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Sufficient and/or Necessary Condition for the Heat Transfer Coefficient on Γ_1 and the Heat Flux on Γ_2 to Obtain a Steady-State Two-Phase Stefan Problem

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1. INTRODUCTION

We consider a material Ω, a bounded domain of $\mathbb{R}^n$ ($n = 1, 2, 3$ in practice), with a sufficiently regular boundary $\Gamma = \Gamma_1 \cup \Gamma_2$ (with meas$(\Gamma_1) > 0$ and meas$(\Gamma_2) > 0$). We assume, without loss of generality that the phase-change temperature is 0°C. We impose a Newton law with coefficient $\alpha > 0$ on Γ_1 with a temperature $b = \text{const.} > 0$ and an outcoming heat flux $q = \text{const.} > 0$ on Γ_2.

If we consider in Ω a steady-state heat conduction problem, then we are interested in finding out sufficient and/or necessary conditions for the heat transfer coefficient α and/or for the heat flux q to obtain a change of phase in Ω, that is, a steady-state two phase Stefan problem in Ω. In others words, we are interested in obtaining the steady-state temperature of non-constant sign in Ω.

Following [3] we study the temperature $\theta = \theta(x)$, defined for $x \in \Omega$. The set Ω can be expressed in the form

$$\Omega = \Omega_1 \cup \Omega_2 \cup l, \tag{1}$$

where

$$\begin{aligned} \Omega_1 &= \{x \in \Omega / \theta(x) < 0\} \\ \Omega_2 &= \{x \in \Omega / \theta(x) > 0\} \\ l &= \{x \in \Omega / \theta(x) = 0\} \end{aligned} \tag{2}$$

are the solid phase, the liquid phase, and the free boundary that separates them, respectively.

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The temperature θ can be represented in Ω by

$$\theta(x)=\begin{cases}\theta_1(x)<0, & x\in\Omega_1\\ 0, & x\in l\\ \theta_2(x)>0, & x\in\Omega_2\end{cases}\tag{3}$$

and satisfies the conditions

$$\begin{aligned}
&\text{(i)} && \Delta\theta_i=0 \quad \text{in } \Omega_i \quad (i=1,2),\\
&\text{(ii)} && \theta_1=\theta_2=0,\quad k_1\frac{\partial\theta_1}{\partial n}=k_2\frac{\partial\theta_2}{\partial n} \quad \text{on } l,\\
&\text{(iii)} && -k_2\frac{\partial\theta_2}{\partial n}\bigg|_{\Gamma_2}=q \quad \text{if } \theta|_{\Gamma_2}>0,\\
& && -k_1\frac{\partial\theta_1}{\partial n}\bigg|_{\Gamma_2}=q \quad \text{if } \theta|_{\Gamma_2}<0,\\
&\text{(iv)} && -k_2\frac{\partial\theta_2}{\partial n}\bigg|_{\Gamma_1}=\alpha(k_2\theta_2-B) \quad \text{if } \theta|_{\Gamma_1}>0,\\
& && -k_1\frac{\partial\theta_1}{\partial n}\bigg|_{\Gamma_1}=\alpha(k_1\theta_1-B) \quad \text{if } \theta|_{\Gamma_1}<0,
\end{aligned}\tag{4}$$

where $k_i>0$ is the thermal conductivity of phase i ($i=1$: solid phase, $i=2$: liquid phase) and $B=k_2 b>0$.

If we defined the function u in Ω as

$$u=k_2\theta^+-k_1\theta^- \qquad \text{in } \Omega,\tag{5}$$

where θ^+ and θ^- represent the positive and the negative parts of the function θ, respectively, then problem (4) is transformed into

$$\begin{aligned}
&\text{(i)} && \Delta u=0 \quad \text{in } D'(\Omega)\\
&\text{(ii)} && -\frac{\partial u}{\partial n}\bigg|_{\Gamma_2}=q\\
&\text{(iii)} && -\frac{\partial u}{\partial n}\bigg|_{\Gamma_1}=\alpha(u-B)
\end{aligned}\tag{6}$$

whose variational formulation is given by ($u\equiv u_{\alpha q}$)

$$a_\alpha(u,v)=L_{\alpha qB}(v), \qquad \forall v\in V, \quad u\in V,\tag{7}$$

where

$$V = H^1(\Omega), \qquad a(u, v) = \int_\Omega \nabla u \cdot \nabla u \, dx,$$

$$a_\alpha(u, v) = a(u, v) + \alpha \int_{\Gamma_1} uv \, d\gamma, \tag{8}$$

$$L_{\alpha q B}(v) = L_q(v) + \alpha B \int_{\Gamma_1} v \, d\gamma, \qquad L_q(v) = -q \int_{\Gamma_2} v \, d\gamma.$$

If we consider the problem (4)(bis) by replacing condition (4)(iv) by

$$\theta|_{\Gamma_1} = b > 0 \qquad \text{(4)(iv bis)}$$

then the function u, defined by (5), satisfies conditions (6)(bis), i.e., (6)(i)(ii) and

$$u|_{\Gamma_1} = B \qquad \text{(6)(iii bis)}$$

whose variational formulation is given by ($u \equiv u_q$)

$$a(u, v - u) = L_q(v - u), \qquad \forall v \in K, \quad u \in K, \tag{9}$$

where

$$V_0 = \{v \in V / v|_{\Gamma_1} = 0\}, \qquad K = \{v \in V / v|_{\Gamma_1} = B\}. \tag{10}$$

The variational inequalities (7) and (9) have unique solutions for all $\alpha > 0$, $q > 0$, and $B > 0$ [1–3, 5]. We suppose that Ω and Γ have the necessary regularity so that these solutions belong to $C^0(\bar{\Omega})$.

For the problem (4)(bis) or (6)(bis) or (9), the following result was given in [6]: If $q > q_0$, where

$$q_0(B) = \frac{B}{C} \operatorname{meas}(\Gamma_2) \qquad (B > 0), \tag{11}$$

then we obtain a steady-state two-phase Stefan problem in Ω; i.e., the function u_q is of non-constant sign in Ω. The constant C is such that

$$a(u_q, u_q) = Cq^2, \qquad \forall q > 0, \tag{12}$$

and it can also be calculated by the expression

$$C = \frac{1}{q} \int_{\Gamma_2} (B - u_q) \, d\gamma = a(u_0, u_0) = \int_{\Gamma_2} u_0 \, d\gamma, \qquad q > 0, \tag{13}$$

where u_0 is the solution of the problem

$$a(u_0, v) = \int_{\Gamma_2} v \, d\gamma, \qquad \forall v \in V_0, \quad u_0 \in V_0. \tag{13)(bis}$$

Moreover, it has the physical dimension $[C] = (Cm)^n$, where n is the dimension of the space $\mathbb{R}^n$.

