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On the Tykhonov Well-Posedness of an Antiplane Shear Problem

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Abstract. We consider a boundary value problem which describes the frictional antiplane shear of an elastic body. The process is static and friction is modeled with a slip-dependent version of Coulomb's law of dry friction. The weak formulation of the problem is in the form of a quasivariational inequality for the displacement field, denoted by $\mathcal{P}$. We associated with problem $\mathcal{P}$ a boundary optimal control problem, denoted by $\mathcal{Q}$. For Problem $\mathcal{P}$, we introduce the concept of well-posedness and for Problem $\mathcal{Q}$ we introduce the concept of weakly and weakly generalized well-posedness, both associated with appropriate Tykhonov triples. Our main results are Theorems 5 and 16. Theorem 5 provides the well-posedness of Problem $\mathcal{P}$ and, as a consequence, the continuous dependence of the solution with respect to the data. Theorem 16 provides the weakly generalized well-posedness of Problem $\mathcal{Q}$ and, under additional hypothesis, its weakly well posedness. The proofs of these theorems are based on arguments of compactness, lower semicontinuity, monotonicity and various estimates. Moreover, we provide the mechanical interpretation of our well-posedness results.

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1. Introduction

The concept of Tykhonov well-posedness was introduced in [27] for a minimization problem and then it has been generalized for different optimization problems, see for instance [7, 8, 14, 15, 18, 29]. The well-posedness in the sense of Tykhonov (well-posedness, for short) has been extended in the recent years to various mathematical problems like inequalities, inclusions, fixed point and saddle point problems. Thus, the well-posedness of variational inequalities was studied for the first time in [16, 17] and the study of well-posedness

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of hemivariational inequalities was initiated in [10]. References in the field include [25,28], among others. A general framework which unifies the view on the well-posedness for abstract problems in metric spaces was recently introduced in [26]. There, the well-posedness concept has been introduced by using families of approximating sets $\{\Omega(\theta)\}_\theta$ indexed upon a positive parameter $\theta > 0$, together with approximating sequences, which are sequences $\{u_n\}_n$ such that $u_n \in \Omega(\theta_n)$ for all $n \in \mathbb{N}$.

Antiplane shear deformations are one of the simplest classes of deformations that solids can undergo: in antiplane shear (or longitudinal shear) of a cylindrical body, the displacement is parallel to the generators of the cylinder and is independent of the axial coordinate. We rarely actually load solids so as to cause them to deform in antiplane shear. However, the governing equations and boundary conditions for antiplane shear problems are beautifully simple and the solution will have many of the features of the more general case and may help us to solve the more complex problem too. For this reason, in the last years a considerable attention has been paid to the analysis of antiplane shear deformation, see for instance [12,13] and the review article [11]. There, modern developments for the antiplane shear model and its applications are described for both linear and nonlinear solid materials, various physical settings (including dynamic effects) where the antiplane shear model shows promise for further development are outlined and some open questions concerning a challenging antiplane shear inverse problem in linear isotropic elastostatics are described. A reference in the study of antiplane contact problems with elastic and viscoelastic materials is the book [24].

In this paper, we consider a mathematical model which describes the antiplane shear of the cylinder in frictional contact. The model was already considered in [24] and, therefore, we skip the mechanical assumptions used to derive the equations and boundary conditions involved. In a weak formulation, the model leads to an elliptic quasivariational inequality in which the unknown is the axial component of the displacement field. We associate with this problem a boundary optimal control problem. Optimal control problems for variational inequalities have been discussed in several works, including [1,2,9,21–23] and, more recently, [3,4,6]. Results on optimal control for antiplane contact problems with elastic materials can be found in [19,20] and the references therein. There, the existence of optimal pairs was obtained, under various assumptions on the material's behaviour.

Our aim in this paper is twofold. The first one is to study the well-posedness of the quasivariational inequality mentioned above. The second one is to study the well-posedness of the associated optimal control problem. In this way, we complete the studies initiated in [24], on one hand, and in [19,20], on the other hand. Our main results are gathered in Theorems 5 and 16. The novelty of these results consists in the fact that we use the concept of Tykhonov triple to define the family of approximating sets and, in this way, provide the well-posedness of two relevant problems by using a functional framework which extends the functional framework used in [26]. As a consequence, we obtain new results, which rise to interesting consequences and mechanical interpretation.

The rest of the manuscript is structured as follows. In Sect. 2, we introduce the antiplane shear problem, list the assumption on the data and state its variational formulation, denoted by $\mathcal{P}$. Then, we associate to Problem $\mathcal{P}$ a boundary optimal control problem, denoted by $\mathcal{Q}$. In Sect. 3, we introduce the concept of well-posedness for Problem $\mathcal{P}$. Moreover, we state and prove our first result, Theorem 5, together with some consequences. In Sect. 4, we introduce the concepts of weakly and weakly generalized well-posedness for Problem $\mathcal{Q}$. Then, we state and prove our second result, Theorem 16. The proofs of the theorems are based on arguments of monotonicity, compactness, lower semicontinuity and various estimates. Finally, we provide various mechanical interpretations of our well-posedness results.

2. The Antiplane Shear Model

We consider the slip-dependent frictional antiplane shear model introduced in [24, Section 9.2]. This model can be formulated as follows.
Problem $\mathcal{M}$. *Find a displacement field* $u : D \to \mathbb{R}$ *such that*

$$div\,(\mu\,\nabla u) + f_0 = 0 \qquad \text{in } D, \tag{2.1}$$

$$u = 0 \qquad \text{on } \Gamma_1, \tag{2.2}$$

$$\mu\,\partial_\nu u = f_2 \qquad \text{on } \Gamma_2, \tag{2.3}$$

$$|\mu\,\partial_\nu\, u| \le g(|u|), \quad \mu\,\partial_\nu u = -g(|u|)\,\frac{u}{|u|} \quad \text{if } u \ne 0 \quad \text{on } \Gamma_3. \tag{2.4}$$

Recall that Problem $\mathcal{M}$ describes the equilibrium of an elastic cylinder of transversal section D under the action of axial body forces of density f_0 and surface traction of density f_2. Here, D is assumed to be a regular domain in $\mathbb{R}^2$ with smooth boundary $\partial D = \Gamma$, composed of three measurable sets $\overline{\Gamma}_1$, $\overline{\Gamma}_2$ and $\overline{\Gamma}_3$, with the mutually disjoint relatively open sets Γ_1, Γ_2 and Γ_3, such that the one-dimensional measure of Γ_1, denoted by $meas\,\Gamma_1$, is strictly positive. Equation (2.1) represents the equation of equilibrium in which μ denotes the Lamé coefficient, (2.2) represents the displacement boundary condition and (2.3) represents the traction boundary condition. Finally, (2.4) is a static version of slip-dependent Coulomb's law in which g is a positive function, the friction bound, assumed to depend on the slip, i.e., $g = g(|u|)$. Note that here, in (2.1)–(2.4), and below in this paper, we skip the dependence of various functions with respect to the spatial variable $\boldsymbol{x} \in D \cup \Gamma$.

For the variational analysis of Problem $\mathcal{M}$, we use standard notation for Lebesque and Sobolev spaces. We use the symbols " $\to$" and " $\rightharpoonup$" to indicate the strong and weak convergence in various spaces which will be indicated below. Moreover, we use the space V given by

$$V = \{\, v \in H^1(D) \,:\, v = 0 \ \text{ on } \Gamma_1 \,\} \tag{2.5}$$

and we denote by $(\cdot,\cdot)_V$, the inner product induced on V by the inner product of $H^1(D)$. In addition, $\|\cdot\|_V$ and 0_V will represent the associated norm and the zero element of V. It is well known that $(V, \|\cdot\|_V)$ is a real Hilbert space. Moreover, by the Friedrichs–Poincaré inequality and the Sobolev trave theorem we have

$$\|v\|_V \le c_0 \, \|\nabla v\|_{L^2(D)^2}, \tag{2.6}$$

$$\|v\|_{L^2(\Gamma_3)} \le c_3 \, \|v\|_V \tag{2.7}$$

for all $v \in V$, respectively. Here, c_0 and c_3 are positive constants which depend on D, Γ_1 and Γ_3.

