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NUMERICAL ANALYSIS OF A MIXED ELLIPTIC PROBLEM WITH SOLUTION OF NON-CONSTANT SIGN

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ABSTRACT: A continuous and its corresponding discrete mixed elliptic differntial problem with solutions of non-constant sign, as functions of the Dirichlet and Neumann data, are studied in a convex polygonal bounded domain Ω of R^n. An inequality for the heat flux is given in order to obtein a continuous and discrete change of phase, that is, a continuous or discrete solution of non-constant sign in Ω (steady-state two-phase continuous or discrete Stefan problem). A convergence for the two inequaliteies, as function of the parameter h of the finite element method, is also obtained.

KEY WORDS: Steady-state Stefan problem, free boundary problems, phase-change problems, variational inequalities, Mixed elliptic problems, Numerical Analysis, Finite Element Method.

AMS SUBJECT CLASSIFICATION: 35R35, 35J85, 65N15, 65N30.

I. INTRODUCTION

We consider a heat conducting material occuping Ω, a convex polygonal bounded domain of $R^n (n = 1, 2, 3$ in practice), with a sufficiently regular boundary $\Gamma = \Gamma_1 \cup \Gamma_2$(with meas $(\Gamma_1) \equiv |\Gamma_1| > 0,\ |\Gamma_2| > 0)$. We assume, without loss of generality, that the phase-change temperature is $0°C$. We impose a temperature $b > 0$ on Γ_1 and autcoming heat flux $q > 0$ on Γ_2. If we consider in Ω a steady-state heat conduction problem, then we are interested in finding sufficient and/or necessary conditions for the heat flux q on Γ_2 to obtain a change of phase in Ω, that is, a steady-state two-phase Stefan problem in Ω (i.e. the temperature is a function of non-constant sign in Ω) [10].

Following [9] we study the temperature $\theta = \theta(x)$, defined for $x \in \Omega$. The set Ω can be expressed in the form

$$(1) \qquad \Omega = \Omega_1 \cup \Omega_2 \cup \mathcal{L}.$$

where

$$(2) \begin{cases} \Omega_1 = \{x \in \Omega \ / \ \theta(x) < 0\}, \\ \Omega_2 = \{x \in \Omega \ / \ \theta(x) > 0\}, \\ \mathcal{L} = \{x \in \Omega \ / \ \theta(x) = 0\}, \end{cases}$$

are the solid phase, the liquid phase and the free boundary (e.g. a surface in R^3) that separates them respectively. The temperature θ can be represented in Ω in the following way:

$$(3) \quad \theta(x) = \begin{cases} \theta_1(x) < 0, & x \in \Omega_1 \ , \\ 0, & x \in \mathcal{L} \ , \\ \theta_2(x) > 0, & x \in \Omega_2 \ , \end{cases}$$

and satisfies the conditions below:

$$(4) \begin{cases} \text{i)} \Delta\theta_i = 0 & \text{in } \Omega_i (i = 1, 2), \\ \text{ii)} \theta_1 = \theta_2 = 0, \ k_1 \frac{\partial \theta_1}{\partial n} = k_2 \frac{\partial \theta_2}{\partial n} & \text{on } \mathcal{L}, \\ \text{iii)} \theta_2 |_{\Gamma_1} = b, & \\ \qquad -k_2 \frac{\partial \theta_2}{\partial n} |_{\Gamma_2} = q & \text{if } \theta|_{\Gamma_2} > 0, \\ \text{iv)} & \\ \qquad -k_1 \frac{\partial \theta_1}{\partial n} |_{\Gamma_2} = q & \text{if } \theta|_{\Gamma_2} < 0, \end{cases}$$

where $k_i > 0$ is the thermal conductivity of phase i ($i = 1$: solid phase, $i = 2$: liquid phase), $b > 0$ is the temperature given on Γ_1 , and $q > 0$ is the heat flux given on Γ_2.