In this paper we shall generalize the above result for problem (4) or (6) or (7), i.e., for the problem with α and q as independent variables. In others words, we shall study the mixed boundary value problem for the Laplace equation (6) with the object of deciding when it exhibits a solution of non-constant sign in Ω.

In Section II we given some results related to the functions $u_{\alpha q}$ and u_q.

In Section III we consider the case where $q > q_0$ (q_0 given by (11)) and we obtain that there exists a steady-state two-phase Stefan problem in Ω (i.e., the function $u_{\alpha q}$ is of non-constant sign in Ω), for all $\alpha > \alpha_0$, where

$$\alpha_0(q, B) = \frac{q \operatorname{meas}(\Gamma_2)}{B \operatorname{meas}(\Gamma_1)} \qquad (q > q_0, B > 0). \tag{14}$$

Moreover, in the case where, because of symmetry, we find that $u_{\alpha q}$ is constant on Γ_2, then the sufficient condition (14) is also necessary.

In Section IV we consider the general case $\alpha > 0$ and $q > 0$ for each $B > 0$. We then obtain that there exists a steady-state two-phase Stefan problem in Ω (i.e., the function $u_{\alpha q}$ is of non-constant sign in Ω) for

$$q_m(\alpha, B) < q < q_M(\alpha, B), \qquad \alpha > 0 \ (B > 0), \tag{15}$$

where the functions q_m and q_M, defined for α, $B > 0$ are given by

$$q_m(\alpha, B) = \frac{B \operatorname{meas}(\Gamma_2)}{A(\alpha)}, \qquad q_M(\alpha, B) = \frac{B \operatorname{meas}(\Gamma_1)\, \alpha}{\operatorname{meas}(\Gamma_2)}, \tag{16}$$

where $A = A(\alpha)$ has an adequate expression. Moreover, we have that (for all $B > 0$)

$$\begin{gathered} q_m(0^+, B) = q_M(0^+, B) = 0, \\ \lim_{\alpha \to +\infty} q_m(\alpha, B) = q_0, \end{gathered} \tag{17}$$

q_m is an increasing monotone function of α.

In Section V we consider a particular case of the one developed in Section IV for which we can obtain more information for the expression of the function $A = A(\alpha)$.

Finally, we remark here that all that we have proved in this paper is still valid if the boundary Γ of Ω is represented by the union of three portions $\Gamma = \Gamma_1 \cup \Gamma_2 \cup \Gamma_2$ such that:

(i) Γ_1 and Γ_2 have the same condition as those previously described in problem (4).

(ii) Γ_3 is as wall impermeable to heat, i.e., we have $(\partial\theta/\partial n)|_{\Gamma_3} = 0$ in (4) and therefore, $(\partial u/\partial n)|_{\Gamma_3} = 0$ in (6).

In Section VI we give three examples in which the solution is explicitly known. Moreover, for all these examples, the sufficient condition (15) is also necessary.

The method employed in this paper is the elliptic variational inequalities theory [1, 2, 5] which has been used in numerous mechanical problems, e.g., free boundary problems, variational principles in elasticity and plasticity (Saint–Venant theory), etc. [1, 2, 5, 7].

We shall denote by $(N-n)$ the formula (n) of Section N and we shall omit N in the same paragraph. We also omit the space variable $x \in \Omega$ for every function defined in Ω.

II. Preliminary Properties of $u_{\alpha q}$ and u_q

We shall use $u_\alpha = u_{\alpha q}$ and $u_\infty = u_q$ when it is necessary for convenience of notation.

Remark 1. The bilinear form a_1 is coercive on V, i.e.,

$$\exists \lambda_1 > 0 / a_1(v, v) = a(v, v) + \int_{\Gamma_1} v^2 \, d\gamma \geqslant \lambda_1 \| v \|_V^2, \qquad \forall v \in V. \tag{1}$$

Moreover, so is the bilinear form a_α and we have

$$a_\alpha(u, v) \geqslant \lambda_\alpha \| v \|_V^2, \qquad \forall v \in V, \quad \lambda_\alpha = \lambda_1 \min(1, \alpha). \tag{2}$$

Lemma 1. *We have the following properties*:

$$\begin{aligned}
&\text{(i)} && u_{\alpha q} \to u_q && \textit{in } V, \textit{ when } \alpha \to +\infty, \forall q > 0,\\
&\text{(ii)} && u_{\alpha q} \leqslant B && \textit{in } \Omega, \forall \alpha > 0, \forall q > 0,\\
&\text{(iii)} && u_{\alpha q} \leqslant u_q \leqslant B && \textit{in } \Omega, \forall \alpha > 0, \forall q > 0,\\
&\text{(iv)} && u_{\alpha_1 q_1} \leqslant u_{\alpha_2 q_2} && \textit{in } \Omega, \forall q_2 \leqslant q_1, \forall \alpha_1 \leqslant \alpha_2.
\end{aligned} \tag{3}$$

Proof. (i) It was given in [3]. We will only prove (iv) because the other cases are analogous. We shall take into account the equivalence

$$u_1 \leqslant u_2 \quad \text{in } \Omega \Leftrightarrow w=0 \quad \text{in } \Omega, \tag{4}$$

where $w=(u_1-u_2)^+$ in Ω and $u_i=u_{\alpha_i q_i}$ $(i=1,2)$.

If we use $v=w \in V$ in the variational equality (I-7) corresponding to u_1, and $v=-w \in V$ in the one corresponding to u_2 and we later add them up, we have

$$a_1(w,w)+(q_1-q_2)\int_{\Gamma_2} w\,d\gamma+(\alpha_2-\alpha_1)\int_{\Gamma_1}(B-u_2)\,w\,d\gamma=0, \tag{5}$$

that is, $w=0$ in Ω.

LEMMA 2. *We have the properties*

$$M_2 \leqslant u_{\alpha q} \leqslant M_1 \quad \text{in } \Omega, \tag{6}$$

where

$$M_2=\operatorname*{Min}_{\Gamma_2} u_{\alpha q}, \qquad M_1=\operatorname*{Max}_{\Gamma_1} u_{\alpha q}. \tag{7}$$

Proof. Let w_1 and w_2 be the functions defined in the following way:

$$w_1=(u_{\alpha q}-M_1)^+, \qquad w_2=(M_2-u_{\alpha q})^+ \qquad \text{in } \Omega. \tag{8}$$

If we use $v=w_1 \in V$ and $v=w_2 \in V$ in the variational equality (I-7) and we take into account that $w_1|_{\Gamma_1}=0$ and $w_2|_{\Gamma_2}=0$, then we obtain

$$\begin{aligned} &a(w_1,w_1)+q\int_{\Gamma_2} w_1\,d\gamma=0,\\ &a(w_2,w_2)+\alpha\int_{\Gamma_1}(B-u_{\alpha q})\,w_2\,d\gamma=0; \end{aligned} \tag{9}$$

that is, $w_1=w_2=0$ in Ω, i.e., (6).