We now list the assumption of the data. First, we assume that the Lamé coefficient and the densities of the body forces and surface traction satisfy the following conditions.

$$\begin{aligned} &\mu \in L^\infty(D) \text{ and there exists } \mu^* > 0 \text{ such that} \\ &\quad \mu(\boldsymbol{x}) \ge \mu^* \text{ a.e. } \boldsymbol{x} \in D. \end{aligned} \tag{2.8}$$

$$f_0 \in L^2(D). \tag{2.9}$$

$$f_2 \in L^2(\Gamma_2). \tag{2.10}$$

Moreover, we assume that the friction bound is such that

$$\left\{ \begin{array}{l} (a)\ g : \Gamma_3 \times \mathbb{R} \to \mathbb{R}_+. \\ (b)\ \text{There exists } L_g > 0 \text{ such that} \\ \quad |g(\boldsymbol{x}, r_1) - g(\boldsymbol{x}, r_2)| \le L_g \, |r_1 - r_2| \\ \qquad \forall\, r_1,\, r_2 \in \mathrm{I\!R}, \ \text{ a.e. } \boldsymbol{x} \in \Gamma_3. \\ (c)\ \text{The mapping } \boldsymbol{x} \mapsto g(\boldsymbol{x},\, r) \\ \qquad\quad \text{is Lebesgue measurable on } \Gamma_3,\, \forall\, r \in \mathbb{R}. \\ (d)\ \text{The mapping } \boldsymbol{x} \mapsto g(\boldsymbol{x}, 0) \text{ belongs to } L^2(\Gamma_3). \end{array} \right. \tag{2.11}$$

Finally, we assume that the following smallness condition holds:

$$L_g c_0^2 c_3^2 < \mu^*. \tag{2.12}$$

Define the bilinear form $a : V \times V \to \mathrm{I\!R}$, the function $j : V \times V \to \mathrm{I\!R}$ and the element $f \in V$ by equalities

$$a(u, v) = \int_D \mu \, \nabla u \cdot \nabla v \, \mathrm{d}x \quad \forall\, u,\, v \in V, \tag{2.13}$$

$$j(u,\, v) = \int_{\Gamma_3} g(|u|) \, |v| \, \mathrm{d}a \quad \forall\, u,\, v \in V, \tag{2.14}$$

$$(f, v)_V = \int_D f_0 v \, \mathrm{d}x + \int_{\Gamma_2} f_2 v \, \mathrm{d}a \quad \forall\, v \in V. \tag{2.15}$$

Then, using standard arguments, it is easy to derive the following variational formulation of Problem $\mathcal{M}$.

Problem $\mathcal{P}$. *Find a displacement field* $u \in V$ *such that*

$$a(u, v - u) + j(u, v) - j(u, u) \ge (f, v - u)_V \quad \forall\, v \in V. \tag{2.16}$$

A function $u \in V$ which satisfies (2.16) is called a *weak solution* to Problem $\mathcal{M}$. Note that the function j in (2.16) depends on the solution u and, for this reason, we refer to (2.16) as a quasivariational inequality.

Now, to formulate a boundary optimal control for Problem $\mathcal{P}$, we assume that μ, f_0 and g are given and, for any $f_2 \in L^2(\Gamma_2)$, we use notation (2.15). Then, the set of admissible pairs for inequality (2.16) is given by

$$\mathcal{V}_{ad} = \{\, (u, f_2) \in V \times L^2(\Gamma_2) \text{ such that (2.16) holds} \,\}. \tag{2.17}$$

We also consider the cost function $\mathcal{L}$ given by

$$\begin{aligned}\mathcal{L}(u, f_2) &= a_0 \int_D |u - \phi|^2 \,\mathrm{d}a + a_2 \int_{\Gamma_2} |f_2|^2 \,\mathrm{d}a \\ &= a_0 \|u - \phi\|^2_{L^2(D)} + a_2 \|f_2\|^2_{L^2(\Gamma_2)} \end{aligned} \tag{2.18}$$

for all $u \in V$, $f_2 \in L^2(\Gamma_2)$. Here, ϕ is a given element in V and a_0, a_2 are strictly positive constants, i.e.,

$$\phi \in V, \tag{2.19}$$

$$a_0,\ a_2 > 0. \tag{2.20}$$

With the notation above, we consider the following optimal control problem.

Problem $\mathcal{Q}$. *Find* $(u^*, f_2^*) \in \mathcal{V}_{ad}$ *such that*

$$\mathcal{L}(u^*, f_2^*) = \min_{(u, f_2) \in \mathcal{V}_{ad}} \mathcal{L}(u, f_2). \tag{2.21}$$

With this choice for $\mathcal{L}$, the mechanical interpretation of Problem $\mathcal{Q}$ is the following: given a frictional antiplane shear described by the quasivariational inequality (2.16) with the data μ, f_0 and g which satisfy condition (2.8), (2.9), (2.11), respectively, we look for a surface traction $f_2 \in L^2(\Gamma_2)$ such that the displacement u of the cylinder solution is as close as possible to the "desired displacement" ϕ. Furthermore, this choice has to fulfill a minimum expenditure condition which is taken into account by the last term in (2.18). In fact, a compromise policy between the two aims ("u close to ϕ" and "minimal data f_2") has to be found and the relative importance of each criterion with respect to the other is expressed by the choice of the weight coefficients a_0, $a_2 > 0$.

The unique solvability of Problem $\mathcal{P}$ as well as the solvability of Problem $\mathcal{Q}$ represents a consequence of assumptions (2.8)–(2.12), (2.19), (2.20) and can be obtained by using standard arguments, already used in [24] and [19,20], respectively. Our aim in what follows is to study the well-posedness of these problems in the sense of Tykhonov and to derive some consequences together with their mechanical interpretation. To this end, we need to prescribe three ingredients: a set of indices, a family of approximating sets and a convergence criterion for the sequences of indices. These three ingredients lead us to introduce the new concept of Tykhonov triple, both for Problem $\mathcal{P}$ and Problem $\mathcal{Q}$, in Sects. 3 and 4, respectively.

3. A Well-Posedness Result

Everywhere in this section, we assume that (2.8)–(2.12) hold. Moreover, we note that the concepts of Tykhonov triple, approximating sequence and well-posedness we introduce below in this section refer to Problem $\mathcal{P}$, even if we do not mention it explicitly. Extending by our previous work [26], we consider the following definitions.

Definition 1. A Tykhonov triple for Problem $\mathcal{P}$ is a mathematical object of the form $\mathcal{T} = (I, \Omega, \mathcal{C})$ where I is a given set, $\Omega : I \to 2^V$ and $\mathcal{C} \subset \mathcal{S}(I)$.

Note that in this definition and below in this paper, $\mathcal{S}(I)$ represents the set of sequences of elements of I and 2^V denotes the set of parts of the space V. A typical element of I will be denoted by θ and a typical element of $\mathcal{S}(I)$ will be denoted by $\{\theta_n\}_n$. For any $\theta \in I$, we refer to the set $\Omega(\theta) \subset V$ as an approximating set and $\mathcal{C}$ represents the so-called convergence criterion.

Definition 2. Given a Tykhonov triple $\mathcal{T} = (I, \Omega, \mathcal{C})$, a sequence $\{u_n\}_n \subset V$ is called an approximating sequence if there exists a sequence $\{\theta_n\}_n \subset \mathcal{C}$ such that $u_n \in \Omega(\theta_n)$ for each $n \in \mathbb{N}$.

Definition 3. The Problem $\mathcal{P}$ is said to be (strongly) well-posedness with respect to the Tykhonov triple $\mathcal{T} = (I, \Omega, \mathcal{C})$ if it has a unique solution and every approximating sequence converges in V to this solution.

Note that the concept of approximating the sequence above depends on the Tykhonov triple $\mathcal{T}$ considered and, for this reason, we sometimes refer to approximating sequences with respect to $\mathcal{T}$. As a result, the concept of well-posedness depends on the Tykhonov triple $\mathcal{T}$. We also remark that the choice of this triple is crucial for the analysis of the well-posedness of Problem $\mathcal{P}$. In what follows, we construct an example of such triple used in the study of Problem $\mathcal{P}$.