Problem (4) represents a free boundary elliptic problem (when $\mathcal{L} \neq \emptyset$) where the free boundary $\mathcal{L}$ (unknown a priori) is characterised by the three conditions (4ii). Following the idea of [1, 3, 4, 9] we shall transform (4) into a new elliptic problem but now without a free boundary. If we define the function u in Ω as follows

$$(5) \qquad u = k_2\theta^+ - k_1\theta^- (\theta = \frac{1}{k_2}u^+ - \frac{1}{k_1}u^-) \quad \text{in } \Omega \ ,$$

where θ^+ and θ^- represent the positive and negative parts of the function θ respectively, then problem (4) is transformed into

$$(6)\begin{cases} i)\Delta u = 0 & \text{in } D'(\Omega), \\ ii)u|_{\Gamma_1} = B, & B = k_2 b > 0 , \\ iii) -\frac{\partial u}{\partial n}|_{\Gamma_2} = q, & \end{cases}$$

whose variational formulation is given by

$$(7)\begin{cases} a(u, v-u) = L(v-u), \quad \forall v \in K, \\ u \in K, \end{cases}$$

where

$$(8)\begin{cases} V = H^1(\Omega), & V_0 = \{v \in V / v|_{\Gamma_1} = 0\}, \\ K = K_B = \{v \in V \ / \ v|_{\Gamma_1} = B\}, & \\ a(u,v) = \int_\Omega \nabla u . \nabla \ u \ dx, & L(v) = L_q(v) = -\int_{\Gamma_2} q \ v \ d\gamma. \end{cases}$$

Under the hypotheses $L \in V_0'$ (e.g. $q \in L^2(\Gamma_2)$)and $B \in H^{\frac{1}{2}}(\Gamma_1)$, there exists a unique solution of (7) which is characterised by the following minimisation problem [1,6]

$$(9)\begin{cases} J(u) \le J(v), \quad \forall v \in K, \\ u \in K, \end{cases}$$

where

$$(10) \qquad J(v) = J_q(v) = \frac{1}{2}a(v,v) - L(v) = \frac{1}{2}a(v,v) + \int_{\Gamma_2} q \ v \ d\gamma.$$

LEMMA 1. If $u = u_{qB}$ is the unique solution of problem (7) for data q on Γ_2 and $B > 0$ on Γ_1, then we have the monotony property:

$$(11) \qquad B_1 \le B_2 \text{ on } \Gamma_1 \text{ and } q_2 \le q_1 \text{ on } \Gamma_2 \Rightarrow u_{q_1B_1} \le u_{q_2B_2} \text{ in } \overline{\Omega}.$$

Moreover,

$$(12) \qquad q > 0 \text{ on } \Gamma_2 \Rightarrow u_{qB} \le \max_{\Gamma_1} B \text{ in } \overline{\Omega},$$

and function $u = u_{qB}$ satisfies the equality

$$(13) \qquad a(u^-, u^-) = \int_{\Gamma_2} qu^- d\gamma \ .$$

COROLLARY 2.- From (13), we deduce

$$(14) \qquad u^- \neq 0 \text{ in } \overline{\Omega} \Leftrightarrow u^- \neq 0 \text{ on } \Gamma_2 ,$$

where $q > 0$ and $B > 0$.

NOTE 1.- We shall denote by (N-n) the formula (n) of Section N and we shall omit N in the same paragraph. Idem for theorems, lemmas, corollaries, remarks and notes. We shall also omit the space variable $x \in \Omega$ for every function defined in Ω.

II. MIXED ELLIPTIC PROBLEMS WITH OR WITHOUT PHASE CHANGE.

We shall give a problem which are related to the mixed elliptic partial differential equations (I-6) or (1-7).

Probem P : For the constant case $B > 0$ and $q > 0$, find a constant $q_0 = q_0(B) > 0$ such that for $q > q_0(B)$ we have a steady-state twő-phase Stefan problem in Ω, that is the solution u of (I-7) is a function of non-constant sign in Ω.

REMARK 1.- From (I-14) we deduce that an answer to problem P is the element q for which u takes negative values on the boundary Γ_2.

LEMMA 1.- Let $u = u_q$ be the unique solution of the variational equality (I-7) for $q > 0$ (for a given $B > 0$). Then

(i) The mappings

$$(1) \qquad q > 0 \rightarrow u_q \in V \quad \text{and} \quad q > 0 \rightarrow \int_{\Gamma_2} u_q d\gamma \in R$$

are strictly decreasing functions.

