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A Penalty Method for Elliptic Variational–Hemivariational Inequalities

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Abstract: We consider an elliptic variational–hemivariational inequality $\mathcal{P}$ in a real reflexive Banach space, governed by a set of constraints K. Under appropriate assumptions of the data, this inequality has a unique solution $u \in K$. We associate inequality $\mathcal{P}$ to a sequence of elliptic variational–hemivariational inequalities $\{\mathcal{P}_n\}$, governed by a set of constraints $\widetilde{K} \supset K$, a sequence of parameters $\{\lambda_n\} \subset \mathbb{R}_+$, and a function ψ. We prove that if, for each $n \in \mathbb{N}$, the element $u_n \in \widetilde{K}$ represents a solution to Problem $\mathcal{P}_n$, then the sequence $\{u_n\}$ converges to u as $\lambda_n \to 0$. Based on this general result, we recover convergence results for various associated penalty methods previously obtained in the literature. These convergence results are obtained by considering particular choices of the set $\widetilde{K}$ and the function ψ. The corresponding penalty methods can be applied in the study of various inequality problems. To provide an example, we consider a purely hemivariational inequality that describes the equilibrium of an elastic membrane in contact with an obstacle, the so-called foundation.

Keywords: elliptic variational–hemivariational inequality; Clarke generalized derivative; penalty method; convergence result; elastic membrane; contact; unilateral constraint

MSC: 47J20; 49J52; 49J45; 47H06; 74K15; 74G22

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1. Introduction

Inequality problems with unilateral constraints arise in mechanics and physics. The famous Signorini contact problem represents a relevant example, among many others. These inequalities are divided into three main classes. The first one is the class of variational inequalities, which are inequalities governed by a convex function. Their analysis (including solvability results and error estimates for numerical schemes) is based on arguments of monotonicity and convexity. References in the field are [1–8], for instance. The second class is the class of hemivariational inequalities, which are inequalities governed by a locally Lipschitz continuous function. Their analysis is based on properties of the subdifferential in the sense of Clarke. Comprehensive references in the field are the books [9–12] as well as the recent papers [13–15], for instance. Finally, the third class is given by the class of the so-called variational–hemivariational inequalities, which are governed by both a convex and a locally Lipschitz function. Such inequalities are more general since they contain as particular cases both the class of variational and the class of hemivariational inequalities. Their study was carried out in various references, including [16–19].

In this paper, we deal with variational–hemivariational inequalities, the problem we are interested in being formulated in a reflexive Banach space X. We denote by $\|\cdot\|_X$ the norm of X, by X^* its dual, and by $\langle\cdot,\cdot\rangle$ the corresponding duality pairing mapping. Let $K \subset X$, $A\colon X \to X^*$, $\varphi\colon X \times X \to \mathbb{R}$, $j : X \to \mathbb{R}$ and $f \in X^*$. We assume that the function j is locally

Lipschitz, and we denote by $j^0(u;v)$ the generalized directional derivative of j at $u \in X$ in the direction $v \in X$. With these notations, we consider the following inequality problem:

Problem 1. *$\mathcal{P}$. Find u such that*

$$u \in K, \qquad \langle Au, v-u\rangle + \varphi(u,v) - \varphi(u,u) + j^0(u, v-u) \geq \langle f, v-u\rangle \qquad \forall\, v \in K. \quad (1)$$

An existence and unique result in the study of Problem $\mathcal{P}$ is recalled in the next section; see Theorem 1. Since the problem is governed by the set of constraints K, for numerical reasons, it is useful to approximate its solution using a penalty method. The main ingredients of any penalty method are the following: (i) we replace the problem we are interested in with a sequence of approximating problems, the so-called penalty problems, in which the constraints are removed or relaxed; (ii) we prove that each penalty problem has a unique solution and the sequence of the solutions obtained in this way converges to the solution of the original problem. Penalty methods have been used in [5,20] and [13,19,21] to approximate various classes of variational inequalities and variational–hemivariational inequalities, respectively.