Example 4. Take $\mathcal{T} = (I, \Omega, \mathcal{C})$, where

$$I = \{\,\theta = (\varepsilon, \widetilde{f}_0, \widetilde{f}_2, \widetilde{g}) \ : \ \varepsilon \geq 0,\ \widetilde{f}_0,\ \widetilde{f}_2,\ \widetilde{g} \text{ satisfy } (2.9), (2.10) \text{ and } (2.11)\,\}, \tag{3.1}$$

$$\begin{aligned}\Omega(\theta) = \{\, u \in V \ : \ & a(u, v-u) + \widetilde{j}(u, v) - \widetilde{j}(u, u) + \varepsilon \|u\|_V \|v-u\|_V \\ & \geq (\widetilde{f}, v-u)_V \quad \forall\, v \in V\,\}\end{aligned} \tag{3.2}$$

where for a given $\theta = (\varepsilon, \widetilde{f}_0, \widetilde{f}_2, \widetilde{g}) \in I$, $\widetilde{j}$ and $\widetilde{f}$ are defined by (2.14), (2.15), replacing g with $\widetilde{g}$, f_0 with $\widetilde{f}_0$ and f_2 with $\widetilde{f}_2$. Moreover, by definition, a sequence $\{\theta_n\}_n \subset \mathcal{S}(I)$ with $\theta_n = (\varepsilon_n, f_{0n}, f_{2n}, g_n)$ belongs to $\mathcal{C}$ if and only if the following hold:

$$\varepsilon_n \to 0, \tag{3.3}$$

$$f_{0n} \rightharpoonup f_0 \quad \text{in } L^2(D), \tag{3.4}$$

$$f_{2n} \rightharpoonup f_2 \quad \text{in } L^2(\Gamma_2) \tag{3.5}$$

as $n \to \infty$ and, in addition,

$$\begin{cases} \text{for any } n \in \mathbb{N} \text{ there exists } \alpha_n \geq 0 \text{ and } \beta_n \geq 0 \ \text{ such that} \\ \text{(a) } |g_n(x, r) - g(x, r)| \leq \alpha_n + \beta_n |r| \quad \forall\, r \in \mathbb{R},\ \text{a.e. } \boldsymbol{x} \in \Gamma_3. \\ \text{(b) } \lim_{n\to\infty} \alpha_n = \lim_{n\to\infty} \beta_n = 0. \end{cases} \tag{3.6}$$

Our main result in this section is the following.

Theorem 5. *Assume that (2.8)–(2.12) hold. Then Problem $\mathcal{P}$ is well posed with respect to the Tykhonov triple $\mathcal{T}$ in Example 4.*

Proof. Following Definition 3, the proof is carried out in two main steps.
(i) *Unique solvability of Problem $\mathcal{P}$.* First, we use assumption (2.8) and inequality (2.6) to see that the bilinear form a is continuous and coercive, i.e.,

$$a(v,v) \geq \frac{\mu^*}{c_0^2}\|v\|_V^2 \quad \forall\, v \in V. \tag{3.7}$$

Next, we use assumption (2.11) and inequality (2.7) to see that for all $\eta \in V$ $j(\eta,\cdot) : V \to \mathbb{R}$ is a continuous seminorm and, moreover,

$$\begin{aligned} &j(\eta_1, v_2) - j(\eta_1, v_1) + j(\eta_2, v_1) - j(\eta_2, v_2) \\ &\quad \leq L_g c_3^2\, \|\eta_1 - \eta_2\|_V \|v_1 - v_2\|_V \quad \forall\, \eta_1,\, \eta_2,\, v_1,\, v_2 \in V. \end{aligned} \tag{3.8}$$

In addition, recall the smallness assumption (2.12). The existence of a unique solution $u \in V$ to Problem $\mathcal{P}$ follows now by using a standard argument of quasivariational inequalities, see, for instance [24, Thorrem 3.7].
(ii) *Convergence of approximating sequences.* To proceed, we consider an approximating sequence for the Problem $\mathcal{P}$, denoted $\{u_n\}_n$. Then, according to Definition 2 it follows that there exists a sequence $\{\theta_n\}_n$ of elements of I, with the general term denoted by $\theta_n = (\varepsilon_n, f_{0n}, f_{2n}, g_n)$, such that $u_n \in \Omega(\theta_n)$ for each $n \in \mathbb{N}$ and, moreover, (3.3)–(3.6) hold. Our aim in what follows is to prove the convergence

$$u_n \to u \quad \text{in} \quad V, \quad \text{as} \quad n \to \infty. \tag{3.9}$$

To this end, we proceed in three intermediate steps that we present below.
(ii-a) *A boundedness result.* Let $n \in \mathbb{N}$ be fixed. Then, exploiting definition (3.2) we deduce that

$$\begin{aligned} &a(u_n, v - u_n) + j_n(u_n, v) - j_n(u_n, u_n) + \varepsilon_n \|u_n\|_V \|v - u_n\|_V \\ &\quad \geq (f_n, v - u_n)_V \quad \forall\, v \in V, \end{aligned} \tag{3.10}$$

where

$$j_n(u,\, v) = \int_{\Gamma_3} g_n(|u|)\, |v|\, \mathrm{d}a \quad \forall\, u,\, v \in V, \tag{3.11}$$

$$(f_n, v)_V = \int_D f_{0n} v\, \mathrm{d}x + \int_{\Gamma_2} f_{2n} v\, \mathrm{d}a \quad \forall\, v \in V. \tag{3.12}$$

We take $v = 0_V$ in (3.10) to obtain that

$$a(u_n, u_n) + j_n(u_n, u_n) \leq \varepsilon_n \|u_n\|_V^2 + (f_n, u_n)_V.$$

Using now inequalities (3.7), $j_n(u_n, u_n) \geq 0$ and $(f_n, u_n)_V \leq \|f_n\|_V \|u_n\|_V$ yields

$$\left(\frac{\mu^*}{c_0^2} - \varepsilon_n\right) \|u_n\|_V \leq \|f_n\|_V.$$

Finally, we use the convergences (3.3)–(3.5) to deduce that the sequence $\{u_n\}_n$ is bounded in V, i.e., there exists k which does not depend on n such that

$$\|u_n\|_V \le k. \tag{3.13}$$

(ii-b) *Weak convergence.* The bound (3.13) allows us to deduce that there exists an element $\widetilde{u} \in V$ and a subsequence of $\{u_n\}_n$, again denoted by $\{u_n\}_n$, such that

$$u_n \rightharpoonup \widetilde{u} \quad \text{in} \quad V \quad \text{as} \quad n \to \infty. \tag{3.14}$$

Let $n \in \mathbb{N}$ and $v \in V$. Then, (3.10) implies that

$$\begin{aligned} &a(u_n, v) + j_n(u_n, v) - j_n(u_n, u_n) + \varepsilon_n \|u_n\|_V \|v - u_n\|_V \\ &\quad \ge (f_n, v - u_n)_V + a(u_n, u_n). \end{aligned} \tag{3.15}$$

We now use the convergence (3.14), the compactness of the trace operator and assumption (3.6) to see that

$$|u_n| \to |\widetilde{u}|, \quad g_n(|u_n|) \to g(|\widetilde{u}|) \quad \text{in} \quad L^2(\Gamma_3) \quad \text{as} \quad n \to \infty$$

and, therefore,

$$j_n(u_n, v) - j_n(u_n, u_n) \to j(\widetilde{u}, v) - j(\widetilde{u}, \widetilde{u}) \quad \text{as} \quad n \to \infty. \tag{3.16}$$

Moreover, the convergences (3.3) and (3.14) imply that

$$\varepsilon_n \|u_n\|_V \|v - u_n\|_V \to 0 \quad \text{as} \quad n \to \infty. \tag{3.17}$$

Finally, the convergences (3.4), (3.5) and (3.14) combined with standard compactness arguments imply that

$$(f_n, v - u_n)_V \to (f, v - \widetilde{u})_V \quad \text{as} \quad n \to \infty. \tag{3.18}$$

On the other hand, using the properties of the form a we have that

$$a(u_n, v) \to a(\widetilde{u}, v) \quad \text{as} \quad n \to \infty, \quad \forall\, v \in V. \tag{3.19}$$

Moreover, since $a(u_n - \widetilde{u}, u_n - \widetilde{u}) \ge 0$, we deduce that

$$a(u_n, u_n) \ge a(\widetilde{u}, u_n) + a(u_n, \widetilde{u}) - a(\widetilde{u}, \widetilde{u}).$$

Using now (3.19) and the convergence (3.14), we find that

$$\liminf_{n \to \infty} a(u_n, u_n) \ge a(\widetilde{u}, \widetilde{u}). \tag{3.20}$$

We now pass to the lower limit in inequality (3.15) and use (3.16)–(3.20) to deduce that

$$a(\widetilde{u}, v - \widetilde{u}) + j(\widetilde{u}, v) - j(\widetilde{u}, \widetilde{u}) \ge (f, v - \widetilde{u})_V. \tag{3.21}$$

Next, it follows from (3.21) that $\widetilde{u}$ is a solution of inequality (2.16) and, by the uniqueness of the solution of this inequality, we obtain that

$$\widetilde{u} = u. \tag{3.22}$$

A careful analysis, based on the arguments above, reveals that u is the weak limit of any weakly convergent subsequence of the sequence $\{u_n\}_n$.