COROLLARY 3. *We have the following properties*:

(a)

$$\operatorname*{Max}_{\Omega} u_{\alpha q}=M_1, \qquad \operatorname*{Min}_{\Omega} u_{\alpha q}=M_2, \tag{10}$$

where the elements M_1 and M_2 are defined in (7).

(b) *The problem* (I-4) *or* (I-6) *or* (I-7) *is a steady-state two-phase Stefan problem in* Ω (*i.e., the function* $u_{\alpha q}$ *is of non-constant sign in* Ω) *iff*

$$\exists x_1 \in \Gamma_1, \qquad x_2 \in \Gamma_2 / u_{\alpha q}(x_1) > 0, \qquad u_{\alpha q}(x_2) < 0. \tag{11}$$

(c) *If* $u_{\alpha q}$ *satisfies the condition*

$$\int_{\Gamma_1} u_{\alpha q}\, d\gamma > 0, \qquad \int_{\Gamma_2} u_{\alpha q}\, d\gamma < 0, \tag{12}$$

then problem (I-7) *is a two-phase problem.*

LEMMA 4. *For all* $B > 0$, *we have the following expression*:

$$\int_{\Gamma_1} u_{\alpha q}\, d\gamma = B \operatorname{meas}(\Gamma_1) - \frac{q}{\alpha} \operatorname{meas}(\Gamma_2), \qquad \forall \alpha, q > 0. \tag{13}$$

Proof. By using (I-7) with $v = 1 \in V$, we obtain

$$0 = a_\alpha(u_{\alpha q}, 1) - L_{\alpha q B}(1) = \alpha \int_{\Gamma_1} u_{\alpha q}\, d\gamma + q \operatorname{meas}(\Gamma_2) - \alpha B \operatorname{meas}(\Gamma_1),$$

that is, (13).

LEMMA 5. *For all* $B > 0$, *we have the following properties*:

$$\begin{aligned} &\text{(i)} && a(u_q, u_q) = L_q(u_q) + Bq \operatorname{meas}(\Gamma_2), && \forall q > 0, \\ &\text{(ii)} && a(u_{\alpha q}, u_q) = a(u_q, u_q), && \forall \alpha, q > 0. \end{aligned} \tag{14}$$

Proof. If we use $v = B \in K$ in (9), we obtain

$$a(u_q, u_q) = a(u_q, u_q - B) = L_q(u_q - B) = {}_q(u_q) - L_q(B),$$

that is, (14)(i).

If we use $v = u_q \in K \subset V$ in (7) and take into account (13) and (14)(i) we obtain

$$\begin{aligned} a(u_{\alpha q}, u_q) &= L_{\alpha q B}(u_q) - \alpha \int_{\Gamma_1} u_{\alpha q} u_q\, d\gamma = L_q(u_q) + \alpha B \int_{\Gamma_1} u_q\, d\gamma \\ &\quad - \alpha B \int_{\Gamma_1} u_{\alpha q}\, d\gamma = L_q(u_q) + Bq \operatorname{meas}(\Gamma_2) = a(u_q, u_q), \end{aligned}$$

that is, (14)(ii).

III. Study of the Problem (I-4) When $q > q_0$

We shall study the problem (I-4) or (I-7) when the heat flux $q = \text{const.} > 0$ on Γ_2 is such that $q > q_0$ (q_0 is given by (I-11)); that is, the problem (I-4)(bis) steady-state two-phase Stefan problem in Ω (i.e., the function u_q is of non-constant sign in Ω). Then, we obtain the following result:

THEOREM 6. *If $q > q_0$, then* (I-4) *is a steady-state two-phase Stefan problem in Ω (i.e., the function $u_{\alpha q}$, solution of* (I-7)*, is of non-constant sign in Ω) for all $\alpha > \alpha_0$, where α_0 is given by* (I-14).

Proof. Owing to $q > q_0$, we have that [6]

$$\operatorname*{Min}_{\Omega} u_q = \operatorname*{Min}_{\Gamma_2} u_q < 0 \tag{1}$$

and therefore, by using (II-3)(iii), we deduce that

$$M_2 < 0, \qquad \forall \alpha > 0. \tag{2}$$

Besides, by using (II-13), we have that

$$\int_{\Gamma_1} u_{\alpha q}\, d\gamma > 0 \Leftrightarrow \alpha > \alpha_0 \qquad (\text{with } q > q_0), \tag{3}$$

then we obtain the thesis.

COROLLARY 7. *In the case where, because of symmetry, we find that function $u_{\alpha q}$ is constant on Γ_1 then the sufficient condition, given by Theorem* 6, *is also necessary for problem* (I-4) *to be a two-phase problem.*

Proof. Since $u_{\alpha q}|_{\Gamma_1} = \text{const.}$, the property follows from Theorem 5 and the following equivalence:

$$\int_{\Gamma_1} u_{\alpha q}\, d\gamma > 0 \Leftrightarrow u_{\alpha q}|_{\Gamma_1} > 0. \tag{4}$$

Remark 2. For the three examples given in Section VI we can apply the above corollary.

IV. Study of the General Problem (I-4)

We shall study the problem (I-4) or (I-7) for $\alpha > 0$ and $q > 0$ (and for an arbitrary but given $b > 0$ or $B > 0$); we shall give sufficient (and in some cases also necessary) conditions for problem (I-4) to be a two-phase problem.

From now on, we will denote $u = u(\alpha, q, B)$ (it was $u_{\alpha q}$ in Sects. I, II, and III) and $u_\infty = u_\infty(q, B)$ (it was u_q in Sects. I, II, and III) as the unique solutions of the variational equalities (I-7) and (I-9), respectively, for $\alpha > 0$, $q > 0$, and $B > 0$.

The solution $u(\alpha, q, B)$ of (I-7) is characterized by the minimum problem [1, 2, 5]

$$J_{\alpha q B}(u(\alpha, q, B)) \leqslant J_{\alpha q B}(v), \qquad \forall v \in V, \qquad u(\alpha, q, B) \in V, \tag{1}$$

where

$$J_{\alpha q B}(v) = \tfrac{1}{2} a_\alpha(v, v) - L_{\alpha q B}(v), \qquad \forall v \in V. \tag{2}$$

Following [6], let $f: (\mathbb{R}^+)^3 \to \mathbb{R}$ be the real function, defined by

$$f(\alpha, q, B) = J_{\alpha q B}(u(\alpha, q, B)) \tag{3}$$

which is equivalent to the expression

$$\begin{aligned} f(\alpha, q, B) &= -\frac{1}{2} a_\alpha(u(\alpha, q, B), u(\alpha, q, B)) = -\frac{1}{2} L_{\alpha q B}(u(\alpha, q, B)) \\ &= \frac{q}{2} \int_{\Gamma_2} u(\alpha, q, B)\, d\gamma - \frac{\alpha B}{2} \int_{\Gamma_1} u(\alpha, q, B)\, d\gamma \leqslant 0, \end{aligned} \tag{4}$$

by using equality (I-7) for $v = u(\alpha, q, B)$.