Consider now a set $\widetilde{K}$ such that $K \subset \widetilde{K} \subset X$, a function $\psi : X \times X \to \mathbb{R}$, and a sequence $\{\lambda_n\} \subset \mathbb{R}_+$. With these data, we associate problem $\mathcal{P}$ to a sequence $\{\mathcal{P}_n\}$ of penalty problems defined, for each $n \in \mathbb{N}$, as follows:

Problem 2. *$\mathcal{P}_n$. Find u_n such that*

$$\begin{aligned} u_n \in \widetilde{K}, \qquad &\langle Au_n, v-u_n\rangle + \frac{1}{\lambda_n}\psi(u_n,v) - \frac{1}{\lambda_n}\psi(u_n,u_n) \\ &+\varphi(u_n,v) - \varphi(u_n,u_n) + j^0(u_n, v-u_n) \geq \langle f, v-u_n\rangle \qquad \forall\, v \in \widetilde{K}. \end{aligned} \quad (2)$$

Note that in contrast with (1), this inequality includes a penalty term governed by the parameter λ_n and the function ψ. Moreover, (2) is obtained from (1) by replacing the constraint set K with the constraint sets $\widetilde{K} \supset K$, and therefore, we refer to it as a penalty inequality of (1). The case $\widetilde{K} = X$ leads to an unconstrained problem and corresponds to the classical penalty problems considered in the literature. Nevertheless, some mathematical models of contact lead to inequalities of the form (2) in which the constraints are only partially relaxed, i.e., $\widetilde{K} \neq X$. An example was provided in [22], where a contact problem that describes the equilibrium of two elastic rods attached to a nonlinear spring was considered. The variational formulation of the problem was in the form of an elliptic variational inequality for the displacement field, i.e., an inequality of the form (1) in which j vanishes. Besides a general convergence result, a penalty method was introduced, and the numerical approximation of the problem was considered based on a finite element scheme. The numerical simulations provided there validate the theoretical convergence result. Nevertheless, considering nonsmooth contact problems leads to genuine variational–hemivariational inequalities, i.e., to inequalities of the form (1) in which j does not vanish. Therefore, in view of various applications, there is a need to extend the results in [22] to such kinds of inequalities.

Our aim in this paper is to fill this gap, and it is three-fold. The first one is to prescribe conditions on the set $\widetilde{K}$ and the function ψ such that, if u_n represents a solution to Problem $\mathcal{P}_n$, then the sequence $\{u_n\}$ converges to the solution u of the variational inequality (1), as $\lambda_n \to 0$. Our second aim is to provide relevant examples of sets $\widetilde{K}$ and functions ψ that satisfy these conditions. Finally, our third aim is to apply our theoretical results in contact mechanics.

Our study shows that the choice of the penalty problems is not unique; i.e., it is possible to construct various penalty problems (which have a different structure) for the same constrained inequality. The penalty method we introduce in this paper is new since, to the best of our knowledge, the penalty problems of (1) studied in the literature are in the form of inequalities governed by a penalty operator and are unconstrained (that is, $\widetilde{K} = X$). In contrast, in our approach, the penalty problems are constructed using the function ψ, and the sequence of penalty inequalities can involve constraints (that is, $\widetilde{K} \neq X$). This represents the main traits of the novelty of the current manuscript.

The outline of this paper is the following. In Section 2, we introduce some preliminary material needed in the rest of the paper. Next, in Section 3, we state and prove our main results, Theorem 2 and Corollary 1. They provide sufficient conditions on the set $\widetilde{K}$ and the function ψ that guarantee the convergence of the sequence $\{u_n\} \subset \widetilde{K}$ of solutions of inequality (2) to the solution u of inequality (1) as $\lambda_n \to 0$. In Section 4, we indicate several relevant particular cases in which these conditions are satisfied, and in this way, we recover well-known convergence results previously obtained in [19,21]. Finally, in Section 5, we apply these abstract results in the study of a bi-dimensional problem of contact. We end our paper with some concluding remarks presented in Section 6, where we provide some ideas for forthcoming research related to the material presented in this manuscript.

2. Preliminaries

We denote by 2^{X^*} the set of parts of X^*. We use "$\to$" and "$\rightharpoonup$" to represent the strong and weak convergences, respectively, and the lower and upper limits below are considered as $n \to \infty$.

Locally Lipschitz functions. We now recall some basic definitions and properties of the generalized subdifferential in the sense of Clarke [23]. For a locally Lipschitz function $j\colon X \to \mathbb{R}$, we define the generalized (Clarke) directional derivative of j at the point u in the direction v by the equality

$$j^0(u;v) = \limsup_{w \to u, \lambda \searrow 0} \frac{j(w + \lambda v) - j(w)}{\lambda} \qquad \forall\, u,\ v \in X.$$

The generalized gradient of j at u is a subset of the dual space X^* given by

$$\partial j(u) := \{\, \xi \in X^* \mid \langle \xi, v \rangle_{X^* \times X} \leq j^0(u;v) \ \text{ for all } v \in X \,\}$$

and $\partial j : X \to 2^{X^*}$ represents the Clarke subdifferential of the function j. A locally Lipschitz function j is said to be regular at $u \in X$ if, for all $v \in X$, the one-sided directional derivative

$$j'(u;v) = \lim_{\lambda \downarrow 0} \frac{j(u + \lambda v) - j(u)}{\lambda}$$

exists and $j^0(u;v) = j'(u;v)$.

