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Modelling and analysis of a viscoelastic contact problem with unilateral constraints

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Abstract
We consider a mathematical model which describes the equilibrium of two viscoelastic membranes situated in parallel plans, submitted to the action of external forces. As a result, the membranes could arrive in contact with two obstacles and could arrive in contact each other, too. The contact is frictionless and is described both with normal compliance and the Signorini unilateral contact condition. We list the assumption on the data, then we derive a variational formulation of the model which is in a form of a history-dependent variational inequality for the displacement field. We prove the unique weak solvability of the contact model, then we consider an associated optimization problem, for which we prove the existence of the solution. We also provide some conclusions and list related problems for further research.

Keywords Viscoelastic membrane · Unilateral contact · Normal compliance · History-dependent variational inequality · Weak solution · Optimal design

Mathematics Subject Classification 74K15 · 74M15 · 74G22 · 74G30 · 49J40 · 35B30

1 Introduction

Contact involving thin structures like membranes and shells arise in industry and everyday life. They lead to interesting mathematical problems and their analysis represents a first step in the study of more complicate problems stated in the three-dimensional setting. For this reason, there is a real interest in the analysis and numerical simulation of such kind or problems, and the literature in the field is extensive. It includes [2–4, 12, 13, 19], for instance and, more recently, [11, 14, 20]. A brief description of the models and results obtained in these papers follows.

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First, [4] deals with study of the shape of an elastic membrane in a confining box, by introducing a repulsive confinement pressure that prevents the membrane from intersecting an obstacle, say a wall. Reference [11] deal with a contact model in which the process is dynamic, the membrane is supposed to have a viscoelastic behaviour and the contact is assumed to be unilateral. For the model studied in [19] the material behaviour is given by Christensen's large deformation viscoelastic law and the contact is with a rigid obstacle. Reference [20] focus on the study of a hyperelastic membrane in contact with rigid obstacles whose geometry admits an analytical description. The numerical methods developed there allow to identify the evolution of the contact interface. The results in [2, 3] concern the finite element discretization of contact models for elastic membranes. In contrast, the references [12, 13] deal with the modelling of contact problems involving membranes. To this end, the authors use arguments of asymptotic analysis which consists to consider contact models for shells with a nontrivial curvature, and to pass to the limit as the thickness converges to zero. In [14] a mathematical model which describes the equilibrium of two elastic membranes fixed on their boundary and attached to an adhesive body, say a glue, was considered. The variational analysis of the model was provided, including existence, uniqueness and convergence results. Numerical simulations have also been obtained, together with their mechanical interpretations. The results in this reference have been obtained by using arguments of elliptic quasivariational inequalities.

The current paper represents a continuation of [14]. In contrast with [14], in the current paper we deal with a model which describes the equilibrium of two viscoelastic membranes which could arrive in contact with two obstacles and/or could arrive in contact, each other, too. We use a constitutive law with long term memory to describe the material behaviour. As a result, the variational formulation of the problem is expressed in terms of an history-dependent variational inequality, for which we state and prove existence, uniqueness and convergence results. The first trait of novelty of our work consists in the contact model we consider, which seams to be new, and for which we provide a rigorous description of the equations and boundary conditions. The second novelty comes from the variational formulation of this model, whose derivation requires the use of nonstandard computations and estimates. Finally, the last novelty is provided by the existence result of an optimal design problem we consider, which could have real-word applications.

The rest of paper is structured as follows. In Sect. 2 we describe the physical setting, then we list the mechanical assumptions and state the corresponding mathematical model (Problem $\mathcal{M}$). It is in the form of a system coupling partial differential equations and inequalities, associated to homogeneous boundary conditions. In Sect. 3 we deduce a variational formulation of the model which is in a form of a history-dependent variational inequality for the displacement field (Problem $\mathcal{P}$). We start Sect. 4 by recalling an abstract general existence and uniqueness result of history-dependent variational inequalities in Hilbert spaces. Then, we use this result and prove the unique solvability of Problem $\mathcal{P}$. In Sect. 5 we prove the solvability of an associated optimization problem (Problem $\mathcal{Q}$). The main ingredient of the proof consists in a continuous dependence result of the solution with respect to the distances which define the physical setting. Finally, in Sect. 6 we provide some concluding remarks.

2 The model

The physical setting of the contact problem we consider is depicted in Fig. 1 and is described as follows.

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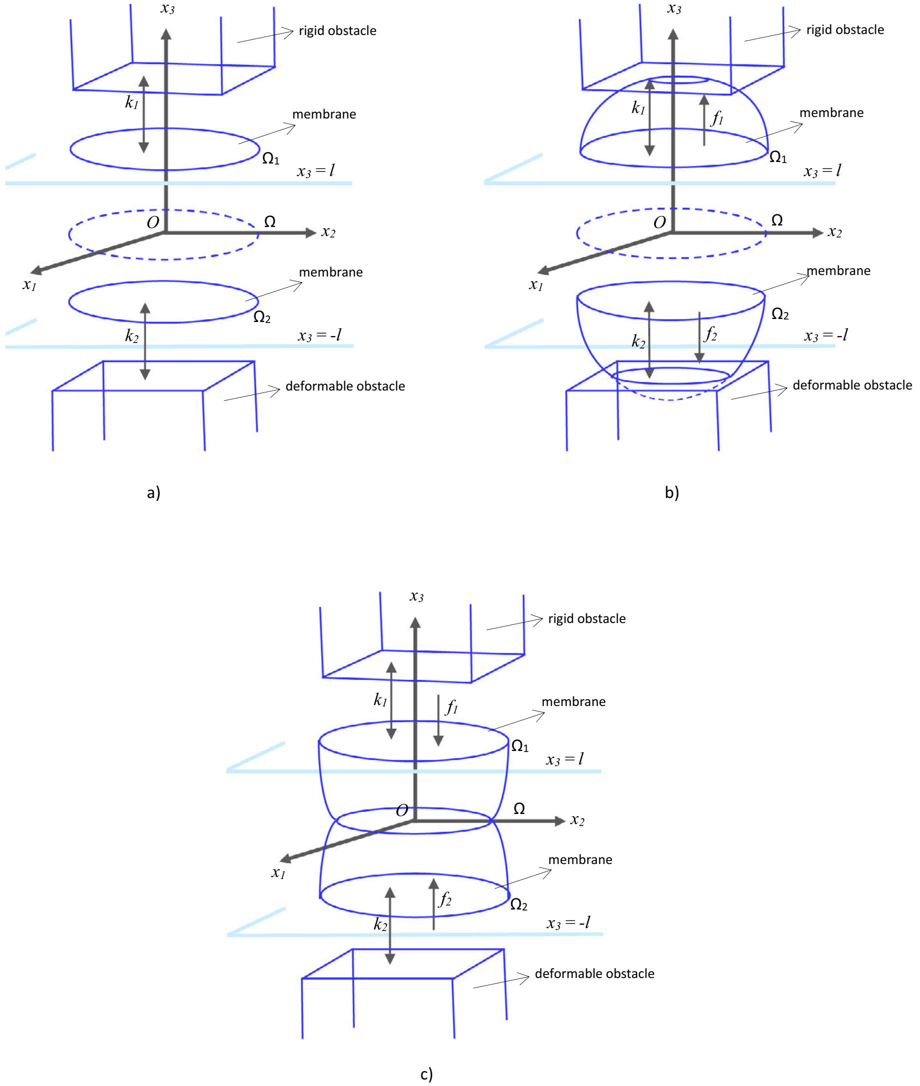

Fig. 1 Physical setting: **a** The reference configuration. **b** Equilibrium configuration—the membranes are in contact with the obstacles. **c** Equilibrium configuration—the membranes are in contact each other