(ii) For all $q > 0$ and $h > 0$ we have the following estimates:

$$(2) \qquad \|\frac{1}{h}(u_{q+h} - u_q)\|_V \leq C_1 = \frac{\|\gamma_0\|}{\alpha_0}|\Gamma_2|^{\frac{1}{2}} ,$$

$$(3) \qquad \|\frac{1}{h}(u_q - u_{q+h})\|_{L^2(\Gamma_2)} \leq C_2 = C_1\|\gamma_0\| ,$$

where γ_0 is the trace operator (linear and continuous, defined on V), and $\alpha > 0$ is the coercivity constant on V_0 of the bilinear a , i.e. :

$$(4) \qquad \exists \alpha > 0 \ / \ a(v,v) = \|v\|_{V_0}^2 \geq \alpha \|v\|_V^2 , \ \forall v \in V_0 .$$

(iii) For all $q > 0$ and $h > 0$ we have

$$(5) \qquad 0 < \int_{\Gamma_2} u_q d\gamma - \int_{\Gamma_2} u_{q+h} d\gamma \leq C_3 h (C_3 = C_2 |\Gamma_2|^{\frac{1}{2}} > 0)$$

and therefore the function $q > 0 \to \int_{\Gamma_2} u_q d\gamma$ is continuous.

PROOF.- If $u_i = u_{q_i}$ is the solution of (I-7) for $q_i > 0 \ (i = 1, 2)$, then we have the following equalities:

$$(6) \qquad a(u_2 - u_1, u_2 - u_1) = (q_1 - q_2) \int_{\Gamma_2} (u_2 - u_1) d\gamma \ ,$$

$$(7) \qquad a(u_2, u_2) - a(u_1, u_1) = a(u_2 + u_1, u_2 - u_1) = (q_1 + q_2) \int_{\Gamma_2} (u_1 - u_2) d\gamma \ ,$$

because we take $v = u_2 \in K$ in the variational equality corresponding to u_1 , and $v = u_1 \in K$ in the one corresponding to u_2, and we add up and subtract both equalities. From (6) and (7) we obtain (2) and (3) [12].

Let $f : R^+ \to R$ be the real function defined by

$$(8) \qquad f(q) = J(u_q) = \frac{1}{2} a(u_q, u_q) + q \int_{\Gamma_2} u_q \, d\gamma \ .$$

REMARK 2.- To solve Problem P it is sufficient to find a value $q > 0$ for which we have $f(q) < 0$. We shall further see that this technique can still be improved.

THEOREM 2.- (i) The function f is differentiable. Moreover, f' is a continuous and strictly decreasing function, and it is given by the following expresion

$$(9) \qquad f'(q) = \int_{\Gamma_2} u_q d\gamma.$$

(ii) There exists a constant $C > 0$ such that

$$(10) \qquad a(u_q, u_q) = C q^2,$$

$$(11) \qquad f(q) = -\frac{C}{2} q^2 + B |\Gamma_2| q.$$

(iii) If

$$(12) \qquad q > q_0(B),$$

then we obtain a two-phase steady-state Stefan problem in Ω (i.e. u_q is a function of non-constant sign in Ω), where

$$(13) \qquad q_0(B) = \frac{B|\Gamma_2|}{C}.$$

(iv.) Constant $C = C(\Omega, \Gamma_1, \Gamma_2) > 0$ is given by

$$(14) \qquad C = a(u_3, u_3) = \int_{\Gamma_2} u_3 \, d\gamma,$$

where u_3 is the solution of the variational equality

$$(15) \begin{cases} a(u_3, v) = \int_{\Gamma_2} v \, d\gamma, & \forall v \in V_0, \\ u_3 \in V_0 \end{cases}$$

PROOF.- We deduce (8) by considering the fact that

$$(16) \qquad \frac{f(q+h) - f(q)}{h} = \frac{1}{2} \int_{\Gamma_2} u_q \, d\gamma + \frac{1}{2} \int_{\Gamma_2} u_{q+h} \, d\gamma$$

which is obtained from (I-7) after elementary manipulations.

Moreover, we have

$$(17) \qquad u_q = B - q \, u_3 \text{ in } \Omega,$$

$$(18) \qquad f'(q_0(B)) = 0.$$

We obtain the thesis by using the fact that if $\int_{\Gamma_2} u_q \, d\gamma < 0$ then $u_q^- \neq 0$ in $\overline{\Omega}$.

REMARK 3.- The sufficient condition $f(q) < 0$, to solve Problem P, was improved by the condition $f'(q) < 0$, which is optional (see examples more later). In the case where, because of symmetry, we find that the function u_q is constant on Γ_2 , the sufficient condition, given by (12), is also necessary to have a steady-state two-phase Stefan problem.