The following result collects some properties of the generalized directional derivative and the generalized gradient.

Proposition 1. *Assume that $j\colon X \to \mathbb{R}$ is a locally Lipschitz function. Then, the following hold.*

(i) *For every $u \in X$, the function $X \ni v \mapsto j^0(u;v) \in \mathbb{R}$ is positively homogeneous (i.e., $j^0(u;\lambda v) = \lambda j^0(u;v)$ for all $\lambda \geq 0$) and subadditive (i.e., $j^0(u;v_1 + v_2) \leq j^0(u;v_1) + j^0(u;v_2)$ for all $v_1, v_2 \in X$).*

(ii) *The function $X \times X \ni (u,v) \mapsto j^0(u;v) \in \mathbb{R}$ is upper semicontinuous; i.e., for all $u, v \in X$, $\{u_n\}, \{v_n\} \subset X$ such that $u_n \to u$ and $v_n \to v$ in X, we have $\limsup j^0(u_n;v_n) \leq j^0(u;v)$.*

(iii) *For every $u, v \in X$, we have $j^0(u;v) = \max\{\, \langle \xi, v \rangle \ : \ \xi \in \partial j(u) \,\}$.*

Additional results on the generalized gradient can be found in [24,25].

An existence and uniqueness result. Existence, uniqueness, and convergence results for variational–hemivariational inequalities of the form (1) can be found in [19,21,26,27]. Here, in the study of (1), we assume the following:

$$K \text{ is a nonempty closed convex subset of } X. \tag{3}$$

$$\begin{cases} A\colon X \to X^* \text{ is pseudomonotone and strongly monotone, i.e.,} \\ \text{(a) } A \text{ is bounded and } u_n \rightharpoonup u \text{ in } X \text{ with } \limsup \langle Au_n, u_n - u\rangle \le 0 \\ \quad \text{implies that } \liminf \langle Au_n, u_n - v\rangle \ge \langle Au, u - v\rangle \ \ \forall\, v \in X. \\ \text{(b) there exists } m_A > 0 \text{ such that} \\ \quad \langle Au - Av, u - v\rangle \ge m_A \|u - v\|_X^2 \ \ \forall\, u,\, v \in X. \end{cases} \tag{4}$$

$$\begin{cases} \varphi\colon X \times X \to \mathbb{R} \text{ is such that} \\ \text{(a) } \varphi(\eta, \cdot)\colon X \to \mathbb{R} \text{ is convex and lower semicontinuous,} \\ \quad \text{for all } \eta \in X. \\ \text{(b) there exists } \alpha_\varphi \ge 0 \text{ such that} \\ \quad \varphi(\eta_1, v_2) - \varphi(\eta_1, v_1) + \varphi(\eta_2, v_1) - \varphi(\eta_2, v_2) \\ \quad\quad \le \alpha_\varphi \|\eta_1 - \eta_2\|_X \, \|v_1 - v_2\|_X \ \ \forall\, \eta_1, \eta_2, v_1, v_2 \in X. \end{cases} \tag{5}$$

$$\begin{cases} j\colon X \to \mathbb{R} \text{ is such that} \\ \text{(a) } j \text{ is locally Lipschitz.} \\ \text{(b) } \|\xi\|_{X^*} \le c_0 + c_1 \|v\|_X \ \ \forall\, v \in X,\ \xi \in \partial j(v) \\ \quad\quad \text{with } c_0, c_1 \ge 0. \\ \text{(c) there exists } \alpha_j \ge 0 \text{ such that} \\ \quad j^0(v_1; v_2 - v_1) + j^0(v_2; v_1 - v_2) \le \alpha_j \|v_1 - v_2\|_X^2 \\ \quad\quad \forall\, v_1, v_2 \in X. \end{cases} \tag{6}$$

$$\alpha_\varphi + \alpha_j < m_A. \tag{7}$$

$$f \in X^*. \tag{8}$$

We now recall the following result, proved in [19] (p. 143).

Theorem 1. *Assume* (3)–(8). *Then, inequality* (1) *has a unique solution* $u \in K$.

Penalty operators. Next, we follow [28] and present some proprieties of a special class of operators, the so-called penalty operators. Let K be a subset of X. An operator $G\colon X \to X^*$ is said to be a penalty operator of K if G is bounded, demi-continuous, and monotone and

$$K = \{\, x \in X \mid Gx = 0_{X^*} \,\}. \tag{9}$$

It can be proved that any penalty operator G of a set K is a pseudomonotone operator, i.e., it satisfies the properties in (4) (a).