Consider a bounded domain $\Omega \subset \mathbb{R}^2$ situated in the plan $x_3 = 0$ of the carthesian system $Ox_1x_2x_3$. The boundary of Ω, supposed to be regular, is denoted by Γ. We also denote by $\overline{\Omega} = \Omega \cup \Gamma$ the adherence of Ω and let $l > 0$, $k_1 > 0$, $k_2 > 0$. Moreover, we consider two membranes situated in the plans $x_3 = l$ and $x_3 = -l$, respectively, such that their orthogonal projection on the plan Ox_1x_2 is Ω. In other words, in the reference configuration, the membranes occupy the sets

$$\Omega_1 = \{\boldsymbol{x} = (x_1, x_2, x_3) \in \mathbb{R}^3 \ \ (x_1, x_2) \in \Omega,\ x_3 = l\,\},$$
$$\Omega_2 = \{\boldsymbol{x} = (x_1, x_2, x_3) \in \mathbb{R}^3 \ \ (x_1, x_2) \in \Omega,\ x_3 = -l\,\},$$

as depicted in Fig. 1a. We refer to the membranes which ocupy the domains Ω_1 and Ω_2 as the first and the second membrane, respectively. We assume that the membranes are fixed on their boundary Γ_1 and Γ_2, defined by

$$\Gamma_1 = \{\boldsymbol{x} = (x_1, x_2, x_3) \in \mathbb{R}^3 \ \ (x_1, x_2) \in \Gamma,\ x_3 = l\,\},$$
$$\Gamma_2 = \{\boldsymbol{x} = (x_1, x_2, x_3) \in \mathbb{R}^3 \ \ (x_1, x_2) \in \Gamma,\ x_3 = -l\,\},$$

respectively. Moreover, they are submitted to the action of time-dependent vertical forces of density f_1 and f_2, respectively. As a result, they deform and they could arrive in frictionless contact with two rectangular obstacles situated at the distance k_1 and k_2 from the plane of the first and second membrane, respectively (see Fig. 1b). The upper obstacle is assumed to be rigid and, therefore, its penetration is not allowed. In contrast, the lower obstacle is assumed to be deformable and, therefore, it allows penetration. In addition, under the action of these applied forces the membranes could approach each other and could arrive in contact (as depicted in Fig. 1c). We assume that their contact is unilateral and frictionless.

We are interested to construct a mathematical model which describes the above mechanical process of contact, in the time interval of interest $I = [0, T]$ with $T > 0$ given. To this end, we denote by u_1, u_2 the vertical displacements field in the two membranes and assume that these are real-valued functions on x_1, x_2 and t, i.e., $u_1 : \overline{\Omega} \times [0, T] \to \mathbb{R}_+$ and $u_2 : \overline{\Omega} \times [0, T] \to \mathbb{R}_+$. Nevertheless, for simplicity, we sometimes skip the dependence of various variables form x_1, x_2 and/or t, that is, for instance, we write u_1 or $u_1(t)$ instead of $u_1(x_1, x_2, t)$. Since the membranes are fixed on their boundary it follows that the functions u_1 and u_2 satisfy the following Dirichlet boundary conditions:

$$u_1(t) = 0, \qquad u_2(t) = 0 \qquad \text{on } \Gamma, \tag{2.1}$$

for any $t \in [0, T]$.

Next, let $u = (u_1, u_2) : \overline{\Omega} \times [0, T] \to \mathbb{R}^2$ and note that in the deformed configuration the distance between the points of the membranes which are situated on the same vertical is given by

$$\theta(u(t)) = 2l + u_1(t) - u_2(t). \tag{2.2}$$

Note that, even if we do not mention explicitly, equality (2.2) as well as the equalities and inequalities below are valid in Ω, at any moment $t \in [0, T]$. Since we assume that there is no penetration between the first membrane and the upper obstacle as well as between the two membranes, we impose the unilateral conditions

$$u_1(t) \le k_1, \tag{2.3}$$
$$\theta(u(t)) \ge 0. \tag{2.4}$$

To proceed, we model the material's behavior of the membranes with a viscoelastic constitutive law with long memory, in which $\mu_1 > 0$ and $\mu_2 > 0$ represent the Lamé coefficients and b_1, b_2 denote the relaxation functions. Following the assumptions above, the resultant forces acting on each membrane is vertical. Therefore, denoting by $F_1(t)$ and $F_2(t)$ their magnitude at the moment t, we deduce that the equilibrium of the membranes is described by the balance equations

$$\mu_1 \Delta u_1(t) + \int_0^t b_1(t-s)\Delta u_1(s)\,ds + F_1(t) = 0, \tag{2.5}$$

$$\mu_2 \Delta u_2(t) + \int_0^t b_2(t-s)\Delta u_2(s)\,ds + F_2(t) = 0. \tag{2.6}$$

These equations are obtained by combining the equilibrium equation, the constitutive law and the antiplane shear arguments. Details can be found in [15, Ch.8].

We now describe in detail the forces F_1 and F_2. First, using the principle of superposition, the force $F_1(t)$ is given by

$$F_1(t) = f_1(t) + R_1(t) + R_1^c(t) \tag{2.7}$$

where $R_1(t)$ denotes the reaction of the rigid obstacle on the first membrane and $R_1^c(t)$ the contact force exerted by the second membrane on the first one, both at the moment $t \in [0, T]$. Note that the time-dependent force f_1 in (2.7) is given. In contrast, the forces R_1 and R_1^c are unknown and depend on the contact process. These forces satisfy the following Signorini-type conditions:

$$R_1(t) = 0 \ \text{ if } \ u_1(t) < k_1 \quad \text{and} \quad R_1(t) \le 0 \ \text{ if } \ u_1(t) = k_1. \tag{2.8}$$

$$R_1^c(t) = 0 \ \text{ if } \ \theta(u(t)) > 0 \quad \text{and} \quad R_1^c(t) \ge 0 \ \text{ if } \ \theta(u(t)) = 0. \tag{2.9}$$

Condition (2.8) shows that the force $R_1(t)$ is inactive when there is no contact, i.e., when $u_1(t) < k_1$. It is active when the contact between the first membrane with the upper obstacle arises and, in this case, it is oriended toward the first membrane. Condition (2.9) concerns the contact between the two membranes and has a similar interpretation.

For the force F_2 we have a similar description. We have

$$F_2(t) = f_2(t) + R_2(t) + R_2^c(t) \tag{2.10}$$

where $R_2(t)$ denotes the reaction of the lower obstacle on the second membrane and $R_2^c(t)$ is the contact force exerced by the first membrane on the second one, both at the instant $t \in [0, T]$. Again, note that in (2.10) the time-dependent force f_2 is given but the forces R_2 and R_2^c are unknown and depend on the contact process. Since the lower obtacle is deformable, we assume that the force R_2 satisfies the so-called normal compliance contact condition, that is,

$$R_2(t) = -p(u_2(t) + k_2), \tag{2.11}$$

in where p is a negative real valued function which vanishes for a positive argument. Therefore, equality (2.11) shows that when the second membrane touch the lower obstacle (i.e., when $u_2(t)+k_2 \le 0$) then the force $R_2(t)$ is in the positive sense of the Ox_3 axis. In contrast, when the second membrane does not touch the lower obstacle (i.e., when $u_2(t)+k_2 > 0$) then the reaction $R_2(t)$ vanishes. The normal compliance condition was introduced for the first time in [10], in the study of a dynamic viscoelastic three dimensional contact problem. Then, it was used in many papers and surveys, as it resuts from [16] and the references therein. The term "normal compliance" was introduced in [8, 9]. Moreover, using the principle of action and reaction, the contact force R_2^c satisfies equality

$$R_2^c(t) = -R_1^c(t) \tag{2.12}$$

and, therefore, (2.9) yields

$$R_2^c(t) = 0 \ \text{ if } \ \theta(u(t)) > 0 \quad \text{and} \quad R_2^c(t) \le 0 \ \text{ if } \ \theta(u(t)) = 0. \tag{2.13}$$