III. NUMERICAL ANALYSIS OF MIXED ELLIPTIC PROBLEMS WITH OR WITHOUT PHASE CHANGE

Now, we consider τ_h , a regular triangulation of polygonal domain Ω with Lagrange triangles of type 1, constituted by affine-equivalent finite element of class C^0, where $h > 0$ is a parameter which goes to zero. We can take h equals to the longest side of the triangles $T \in \tau_h$ and we can approximated V_0 by [2]:

$$(1) \qquad V_h = \{v_h \in C^0(\overline{\Omega}) \ / \ v_h|_T \in P_1(T), \ \ \forall T \in \tau_h, v_h|_{\Gamma_1} = 0\},$$

where P_1 is the set of the polynomials of degree less or equals than 1. Let π_h be the corresponding linear interpolation operator. Then, we can consider that there exists a constant $C_0 > 0$ (independent of the parameter h) such that

$$(2) \qquad ||v - \pi_h v||_V \leq C_0 h^{r-1} ||v||_{r,\Omega}, \forall v \in H^r(\Omega), \ \text{with } 1 < r \leq 2.$$

We consider the following finite dimensional approximate variational problem, corresponding to the continuous variational problem (I-7), given by:

$$(3) \begin{cases} a(u_h, v_h) = L(v_h), & \forall v_h \in V_h, \\ u_h \in K_h = B + V_h, \end{cases}$$

and we can obtain the following results.

LEMMA 1.- We have

$$(4) \qquad \lim_{h \to o^+} ||u_h - u||_V = 0,$$

where u is the unique solution of the variational equality (I-7).

PROOF.- Owing to meas $(\Gamma_1) > 0$, we have that the bilinear form a is coercitivity over V_0 and therefore $||.||_{V_0}$ and $||.||_V$ are two equivalents norms in V_0 . We follow a similar method developped in [2].

COROLLARY 2.- If we define

$$(5) \qquad \theta_h = \frac{1}{k_2} u_h^+ - \frac{1}{k_1} u_h^- \in V \ , \ \theta = \frac{1}{k_2} u^+ - \frac{1}{k_1} u^- \in V$$

then we have

$$(6) \qquad \lim_{h\to 0^+} \|\theta_h - \theta\|_H = 0 \ ,$$

where $H = L^2(\Omega)$.

PROOF.- If we consider the scalar product in H, defined by

$$(7) \qquad (u, v) = \int_\Omega u \ v \ dx,$$

then, we deduce

$$(8) \qquad \|u_h - u\|_H^2 = \|u_h^+ - u^+\|_H^2 + \|u_h^- - u^-\|_H^2 + 2(u_h^+, u^-) +$$

$$+2(u_h^+, u^+) \geq \|u_h^+ - u^+\|_H^2 + \|u_h^- - u^-\|_H^2 \ ,$$

that is

$$(9) \qquad \max(\|u_h^+ - u^+\| \ , \ \|u_h^- - u^-\|) \leq \|u_h - u\|_H \ .$$

From (5) we obtain:

$$(10) \qquad \|\theta_h - \theta\|_H \leq \frac{1}{k_2}\|u_h^+ - u^+\|_H + \frac{1}{k_1}\|u_h^- - u^-\|_H \leq$$

$$\leq (\frac{1}{k_1} + \frac{1}{k_2})\|u_h - u\|_H \ ,$$

i.e. (6).

The goal of this part is to consider the numerical analysis of the inequality (II - 12). We study sufficient (and/or necessary) conditions for the constant heat flux q on Γ_2 to obtain a change of phase (steady-state two-phase discretised Stefan problem) into the corresponding discretised domain, that is a discrete temperature of non-constant sign in Ω. We obtain that (similarly to the continuous problem):
(i) there exists a constant $C_h > 0$ (which depends only of the geometry of the domain Ω for each $h > 0$ and it is characterised by a variational problem) such that if $q > q_{0h}(B) = B|\Gamma_2|/C_h$ then the steady-state discretized problem presents two phases.
(ii) we have the estimations $C_h < C$ and $q_0(B) < q_{0h}(B)$ where C and

$q_0(B)$ have been obtained for the continuous problem by (II-14) and (II-13) respectively.
(iii) we deduce an error bounds for $C - C_h$ and $q_{0h}(B) - q_0(B)$ as a function of the parameter h.