Remark 3. The function

$$(\alpha, q, B) \in (\mathbb{R}^+)^3 \to \int_{\Gamma_1} u(\alpha, q, B)\, d\gamma \tag{5}$$

is continuous because of expression (II-13).

Let be $h > 0$. If we define $\Delta_h u(\alpha)$, $\Delta_h u(q)$, and $\Delta_h u(B)$, for all $\alpha, q, B > 0$, as

$$\begin{aligned} \Delta_h u(\alpha) &= u(\alpha + h, q, B) - u(\alpha, q, B) \\ \Delta_h u(q) &= u(\alpha, q + h, B) - u(\alpha, q, B) \\ \Delta_h u(B) &= u(\alpha, q, B + h) - u(\alpha, q, B) \end{aligned} \tag{6}$$

then we obtain the following estimates:

LEMMA 8. *We have*

$$\|\Delta_h u(\alpha)\|_V \leqslant C_1 h \tag{7}$$

$$\|\Delta_h u(\alpha)\|_{L^2(\Gamma_1)} \leqslant C_2 h, \tag{8}$$

where

$$C_1 = \frac{\|\gamma_0\|}{\lambda_\alpha} \|B - u(\alpha, q, B)\|^2_{L^2(\Gamma_1)}, \qquad C_2 = C_1 \|\gamma_0\|, \tag{9}$$

and γ_0 is the trace operator (linear and continuous, defined on V).

Proof. If we use $v = \Delta_h u(\alpha)$ and $v = -\Delta_h u(\alpha)$ in (I-7) with the parameters $\alpha + h$ and α, respectively, add them up, and take into account (II-2), (II-3), the Cauchy–Schwarz inequality, and the continuity of γ_0, we deduce

$$\lambda_\alpha \|\Delta_h u(\alpha)\|^2_V \leqslant a_\alpha(\Delta_h u(\alpha), \Delta_h u(\alpha)) = h \int_{\Gamma_1} (B - u(\alpha + h))\, \Delta_h u(\alpha))\, d\gamma$$

$$= h \int_{\Gamma_1} (B - u(\alpha + h, q, B))\, \Delta_h u(\alpha)\, d\gamma$$

$$\leqslant h \|B - u(\alpha, q, B)\|^2_{L^2(\Gamma_1)}, \|\Delta_h u(\alpha)\|_V \|\gamma_0\|,$$

that is, (7). Taking into account (7) and the continuity of γ_0 we obtain (8).

COROLLARY 9. *For all $\alpha > 0$, $q > 0$, and $B > 0$, we have*

$$\lim_{h \to 0^+} \int_{\Gamma_1} u(\alpha + h, q, B)\, u(\alpha, q, B)\, d\gamma = \int_{\Gamma_1} u^2(\alpha, q, B)\, d\gamma. \tag{10}$$

Let be $h > 0$. We define $\Delta_h f(\alpha)$, $\Delta_h f(q)$, and $\Delta_h f(B)$, for all α, q, $B > 0$, as follows:

$$\begin{aligned} \Delta_h f(\alpha) &= f(\alpha + h, q, B) - f(\alpha, q, B) \\ \Delta_h f(q) &= f(\alpha, q + h, B) - f(\alpha, q, B) \\ \Delta_h f(B) &= f(\alpha, q, B + h) - f(\alpha, q, B). \end{aligned} \tag{11}$$

Then we obtain the following properties:

LEMMA 10. *We have*

$$\frac{\Delta_h f(\alpha)}{h} = \frac{1}{2} \int_{\Gamma_1} (u(\alpha + h, q, B)\, u(\alpha, q, B) - B(u(\alpha + h, q, B) + u(\alpha, q, B)))\, d\gamma, \tag{12}$$

$$\frac{\Delta_h f(q)}{h} = \frac{1}{2} \int_{\Gamma_2} (u(\alpha, q+h, B) + u(\alpha, q, B))\, d\gamma, \tag{13}$$

$$\frac{\Delta_h f(B)}{h} = -\frac{\alpha}{2} \int_{\Gamma_1} (u(\alpha, q, B+h) + u(\alpha, q, B))\, d\gamma. \tag{14}$$

Proof. We obtain (12)–(14) after some manipulations in the variational equality (I-7) by choosing different test functions and parameters. Moreover, we deduce

$$\int_{\Gamma_2} \Delta_h u(q)\, d\gamma = \frac{h}{q} \int_{\Gamma_2} u(\alpha, q, B)\, d\gamma + \frac{\alpha B}{q} \int_{\Gamma_1} \Delta_h u(q)\, d\gamma \tag{15}$$

before obtaining (13).

Remark 4. Taking into account that $\int_{\Gamma_1} u(\alpha, q, B)\, d\gamma$ is a continuous function with respect to $\alpha, q, B > 0$, we deduce that

$$q > 0 \to \int_{\Gamma_2} u(\alpha, q, B)\, d\gamma \tag{16}$$

is also a continuous function for all α, $B > 0$.

Finally, we obtain the

THEOREM 11. *Function f has partial derivatives with respect to variables α, q, and B, and they are given by the following expressions for all α, q, $B > 0$:*

$$\frac{\partial f}{\partial \alpha}(\alpha, q, B) = \int_{\Gamma_1} \left[\frac{1}{2} u^2(\alpha, q, B) - B u(\alpha, q, B)\right] d\gamma, \tag{17}$$

$$\frac{\partial f}{\partial q}(\alpha, q, B) = \int_{\Gamma_2} u(\alpha, q, B)\, d\gamma, \tag{18}$$

$$\frac{\partial f}{\partial B}(\alpha, q, B) = -\alpha \int_{\Gamma_1} u(\alpha, q, B)\, d\gamma. \tag{19}$$

Proof. Taking into account (5) (Remark 3), (10), and (16) (Corollary 10), it is enough to take the limit $h \to 0$ in (12)–(14) to obtain (17)–(19).

By using (II-13) and (19), we obtain the

COROLLARY 12. *For all α, q, $B > 0$, we have*

$$\frac{\partial f}{\partial B}(\alpha, q, B) = q \operatorname{meas}(\Gamma_2) - \alpha B \operatorname{meas}(\Gamma_1) \tag{20}$$

which is an affine function in each variable $\alpha > 0$, $q > 0$, and $B > 0$.