The contact model we consider have as the unknowns the displacement field $u = (u_1, u_2)$. Therefore, to state it there is a need to eliminate the unknown functions R_1, R_2 and R_1^c and R_2^c described above. To this end, we proceed as follows. First, we add the equations (2.5) and (2.6), use equalities (2.7), (2.10), (2.11) and (2.12) to find that

$$\mu_1 \Delta u_1(t) + \mu_2 \Delta u_2(t) + \int_0^t \big[b_1(t-s)\Delta u_1(s) + b_2(t-s)\Delta u_2(s)\big]\,ds \\ + f_1(t) + f_2(t) + R_1(t) - p(u_2(t) + k_2) = 0. \tag{2.14}$$

It follows from here that

$$R_1(t) = p(u_2(t) + k_2) - \big[\mu_1 \Delta u_1(t) + \mu_2 \Delta u_2(t)\big] \\ - \int_0^t \big[b_1(t-s)\Delta u_1(s) + b_2(t-s)\Delta u_2(s)\big]\,ds - \big[f_1(t) + f_2(t)\big].$$

and, therefore, conditions (2.8) imply that

$$p(u_2(t) + k_2) - \big[\mu_1 \Delta u_1(t) + \mu_2 \Delta u_2(t)\big] \\ - \int_0^t \big[b_1(t-s)\Delta u_1(s) + b_2(t-s)\Delta u_2(s)\big]\,ds - \big[f_1(t) + f_2(t)\big] \le 0, \tag{2.15}$$

$$\Big(p(u_2(t) + k_2) - \big[\mu_1 \Delta u_1(t) + \mu_2 \Delta u_2(t)\big] \\ - \int_0^t \big[b_1(t-s)\Delta u_1(s) + b_2(t-s)\Delta u_2(s)\big]\,ds - \big[f_1(t) + f_2(t)\big]\Big)(u_1(t) - k_1) = 0 \tag{2.16}$$

On the other hand, from (2.6), (2.10) and (2.11) we deduce that

$$R_2^c(t) = p(u_2(t) + k_2) - \mu_2 \Delta u_2(t) - \int_0^t b_2(t-s)\Delta u_2(s)\,ds - f_2(t)$$

and, therefore, conditions (2.13) imply that

$$p(u_2(t) + k_2) - \mu_2 \Delta u_2(t) - \int_0^t b_2(t-s)\Delta u_2(s)\,ds - f_2(t) \le 0, \tag{2.17}$$

$$\Big(p(u_2(t) + k_2) - \mu_2 \Delta u_2(t) - \int_0^t b_2(t-s)\Delta u_2(s)\,ds - f_2(t)\Big)\theta(u(t)) = 0. \tag{2.18}$$

We now gather the above equations and inequalities to deduce the following mathematical model which describes the equilibrium of the two membranes in the physical setting described above.

Problem $\mathcal{M}$. *Find a displacement field $u = (u_1, u_2) : \overline{\Omega} \times [0, T] \to \mathbb{R}^2$ such that, relations (2.3), (2.4), (2.15)–(2.18) hold in Ω at any $t \in [0, T]$ and the boundary conditions (2.1) are satisfied on Γ, at any $t \in [0, T]$, with θ being given by equality (2.2), valid in Ω, at any $t \in [0, T]$.*

We end this section with the remark that Problem $\mathcal{M}$ represents a non-standard problem stated in a form of a non-linear system which includes partial differential equations and inequalities, associated to homogeneous boundary conditions and unilateral constraints. It represents a free boundary problem. For this reason, its analysis will be performed by using a variational formulation that we derive in the next section.

3 Variational formulation

We start this section with a description of the function spaces we use, then we turn to the variational formulation of Problem $\mathcal{M}$.

Function spaces. Everywhere in the paper we denote by "$\cdot$" the canonical inner product on the space $\mathbb{R}^2$ and by "$\|\cdot\|$" the associated Euclidean norm. We use the standard notation for Lebesgue and Sobolev spaces associated to Ω and Γ. Moreover, we use the symbol "$\to$" to represent the convergence in various space will be specified. We recall that

$$\|\xi\|^2_{H^1(\Omega)} = \|\xi\|^2_{L^2(\Omega)} + \|\nabla\xi\|^2_{L^2(\Omega)^2} \qquad \forall\, \xi \in H^1(\Omega)$$

and, therefore,

$$\|\xi\|_{L^2(\Omega)} \le \|\xi\|_{H^1(\Omega)} \qquad \forall\, \xi \in H^1(\Omega). \tag{3.1}$$

In addition, we have the Friedrichs-Poincaré inequality

$$\|\xi\|_{H^1(\Omega)} \le c_0\|\nabla\xi\|_{L^2(\Omega)^2} \qquad \forall\, \xi \in H^1_0(\Omega) \tag{3.2}$$

with c_0 being a positive constant which depends on Ω. As a consequence of this inequality it follows that $H^1_0(\Omega)$ is a Hilbert space endowed with the inner product

$$(\eta, \xi)_{H^1_0(\Omega)} = (\nabla\eta, \nabla\xi)_{L^2(\Omega)^2} \qquad \forall\, \xi \in H^1_0(\Omega). \tag{3.3}$$

and the associated norm $\|\cdot\|_{H^1_0(\Omega)}$.

Next, we need the product Hilbert space

$$V = H^1_0(\Omega) \times H^1_0(\Omega). \tag{3.4}$$

It follows from above that V is a real Hilbert space with the inner product

$$(u, v)_V = \int_\Omega \nabla u_1 \cdot \nabla v_1\, dx + \int_\Omega \nabla u_2 \cdot \nabla v_2\, dx \quad \forall\, u = (u_1, u_2),\ v = (v_1, v_2) \in V$$

and the associated norm denoted by $\|\cdot\|_V$. Thus,

$$\|\xi\|^2_V = \|\nabla\xi_1\|^2_{L^2(\Omega)^2} + \|\nabla\xi_2\|^2_{L^2(\Omega)^2} \qquad \forall\, \xi = (\xi_1, \xi_2) \in V. \tag{3.5}$$

Moreover, we denote by 0_V the zero element of V.

Finally, for any Hilbert space X and we denote by $C([0, T]; X)$ the space of continuous functions defined on $[0, T]$ with values in X. It is well known that $C([0, T]; X)$ is a Banach space equipped with the norm of the uniform convergence given by

$$\|v\|_{C([0,T];X)} = \max_{t\in[0,T]} \|v(t)\|_X. \tag{3.6}$$

Variational formulation of Problem $\mathcal{M}$. We now turn to the variational formulation of Problem $\mathcal{M}$ and, to this end, we start with the list of conditions we impose on the data. First, we assume that the densities of applied forces and the relaxation functions have the regularity

$$f_1 \in C([0, T]; L^2(\Omega)), \qquad f_2 \in C([0, T]; L^2(\Omega)), \tag{3.7}$$

$$b_1 \in C([0, T]), \qquad b_2 \in C([0, T]). \tag{3.8}$$

Here and below $C([0, T])$ represents the space of real-valued continuous functions defined on the interval I, that is, $C([0, T]) = C([0, T]; \mathbb{R})$. We also recall the inequalities

$$\mu_1 > 0, \qquad \mu_2 > 0, \tag{3.9}$$

$$l > 0, \tag{3.10}$$

$$k_1 > 0, \qquad k_2 > 0 \tag{3.11}$$

and we assume that the normal compliance function p satisfies the following condition.

$$\left\{\begin{array}{l} \text{(a)}\ \ p\colon \mathbb{R} \to \mathbb{R}. \\ \text{(b)}\ \ \text{There exists } L_p > 0 \text{ such that} \\ \qquad |p(r) - p(s)| \le L_p |r - s| \quad \forall\, r,\, s \in \mathbb{R}. \\ \text{(c)}\ \ p(r) \le 0 \quad \forall\, r \in \mathbb{R} \ \text{ and } \ p(r) = 0 \quad \forall\, r > 0. \end{array}\right. \tag{3.12}$$

Note that conditions (3.12)(c) are imposed by physical reasons. Indeed, combined with equality (2.11), these conditions guarantee that the reaction of the lower obstacle is towards the second membrane and vanishes when there is no contact. An example of function p which satisfies condition (3.12) is given by $p(r) = -\alpha r_-$ where α is a positive stiffness coefficient and r_- represents the negative part of r, that is $r_- = \max\{-r, 0\}$.