Therefore, using a standard argument we deduce that the whole sequence $\{u_n\}_n$ converges weakly in V to u as $n \to \infty$, i.e.,

$$u_n \rightharpoonup u \quad \text{in} \quad V \quad \text{as } n \to \infty. \tag{3.23}$$

(ii-c) *Strong convergence.* Let $n \in \mathbb{N}$ be given. We take $v = u$ in inequality (3.10) to see that

$$a(u_n, u_n - u) \le j_n(u_n, u) - j_n(u_n, u_n) + \varepsilon_n \|u_n\|_V \|u - u_n\|_V + (f_n, u_n - u)_V. \tag{3.24}$$

Next, we use inequalities (3.7) and (3.24) to find that

$$\begin{aligned} \frac{\mu^*}{c_0^2} \|u_n - u\|_V^2 \le a(u_n - u, u_n - u) &= a(u_n, u_n - u) - a(u, u_n - u)_V \\ &\le j_n(u_n, u) - j_n(u_n, u_n) + \varepsilon_n \|u_n\|_V \|u - u_n\|_V \\ &\quad + (f_n, u_n - u)_V - a(u, u_n - u). \end{aligned}$$

We now pass to the upper limit in this inequality and use the convergences (3.16)–(3.19), (3.23) and equality (3.22) to deduce that

$$\|u_n - u\|_V \to 0 \quad \text{as} \quad n \to \infty.$$

This convergence concludes the proof. □

Theorem 5 provides a useful tool in the study of the continuous dependence of the solution with respect to the data. Various examples can be considered but, for simplicity, we restrict ourselves to present two examples which make the object of the corollaries below.

Corollary 6. *Assume that (2.8)–(2.12) hold and, for each $n \in \mathbb{N}$, assume that $f_{0n} \in L^2(D)$, $f_{2n} \in L^2(\Gamma_2)$ and g_n are given, g_n satisfies condition (2.11) and, moreover, (3.4), (3.5) and (3.6) hold. Then the solution of Problem $\mathcal{P}$ with the data f_{0n}, f_{2n} and g_n, denoted by u_n, converge to the solution of Problem $\mathcal{P}$, i.e., (3.9) holds.*

Proof. We use the Tykhonov triple $\mathcal{T} = (I, \Omega, \mathcal{C})$ in Example 4. For each $n \in \mathbb{N}$, denote $\theta_n = (0, f_{0n}, f_{2n}, g_n) \in I$. Then it is easy to see that the sequence $\{\theta_n\}_n$ belongs to the set $\mathcal{C}$. Moreover, by definition, it follows that for each $n \in \mathbb{N}$ the element $u_n \in V$ satisfies the quasivariational inequality

$$a(u_n, v - u_n) + j_n(u_n, v) - j_n(u_n, u_n) \ge (f_n, v - u_n)_V \quad \forall\, v \in V$$

where, recall, j_n is defined by (3.11) and f_n is defined by (3.12). We now use (3.2) to deduce that $u_n \in \Omega(\theta_n)$ for each $n \in \mathbb{N}$ and, hence, Definition 2 shows that $\{u_n\}_n$ is an approximating sequence for Problem $\mathcal{P}$. Using now Theorem 5 and Definition 3, we deduce that the convergence (3.9) holds, which concludes the proof. □

Corollary 7. *Assume that (2.8)–(2.12) hold and, for each $n \in \mathbb{N}$, assume that μ_n is given, satisfying condition (2.8) and, moreover,*

$$\mu_n \to \mu \quad \text{in} \quad L^\infty(D) \quad \text{as} \quad n \to \infty. \tag{3.25}$$

Then, the solution of Problem $\mathcal{P}$ with the data μ_n, denoted u_n, converges to the solution of Problem $\mathcal{P}$, i.e., (3.9) holds.

Proof. Let $n \in \mathbb{N}$ and $v \in V$ be given. Then,

$$a_n(u_n, v - u_n) + j(u_n, v) - j(u_n, u_n) \geq (f, v - u_n)_V \quad \forall\, v \in V, \tag{3.26}$$

where a_n is defined by equality (2.13) in which μ is replaced by μ_n, that is

$$a_n(u, v) = \int_D \mu_n \,\nabla u \cdot \nabla v \,\mathrm{d}x \quad \forall\, u,\, v \in V. \tag{3.27}$$

We now combine (3.26) and (3.27) to see that

$$\begin{aligned} &a(u_n, v - u_n) + \int_D (\mu_n - \mu)\,\nabla u_n \cdot \nabla(v - u_n)\,\mathrm{d}x + j(u_n, v) - j(u_n, u_n) \\ &\quad \geq (f, v - u_n)_V \end{aligned}$$

and, since

$$\begin{aligned} &\int_D (\mu_n - \mu)\,\nabla u_n \cdot \nabla(v - u_n)\,\mathrm{d}x \\ &\quad \leq \|\mu_n - \mu\|_{L^\infty(D)} \|\nabla u_n\|_{L^2(D)^2} \|\nabla(v - u_n)\|_{L^2(D)^2} \\ &\quad \leq \|\mu_n - \mu\|_{L^\infty(D)} \|u_n\|_V \|v - u_n\|_V, \end{aligned}$$

we deduce that

$$\begin{aligned} &a(u_n, v - u_n) + j(u_n, v) - j(u_n, u_n) + \varepsilon_n \|u_n\|_V \|v - u_n\|_V \\ &\quad \geq (f, v - u_n)_V, \end{aligned} \tag{3.28}$$

where

$$\varepsilon_n = \|\mu_n - \mu\|_{L^\infty(D)}. \tag{3.29}$$

We use the Tykhonov triple $\mathcal{T} = (I, \Omega, \mathcal{C})$ in Example 4. For each $n \in \mathbb{N}$, denote $\theta_n = (\varepsilon_n, f_0, f_2, g) \in I$. Then, using (3.29) and (3.25), it follows that (3.3) holds and, moreover, the sequence $\{\theta_n\}_n$ belongs to the set $\mathcal{C}$. In addition, inequality (3.28) and (3.2) show that $u_n \in \Omega(\theta_n)$ for each $n \in \mathbb{N}$. Therefore, using Definition 2, we deduce that $\{u_n\}_n$ is an approximating sequence for Problem $\mathcal{P}$. Finally, using Theorem 5 and Definition 3, we deduce that the convergence (3.9) holds, which concludes the proof. □

In additional to the mathematical interest in the convergence result in Corollary 6, it is important from mechanical point of view since it shows that small perturbations of the density of body forces and surface tractions together with small perturbation of the friction bound give rise to small perturbations of the weak solution of the antiplane contact problem $\mathcal{M}$. A similar comment can be made on Corollary 7 which shows the continuous dependence of the weak solution of Problem $\mathcal{M}$ with respect to the Lamé coefficient μ.

We proceed with an elementary example in which an analytic expression of the solution of Problem $\mathcal{P}$ can be given.