THEOREM 13. *There exists a function $A = A(\alpha)$, defined for $\alpha > 0$, such that*

$$f(\alpha, q, B) = -\frac{A(\alpha)}{2} q^2 + Bq \operatorname{meas}(\Gamma_2) - \frac{B^2 \alpha}{2} \operatorname{meas}(\Gamma_1). \tag{21}$$

Proof. By using (II-13), (4), and (18), we obtain

$$f(\alpha, q, B) = +\frac{q}{2} f_q(\alpha, q, B) + \frac{qB}{2} \operatorname{meas}(\Gamma_2) - \frac{B^2 \alpha}{2} \operatorname{meas}(\Gamma_1). \tag{22}$$

By differentiating (22) with respect to the variable q, we deduce

$$q f_{qq}(\alpha, q, B) - f_q(\alpha, q, B) = -B \operatorname{meas}(\Gamma_2) \tag{23}$$

and therefore $\partial^3 f / \partial q^3 \equiv 0$. Then, function f can be written in the form

$$f(\alpha, q, B) = -\frac{A(\alpha, B)}{2} q^2 + A_1(\alpha, B)\, q + A_2(\alpha, B). \tag{24}$$

By some manipulations with (22)–(24), we obtain

$$\begin{aligned} A_1(\alpha, B) &= B \operatorname{meas}(\Gamma_2), \\ A_2(\alpha, B) &= -\frac{B^2 \alpha}{2} \operatorname{meas}(\Gamma_1), \\ \frac{\partial A}{\partial B}(\alpha, B) &= 0, \end{aligned} \tag{25}$$

that is, (21).

By using (18) and (21), we deduce

COROLLARY 14. *For all $\alpha, q, B > 0$, we have*

$$\int_{\Gamma_2} u(\alpha, q, B)\, d\gamma = B \operatorname{meas}(\Gamma_2) - A(\alpha)\, q. \tag{26}$$

COROLLARY 15. *Function $A = A(\alpha)$ satisfies*

$$\begin{aligned} &\text{(i)} \qquad A(\alpha) > 0, \qquad \forall \alpha > 0, \\ &\text{(ii)} \qquad A \textit{ is a decreasing function of } \alpha. \end{aligned} \tag{27}$$

Proof. By using (II-3)(ii), (26), and by the fact that $\operatorname{meas}(\Gamma_2) > 0$ we deduce $A(\alpha) \geqslant 0$, $\forall \alpha > 0$. The case $A(\alpha) = 0$ is not possible for $\alpha > 0$ because we have

$$A(\alpha) = 0 \Rightarrow u(\alpha, q, B)|_{\Gamma_2} \equiv B \Rightarrow u(\alpha, q, B) \equiv B \qquad \text{in } \Omega \Rightarrow q = 0. \tag{28}$$

(ii) follows from (II-3)(iv) and (4) or (26).

Remark 5. By the definition of function f, we have $f(\alpha, q, B) \leqslant 0$ for all , q, $B > 0$. We can obtain the following limit cases:

$$\begin{aligned} &\text{(i)} \qquad f(\alpha, q, B) = 0, \qquad \text{for some } \alpha > 0, \Rightarrow q = B = 0, \\ &\text{(ii)} \qquad q = B = 0 \Leftrightarrow f(\alpha, q, B) = 0, \qquad \forall \alpha > 0. \end{aligned} \tag{29}$$

Moreover, we have

$$f(\alpha, q, B) < 0, \qquad \forall \alpha, q, B > 0. \tag{30}$$

Proof. (i) We suppose $f(\alpha_1, q, B) = 0$ for some $\alpha = \alpha_1 > 0$. By using (4), we deduce $u(\alpha_1, q, B) = 0$ in Ω, and therefore, we obtain $B = q = 0$ because of (I-6). (ii) follows from (i) and (21).

We shall improve (27).

LEMMA 16. *We suppose that* $\operatorname{meas}(\Gamma_1) > 0$, $q > 0$, *and* $B > 0$. *Then, the following propositions are equivalent*:

$$\begin{aligned} &\text{(i) } \operatorname{meas}(\Gamma_2) > 0; \qquad \text{(ii) } A(\alpha) > \frac{(\operatorname{meas}(\Gamma_2))^2}{\operatorname{meas}(\Gamma_1)} \frac{1}{\alpha}, \forall \alpha > 0; \\ &\text{(iii) } A(\alpha) > 0, \forall \alpha > 0; \qquad \text{(iv) } A(\alpha) > 0, \textit{for some } \alpha > 0. \end{aligned} \tag{31}$$

Proof. (ii) $\Rightarrow$ (iii) and (iii) $\Rightarrow$ (iv) are evident. (iv) $\Rightarrow$ (i) follows from (26). To prove (i) $\Rightarrow$ (ii) it is enough to use $B = \operatorname{meas}(\Gamma_2)/\operatorname{meas}(\Gamma_1)(q/\alpha)$ in (21), because of (30).

Remark 6. As (31)(i) is one of our general hypotheses, we have obtained the inequality (31)(ii) for function $A = A(\alpha)$.

COROLLARY 17. *For all* $B > 0$, *we have that the set*

$$S^{(2)} = \{(\alpha, q) \in (\mathbb{R}^+)^2 / q_m(\alpha, B) < q < q_M(\alpha, B), \alpha > 0\} \neq \varnothing, \tag{32}$$

where q_m *and* q_M *are defined by* (I-16).

Proof. By using (31)(ii) we have $q_m(\alpha, B) < q_M(\alpha, B)$ for all $\alpha > 0$, $B > 0$, that is, (32).

Now, we shall give a sufficient condition in order to have a two-phase problem for (I-7).

THEOREM 18. *If* $(\alpha, q) \in S^{(2)}$, *then* (I-4) *or* (*I*-7) *is a steady-state two-phase Stefan problem in* Ω; *that is, the function* $u(\alpha, q, B)$ *is of non-constant sign in* Ω.

Proof. By using (II-13) and (26), we deduce the equivalences

$$\begin{aligned} &\text{(i)} \qquad \int_{\Gamma_1} u(\alpha, q, B)\, d\gamma > 0 \Leftrightarrow q < q_M(\alpha, B), \\ &\text{(ii)} \qquad \int_{\Gamma_2} u(\alpha, q, B)\, d\gamma < 0 \Leftrightarrow q > q_m(\alpha, B), \end{aligned} \tag{33}$$

which yield the theorem because of Corollary 3(c).

COROLLARY 19. *In the case where, because of symmetry, we find that function* $u(\alpha, q, B)$ *is constant on* Γ_1 *and* Γ_2, *respectively, then the sufficient condition, given by Theorem* 18, *is also necessary for problem* (I-4) *to be a two-phase problem. Moreover, for the three examples given in Section* VI, *we can apply this fact.*

LEMMA 20. *We have*

$$\lim_{\alpha \to +\infty} A(\alpha) = C > 0, \tag{34}$$

where C is the constant defined by (I-13), *which is independent of* q, $B > 0$.