Finally, we assume that

$$c_0^2 L_p < \min\{\mu_1, \mu_2\} \tag{3.13}$$

and we interpret this assumption as a smallness condition for the constant L_p. Note that in this condition c_0 is the positive constant given by the Freiderichs-Poincaré inequality (3.2).

In the study of Problem $\mathcal{M}$ we use the Hilbert space V given by (3.4). In addition, we define the operators $\theta : V \to L^2(\Omega)$, $A : V \to V$, $\mathcal{S} : C([0, T]; V) \to C([0, T]; V)$, the set K, the function $j : V \times V \to \mathbb{R}$ and the element $f \in V$ by equalities

$$\theta(v) = 2l + v_1 - v_2 \qquad \forall\, v = (v_1, v_2) \in V, \tag{3.14}$$

$$K = \{\, v = (v_1, v_2) \in V \,:\, v_1 \le k_1,\ \ \theta(v) \ge 0 \ \text{ a.e. in } \Omega\,\}, \tag{3.15}$$

$$(Au, v)_V = \mu_1 \int_\Omega \nabla u_1 \cdot \nabla v_1\, dx + \mu_2 \int_\Omega \nabla u_2 \cdot \nabla v_2\, dx$$
$$\forall\, u = (u_1, u_2),\ v = (v_1, v_2) \in V, \tag{3.16}$$

$$(\mathcal{S}u(t), v)_V = \int_\Omega \Big(\int_0^t b_1(t - s) \nabla u_1(s)\, ds \Big) \cdot \nabla v_1\, dx \tag{3.17}$$
$$+ \int_\Omega \Big(\int_0^t b_2(t - s) \nabla u_2(s)\, ds \Big) \cdot \nabla v_2\, dx$$
$$\forall\, u = (u_1, u_2) \in C([0, T]; V),\ v = (v_1, v_2) \in V,\ t \in [0, T],$$

$$j(u, v) = \int_\Omega p(u_2 + k_2) v_2\, dx \qquad \forall\, u = (u_1, u_2),\ v = (v_1, v_2) \in V, \tag{3.18}$$

$$(f, v)_V = \int_\Omega (f_1 v_1 + f_2 v_2)\, dx \qquad \forall\, v = (v_1, v_2) \in V. \tag{3.19}$$

Note that the definition of the operator $\mathcal{S}$ follows from the Riesz representation theorem since, for any $u = (u_1, u_2)$ and $t \in [0, T]$, the functional

$$v \mapsto \int_\Omega \Big(\int_0^t b_1(t - s) \nabla u_1(s)\, ds \Big) \cdot \nabla v_1\, dx + \int_\Omega \Big(\int_0^t b_2(t - s) \nabla u_2(s)\, ds \Big) \cdot \nabla v_2\, dx$$

is linear and continuous on the space V. A similar argument is used in order to define the operator A by equality (3.16). Also, note that in (3.17) and below in this paper we use the shorthand notation $\mathcal{S}u(t)$ to represent the value of the function $\mathcal{S}u$ at the point t, i.e., $\mathcal{S}u(t) = (\mathcal{S}u)(t)$, for all $t \in [0, T]$.

With these preliminaries we are in a position to derive the variational formulation of Problem $\mathcal{M}$. Assume that $u = (u_1, u_2)$ is a solution to Problem $\mathcal{M}$, $t \in [0, T]$ and $v = (v_1, v_2) \in K$. Then, using (2.1), (2.3) and (2.4) we deduce that

$$u(t) \in K. \tag{3.20}$$

On the other hand, using (3.14) we see that

$$(v_2 - u_2(t)) = (v_1 - u_1(t)) - (\theta(v) - \theta(u(t))) \tag{3.21}$$

and, using this identity we find that

$$\begin{aligned}
&\mu_1 \Delta u_1(t)(v_1 - u_1(t)) + \Big(\int_0^t b_1(t-s)\Delta u_1(s)\,ds\Big)(v_1 - u_1(t)) \\
&\quad + \mu_2 \Delta u_2(t)(v_2 - u_2(t)) + \Big(\int_0^t b_2(t-s)\Delta u_2(s)\,ds\Big)(v_2 - u_2(t)) \\
&= \Big(\mu_1 \Delta u_1(t) + \mu_2 \Delta u_2(t) + \int_0^t \big[b_1(t-s)\Delta u_1(s) + b_2(t-s)\Delta u_2(s)\big]\,ds\Big) \\
&\quad (v_1 - u_1(t)) - \Big(\mu_2 \Delta u_2(t) + \int_0^t b_2(t-s)\Delta u_2(s)\,ds\Big)\Big(\theta(v) - \theta(u(t))\Big) \\
&= E(t)(v_1 - u_1(t)) - F(t)\Big(\theta(v) - \theta(u(t))\Big)
\end{aligned} \tag{3.22}$$

where, here and below, we employ the short-hand notation

$$E(t) = \mu_1 \Delta u_1(t) + \mu_2 \Delta u_2(t) + \int_0^t \big[b_1(t-s)\Delta u_1(s) + b_2(t-s)\Delta u_2(s)\big]\,ds, \tag{3.23}$$

and

$$F(t) = \mu_2 \Delta u_2(t) + \int_0^t b_2(t-s)\Delta u_2(s)\,ds. \tag{3.24}$$

We now write

$$\begin{aligned}
E(t)(v_1 - u_1(t)) &= \Big(E(t) + f_1(t) + f_2(t) - p(u_2(t) + k_2)\Big)(v_1 - u_1(t)) \\
&\quad + p(u_2(t) + k_2)(v_1 - u_1(t)) - (f_1(t) + f_2(t))(v_1 - u_1(t)) \\
&= \Big(E(t) + f_1(t) + f_2(t) - p(u_2(t) + k_2)\Big)(v_1 - k_1) \\
&\quad + \Big(E(t) + f_1(t) + f_2(t) - p(u_2(t) + k_2)\Big)(k_1 - u_1(t)) \\
&\quad + p(u_2(t) + k_2)(v_1 - u_1(t)) - (f_1(t) + f_2(t))(v_1 - u_1(t))
\end{aligned}$$

and, using notation (3.23), inequalities (2.15), $v_1 \le k_1$ and equality (2.16) we deduce that

$$E(t)(v_1 - u_1(t)) \le p(u_2(t) + k_2)(v_1 - u_1(t)) - (f_1(t) + f_2(t))(v_1 - u_1(t)). \tag{3.25}$$

Similarly, we write

$$\begin{aligned}
&F(t)\Big(\theta(v) - \theta(u(t))\Big) \\
&\quad = \Big(F(t) + f_2(t) - p(u_2(t) + k_2)\Big)\theta(v) - \Big(F(t) + f_2(t) - p(u_2(t) + k_2)\Big)\theta(u(t)) \\
&\quad + p(u_2(t) + k_2)\Big(\theta(v) - \theta(u(t))\Big) - f_2(t)\Big(\theta(v) - \theta(u(t))\Big)
\end{aligned}$$

and, using notation (3.24), inequalities (2.17), $\theta(v) \geq 0$ and equality (2.18) we deduce that

$$F(t)(\theta(v) - \theta(u(t)) \geq p(u_2(t) + k_2)\Big(\theta(v) - \theta(u(t))\Big) - f_2(t)\Big(\theta(v) - \theta(u(t))\Big).$$

and, therefore,

$$-F(t)\Big(\theta(v) - \theta(u(t))\Big) \leq -p(u_2(t) + k_2)\Big(\theta(v) - \theta(u(t))\Big) + f_2(t)\Big(\theta(v) - \theta(u(t))\Big). \tag{3.26}$$