Example 8. Consider the one-dimensional version of Problem $\mathcal{M}$ in which $D = (0, 1)$, $\Gamma_1 = \{0\}, \Gamma_2 = \emptyset$, $\Gamma_3 = \{1\}$, $\mu > 0$, $f_0 \in \mathbb{R}$ and $g > 0$. With these data, the problem consists in finding a function $u : [0, 1]$ such that

$$-\,\mu u'' = f_0, \qquad u(0) = 0, \qquad |u'(1)| \leq g, \qquad u'(1) = -g\,\frac{u(1)}{|u(1)|} \;\text{ if }\; u(1) \neq 0,$$

(3.30)

where, here and below, the prime represents the derivative with respect to the spatial variable x. A simple calculus shows that the solution of the nonlinear boundary value problem (3.30) is given by

$$u(x) = \begin{cases} -\frac{f_0}{2\mu}\,x^2 + \left(\frac{f_0}{\mu} + g\right)x & \text{if} \quad f_0 < -2\mu g \\ -\frac{f_0}{2\mu}\,x^2 + \frac{f_0}{2\mu}\,x & \text{if} \quad -2\mu g \le f_0 \le 2\mu g \\ -\frac{f_0}{2\mu}\,x^2 + \left(\frac{f_0}{\mu} - g\right)x & \text{if} \quad f_0 > 2\mu g \end{cases} \tag{3.31}$$

Moreover, note that this solution is the solution of the corresponding Problem $\mathcal{P}$ which, in this particular case, is stated as follows: find $u \in V$ such that

$$\mu \int_0^1 u'(v' - u')\,\mathrm{d}x + g(|v(1)|) - g(|u(1)|) \ge \int_0^1 f_0(v - u)\,\mathrm{d}x \quad \forall\, v \in V,$$

where $V = \{\, v \in H^1(0,1) \;:\; v(0) = 0 \,\}$. It is easy to see that the function (3.31) depends continuously on the data μ, f_0 and g, which represents a validation of the convergence results in Corollaries 6 and 7.

We end this section with an example which illustrates the sensitivity of the well-posedness concept of Problem $\mathcal{P}$ with respect to the choice of the Tykhonov triple.

Example 9. Let $\overline{f}_0 \in L^2(D)$ be given and take $\mathcal{T} = (I, \Omega, \mathcal{C})$, where

$$I = L^2(\Omega), \tag{3.32}$$

$$\Omega(\theta) = \{\, u \in V \;:\; a(u, v - u) + j(u, v) - j(u, u) \ge (\widetilde{f}, v - u)_V \quad \forall\, v \in V \,\}, \tag{3.33}$$

where, for a given $\theta = \widetilde{f}_0 \in I$, $\widetilde{f}$ is defined by (2.15), replacing f_0 with $\widetilde{f}_0$. Moreover, let

$$\mathcal{C} = \{\, \{f_{0n}\}_n \subset \mathcal{S}(L^2(\Omega)) \;:\; f_{0n} \rightharpoonup \overline{f}_0 \ \text{ in } L^2(D) \quad \text{as} \quad n \to \infty. \,\}. \tag{3.34}$$

We have the following result.

Proposition 10. *Assume that (2.8)–(2.12) hold. Then Problem $\mathcal{P}$ is well posed with respect to the Tykhonov triple $\mathcal{T}$ in Example 9 if and only if $f_0 = \overline{f}_0$.*

Proof. Assume that $f_0 = \overline{f}_0$ and let $\{u_n\}_n$ be an approximating sequence for Problem $\mathcal{P}$ with respect to the Tykhonov triple $\mathcal{T}$ in Example 9. Then, there exists a sequence $\{f_{0n}\}_n \subset L^2(D)$ such that $f_{0n} \rightharpoonup f_0$ in $L^2(D)$ as $n \to \infty$ and, moreover, for each $n \in \mathbb{N}$, u_n satisfies inequality

$$a(u_n, v - u_n) + j(u_n, v) - j(u_n, u_n) \ge (f_n, v - u_n)_V \quad \forall\, v \in V, \tag{3.35}$$

in which f_n is defined by (2.15), replacing f_0 with f_{0n}. It follows from here that $\{u_n\}_n$ is an approximating sequence for Problem $\mathcal{P}$ with respect to the Tykhonov triple $\mathcal{T}$ in Example 4, corresponding to the sequence $\{\theta_n\}_n$ with $\theta_n = (0, f_{0n}, f_2, g)$ for each $n \in \mathbb{N}$. Therefore, using Theorem 5 and Definition 3, we deduce that $u_n \to u$ in V as $n \to \infty$, where u denotes the solution of $\mathcal{P}$. This shows that Problem $\mathcal{P}$ is well posed with the Tykhonov triple (3.32)–(3.34).

Conversely, assume that Problem $\mathcal{P}$ is well posed with the Tykhonov triple in Example 9 and denote by $\overline{\mathcal{P}}$ the quasivariational inequality in Problem $\mathcal{P}$ obtained by replacing f_0 with $\overline{f}_0$.

Let $\{u_n\}_n$ be an approximating sequence for Problem $\mathcal{P}$ with respect to the Tykhonov triple in Example 9. Then, since $\mathcal{P}$ is well posed with $\mathcal{T}$, we deduce that

$$u_n \to u \quad \text{in} \quad V \quad \text{as} \quad n \to \infty, \tag{3.36}$$

where, recall, u denotes the solution to Problem $\mathcal{P}$. On the other hand, by Definition 2, we know that there exists a sequence $\{f_{0n}\}_n \subset L^2(D)$ such that $f_{0n} \rightharpoonup \overline{f}_0$ in $L^2(D)$ as $n \to \infty$ and, moreover, for each $n \in \mathbb{N}$, u_n satisfies inequality (3.35) where, again, f_n is defined by (2.15), replacing f_0 with f_{0n}. Next, for each $n \in \mathbb{N}$ denote $\theta_n = (0, f_{0n}, f_2, g) \in I$. Then, from (3.35) it follows that $\{u_n\}_n$ is an approximating sequence from the Problem $\overline{\mathcal{P}}$ and, therefore, using Theorem 5 we deduce that

$$u_n \to \overline{u} \quad \text{in} \quad V \quad \text{as} \quad n \to \infty, \tag{3.37}$$

where $\overline{u}$ denotes the solution to Problem $\overline{\mathcal{P}}$.

We now combine (3.36) and (3.37) to deduce that $\overline{u} = u$. Moreover, we recall that u denotes the solution to Problem $\mathcal{P}$ with the data μ, f_0, f_2 and g while $\overline{u}$ is the solution to Problem $\mathcal{P}$ with the data μ, $\overline{f}_0$, f_2 and g. Therefore, besides (2.16), u satisfies the inequality

$$a(u, v-u) + j(u,v) - j(u,u) \geq (\overline{f}, v-u)_V \quad \forall\, v \in V, \tag{3.38}$$

where

$$(\overline{f}, v)_V = \int_D \overline{f}_0 v \,\mathrm{d}x + \int_{\Gamma_2} f_2 v \,\mathrm{d}a \quad \forall\, v \in V. \tag{3.39}$$

Let $w \in C_0^\infty(\Omega)$ be given. We take $v = u + w$ in (2.16) then $v = u - w$ in (3.38) and add the resulting inequalities to obtain that

$$j(u, u+w) - j(u,u) + j(u, u-w) - j(u,u) \geq (f - \overline{f}, w)_V. \tag{3.40}$$

Next, since w vanishes on Γ_3, it follows that $j(u, u+w) = j(u, u-w) = j(u,u)$ and, therefore, (3.40) yields $(f - \overline{f}, w)_V \leq 0$. A similar argument shows that $(f - \overline{f}, w)_V \geq 0$. We conclude from here that $(f - \overline{f}, w)_V = 0$ and, using definitions (2.15), (3.39) it follows that

$$\int_D (f_0 - \overline{f}_0) w \,\mathrm{d}x = 0.$$

We now use a density argument to find that $f_0 - \overline{f}_0$ which concludes the proof. □

4. Weakly Well-Posedness Results

Everywhere in this section, we assume that (2.8), (2.9), (2.11), (2.12), (2.19) and (2.20) hold. Moreover, we underlie that the concept of Tykhonov triple, approximating sequence and generalized well-posedness we introduce below in this section refer to Problem $\mathcal{Q}$, even if we do not mention it explicitly.

Definition 11. A Tykhonov triple for Problem $\mathcal{Q}$ is a mathematical object of the form $\mathcal{T} = (I, \Omega, \mathcal{C})$ where I is a given set, $\Omega : I \to 2^{V \times L^2(\Gamma_2)}$ and $\mathcal{C} \subset \mathcal{S}(I)$.

Recall that in this definition, $\mathcal{S}(I)$ represents the set of sequences of elements of I and $2^{V \times L^2(\Gamma_2)}$ denotes the set of parts of $V \times L^2(\Gamma_2)$. For any $\theta \in I$, we refer to the set $\Omega(\theta) \subset V \times L^2(\Gamma_2)$ as an approximating set and $\mathcal{C}$ represents the so-called convergence criterion.

Definition 12. Given a Tykhonov triple $\mathcal{T} = (I, \Omega, \mathcal{C})$, a sequence $\{(u_n, f_{2n})\}_n \subset V \times L^2(\Gamma_2)$ is called an approximating sequence if there exists a sequence $\{\theta_n\}_n \subset \mathcal{C}$, such that $(u_n, f_{2n}) \in \Omega(\theta_n)$ for each $n \in \mathbb{N}$.