Proof. By using (3)(i) we deduce that

$$\lim_{\alpha \to +\infty} \int_{\Gamma_2} u(\alpha, q, B)\, d\gamma = \int_{\Gamma_2} u_\infty(q, B)\, d\gamma \tag{35}$$

and therefore we have (34) because of (I-13) and (26). Moreover, constant C is independent of q, $B > 0$ and positive because $A = A(\alpha)$ is also independent of q, $B > 0$ and $u_\infty(q, B)|_{\Gamma_2} < B$, respectively.

COROLLARY 21. *Function* $q_m = q_m(\alpha, B)$ *is an increasing monotone function of variable* α *and satisfies*

$$\lim_{\alpha \to +\infty} q_m(\alpha, B) = q_0(B) > 0, \qquad q_m(0^+, B) = 0, \tag{36}$$

where $q_0 = q_0(B)$ *is defined by* (I-11).

We shall give a new proof of the result of [6] by passing to the limit $\alpha \to +\infty$ for the above results.

THEOREM 22. *If q, $B>0$ are such that $q>q_0(B)$ then* (I-4)(bis) or (I-9) *is a steady-state two-phase Stefan problem in Ω; that is, the function $u_\infty(q, B)$ is of non-constant sign in Ω.*

Proof. By using (II-3)(i), (27)(ii), and (34), we have

$$\int_{\Gamma_2} u(\alpha, q, B)\, d\gamma = B\,\text{meas}(\Gamma_2) - A(\alpha)\, q \leqslant M < 0, \qquad \forall \alpha > 0, \tag{37}$$

where

$$M = B\,\text{meas}(\Gamma_2) - Cq < B\,\text{meas}(\Gamma_2) - Cq_0 = 0. \tag{38}$$

Therefore, by passing to the limit $\alpha \to +\infty$, we obtain

$$\int_{\Gamma_2} u_\infty(q, B)\, d\gamma \leqslant M < 0; \tag{39}$$

that is, problem (I-4)(bis) or (I-9) iss a two-phase problem [6].

We shall obtain new properties of the function $A = A(\alpha)$.

LEMMA 23. *We have*

$$(\alpha A(\alpha))' = \frac{1}{q^2}\, a(u(\alpha, q, B), u(\alpha, q, B)) > 0, \qquad \forall \alpha > 0, \tag{40}$$

where a is the bilinear form, defined in (I-8), *and* $(\ )' = d(\ )/d\alpha$.

Proof. By using (II-13), (4), (17), and (21), we obtain (omitting variables α, q, B)

$$\begin{aligned} a(u, u) &= -2f - \alpha \int_{\Gamma_1} u^2\, d\gamma = -2f - 2\alpha f_\alpha - 2\alpha B \int_{\Gamma_1} u\, d\gamma \\ &= -2f + \alpha A'(\alpha)\, q^2 + B^2 \alpha\, \text{meas}(\Gamma_1) - 2\alpha B \int_{\Gamma_1} u\, d\gamma \\ &= q^2(A + \alpha A') = q^2(\alpha A)', \end{aligned}$$

that is, (40).

COROLLARY 24. *We have*

$$\lim_{\alpha \to +\infty} \alpha A'(\alpha) = 0. \tag{41}$$

Proof. By using (II-3)(i) and (34), we obtain

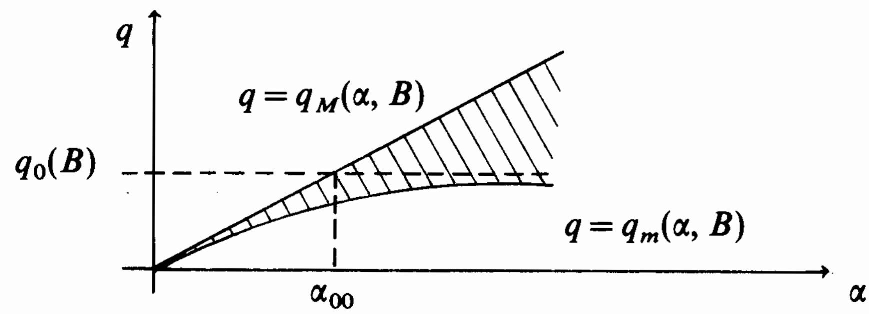
$$\lim_{\alpha \to +\infty} \alpha A'(\alpha) = \lim_{\alpha \to +\infty} \frac{1}{q^2} a(u(\alpha, q, B), u(\alpha, q, B)) - \lim_{\alpha \to +\infty} A(\alpha) = C - C = 0,$$

that is, (41).

COROLLARY 25. *For all $B > 0$, function $q_m = q_m(\alpha, B)$ satisfies*

$$\frac{\partial q_m}{\partial \alpha}(\alpha, B) = -\frac{B \operatorname{meas}(\Gamma_2)\, A'(\alpha)}{A^2(\alpha)} > 0, \qquad \forall \alpha > 0,$$
$$\lim_{\alpha \to +\infty} \frac{\partial q_m}{\partial \alpha}(\alpha, B) = 0. \tag{42}$$

Remark 7. In the plane α, q (for a given $B > 0$), we represent with dashes the region where a two-phase Stefan problem is obtained for problem (I-4), (I-6), or (I-7).



The number α_{00} is defined by

$$\alpha_{00} = \alpha_0(q_0, B) = \frac{[\operatorname{meas}(\Gamma_2)]^2}{C \operatorname{meas}(\Gamma_1)}, \tag{43}$$

which is the α-component of the intersection point between the two straight lines $q = q_0(B)$ and $q = q_M(B)$. We remark that α_{00} is independent of B.

The function $A(\alpha)$, defined for $\alpha > 0$, is not explicitly known but has the properties (27)(i)(ii), (31)(ii), (34), (40), and (41). We shall now consider a particular case for which we can obtain more information about the expression of $A(\alpha)$.

V. A PARTICULAR CASE

We consider the case when $u = u(\alpha, q, B)$ satisfies the condition

$$\frac{1}{q^2} a(u(\alpha, q, B), u(\alpha, q, B)) = \text{const.}, \tag{1}$$

or equivalently

$$(\alpha A(\alpha))' = A(\alpha) + \alpha A'(\alpha) = \text{const.} \tag{2}$$

By using Lemma 23 and Corollary 24, necessarily we have that

$$\text{const.} = \text{const.}(\alpha) = C > 0, \qquad \forall \alpha > 0, \tag{3}$$

where $C > 0$ is the constant defined before by (I-13).

LEMMA 26. *We have the equivalence*

$$u_\infty - u_\alpha = \text{const.} \quad \text{in} \quad \Omega \Leftrightarrow (\alpha A(\alpha))' = C, \tag{4}$$

where we note $u_\infty = u_\infty(q, B)$ *and* $u_\alpha = u(\alpha, q, B)$.