We now combine identity (3.22) with inequalities (3.25) and (3.26) to deduce that

$$\begin{aligned} &\mu_1 \Delta u_1(t)(v_1 - u_1(t)) + \Big(\int_0^t b_1(t-s)\Delta u_1(s)\,ds\Big)(v_1 - u_1(t)) \\ &\quad + \mu_2 \Delta u_2(t)(v_2 - u_2(t)) + \Big(\int_0^t b_2(t-s)\Delta u_2(s)\,ds\Big)(v_2 - u_2(t)) \\ &\leq p(u_2(t) + k_2)(v_1 - u_1(t)) - (f_1(t) + f_2(t))(v_1 - u_1(t)) \\ &\quad - p(u_2(t) + k_2)\Big(\theta(v) - \theta(u(t))\Big) + f_2(t)\Big(\theta(v) - \theta(u(t))\Big) \end{aligned}$$

and, using identity (3.21) we see that

$$\begin{aligned} &\mu_1 \Delta u_1(t)(v_1 - u_1(t)) + \Big(\int_0^t b_1(t-s)\Delta u_1(s)\,ds\Big)(v_1 - u_1(t)) \\ &\quad + \mu_2 \Delta u_2(s)(v_2 - u_2(t)) + \Big(\int_0^t b_2(t-s)\Delta u_2(s)\,ds\Big)(v_2 - u_2(t)) \\ &\leq p(u_2(t) + k_2)(v_2 - u_2(t)) - f_1(t)(v_1 - u_1(t)) - f_2(t)(v_2 - u_2(t)). \end{aligned}$$

We now integrate this inequality on Ω, then we use the definitions (3.18) and (3.19) to deduce that

$$\begin{aligned} &\mu_1 \int_\Omega \Delta u_1(t)(v_1 - u_1(t))\,dx + \int_\Omega \Big(\int_0^t b_1(t-s)\Delta u_1(s)\,ds\Big)(v_1 - u_1(t))\,dx \\ &\quad + \mu_2 \int_\Omega \Delta u_2(t)(v_2 - u_2(t))\,dx + \int_\Omega \Big(\int_0^t b_2(t-s)\Delta u_2(s)\,ds\Big)(v_2 - u_2(t))\,dx \\ &\leq j(u(t), v) - j(u(t), u(t)) - (f(t), v - u(t))_V. \end{aligned}$$

Next, using the boundary conditions (2.1), identity

$$\int_\Omega \Delta u\, v\,dx = -\int_\Omega \nabla u \cdot \nabla v\,dx, \tag{3.27}$$

valid for any $u,\ v \in H_0^1(\Omega)$, and the definition (3.16) of the operator A we find that

$$\begin{aligned} &\int_\Omega \Big(\int_0^t b_1(t-s)\Delta u_1(s)\,ds\Big)(v_1 - u_1(t))\,dx \\ &\quad + \int_\Omega \Big(\int_0^t b_2(t-s)\Delta u_2(s)\,ds\Big)(v_2 - u_2(t))\,dx \\ &\leq j(u(t), v) - j(u(t), u(t)) - (f(t), v - u(t))_V + (Au(t), v - u(t)_V. \end{aligned} \tag{3.28}$$

On the other hand, using the Fubini theorem and identity (3.27) we see that

$$\begin{aligned}
&\int_\Omega \Big(\int_0^t b_1(t-s)\Delta u_1(s)\,ds\Big)(v_1-u_1(t))\,dx\\
&\quad=\int_0^t b_1(t-s)\Big(\int_\Omega \Delta u_1(s)(v_1-u_1(t))\,dx\Big)\,ds\\
&\quad=-\int_0^t b_1(t-s)\Big(\int_\Omega \nabla u_1(s)\cdot\nabla(v_1-u_1(t))\,dx\Big)\,ds\\
&\quad=-\int_\Omega \Big(\int_0^t b_1(t-s)\nabla u_1(s)\,ds\Big)\cdot\nabla(v_1-u_1(t))\,dx
\end{aligned}$$

and, similarly,

$$\begin{aligned}
&\int_\Omega \Big(\int_0^t b_2(t-s)\Delta u_2(s)\,ds\Big)(v_2-u_2(t))\,dx\\
&\qquad=-\int_\Omega \Big(\int_0^t b_2(t-s)\nabla u_2(s)\,ds\Big)\cdot\nabla(v_2-u_2(t))\,dx
\end{aligned}$$

We now add these eaualities and use the definition (3.17) to deduce that

$$\begin{aligned}
&\int_\Omega \Big(\int_0^t b_1(t-s)\Delta u_1(s)\,ds\Big)(v_1-u_1(t))\,dx\\
&\quad+\int_\Omega \Big(\int_0^t b_2(t-s)\Delta u_2(s)\,ds\Big)(v_2-u_2(t))\,dx=(\mathcal{S}u(t),v-u(t))_V.
\end{aligned}\tag{3.29}$$

Finally, we substitute identity (3.29) in (3.28) and use the regularity (3.20) to deduce the following variational formulation of Problem $\mathcal{M}$.

Problem $\mathcal{P}$. *Find a displacement field* $u=(u_1,u_2)$ *such that, for all* $t\in[0,T]$ *the following inequality holds:*

$$\begin{aligned}
&u(t)\in K,\quad (Au(t),v-u(t))_V+(\mathcal{S}u(t),v-u(t))_V\\
&\qquad+j(u(t),v)-j(u(t),u(t))\ge (f(t),v-u(t))_V\quad\forall\, v\in K.
\end{aligned}\tag{3.30}$$

The unique solvability of Problem $\mathcal{P}$ will be provided in the next section, under assumptions (3.7)–(3.13). We refer to the solution to this problem as a weak solution to the contact problem $\mathcal{M}$.

4 Weak solvability

We start this sction with an abstract existence and uniqueness result for history-dependent variational inequalities, then we focus on the weak solvability of Problem $\mathcal{P}$.

An existence and uniqueness result. Let X ba a real Hilbert space and $T>0$. Moreover, consider a subset K, the operators A, $\mathcal{S}$ and the functions j and f such that the conditions below hold.

$$K \text{ is a nonempty closed convex subset of } X. \tag{4.1}$$

$$\begin{cases} A\colon X \to X \text{ is strongly monotone and Lipschitz continuous, i.e.:} \\ \text{(a) there exists } m_A > 0 \text{ such that} \\ \qquad (Au - Av, u - v)_X \geq m_A \|u - v\|_X^2 \quad \text{for all } \ u,\ v \in X, \\ \text{(b) there exists } M_A > 0 \text{ such that} \\ \qquad \|Au - Av\|_X \leq M_A \|u - v\|_X \quad \text{for all } \ u,\ v \in X. \end{cases} \tag{4.2}$$

$$\begin{cases} \mathcal{S} : C([0,T];X) \to C([0,T];X) \text{ is a history-dependent operator,} \\ \text{i.e., there exists } \ L_S > 0 \ \text{ such that} \\ \|\mathcal{S}u(t) - \mathcal{S}v(t)\|_X \leq L_S \displaystyle\int_0^t \|u(s) - v(s)\|_X \, ds \\ \qquad \text{for all } \ u,\ v \in C([0,T];X), \quad t \in [0,T]. \end{cases} \tag{4.3}$$

$$\begin{cases} j\colon X \times X \to \mathbb{R} \text{ is such that:} \\ \text{(a) } j(u,\cdot)\colon X \to \mathbb{R} \text{ is convex and lower semicontinuous} \\ \qquad \text{for all } u \in X, \\ \text{(b) there exists } \alpha_j \geq 0 \text{ such that} \\ \qquad j(u,\widetilde{v}) - j(u,v) + j(\widetilde{u},v) - j(\widetilde{u},\widetilde{v}) \\ \qquad\quad \leq \alpha_j \|u - \widetilde{u}\|_X \, \|v - \widetilde{v}\|_X \quad \text{for all } \ u,\ \widetilde{u},\ v,\ \widetilde{v} \in X. \end{cases} \tag{4.4}$$

$$\alpha_j < m_A. \tag{4.5}$$

$$f \in C([0,T];X). \tag{4.6}$$

Example of operators which satisfy condition (4.3) are the integral operators and various Volterra-type operators. Details can be found in [16], where various properties of history-dependent operators have been studied. Here we restrict ourselves to recall that such kind of operators arise in the statement of constitutive laws for solids, slip-dependent friction laws, as well as in study of various mathematical models of contact with elastic and viscoplastic materials.