Definition 13. The Problem $\mathcal{Q}$ is said to be weakly well posed with respect to the Tykhonov triple $\mathcal{T} = (I, \Omega, \mathcal{C})$ if it has a unique solution and every approximating sequence converges weakly in $V \times L^2(\Gamma_2)$ to this solution.

Definition 14. The Problem $\mathcal{Q}$ is said to be weakly generalized well posed with respect to the Tykhonov triple $\mathcal{T} = (I, \Omega, \mathcal{C})$ if it has at least one solution and every approximating sequence contains a subsequence which converges weakly in $V \times L^2(\Gamma_2)$ to some point of the solution set.

As in the previous section, we remark that the concept of the approximating sequence above depends on the Tykhonov triple $\mathcal{T}$ and, for this reason, we sometimes refer to approximating sequences with respect to $\mathcal{T}$. As a result, the concepts of weakly and weakly generalized well-posedness depend on the Tykhonov triple $\mathcal{T}$. In what follows, we construct a relevant example of such triple that we use in the study of Problem $\mathcal{Q}$.

Example 15. Take $\mathcal{T} = (I, \Omega, \mathcal{C})$, where

$$I = \{\, \theta = (\varepsilon, \widetilde{f}_0, \widetilde{g}, \widetilde{\phi}) \ : \ \varepsilon \geq 0,\ \widetilde{f}_0,\ \widetilde{g},\ \widetilde{\phi} \text{ satisfy (2.9), (2.11) and (2.19)} \,\}, \tag{4.1}$$

$$\Omega(\theta) = \left\{ (u^*, f_2^*) \in \mathcal{V}_{ad}(\theta) \ : \ \mathcal{L}_\theta(u^*, f_2^*) = \min_{(u,f_2) \in \mathcal{V}_{ad}(\theta)} \mathcal{L}_\theta(u, f_2) \right\}, \tag{4.2}$$

where, for a given $\theta = (\varepsilon, \widetilde{f}_0, \widetilde{g}, \widetilde{\phi}) \in I$, the cost functional $\mathcal{L}_\theta : V \times L^2(\Gamma_2) \to \mathbb{R}$ and the set $\mathcal{V}_{ad}(\theta)$ and are defined as follows:

$$\mathcal{L}_\theta(u, f_2) = a_0 \|u - \widetilde{\phi}\|^2_{L^2(D)} + a_2 \|f_2\|^2_{L^2(\Gamma_2)}, \tag{4.3}$$

$$\begin{aligned} &(u, f_2) \in \mathcal{V}_{ad}(\theta) \iff \quad u \in V, \quad f_2 \in L^2(\Gamma_2) \ \text{ and} \\ &\quad a(u, v-u) + \widetilde{j}(u,v) - \widetilde{j}(u,u) + \varepsilon \|u\|_V \|v-u\|_V \geq (\widetilde{f}, v-u)_V \quad \forall\, v \in V. \end{aligned} \tag{4.4}$$

Here, $\widetilde{j}$ and $\widetilde{f}$ are defined by (2.14), (2.15), replacing g with $\widetilde{g}$ and f_0 with $\widetilde{f}_0$. Finally, by definition, a sequence $\{\theta_n\}_n \subset \mathcal{S}(I)$ with $\theta_n = (\varepsilon_n, f_{0n}, g_n, \phi_n)$ belongs to $\mathcal{C}$ if and only if (3.3), (3.4) and (3.6) hold and, moreover,

$$\phi_n \rightharpoonup \phi \ \text{ in } V. \tag{4.5}$$

Our main result in this section is the following.

Theorem 16. *Assume that* (2.8)–(2.12), (2.19) *and* (2.20) *hold. Then Problem* $\mathcal{Q}$ *is weakly generalized well posed with respect to the Tykhonov triple in Example* 15.

Proof. Following Definition 14, the proof is carried out in two main steps.
(i) *Solvability of Problem* $\mathcal{Q}$. The existence of solutions to Problem $\mathcal{Q}$ follows from standard arguments that we resume in the following. Let

$$\omega = \inf_{(u,f_2)\in\mathcal{V}_{ad}} \mathcal{L}(u,f) \in \mathbb{R} \tag{4.6}$$

and let $\{(u_n, f_{2n})\}_n \subset \mathcal{V}_{ad}$ be a minimizing sequence for the functional $\mathcal{L}$, i.e.,

$$\lim_{n\to\infty} \mathcal{L}(u_n, f_{2n}) = \omega. \tag{4.7}$$

We claim that the sequence $\{f_{2n}\}_n$ is bounded in $L^2(\Gamma_2)$. Arguing by contradiction, assume that $\{f_{2n}\}_n$ is not bounded in $L^2(\Gamma_2)$. Then, passing to a subsequence still denoted $\{f_{2n}\}_n$, we have

$$\|f_{2n}\|_{L^2(\Gamma_2)} \to +\infty \quad \text{as} \quad n \to +\infty \tag{4.8}$$

and, using (2.18) it turns that

$$\lim_{n\to+\infty} \mathcal{L}(u_n, f_n) = +\infty. \tag{4.9}$$

Equalities (4.7) and (4.9) imply that $\omega = +\infty$ which is in contradiction with (4.6).

We conclude from the above that the sequence $\{f_{2n}\}_n$ is bounded in $L^2(\Gamma_2)$, as claimed. Therefore, we deduce that there exists $f_2^* \in L^2(\Gamma_2)$ such that, passing to a subsequence still denoted $\{f_{2n}\}$, we have

$$f_{2n} \rightharpoonup f_2^* \quad \text{in} \quad L^2(\Gamma_3) \quad \text{as} \quad n \to +\infty. \tag{4.10}$$

Let u^* be the solution of the quasivariational inequality (2.16) for $f_2 = f_2^*$. Then, by the definition (2.17) of the set $\mathcal{V}_{ad}$, we have

$$(u^*, f_2^*) \in \mathcal{V}_{ad}. \tag{4.11}$$

Moreover, using the convergence (4.10) and Corollary 6 it follows that

$$u_n \to u^* \quad \text{in} \quad V \quad \text{as} \quad n \to +\infty. \tag{4.12}$$

We now use the convergences (4.10), (4.12) and the weakly lower semicontinuity of the functional $\mathcal{L}$, guaranteed by the fact that $\mathcal{L}$ is a convex continuous functional, to deduce that

$$\liminf_{n\to+\infty} \mathcal{L}(u_n, f_{2n}) \geq \mathcal{L}(u^*, f_2^*). \tag{4.13}$$

It follows from (4.7) and (4.13) that

$$\omega \geq \mathcal{L}(u^*, f_2^*). \tag{4.14}$$

In addition, (4.11) and (4.6) yield

$$\omega \leq \mathcal{L}(u^*, f_2^*). \tag{4.15}$$

We now combine inequalities (4.14) and (4.15) to see that (2.21) holds, which shows that (u^*, f_2^*) is a solution to Problem $\mathcal{Q}$.

(ii) *Convergence of approximating sequences.* To proceed, we consider an approximating sequence for the Problem $\mathcal{Q}$, denoted by $\{(u_n^*, f_{2n}^*)\}_n$. Then, according to Definition 12, it follows that there exists a sequence $\{\theta_n\}_n$ of elements of I, with $\theta_n = (\varepsilon_n, f_{0n}, g_n, \phi_n)$, such that $(u_n, f_{2n}) \in \Omega(\theta_n)$ for each $n \in \mathbb{N}$ and, moreover, (3.3), (3.4), (3.6) and (4.5) hold. To simplify the notation, for each $n \in \mathbb{N}$, we write $\mathcal{V}_{ad}^n$ and $\mathcal{L}_n$ instead of $\mathcal{V}_{ad}^n(\theta_n)$ and $\mathcal{L}_{\theta_n}$, respectively. Then, exploiting the definition (4.2) we deduce that $(u_n^*, f_{2n}^*) \in \mathcal{V}_{ad}^n$ and

$$\mathcal{L}_n(u_n^*, f_{2n}^*) \leq \mathcal{L}_n(u_n, f_{2n}) \tag{4.16}$$

for each couple of functions $(u_n, f_{2n}) \in \mathcal{V}_{ad}^n$, i.e., for each couple of functions $(u_n, f_{2n}) \in V \times L^2(\Gamma_2)$ which satisfies inequality (3.10) in which, recall, j_n and f_n are defined by (3.11) and (3.12), respectively, and for each $n \in \mathbb{N}$.