In other words, we obtain for the mixed elliptic discretised probem, defined by u_h, analogous conditions to the ones obtained for the corresponding continuous problem [12], defined by u .

For each $q > 0$ we consider the functions $u(q) \in K$ and $u_h(q) \in K_h$, as the unique solution of the variational equalities (I-7) (continuous problem) and (3) (discrete problem) respectively. We define the real function $f_h : R^+ \to R$, for each $h > 0$, in the following way

$$(11) \qquad f_h(q) = J_q(u_h(q)) = \frac{1}{2}a(u_h(q), u_h(q)) + q\int_{\Gamma_2} u_h(q)d\gamma, \quad q > 0.$$

Therefore, we obtain the following properties:

THEOREM 3.- (i) If $u_i = u_h(q_i)$ is the solution of (3) for $q_i > 0$ $(i = 1, 2)$, then we have the following equalities:

$$(12) \qquad a(u_2 - u_1, u_2 - u_1) = (q_1 - q_2)\int_{\Gamma_2}(u_2 - u_1)d\gamma,$$

$$(13) \qquad a(u_2, u_2) - a(u_1, u_1) = a(u_2 + u_1, u_2 - u_1) = (q_1 + q_2)\int_{\Gamma_2}(u_1 - u_2)d\gamma\,.$$

(ii) For all real numbers $q > 0$ and Δ such that $(q + \Delta) > 0$, we obtain the following estimations:

$$(14) \qquad \left\|\frac{1}{\Delta}[u_h(q) - u_h(q+\Delta)]\right\|_V \leq D_1 = \frac{\|\gamma_0\|}{\alpha}|\Gamma_2|^{\frac{1}{2}},$$

$$(15) \qquad \left\|\frac{1}{\Delta}[u_h(q) - u_h(q+\Delta)]\right\|_{L^2(\Gamma_2)} \leq D_2 = D_1\|\gamma_0\|\,,$$

where γ_0 is the linear and continuous trace operator, defined over V . Moreover, the function

$$(16) \qquad q > 0 \;\to\; \int_{\Gamma_2} u_h(q)d\gamma \in R,$$

is a continuous and strictly decreasing function.
(iii) Function $f_h = f_h(q)$is derivable. Moreover, $f_h{}'$ is a continuous and strictly decreasing function and given by the following expression

$$(17) \qquad f_h{}'(q) = \int_{\Gamma_2} u_h(q) d\gamma.$$

PROOF.- (i) If we take $v = u_2 - u_1 \in V_h$ in the variational equality corresponding to u_1 and $v = u_1 - u_2 \in V_h$ in the one corresponding to u_2, and we add up and subtract both equalities, then we obtain (12) and (13) respectively.
(ii) Taking into account (II-4), the Cauchy-Schwarz inequality and the continuity of the operator γ_0 we deduce (14). From (14) and the continuity of γ_0 we have (15). Therefore we have (16) because

$$(18) \qquad \left| \int_{\Gamma-2} [u_h(q) - u_h(q+\Delta)] d\gamma \right| \leq D_2 |\Gamma_2|^{\frac{1}{2}} \Delta.$$

Moreover, the monotony property is a consequence of (12).
(iii) From (11) and elementary computations, we deduce

$$(19) \qquad \frac{1}{\Delta}[f_h(q+\Delta) - f_h(q)] = \frac{1}{2} \int_{\Gamma_2} [u_h(q) + u_h(q+\Delta)] d\gamma,$$

that is (17), by using (16).

THEOREM 4.- (i) The element $u_h = u_h(q) \in V_h$ can be written by

$$(20) \qquad u_h(q) = B - q\, u_{3h}$$

where u_{3h} is the unique solution of the variational equality

$$(21) \qquad \begin{cases} a(u_{3h}, v_h) = \int_{\Gamma_2} v_h d\gamma, & \forall v_h \in V_h, \\ u_{3h} \in V_h. \end{cases}$$