The following result represents an existence and uniqueness result in the study of variational inequalities with history-dependent operators, the so-called history-dependent variational inequalities.

Theorem 1 *Assume* (4.1)–(4.6). *Then, there exists a unique function* $u \in C([0,T];X)$ *such that, for all* $t \in [0,T]$ *the the following inequality holds:*

$$\begin{aligned} u(t) \in K, \quad & (Au(t), v - u(t))_X + (\mathcal{S}u(t), v - u(t))_X \\ & + j(u(t), v) - j(u(t), u(t)) \geq (f(t), v - u(t))_X \quad \forall\, v \in K. \end{aligned} \tag{4.7}$$

Theorem 1 represents a particular case of a more general existence and uniqueness result proved in [18]. It's proof is based on arguments on elliptic variational inequalities and a fixed point property of history-dependent operators. We also mention that inequalities of the form (4.7) have been studied in the recent paper [17], where a convergence criterion to the solution was proved.

Weak solvability of Problem $\mathcal{M}$. Our existence and uniqueness result in the study of Problem $\mathcal{P}$ is the following.

Theorem 2 *Assume* (3.7)–(3.13). *Then Problem* $\mathcal{P}$ *has a unique solution* $u \in C([0,T];V)$.

Proof We use Theorem 1 on the space $X = V$. To this end we use definition (3.15) to see that condition (4.1) is satisfied. Moreover, for any $u = (u_1, u_2)$, $v = (v_1, v_2) \in V$, using (3.5) we have

$$
\begin{aligned}
(Au, v)_V &= \mu_1 \int_\Omega \nabla u_1 \cdot \nabla v_1 \, dx + \mu_2 \int_\Omega \nabla u_2 \cdot \nabla v_2 \, dx \\
&\leq \mu_1 \|\nabla u_1\|_{L^2(\Omega)^2} \|\nabla v_1\|_{L^2(\Omega)^2} + \mu_2 \|\nabla u_2\|_{L^2(\Omega)^2} \|\nabla v_2\|_{L^2(\Omega)^2} \\
&\leq (\mu_1 + \mu_2) \|u\|_V \|v\|_V, \\
(Av, v) &= \mu_1 \int_\Omega \nabla v_1 \cdot \nabla v_1 \, dx + \mu_2 \int_\Omega \nabla v_2 \cdot \nabla v_2 \, dx \\
&= \mu_1 \|\nabla v_1\|^2_{L^2(\Omega)^2} + \mu_2 \|\nabla v_2\|^2_{L^2(\Omega)^2} \geq \min\{\mu_1, \mu_2\} \|v\|_V^2,
\end{aligned}
$$

which implies that

$$
\begin{aligned}
\|Au\|_V &\leq M_A \|u\|_V, && (4.8) \\
(Av, v)_V &\geq m_A \|v\|_V^2, && (4.9)
\end{aligned}
$$

with

$$
M_A = \mu_1 + \mu_2, \qquad m_A = \min\{\mu_1, \mu_2\}. \tag{4.10}
$$

It follows from here that the linear operator $A : V \times V \to \mathbb{R}$ satisfies condition (4.2).

Let $u = (u_1, u_2)$, $v = (v_1, v_2) \in C([0, T]; V)$, $t \in [0, T]$ and let $w = (w_1, w_2) \in V$. We use assumption (3.8) to consider the positive numbers B_1 and B_2 defined by

$$
B_1 = \max_{r \in [0,T]} |b_1(r)|, \qquad B_2 = \max_{r \in [0,T]} |b_2(r)|. \tag{4.11}
$$

Then, an elementary calculus base on the definition (3.17), the Cauchy-Schwarz inequality, the properties of the integral and notation (3.5) show that

$$
\begin{aligned}
(Su(t) - Sv(t), w)_V &= \int_\Omega \left(\int_0^t b_1(t-s) \nabla(u_1(s) - v_1(s)) \, ds \right) \cdot \nabla w_1 \, dx \\
&\quad + \int_\Omega \left(\int_0^t b_2(t-s) \nabla(u_2(s) - v_2(s)) \, ds \right) \cdot \nabla w_2 \, dx \\
&\leq \int_\Omega \left(\int_0^t |b_1(t-s)| \|\nabla(u_1(s) - v_1(s))\| \, ds \right) \|\nabla w_1\| \, dx \\
&\quad + \int_\Omega \left(\int_0^t |b_2(t-s)| \|\nabla(u_2(s) - v_2(s))\| \, ds \right) \|\nabla w_2\| \, dx \\
&\leq B_1 \int_0^t \left(\int_\Omega \|\nabla(u_1(s) - v_1(s))\| \|\nabla w_1\| \, dx \right) ds \\
&\quad + B_2 \int_0^t \left(\int_\Omega \|\nabla(u_2(s) - v_2(s))\| \|\nabla w_2\| \, dx \right) ds \\
&\leq B_1 \int_0^t \|\nabla(u_1(s) - v_1(s))\|_{L^2(\Omega)^2} \|\nabla w_1\|_{L^2(\Omega)^2} \Big) \, ds \\
&\quad + B_2 \int_0^t \|\nabla(u_2(s) - v_2(s))\|_{L^2(\Omega)^2} \|\nabla w_2\|_{L^2(\Omega)^2} \Big) \, ds \\
&\leq (B_1 + B_2) \left(\int_0^t \|u(s) - v(s)\|_V \, ds \right) \|w\|_V.
\end{aligned}
$$

We conclude from here that the operator $\mathcal{S}$ satisfies condition (4.3) with $L_S = B_1 + B_2$.

Next, it is easy to see that the functional j given by (3.18) satisfies condition (4.4)(a). Assume now that $u=(u_1,u_2)$, $\widetilde{u}=(\widetilde{u}_1,\widetilde{u}_2)$, $v=(v_1,v_2)$, $\widetilde{v}=(\widetilde{v}_1,\widetilde{v}_2)\in V$. Then, using (3.18) and assumption (3.12)(b) it follows that

$$j(u,\widetilde{v})-j(u,v)+j(\widetilde{u},v)-j(\widetilde{u},\widetilde{v})=\int_\Omega \big((p(u_2+k_2)-p(\widetilde{u}_2+k_2)\big)(\widetilde{v}_2-v_2)\,dx$$
$$\le L_p\int_\Omega |u_2-\widetilde{u}_2||v_2-\widetilde{v}_2|\,dx\le L_p\|u_2-\widetilde{u}_2\|_{L^2(\Omega}\|u_2-\widetilde{u}_2\|_{L^2(\Omega)}.$$

Therefore, using inequality (3.2) and equality (3.5) we deduce that

$$j(u,\widetilde{v})-j(u,v)+j(\widetilde{u},v)-j(\widetilde{u},\widetilde{v})\le c_0^2L_p\|u-\widetilde{u}\|_V\|v-\widetilde{v}\|_V, \tag{4.12}$$

which shows that the function $j:V\times V\to\mathbb{R}$ satisfies condition (4.4)(b) with

$$\alpha_j=c_0^2L_p. \tag{4.13}$$

We now combine (4.10), (4.13) with the smallness assumption (3.13) to see that condition (4.5) is satisfied. Note also that condition (4.6) holds, too. Theorem 2 is now a direct consequence of Theorem 1. □

We end this section with two consequences of inequalities (4.3) and (4.4)(b) obtained in the proof of Theorem 2, which will be used in the next section.