Proof. By using (II-14)(ii), we obtain

$$\begin{aligned} u_\infty - u_\alpha = \text{const.} \quad \text{in} \quad \Omega &\Leftrightarrow a(u_\infty - u_\alpha, u_\infty - u_\alpha) = 0 \\ &\Leftrightarrow a(u_\alpha, u_\alpha) = a(u_\infty, u_\infty) \\ &\Leftrightarrow \frac{a(u_\alpha, u_\alpha)}{q^2} = \frac{a(u_\infty, u_\infty)}{q^2} \quad (= C) \Leftrightarrow (\alpha A(\alpha))' = C, \end{aligned}$$

that is, (4).

THEOREM 27. *For* q, $B > 0$, *we have that the following propositions are equivalent* $(\alpha, \beta > 0)$:

$$\begin{aligned} &\text{(i)} && u_\infty - u_\alpha = C_1 (\text{const.}) && \text{in } \Omega, \\ &\text{(ii)} && u_\infty - u_\alpha = \frac{q}{\alpha} \frac{\text{meas}(\Gamma_2)}{\text{meas}(\Gamma_1)} && \text{in } \Omega, \\ &\text{(iii)} && u_\beta - u_\alpha = \frac{\beta - \alpha}{\beta\alpha} q \frac{\text{meas}(\Gamma_2)}{\text{meas}(\Gamma_1)} && \text{in } \Omega, \\ &\text{(iv)} && u_\beta - u_\alpha = C_2 (\text{const.}) && \text{in } \Omega, \\ &\text{(v)} && \left.\frac{\partial u_\beta}{\partial n}\right|_{\Gamma_1} = \left.\frac{\partial u_\alpha}{\partial n}\right|_{\Gamma_1} && \text{on } \Gamma_1, \\ &\text{(vi)} && u_\alpha|_{\Gamma_1} = B - \frac{q}{\alpha} \frac{\text{meas}(\Gamma_2)}{\text{meas}(\Gamma_1)} && \text{on } \Gamma_1, \\ &\text{(vii)} && \left.\frac{\partial u_\alpha}{\partial n}\right|_{\Gamma_1} = q \frac{\text{meas}(\Gamma_2)}{\text{meas}(\Gamma_1)} && \text{on } \Gamma_1, \end{aligned} \tag{5}$$

(viii) $$\left.\frac{\partial u_\alpha}{\partial n}\right|_{\Gamma_1} = C_3 \text{ (const.)} \qquad on\ \Gamma_1,$$

(ix) $$\left.\frac{\partial u_\infty}{\partial n}\right|_{\Gamma_1} = C_3 \text{ (const)} \qquad on\ \Gamma_1.$$

Proof. To prove the above equivalences we shall use the following plan:

$$\text{(i)} \Rightarrow \text{(ii)} \Rightarrow \text{(iii)} \Leftrightarrow \text{(iv)} \Leftrightarrow \text{(v)} \Rightarrow \text{(vi)} \Rightarrow \text{(vii)} \Rightarrow \text{(viii)} \Rightarrow \text{(ix)} \Rightarrow \text{(i)}.$$

The conditions (ii) ⇒ (iii) ⇒ (iv) ⇒ (v) and (vii) ⇒ (viii) are evident taking

$$C_2 = \frac{\beta-\alpha}{\beta\alpha} q \frac{\text{meas}(\Gamma_2)}{\text{meas}(\Gamma_1)}, \qquad C_3 = q \frac{\text{meas}(\Gamma_1)}{\text{meas}(\Gamma_1)}. \tag{6}$$

(i) ⇒ (ii). By using (II-13), we obtain

$$C_1 \text{ meas}(\Gamma_1) = \int_{\Gamma_1} (u_\infty - u_\alpha)\, d\gamma = B \text{ meas}(\Gamma_1) - \int_{\Gamma_1} u_\alpha \, d\gamma = \frac{q}{\alpha} \text{meas}(\Gamma_2),$$

that is,

$$C_1 = \frac{q}{\alpha} \frac{\text{meas}(\Gamma_2)}{\text{meas}(\Gamma_1)}. \tag{7}$$

(iv) ⇒ (iii). It follows from (II-13).
(v) ⇒ (iv). Owing to the Green formula and the fact that

$$\left.\frac{\partial u_\alpha}{\partial n}\right|_{\Gamma_2} = \left.\frac{\partial u_\beta}{\partial n}\right|_{\Gamma_2} = -q,$$

we obtain $a(u_\beta - u_\alpha, u_\beta - u_\alpha) = 0$, that is, (iv).
(iii) and (v) ⇒ (vi). By using (I-6)(iii) we obtain

$$\frac{\beta-\alpha}{\beta\alpha} q \frac{\text{meas}(\Gamma_2)}{\text{meas}(\Gamma_1)} = (u_\beta - u_\alpha)|_{\Gamma_1} = \frac{\beta-\alpha}{\beta\alpha}(B - u_\alpha)|_{\Gamma_1},$$

that is, (vi).
(vi) ⇒ (vii). It follows from (I-6)(iii).
(viii) ⇒ (ix). Let v be the function defined by

$$v = u_\alpha + \frac{1}{\alpha} C_3 \qquad \text{in } \Omega \left(\left.\frac{\partial u_\alpha}{\partial n}\right|_{\Gamma_1} = C_3 \right) \tag{8}$$

which satisfies the problem

$$\Delta v = 0 \qquad \text{in } \Omega,$$
$$v|_{\Gamma_1} = B, \qquad -\left.\frac{\partial v}{\partial n}\right|_{\Gamma_2} = q; \tag{9}$$

i.e., $v \equiv u_\infty$ by the uniqueness of problem (I-6)(bis). Therefore, we obtain

$$\left.\frac{\partial u_\infty}{\partial n}\right|_{\Gamma_1} = \left.\frac{\partial v}{\partial n}\right|_{\Gamma_1} = \left.\frac{\partial u_\alpha}{\partial n}\right|_{\Gamma_1} = C_3,$$

that is, (ix).

(ix) $\Rightarrow$ (i). Let w be the function defined by

$$w = u_\infty - \frac{1}{\alpha} C_3 \qquad \text{in } \Omega \left(C_3 = \left.\frac{\partial u_\infty}{\partial n}\right|_{\Gamma_1} = \left.\frac{\partial u_\alpha}{\partial n}\right|_{\Gamma_1} \right). \tag{10}$$

This function satisfies the problem

$$\Delta w = 0 \qquad \text{in } \Omega,$$
$$-\left.\frac{\partial w}{\partial n}\right|_{\Gamma_1} = -\left.\frac{\partial u_\infty}{\partial n}\right|_{\Gamma_1} = -C_3 = \alpha(w - B), \tag{11}$$
$$-\left.\frac{\partial w}{\partial n}\right|_{\Gamma_2} = -\left.\frac{\partial u_\infty}{\partial n}\right|_{\Gamma_2} = q;$$

i.e., $w = u_\alpha$ by the uniqueness of problem (I-6). Therefore, we obtain (i) with $C_1 = C_3/\alpha$.