(ii) There exists a constant $C_h > 0$ such that

$$(22) \qquad f_h(q) = qB|\Gamma_2| - \frac{1}{2} C_h q^2, \quad \forall q > 0,$$

$$(23) \qquad a(u_h(q), u_h(q)) = C_h q^2, \quad \forall q > o.$$

where the constant C_h is given by

$$(24) \qquad C_h = a(u_{3h}, u_{3h}) = \int_{\Gamma_2} u_{3h}\, d\gamma.$$

(iii) If

$$(25) \qquad q > q_{0h}(B),$$

then problem (3) represents a discretized steady-state two-phase Stefan problem (i.e. $u_h(q)$ is a funtion on non-constant sign in Ω), where

$$(26) \qquad q_{0h}(B) = \frac{B|\Gamma_2|}{C_h}.$$

PROOF.- (i) It follows from (3), (11) and (20) by uniqueness of the variational equalities (3) and (21).
(ii) It follows from (11) and (20); (iii) It follows taking into account

$$(27) \qquad f_h{}'(q_{0h}(B)) = 0,$$

and the monotony property of function $f_h{}'$.

THEOREM 5.- (i) We have the following equality:

$$(28) \qquad a(u(q), u_h(q)) = C_h q^2, \quad \forall q > 0.$$

(ii) We have the following inequalities:

$$(29) \qquad (a) C_h < C, \quad (b) q_{0h}(B).$$

PROOF.- (i) If we take $v = u_h(q) \in K_h = B + V_h \subset B + V_0 = K$ in the variational equality (I-7), and we take into account the expressions (II-10) and (22), then we obtain (28).
(ii) On the other hand, from (II-4) and (28) we have

$$(30) \qquad \alpha \|u(q) - u_h(q)\|_V{}^2 \le a(u(q) - u_h(q), u(q) - u_h(q)) = (C - C_h) q^2,$$

that is (29a). Moreover, (29b) follows from (II-13), (26) and (29a).

Now, we shall use the interpolation result (2) for the funtion $u_3 \in H^r(\Omega)$, as a hipotesis of regularity of the continuous problem (7) (in general, $1 < r < \frac{3}{2}$[5, 7, 8]). In [11], we present three examples with explicit solution. In these cases, we have $u(q), u_3 \in C^\infty(\Omega)$.

THEOREM 6.- We have the following relations and estimations:

$$(31) \qquad a(u(q)-u_h(q), v_h) = 0, \quad \forall v_h \in V_h,$$

$$(32) \qquad (C-C_h)q^2 = a(u(q)-u_h(q), u(q)-u_h(q)) \leq$$
$$\leq \inf_{v_h \in V_h} a(u(q)-v_h, u(q)-v_h),$$

$$(33) \qquad 0 < C - C_h \leq C_0^2 \ h^{2(r-1)} |u_3|_{r,\Omega}^2 \ ,$$

$$(34) \qquad 0 < q_{0h}(B) - q_0(B) \leq \frac{C_0^2 \ h^{2(r-1)}}{C} |u_3|_{r,\Omega}^2 \ q_{0h}(B).$$

PROOF.- If we take $v = v_h \in V_h \subset V_0$ in the variational equality (I-7) and we subtract it with the variational equality (3), we obtain (31). By using (28), (30) and (31) we deduce

$$(35) \qquad a(u(q)-u_h(q), u(q)-u_h(q)) = a(u(q)-u_h(q), u(q))-$$
$$-a(u(q)-u_h(q), u_h(q)) = a(u(q)-u_h(q), u(q)) - a(u(q)-u_h(q), v_h) =$$
$$= a(u(q)-u_h(q), u(q)-v_h) \leq [a(u(q)-u_h(q), u(q)-u_h(q))]^{\frac{1}{2}}.$$
$$.[a(u(q)-v_h, u(q)-v_h)]^{\frac{1}{2}}, \quad \forall v_h \in V_h,$$

because a(.,.) is a escalar product in V_0, then we obtain (32).

By using (32), the facts that

$$(36) \qquad \Pi_h(u(q)) \in B + V_h \ , \qquad u(q) - \Pi_h(u(q)) \in V_h$$

and the interpolation result (2), we deduce (33). The relation (34) is obtained by using the definition of $q_{0h}(B)$ and $q_0(B)$, and (33).

REMARK 1.- If we only have $u(q) \in V$ (i.e. $u_3 \in V$) we can obtain

$$(37) \qquad 0 < C - C_h \leq \frac{1}{q^2} \|u(q) - \Pi_h(u(q))\|_V^2 = \|u_3 - \Pi_h(u_3)\|_V^2,$$

where the second term converges to zero when $h \to 0^+$ [2], but we cannot give an order of convergence.