Corollary 1 *The operator $\mathcal{S}$ and function j defined by* (3.17) *and* (3.18)*, respectively, satisfy the following inequalities:*

$$\|\mathcal{S}u(t)\|_V\le L_S\int_0^t\|u(s)\|_V ds\qquad\forall\,u\in C([0,T];V),\ t\in[0,T], \tag{4.14}$$
$$|j(u,\widetilde{v})-j(u,v)|\le c_0^2L_p\|u\|_V\|\widetilde{v}-v\|_V\quad\forall\,u,\ \widetilde{v},\ v\in V. \tag{4.15}$$

Proof Let $t\in[0,T]$. We use definition (3.17) to see that $(\mathcal{S}0_V(t),u(t))_V=0$. Inequality (4.14) is now obtained by taking $v\equiv 0_V$ in (4.3). Next, we use the definition (3.18) combined with assumptions $k_2\ge 0$ and (3.12)(c) to see that $j(0_V,v)=0$ for all $v\in V$. We use this equality and inequality (4.12) with $\widetilde{u}=0_V$ to see that (4.15) holds. □

5 An optimization problem

Note that the solution u of Problem $\mathcal{P}$ depends on the data k_1, k_2 and l, among others. Therefore, using the notation $g=(k_1,k_2,l)$ we denote in what follows this solution by $u^g=(u_1^g,u_2^g)$ and we recall, that, under the assumptions of Theorem 2 we have $u^g\in C([0,T];V)$. In this section we consider the problem of finding a parameter g such that the norm of the solution u^g in the space $C([0,T];V)$ is as small as possible. In other words, denoting by G the set given by $G=\mathbb{R}_+^*\times\mathbb{R}_+^*\times\mathbb{R}_+^*$ with $\mathbb{R}_+^*=(0,+\infty)$, the problem we consider is the following.

Problem $\mathcal{Q}$. *Given a nonempty set $G_0\subset G$, find $g^*=(k_1^*,k_2^*,l^*)\in G_0$ such that*

$$\|u^{g^*}\|_{C([0,T];V)}\le\|u^g\|_{C[0,T];V)}\qquad\forall\,g=(k_1,k_2,l)\in G_0. \tag{5.1}$$

The mechanical interpretation of this problem is the following: given a viscoelastic contact process described by Problem $\mathcal{M}$, we are looking for a triple of optimal distances representing the distance of the first membrane to the upper obstacle, the distance of the second membrane to the lower obstacle and the distance between the two membranes such that this triple belongs in a given set G_0 and, in the equilibrium configuration, the resulting displacement field of the two membranes is minimal, at any time interval. To conclude, Problem $\mathcal{Q}$ represents an optimization problem and its solution governs the geometry of the physical setting.

Our main result in this section is the following.

Theorem 3 *Assume (3.7)–(3.9), (3.12), (3.13) and, moreover, assue that G_0 is a nonempty compact subset of $\mathbb{R}^3$. Then Problem $\mathcal{Q}$ has at least one solution solution $g^* \in G$.*

The proof of Theorem 3 is based on two preliminary results that we present in what follows. To this end, below, we assume that (3.7)–(3.9), (3.12) and (3.13) hold, even if we do not mention it explicitly.

Lemma 1 *Then there exists a constant $M > 0$ such that*

$$\|u^g(t)\|_V \le M, \quad \|Au^g(t)\|_V \le M, \quad \|\mathcal{S}u^g(t)\|_V \le M \quad \forall\, t \in [0, T],\ g \in G. \tag{5.2}$$

Proof Let $t \in [0, T]$, $g = (k_1, k_2, l) \in G$ and, for simplicity, denote u instead of u^g. We note that definition (3.15) implies that $0_V \in K$. Thus, taking $v = 0_V$ in (3.30) we deduce that

$$(Au(t), u(t))_V \le (f(t), u(t)_V - (\mathcal{S}u(t), u(t))_V + j(u(t), 0_V) - j(u(t), u(t)).$$

Then, using inequalities (4.9), (4.14) and (4.15) we find that

$$m_A\|u(t))\|_V^2 \le \|f(t)\|_V\|u(t)\|_V + L_S\Big(\int_0^t \|u(s)\|_V ds\Big)\|u(t)\|_V + c_0^2 L_p\|u(t))\|_V^2.$$

Next, we use the smallness assumption (3.13), equality (4.10) and the regularity (4.6) to see that there exists a constant $C > 0$ which does not depend on g such that

$$\|u(t)\|_V \le C + C\int_0^t \|u(s)\|_V ds.$$

Then, exploiting the Gronwall argument we find that

$$\|u(t)\|_V \le Ce^{Ct} \le Ce^{CT}$$

which proves the first inequality in (5.2). The last two inequalities in (5.2) are a direct consequence of (4.8) and (4.14). □

Lemma 2 *The function $g \mapsto u^g$ is continuous from G to $C([0, T]; V)$.*

Proof Let $g = (k_1, k_2, l) \in G$ and consider a sequence $\{g_n\} \subset G$ such that $g_n \to g$ in $\mathbb{R}^3$. Then, if $g_n = (k_1^n, k_2^n, l^n)$, we have

$$k_1^n \to k_1, \qquad k_2^n \to k_2, \qquad l^n \to l, \qquad \text{as } n \to \infty. \tag{5.3}$$

Let $n \in \mathbb{N}$, $t \in [0, T]$ and, for simplicity denote by u_n and u the functions u^{g_n} and u^g, respectively. Then u is the solution of the variational inequality (3.30) and u_n is the solution of the variational inequality

$$\begin{aligned} u_n(t) \in K_n, \quad & (Au_n(t), v - u_n(t))_V + (\mathcal{S}u_n(t), v - u_n(t))_V \\ & + j_n(u_n(t), v) - j_n(u_n(t), u_n(t)) \ge (f(t), v - u_n(t))_X \quad \forall\, v \in K_n, \end{aligned} \tag{5.4}$$

in which the set K_n and the function j_n are defined by equalities

$$\theta^n(v) = 2l^n + v_1 - v_2 \qquad \forall\, v = (v_1, v_2) \in V, \tag{5.5}$$

$$K_n = \{\, v = (v_1, v_2) \in V : \; v_1 \le k_1^n, \;\; \theta^n(v) \ge 0 \;\text{ a.e. in } \Omega \,\}, \tag{5.6}$$

$$j_n(u, v) = \int_\Omega p(u_2 + k_2^n) v_2 \, dx \qquad \forall\, u = (u_1, u_2), \; v = (v_1, v_2) \in V. \tag{5.7}$$

Let a_n and b_n be the positive reals defined by

$$a_n = \min\left\{ \frac{k_1^n}{k_1}, \, \frac{l^n}{l} \right\}, \qquad b_n = \min\left\{ \frac{k_1}{k_1^n}, \, \frac{l}{l^n} \right\}. \tag{5.8}$$

Then, it is easy to see that $a_n v \in K_n$ if $v \in K$, and $b_n v \in K$ if $v \in K_n$. This property allows us to test with $a_n u(t)$ in (5.4) and with $b_n u_n(t)$ in (3.30) to obtain that

$$(Au_n(t), a_n u(t) - u_n(t))_V + (\mathcal{S}u_n(t), a_n u(t) - u_n(t))_V \tag{5.9}$$
$$+ j_n(u(t), a_n u(t)) - j_n(u_n(t), u(t)) \ge (f(t), a_n u(t) - u_n(t))_V,$$

$$(Au(t), b_n u_n(t) - u(t))_V + (\mathcal{S}u(t), b_n u_n(t) - u(t))_V$$
$$+ j(u(t), b_n u_n(t)) - j(u(t), u(t)) \ge (f(t), b_n u_n(t) - u(t))_V. \tag{5.10}$$

We now substitute the identities

$$a_n u(t) - u_n(t) = u(t) - u_n(t) + (a_n - 1) u(t) \tag{5.11}$$

$$b_n u_n(t) - u(t) = u_n(t) - u(t) + (b_n - 1) u_n(t) \tag{5.12}$$

in (5.9) and (5.10), then we add the resulting inequalities to find that

$$\begin{aligned}(Au_n(t) - Au(t), u_n(t) - u(t))_V \le\; & (a_n - 1)(Au_n(t), u(t))_V + (b_n - 1)(Au(t), u_n(t))_V \\ & + (\mathcal{S}u_n(t) - \mathcal{S}u(t), u(t) - u_n(t))_V \\ & + (a_n - 1)(\mathcal{S}u_n(t), u(t))_V + (b_n - 1)(\mathcal{S}u(t), u_n(t))_V \\ & + j_n(u_n(t), a_n u(t)) - j_n(u_n(t), u_n(t)) \\ & + j(u(t), b_n u_n(t)) - j(u(t), u(t)) \\ & + (1 - a_n)(f(t), u(t))_V + (1 - b_n)(f(t), u_n(t))_V\end{aligned}$$