Now we can obtain an expression for $A = A(\alpha)$ in this particular case.

THEOREM 28. *We have the following equivalence*:

$$\text{(i) of Theorem 27} \Leftrightarrow A(\alpha) = C + \frac{1}{\alpha}\frac{(\text{meas}(\Gamma_2))^2}{\text{meas}(\Gamma_1)}. \tag{12}$$

Proof. ($\Leftarrow$) We have $\alpha A(\alpha) = \text{const.} + C\alpha$, i.e., $(\alpha A(\alpha))' = C$. Therefore, (i) of Theorem 27 follows from Lemma 26.

($\Rightarrow$) From Lemma 26 we have that $(\alpha A(\alpha))' = C$, i.e., by integration,

$$A(\alpha) = C + \frac{1}{\alpha} C_4 \qquad (C_4 = \text{const.}). \tag{13}$$

As $u_\alpha|_{\Gamma_1} = B - C_1$ we can evaluate $\int_{\Gamma_1} u_\alpha^2\, d\gamma$. By using (1) we obtain that

$$f(\alpha, q, B) = -\frac{1}{2} a(u_\alpha, u_\alpha) - \frac{\alpha}{2}\int_{\Gamma_1} u_\alpha^2\, dy$$
$$= -\frac{q^2}{2}\left(C + \frac{1}{\alpha}\frac{(\text{meas}(\Gamma_2))^2}{\text{meas}(\Gamma_1)}\right) + Bq\,\text{meas}(\Gamma_2) - \frac{\alpha B^2}{2}\text{meas}(\Gamma_1),$$

i.e.,

$$C_4 = \frac{(\text{meas}(\Gamma_2))^2}{\text{meas}(\Gamma_1)}. \tag{14}$$

Remark 8. (Continuation of Corollary 19). If $u(\alpha, q, B)|_{\Gamma_1} = \text{const.}$ we can also compute the function $A = A(\alpha)$ because of the above results.

VI. Some Examples

We shall give some examples in which the solution is explicitly known [4].

(i) We consider the data

$$\begin{gathered} n = 2, \qquad \Omega = (0, x_0) \times (0, y_0),\ x_0 > 0,\ y_0 > 0, \\ \Gamma_1 = \{0\} \times [0, y_0], \qquad \Gamma_2 = \{x_0\} \times [0, y_0], \\ \Gamma_3 = (0, x_0) \times \{0\} \cup (0, x_0) \times \{y_0\} \end{gathered} \tag{1}$$

and then we obtain

$$u_\infty(x, y) = B - qx, \qquad u_\alpha(x, y) = B - \frac{q}{\alpha} - qx,$$

$$\begin{gathered} C = x_0\, y_0, \qquad q_0(B) = \frac{B}{x_0}, \\ \alpha_0(q, B) = \frac{q}{B}, \qquad \alpha_{00} = \frac{1}{x_0}, \qquad A(\alpha) = y_0\left(x_0 + \frac{1}{\alpha}\right), \end{gathered} \tag{2}$$

$$q_m(\alpha, B) = \frac{B}{x_0 + (1/\alpha)}, \qquad q_M(\alpha, B) = B\alpha.$$

(ii) Next we consider

$$\begin{gathered} n = 2, \qquad 0 < r_1 < r_2, \\ \Omega\text{: annulus of radii } r_1 \text{ and } r_2, \text{ centered at } (0, 0), \\ \Gamma_1\text{: circle of radius } r_1 \text{ and center } (0, 0), \\ \Gamma_2\text{: circle of radius } r_2 \text{ and center } (0, 0), \end{gathered} \tag{3}$$

and we arrive at

$$
\begin{gathered}
u_\infty(x, y) = B - qr_2 \log\left(\frac{r}{r_1}\right), \qquad r = (x^2 + y^2)^{1/2}, \\
u_\alpha(x, y) = B - \frac{q}{\alpha}\frac{r_2}{r_1} - qr_2 \log\left(\frac{r}{r_1}\right), \\
C = 2\pi r_2^2 \log\left(\frac{r_2}{r_1}\right), \qquad q_0(B) = \frac{B}{r_2 \log(r_2/r_1)}, \\
A(\alpha) = 2\pi r_2^2 \left(\frac{1}{\alpha r_1} + \log\left(\frac{r_2}{r_1}\right)\right), \\
\alpha_0(q, B) = \frac{qr_2}{Br_1}, \qquad \alpha_{00} = \frac{1}{r_1 \log(r_2/r_1)}, \\
q_m(\alpha, B) = \frac{1}{r_2((1/\alpha r_1) + \log(r_2/r_1))}, \qquad q_M(\alpha, B) = B\alpha \frac{r_1}{r_2}.
\end{gathered} \tag{4}
$$

For the numerical approximation and owing to the symmetry of the problem, it is convenient to solve it for a quarter of the circular crown (the one corresponding to the first quadrant), bearing in mind that in this case a new portion Γ_3 of the boundary appears , which is given by

$$\Gamma_3 = \{0\} \times (r_1, r_2) \cup (r_1, r_2) \times \{0\}. \tag{5}$$

Therefore, the values for meas(Γ_1), meas(Γ_2), and C are modified by a $\frac{1}{4}$ factor, but the expressions of q_0 and α_0, which are the values or our interest, do not vary.

(iii) Finally, we take into account the same information of example (ii) but now for the case $n = 3$; by doing this, we reach the following results:

$$
\begin{gathered}
u_\infty(x, y, z) = B - qr_2^2\left(\frac{1}{r_1} - \frac{1}{r}\right), \qquad r = (x^2 + y^2 + z^2)^{1/2}, \\
u_\alpha(x, y, z) = B - \frac{q}{\alpha}\frac{r_2^2}{r_1^2} - qr_2^2\left(\frac{1}{r_1} - \frac{1}{r}\right), \\
C = 4\pi \frac{r_2^3(r_2 - r_1)}{r_1}, \qquad \alpha_0(q, B) = \frac{qr_2^2}{Br_1^2}, \\
A(\alpha) = 4\pi r_2^4\left(\frac{1}{\alpha r_1^2} + \frac{1}{r_1} - \frac{1}{r_2}\right), \\
q_0(B) = \frac{Br_1}{r_2(r_2 - r_1)}, \qquad \alpha_{00} = \frac{r_2}{r_1(r_2 - r_1)}, \\
q_m(\alpha, B) = \frac{B}{r_2^2((1/\alpha r_1^2) + (1/r_1) - (1/r_2))}, \qquad q_M(\alpha, B) = B\alpha \frac{r_1^2}{r_2^2}.
\end{gathered} \tag{6}
$$

Remark 9. We remark that for the three above examples we can directly verify all the theoretical results obtained in this paper.

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