1967), 112–114.
7. J. R. CANNON, J. DOUGLAS, JR., AND B. F. JONES, JR., Determination of the diffusivity of an isotropic medium, *Internat. J. Eng. Sci.* **1** (1963), 453–455.
8. J. R. CANNON AND P. C. DUCHATEAU, Determining unknown coefficients in a nonlinear heat conduction problem, *SIAM J. Appl. Math.* **24** (1973), 298–314.
9. J. R. CANNON AND P. C. DUCHATEAU, Determination of unknown coefficients in parabolic operators from overspecified initial–boundary data, *J. Heat Transfer* **100** (1978), 503–507.

10. J. R. CANNON AND D. L. FILMER, The determination of unknown parameters in analytic systems of ordinary differential equations, *SIAM J. Appl. Math.* **15** (1967), 799–809.
11. J. R. CANNON AND J. H. HALTON, The irrotational solution of an elliptic differential equation with an unknown coefficient, *Proc. Cambridge Philos. Soc.* **59** (1963), 680–682.
12. J. R. CANNON AND B. F. JONES, JR., Determination of the diffusivity of an anisotropic medium, *Internat. J. Eng. Sci.* **1** (1963), 457–460.
13. H. S. CARSLAW AND J. C. JAEGER, "Conduction of Heat in Solids," Oxford Univ. Press (Clarendon), London, 1959.
14. J. DOUGLAS, JR. AND B. F. JONES, JR., The determination of a coefficient in a parabolic differential equation. II. Numerical approximation, *J. Math. Mech.* **11** (1962), 919–926.
15. B. F. JONES, JR., The determination of a coefficient in a parabolic differential equation. I. Existence and uniqueness, *J. Math. Mech.* **11** (1962), 907–918.
16. B. F. JONES, JR., Various methods for finding unknown coefficients in parabolic differential equations, *Comm. Pure Appl. Math.* **16** (1963), 33–44.
17. G. LAMÉ AND B. P. CLAPEYRON, Memoire sur la solidification par refroidissement d'un globe liquide, *Ann. Chim. Phys.* **47** (1831), 250–256.
18. L. I. RUBINSTEIN, "The Stefan Problem," Translations of Mathematical Monographs, Vol. 27, Amer. Math. Soc., Providence, R.I., 1971.
19. D. A. TARZIA, Sobre el caso estacionario del problema de Stefan a dos fases, *Math. Notae* **28** (1980–1981), 73–89.
20. D. A. TARZIA, An inequality for the coefficient σ of the free boundary $s(t) = 2\sigma\sqrt{t}$ of the Neumann solution for the two-phase Stefan problem, *Quart. Appl. Math.*, in press.
21. H. WEBER, "Die partiellen differential–Gleinchungen der mathematischen physik," nach Riemmans Vorlesungen, t. II, Braunwschweig, 1901.