We shall prove that there exists a subsequence of the sequence $\{(u_n^*, f_{2n}^*)\}_n$, again denoted by $\{(u_n^*, f_{2n}^*)\}_n$, and an element $(u^*, f_2^*) \in V \times L^2(\Gamma_3)$ such that

$$f_{2n}^* \rightharpoonup f_2^* \quad \text{in} \quad L^2(\Gamma_3) \quad \text{as} \quad n \to \infty, \tag{4.17}$$

$$u_n^* \to u^* \quad \text{in} \quad V \quad \text{as} \quad n \to \infty, \tag{4.18}$$

$$(u^*, f_2^*) \quad \text{is a solution of Problem } \mathcal{Q}. \tag{4.19}$$

To this end, we proceed in four intermediate steps that we present below.
(ii-a) *A boundedness result.* We claim that the sequence $\{f_{2n}^*\}_n$ is bounded in $L^2(\Gamma_2)$. Arguing by contradiction, assume that $\{f_n^*\}_n$ is not bounded in $L^2(\Gamma_2)$. Then, passing to a subsequence still denoted $\{f_n^*\}_n$, we have

$$\|f_{2n}^*\|_{L^2(\Gamma_2)} \to +\infty \quad \text{as} \quad n \to +\infty. \tag{4.20}$$

We use definition (4.3) of the cost functional $\mathcal{L}_n = \mathcal{L}_{\theta_n}$ to deduce that

$$\lim_{n\to\infty} \mathcal{L}_n(u_n^*, f_{2n}^*) = +\infty. \tag{4.21}$$

Next, let $\overline{u}_n$ be the solution of the variational inequality

$$\overline{u}_n \in V, \quad a(\overline{u}_n, v - \overline{u}_n) + j_n(\overline{u}_n, v) - j_n(\overline{u}_n, \overline{u}_n) \geq (\overline{f}_n, v - \overline{u}_n)_V \quad \forall\, v \in V, \tag{4.22}$$

in which

$$(\overline{f}_n, v) = \int_D f_{0n} v \,\mathrm{d}x + \int_{\Gamma_3} f_2 v \,\mathrm{d}a \quad \forall\, v \in V. \tag{4.23}$$

Note that Theorem 5 guarantees that this solution exists and is unique, for each $n \in \mathbb{N}$. Moreover, using (3.4) and (3.6) it follows that the sequence $\{\overline{u}_n\}_n$ is an approximating sequence from Problem $\mathcal{P}$, with respect to the Tykhonov triple in Example 4, corresponding to the sequence $\{\theta_n\}_n$ with $\theta_n = (0, f_{0n}, f_2, g_n)$ for all $n \in \mathbb{N}$. Therefore, using Theorem 5, it follows that

$$\overline{u}_n \to u \quad \text{in} \quad V \quad \text{as} \quad n \to \infty \tag{4.24}$$

and, using (4.5) and the definition of the functional $\mathcal{L}_n$, we deduce that

$$\mathcal{L}_n(\overline{u}_n, f_2) \to \mathcal{L}(u, f_2) \quad \text{as} \quad n \to \infty. \tag{4.25}$$

On the other hand, (4.22) and (4.23) imply that the pair $(\overline{u}_n, f_2)$ satisfies inequality (3.10) with $f_{2n} = f_2$, i.e., $(\overline{u}_n, f_2) \in \mathcal{V}_{ad}^n$. Therefore, (4.16) implies that

$$\mathcal{L}_n(u_n^*, f_{2n}^*) \leq \mathcal{L}_n(\overline{u}_n, f_2) \quad \forall\, n \in \mathbb{N}. \tag{4.26}$$

We now pass to the limit in (4.26) as $n \to \infty$ and use (4.21) and (4.25) to obtain a contradiction.

(ii-b) *Two convergence results.* We conclude from step ii-a) that the sequence $\{f_{2n}^*\}_n$ is bounded in $L^2(\Gamma_2)$ and, therefore, we can find a subsequence again denoted by $\{f_{2n}^*\}_n$ and an element $f_2^* \in L^2(\Gamma_2)$ such that (4.17) holds. Denote by u^* the solution of Problem $\mathcal{P}$ for $f_2 = f_2^*$ and note that definition (2.17) implies that

$$(u^*, f_2^*) \in \mathcal{V}_{ad}. \tag{4.27}$$

Moreover, assumptions (3.4), (3.6) and the convergence (4.17) show that $\{u_n^*\}_n$ is an approximating sequence with the Tykhonov triple in Example 4, corresponding to the sequence $\{\theta_n\}_n$ with $\theta_n = (0, f_{0n}, f_{2n}^*, g_n)$ for all $n \in \mathbb{N}$. Therefore, the well-posedness of Problem $\mathcal{P}$, guaranteed by Theorem 5, imply that (4.18) holds, too.

(ii-c) *Optimality of the limit.* We now prove that (u^*, f_2^*) is a solution to the optimal control problem $\mathcal{Q}$. To this end we use the convergences (4.17), (4.18), (4.5) and the weakly lower semicontinuity of the functional $z \to \|z\|_{L^2(\Gamma_2)}^2$ on the space $L^2(\Gamma_2)$ to see that

$$\mathcal{L}(u^*, f_2^*) \leq \liminf_{n\to\infty} \mathcal{L}_n(u_n^*, f_{2n}^*). \tag{4.28}$$

Next, we fix a solution (u_0^*, f_{02}^*) of Problem $\mathcal{Q}$ and for each $n \in \mathbb{N}$ we denote by u_n^0 be the unique element of V which satisfies the inequality

$$a(u_n^0, v - u_n) + j_n(u_n^0, v) - j_n(u_n^0, u_n^0) \geq (f_n^0, v - u_n)_V \quad \forall\, v \in V,$$

in which

$$(f_n^0, v)_V = \int_D f_{0n} v \,\mathrm{d}x + \int_{\Gamma_3} f_{02}^* v \,\mathrm{d}a \quad \forall\, v \in V.$$

It follows from here that u_n^0 satisfies the inequality

$$\begin{aligned} &a(u_n^0, v - u_n) + j_n(u_n^0, v) - j_n(u_n^0, u_n^0) + \varepsilon_n \|u_n^0\|_V \|v - u^0\|_V \\ &\quad \geq (f_n^0, v - u_n)_V \quad \forall\, v \in V, \end{aligned}$$

too. Therefore, $(u_n^0, f_{02}^*) \in \mathcal{V}_{ad}^n$ and, using the optimality of the pair (u_n^*, f_{n2}^*), (4.16), we find that

$$\mathcal{L}_n(u_n^*, f_{2n}^*) \leq \mathcal{L}_n(u_n^0, f_{02}^*) \quad \forall\, n \in \mathbb{N}.$$

We pass to the upper limit in this inequality to see that

$$\limsup_{n\to\infty} \mathcal{L}_n(u_n^*, f_{2n}^*) \leq \limsup_{n\to\infty} \mathcal{L}_n(u_n^0, f_{02}^*). \tag{4.29}$$

Next, remark that $\{u_n^0\}_n$ is an approximating sequence for Problem $\mathcal{P}$ with respect to the Tykhonov triple in Example 4 corresponding to the sequence $\{\theta_n\}_n$ with $\theta_n = (0, f_{0n}, f_{02}^*, g_n)$ for all $n \in \mathbb{N}$. Therefore, Theorem 5 guarantees that

$$u_n^0 \to u_0^* \quad \text{in} \quad V \quad \text{as} \quad n \to \infty.$$

Using now this convergence and assumption (4.5) yields

$$\lim_{n\to\infty} \mathcal{L}_n(u_n^0, f_{02}^*) = \mathcal{L}(u_0^*, f_{02}^*). \tag{4.30}$$

We now use (4.28)–(4.30) to see that

$$\mathcal{L}(u^*, f_2^*) \leq \mathcal{L}(u_0^*, f_{02}^*). \tag{4.31}$$

On the other hand, since (u_0^*, f_{02}^*) is a solution of Problem $\mathcal{Q}$, we have

$$\mathcal{L}(u_0^*, f_{02}^*) = \min_{(u,f_2)\in\mathcal{V}_{ad}} \mathcal{L}(u, f_2), \tag{4.32}$$

and, therefore, inclusion (4.27) implies that

$$\mathcal{L}(u_0^*, f_{02}^*) \leq \mathcal{L}(u^*, f_2^*). \tag{4.33}$$

We now combine the inequalities (4.31) and (4.33) to see that

$$\mathcal{L}(u^*, f_2^*) = \mathcal{L}(u_0^*, f_{02}^*). \tag{4.34}$$

Next, we use relations (4.27), (4.34) and (4.32) to see that (4.19) holds.
(ii-d) *End of proof.* We remark that the convergences (4.17) and (4.18) imply the weak convergence (in the product Hilbert space $V \times L^2(\Gamma_2)$) of the sequence $\{(u_n^*, f_{2n}^*)\}_n$ to the element (u^*, f_2^*). Theorem 16 is now a direct consequence of Definition 14. □

A direct consequence of Theorem 16 is the following.