REMARK 2.- If the constant heat flux on Γ_2 verifies the inequality $q > q_{0_h}(B)$, then both discrete and continuous problem represent a steady-state two-phase Stefan problem, that is, their temperatures are of non-constant sign in Ω.

REMARK 3.- When the function $u_h(q)$ is constant on Γ_2 (as a funtion of $x \in \Gamma_2$), then the sufficient condition, given by (25), is also a necessary condition to have a two-phase discrete problem, because

$$(38) \qquad \int_{\Gamma_2} u_h(q) d\gamma < 0 \Leftrightarrow u_h(q) < 0 \text{ on } \Gamma_2.$$

THEOREM 7.- If we consider $h, B > 0$,and $0 < \epsilon_0 < 1$ (ϵ_0 is a parameter to be chosen arbitrarily), then we have the following estimations:

$$(39) \qquad q_0(B) < q_{0_h}(B) \le \frac{q_0(B)}{\epsilon_0} \quad \text{and} \quad C_h \ge C \ \ \epsilon_0 \ , \quad \forall h \le h_r(\epsilon_0),$$

$$(40) \qquad 0 < q_{0_h}(B) - q_0(B) \le \frac{C_0^2 \ \ |u_3|^2_{r,\Omega}}{C \ \ \epsilon_0} q_0(B) \ h^{2(r-1)}, \quad \forall h \le h_r(\epsilon_0),$$

where

$$(41) \qquad h_r(\epsilon_0) = \left(\frac{C(1-\epsilon_0)}{C_0^2 |u_3|^2_{r,\Omega}} \right)^{\frac{1}{2(r-1)}} .$$

PROOF.- From (34) we deduce

$$(42) \qquad A(h) q_{0_h}(B) \le q_0(B),$$

where

$$(43) \qquad A(h) = 1 - \frac{C_0^2 \ \ |u_3|^2_{r,\Omega}}{C} \ h^{2(r-1)} < 1.$$

If we consider, for each parameter $0 < \epsilon_0 < 1$ the following equivalence:

$$(44) \qquad 0 < \epsilon_0 < A(h) < 1 \Leftrightarrow 0 < h < h_r(\epsilon_0),$$

we can deduce the inequalities (39) and (40).

COROLLARY 8.- If $B > 0$, then we have the following limit

$$(45) \qquad \lim_{h\to 0+} q_{0h}(B) = q_0(B).$$

REMARK 4.- Every thing we proved in this paper is still valid if the boundary Γ of the bounded domain Ω is represented by the union of the portions ($\Gamma = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3$) such that they have the following characteristics:
(i) Γ_1 and Γ_2 have the same conditions as the ones previously described in (I-4).
(ii) Γ_3 is a wall impermeable to heat, i.e. we have $\frac{\partial \theta}{\partial n}|_{\Gamma_3} = 0$ in (I-4) and therefore $\frac{\partial u}{\partial n}|_{\Gamma_3} = 0$ in (I-6).

Moreover, the first example considered (see bellow) verifies this condition.

We shall give three examples in which the solution is explicity known [11] so that we can verify all the theoretical results obtained in this work.

Example 1.- We consider the following data

$$(46) \qquad \begin{cases} n = 2\,, \quad \Omega = (0, x_0) \times (0, y_0)\,, \quad x_0 > 0\,, \quad y_0 > 0, \\ \Gamma_1 = \{0\} \times [0, y_0]\,, \quad \Gamma_2 = \{x_0\} \times [0, y_0], \\ \Gamma_3 = (0, x_0) \times \{0\} \cup (0, x_0) \times \{y_0\} \end{cases}$$

Example 2.- Next we consider

$$(47) \qquad \begin{cases} n = 2\,, \quad 0 < r_1 < r_2\,, \quad \Gamma_3 = \emptyset, \\ \Omega: \text{ annulus of radius } r_1 \text{ and } r_2, \text{ centered at } (0,0), \\ \Gamma_1: \text{ circunference of radius } r_1 \text{ and center } (0,0), \\ \Gamma_2: \text{ circunference of radius } r_2 \text{ and center } (0,0). \end{cases}$$

Example 3.- Finally, we take into account the same information of Example 2 but now for the case $n = 3$.

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