Next, we use the bounds (5.2) and the regularity $f \in C([0, T]; V)$ to see that

$$\begin{aligned}(Au_n(t) - Au(t), u_n(t) - u(t))_V \le\; & C_1\big(|a_n - 1| + |b_n - 1|\big) \\ & + (\mathcal{S}u_n(t) - \mathcal{S}u(t), u(t) - u_n(t))_V \\ & + j_n(u_n(t), a_n u(t)) - j_n(u_n(t), u_n(t)) \\ & + j(u(t), b_n u_n(t)) - j(u(t), u(t))\end{aligned}$$

where, here and below, C_i $(i = 1, 2, \ldots)$ represent various positive constants which does not depend on n ant t and whose value may change from place to place. Then, using the properties (4.2) and (4.3) of the operators A and $\mathcal{S}$ it follows that

$$\begin{aligned}m_A \|u_n(t) - u(t)\|_V^2 \le\; & C_2\big(|a_n - 1| + |b_n - 1|\big) \\ & + L_S\left(\int_0^t \|u_n(s) - u(s)\|_V ds \right) \|u_n(t) - u(t)\|_V \\ & + j_n(u(t), a_n u(t)) - j_n(u_n(t), u(t)) \\ & + j(u(t), b_n u_n(t)) - j(u(t), u(t))\end{aligned} \tag{5.13}$$

where, recall, $m_A = \min\{\mu_1, \mu_2\}$ and $L_S = B_1 + B_2$ with B_1 and B_2 given by (4.11).

On the other hand, an elemetary calcul based on the definitions (3.18) and (5.7) show that

$$\begin{aligned}&j_n(u_n(t), a_n u(t)) - j_n(u_n(t), u_n(t)) + j(u(t), b_n u_n(t)) - j(u(t), u(t))\\&\quad = \int_\Omega \Big[p(u_2^n(t) + k_2^n)(a_n u_2(t) - u_2^n(t)) + p(u_2(t) + k_2)(b_n u_2^n(t) - u_2(t))\Big]\, dx\end{aligned}$$

and, using identities (5.11), (5.12) we find that

$$\begin{aligned}&j_n(u_n(t), a_n u(t)) - j_n(u_n(t), u_n(t)) + j(u(t), b_n u_n(t)) - j(u(t), u(t))\\&\quad = \int_\Omega \Big[p(u_2^n(t) + k_2^n) - p(u_2(t) + k_2)\Big](u_2(t) - u_2^n(t))\, dx\\&\qquad + (a_n - 1)\int_\Omega p(u_2^n(t) + k_2^n) u_2(t)\, dx + (b_n - 1)\int_\Omega p(u_2(t) + k_2) u_2^n(t)\, dx\\&\quad \le \int_\Omega |p(u_2^n(t) + k_2^n) - p(u_2(t) + k_2)||u_2(t) - u_2^n(t)|\, dx\\&\qquad + |a_n - 1|\int_\Omega |p(u_2^n(t) + k_2^n)||u_2(t)|\, dx + |b_n - 1|\int_\Omega |p(u_2(t) + k_2)||u_2^n(t)|\, dx.\end{aligned}$$

Next, we use equalities $p(k_2^n) = 0$, $p(k_2) = 0$, assumption (3.12)(b), inequalities (3.1), (3.2), definition (3.5) and the bound (5.2) to deduce that

$$\begin{aligned}&j_n(u_n(t), a_n u(t)) - j_n(u_n(t), u_n(t)) + j(u(t), b_n u_n(t)) - j(u(t), u(t))\\&\quad \le c_0^2 L_p \|u_n(t) - u(t)\|_V^2 + C_3\big(|a_n - 1| + |b_n - 1|\big) + C_4|k_2^n - k_2|\|u_n(t) - u(t)\|_V.\end{aligned} \tag{5.14}$$

We now combine inequalities (5.13) and (5.14) then we use the smallness assumption (3.13) to find that

$$\begin{aligned}\|u_n(t) - u(t)\|_V^2 &\le C_5\big(|a_n - 1| + |b_n - 1|\big)\\&\quad + C_6\Big(\int_0^t \|u_n(s) - u(s)\|_V ds + |k_2^n - k_2|\Big)\|u_n(t) - u(t)\|_V.\end{aligned}$$

Then, using the elementary inequality

$$x^2 \le ax + b \quad \Longrightarrow \quad x \le a + \sqrt{b} \qquad \forall\, x,\ a,\ b \in \mathbb{R}_+$$

we find that

$$\|u_n(t) - u(t)\|_V \le \sqrt{C_5\big(|a_n - 1| + |b_n - 1|\big)} + C_6|k_2^n - k_2| + C_6\int_0^t \|u_n(s) - u(s)\|_V ds.$$

We now use the Gronwall argument to see that

$$\|u_n(t) - u(t)\|_V \le \Big(\sqrt{C_5\big(|a_n - 1| + |b_n - 1|\big)} + C_6|k_2^n - k_2|\Big)\, e^{C_6 t}$$

which, combined with (3.6), implies that

$$\|u_n - u\|_{C([0,T];V)} \le \Big(\sqrt{C_5\big(|a_n - 1| + |b_n - 1|\big)} + C_6|k_2^n - k_2|\Big)\, e^{C_6 T}. \tag{5.15}$$

Finally, we note that equalities (5.8) and convergences (5.3) show that $a_n \to 1$, $b_n \to 1$ and $k_2^n \to k_2$. Therefore, inequality (5.15) implies that $u_n \to u$ in $C([0, T]; V)$, which concludes the proof. □

We now have all the ingredients to provide the proof of Theorem 3.

Proof of Theorem 3 We use Lemma 2 to see that the real valued function $g \mapsto \|u^g\|_{C([0,T];V)}$ is continuous. Recall also that G_0 is supposed to be a non-empty compact subset of $\mathbb{R}^3$. Theorem 3 is now a direct consequence of the well-known Weierstrass theorem. □

6 Conclusion

In this paper we described a mathematical model for the equilibrium of two viscoelastic membranes in contact. The novelty of the model consists in its non-standard structure, given by a non-linear system involving partial differential equations and inequalities. In the variational formulation, the problem leads to a history-dependent variational inequality with unilateral constraints. We used an abstract existence and uniqueness result for such kind of inequalities in order to prove the unique weak solvability of the contact model. Then, we considered an optimization problem and proved its solvability.

Our research in this paper could be continued in several directions. First, it could be interesting to obtain additional convergence results of the solution which show its continuous dependence with respect to the rest of the data, including the forces f_1 and f_2. Such kind of results could be used in the study of associated optimal control and optimization problems. Second, it is possible to use the recent result in [17] in order to derive two equivalent formulations of Problem $\mathcal{P}$. These formulations are given by a nonlinear equation or a minimization problem, both governed by the so-called gap function, introduced in [1] in the case of elliptic variational inequalities. Moreover, it would be useful to follow the methods described in [5–7] in order to construct discrete schemes in the study of Problem $\mathcal{P}$ and to provide the corresponding error estimates.

Another direction of research would be to change the contact model. Thus, replacing equality (2.11) with a multivalued condition of the form

$$-R_2 \in \partial h(h_2 + k_2)$$

where $h : \mathbb{R} \to \mathbb{R}$ is a locally Lipschitz function and ∂h denotes its Clarke subdifferential would lead to a variational formulation expressed in terms of a history-dependent variational-hemivariational inequality. Then, the results presented in this paper could be easily transposed in this case, following the arguments and the results presented in [16], for instance. Further, an extension of the results obtained in this current paper could be considered in the case when the membrane are supposed to be viscoelastic with short memory or with both short and long memory.

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Declarations

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