Corollary 17. *Assume that* (2.8)–(2.12), (2.19), (2.20) *hold and, moreover, assume that Problem $\mathcal{Q}$ has a unique solution. Then Problem $\mathcal{Q}$ is weakly well posed with respect to the Tykhonov triple in Example* 15.

Proof. Let (u^*, f_2^*) be the unique solution to Problem $\mathcal{Q}$ and let $\{(u_n^*, f_{2n}^*)\}_n$ be an approximating sequence for the Problem $\mathcal{Q}$, with respect to the Tykhonov triple in Example 15. First, it follows from the proof of Theorem 16 that the sequence $\{f_{2n}^*\}_n$ is bounded in $L^2(\Gamma_2)$. Therefore, using arguments similar to those used in step i) of the proof of Theorem 5, we deduce that the sequence $\{u_n^*\}_n$ is bounded in V. We conclude from here that the sequence $\{(u_n^*, f_{2n}^*)\}_n$ is bounded in the product space $V \times L^2(\Gamma_2)$. Second, a careful analysis of the proof of Theorem 16 reveals that (u^*, f_2^*) is the weak limit (in $V \times L^2(\Gamma_2)$) of any weakly convergent subsequence of the sequence $\{(u_n^*, f_{2n}^*)\}_n$. The two properties above allow us to use a standard argument to deduce that the whole sequence $\{(u_n^*, f_{2n}^*)\}_n$ converges weakly in $V \times L^2(\Gamma_2)$ to (u^*, f_2^*), as $n \to \infty$. Corollary 17 is now a direct consequence of Definition 13. □

We now proceed with the following convergence result.

Corollary 18. *Assume that* (2.8)–(2.12), (2.19), (2.20) *hold and, moreover, assume that for each* $\widetilde{\phi} \in V$*, Problem* $\mathcal{Q}$ *has a unique solution. In addition, for each* $n \in \mathbb{N}$*, assume that* ϕ_n *is given and the convergence and* (3.6) *holds. Then the solution of Problem* $\mathcal{Q}$ *with the data* ϕ_n*, denoted by* (u_n^*, f_{2n}^*)*, converges weakly to the solution of Problem* $\mathcal{Q}$*, in the space* $V \times L^2(\Gamma_2)$.

Proof. We use the Tykhonov triple $\mathcal{T} = (I, \Omega, \mathcal{C})$ in Example 15, again. For each $n \in \mathbb{N}$, denote $\theta_n = (0, f_0, g, \phi_n) \in I$. Then, using (4.5) it is easy to see that the sequence $\{\theta_n\}_n$ belongs to the set $\mathcal{C}$. Moreover, note that in this case $\mathcal{V}_{ad}^n = \mathcal{V}_{ad}$, for all $n \in \mathbb{N}$. In addition, using the statement of Problem $\mathcal{Q}$ we see that

$$(u_n^*, f_{2n}^*) \in \mathcal{V}_{ad} \quad \text{and} \quad \mathcal{L}_n(u_n^*, f_{2n}^*) \leq \mathcal{L}_n(u_n, f_{2n}) \quad \forall\, (u_n, f_{2n}) \in \mathcal{V}_{ad}.$$

It follows now from the definition (4.2) that $(u_n^*, f_{2n}^*) \in \Omega(\theta_n)$ for each $n \in \mathbb{N}$ and, therefore, Definition 12 shows that $\{(u_n^*, f_{2n}^*)\}_n$ is an approximating sequence for Problem $\mathcal{Q}$. We now use Corollary 17 and Definition 13 to conclude the proof. □

We end this section with the following comments, remarks and mechanical interpretation of Theorem 16 and Corollaries 17 and 18.

(i) First, recall that the convergence result (4.18) represents the strong convergence of the sequence $\{u_n\}_n$ to the element u, in the space V. Therefore, Theorem 16 provides more than the weakly generalized well-posedness of Problem $\mathcal{Q}$ with respect to the Tykhonov triple in Example 15. Indeed, according to Definition 14, to obtain the weakly generalized well-posedness of Problem $\mathcal{Q}$ we need only the weak convergence $u_n \rightharpoonup u$ in V as $n \to \infty$, which is obviously implied by the strong convergence (4.18). A similar comment can be made concerning Corollaries 17 and 18.

(ii) Second, recall that, in general, Problem $\mathcal{Q}$ does not have a unique solution. The reason arises in the fact that the optimal control $\mathcal{Q}$ is equivalent to the problem of finding $u^* \in V$ and $f_2^* \in L^2(\Gamma_3)$ such that

$$u^* = u(f_2^*) \quad \text{and} \quad J(f_2^*) = \min_{f_2 \in L^2(\Gamma_2)} J(f_2), \tag{4.35}$$

where $u(f_2)$ represents the solution of Problem $\mathcal{P}$ with the data μ, f_0, f_2, g and $J : L^2(\Gamma_2) \to \mathbb{R}$ is the functional defined by

$$J(f_2) = \mathcal{L}(u(f_2), f_2) \quad \forall\, f_2 \in L^2(\Gamma_2). \tag{4.36}$$

Note that, in general the functional J is not strictly convex. This implies that the solution of the optimization problem in (4.35) is not unique and, therefore, Corollaries 17, 18 cannot be applied. Nevertheless, we stress that these results can be applied in the particular case when $\Gamma_3 = \emptyset$. Indeed, in this case, problem $\mathcal{P}$ consists in finding an element $v \in V$ such that

$$a(u, v) = (f, v)_V \quad \forall\, v \in V.$$

Then, it is easy to see that the operator $f_2 \mapsto u(f_2) : L^2(\Gamma_2) \to V$ is linear and continuous and, therefore, the functional J is strictly convex, which implies the unique solvability of Problem $\mathcal{Q}$. We conclude that in this case Corollaries 17 and 18can be applied. We have a similar conclusion in the case when the function g vanishes, i.e., $g \equiv 0$. Another important case when Corollaries 17 and 18 is when equality $\phi = 0_V$ holds. Indeed, in this case the cost function $\mathcal{L}$ becomes

$$\mathcal{L}(u, f_2) = a_0 \|u\|^2_{L^2(D)} + a_2 \|f_2\|^2_{L^2(\Gamma_2)}$$

for all $u \in V$, $f_2 \in L^2(\Gamma_2)$ and it was proved in [5] that the corresponding functional J defined by (4.36) is strictly convex.

(iii) Theorem 16 establishes a link between the solutions of the optimal control problem $\mathcal{Q}$ and the solutions of the optimal control problem of finding an element $(u^*, f_2^*) \in \mathcal{V}_{ad}(\theta)$ such that

$$\mathcal{L}_\theta(u^*, f_2^*) = \min_{(u,f_2)\in\mathcal{V}_{ad}(\theta)} \mathcal{L}_\theta(u, f_2). \tag{4.37}$$

A short comparison between the optimal control problems $\mathcal{Q}$ and (4.37) shows that in problem (4.37), both the state equation (and, therefore, the set of admissible pairs) and the cost functional are different to those in Problem $\mathcal{Q}$. The importance of Theorem 16 is that it provides a convergence result between the solutions of these optimal control problems which have a different structure. In particular, Corollary 18 provides a continuous dependence result for the solutions of the optimal control $\mathcal{Q}$ with respect to the target displacement ϕ. This property has an important mechanical interpretation, since it shows that, in the context of antiplane shear with elastic materials, small perturbation in the target displacement field gives rise to small perturbation in the corresponding optimal pairs, i.e., in the optimal solution of Problem $\mathcal{Q}$.

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