On reflexive Banach spaces, penalty operators associated to a set K that satisfies condition (3) always exist. Indeed, using a classical renorming theorem, we can assume that X is strictly convex space and, therefore, the duality map $\mathcal{J}\colon X \to 2^{X^*}$, defined by

$$\mathcal{J}u = \{\, u^* \in X^* \mid \langle u^*, u\rangle = \|u\|_X^2 = \|u^*\|_{X^*}^2 \,\} \quad \forall\, u \in X, \tag{10}$$

is a single-valued operator. Details can be found in [29] (Proposition 1.3.27) and [30] (Proposition 32.22). Moreover, since X is strictly convex, using a Weierstrass-type argument, it follows that for any nonempty convex closed set $K \subset X$, we are in a position to define the projection operator $P_K : X \to K$ by the equality

$$u = P_K f \quad \Longleftrightarrow \quad u \in K \ \text{ and } \ \|u - f\|_X = \min_{v \in K} \|v - f\|_X \quad \forall\, f \in X. \tag{11}$$

Denote by $I_X: X \to X$ the identity map on X. Then, the operator

$$G = \mathcal{J}(I_X - P_K): X \to X^* \tag{12}$$

is a penalty operator of K. A proof of this result can be found in [28] (p. 267).

3. A Convergence Result

Everywhere in this section, we assume that (3)–(8) hold, and we denote by u the solution of inequality (1) provided by Theorem 1. Moreover, we consider the following assumptions:

$$K \subset \widetilde{K} \subset X. \tag{13}$$

$$\widetilde{K} \text{ is a nonempty closed convex subset of } X. \tag{14}$$

$$\left\{\begin{array}{l} \psi: X \times X \to \mathbb{R} \text{ is such that} \\ \text{(a)}\ \ \psi(u,v) \le \psi(u,u) \quad \forall\, u \in \widetilde{K},\ v \in K. \\ \text{(b)}\ \ \text{One of the two implications below holds:} \\ \qquad (\text{b}_1)\ \ u \in \widetilde{K},\ \psi(u,v) \ge \psi(u,u)\ \ \forall\, v \in K\ \text{ implies that } u \in K. \\ \qquad (\text{b}_2)\ \ \widetilde{K} = X,\ u \in X,\ \psi(u,v) \ge \psi(u,u)\ \ \forall\, v \in X \text{ implies that } u \in K. \\ \text{(c) For all sequence } \{u_n\} \subset \widetilde{K}\ \text{ such that } u_n \rightharpoonup u \text{ in } X, \\ \qquad \text{the inequality } \limsup\, \big[\psi(u_n,u_n) - \psi(u_n,u)\big] \le 0 \text{ implies that} \\ \qquad \liminf\, \big[\psi(u_n,u_n) - \psi(u_n,v)\big] \ge \psi(u,u) - \psi(u,v)\ \ \forall\, v \in X. \end{array}\right. \tag{15}$$

$$\lambda_n > 0. \tag{16}$$

$$\lambda_n \to \infty. \tag{17}$$

$$\left\{\begin{array}{l} \text{For all } u \in X, \text{ there exists } L_u > 0 \text{ such that} \\ \varphi(u,v_1) - \varphi(u,v_2) \le L_u \|v_1 - v_2\|_X \quad \forall\, u,\ v_1,\ v_2 \in X. \end{array}\right. \tag{18}$$

$$\left\{\begin{array}{l} \text{For all sequence } \{u_n\} \subset X \text{ such that} \\ u_n \rightharpoonup u \text{ in } X \text{ and for all } v \in K, \text{we have} \\ \limsup\, \big[\varphi(u_n,v) - \varphi(u_n,u_n)\big] \le \varphi(u,v) - \varphi(u,u). \end{array}\right. \tag{19}$$

$$\left\{\begin{array}{l} \text{For all sequence } \{u_n\} \subset X \text{ such that} \\ u_n \rightharpoonup u \text{ in } X \text{ and for all } v \in X, \text{we have} \\ \limsup\, j^0(u_n; v - u_n) \le j^0(u; v - u). \end{array}\right. \tag{20}$$

Our main result in this section is the following.

Theorem 2. *Assume (3)–(8) and (13)–(20) and assume that, for each $n \in \mathbb{N}$, u_n is a solution of inequality (2). Then, $u_n \to u$ in X as $n \to \infty$.*

Proof. The proof is structured in four steps as follows.

(i) *In the first step, we prove that there is an element $\widetilde{u} \in X$ and a subsequence of $\{u_n\}$, still denoted by $\{u_n\}$, such that $u_n \rightharpoonup \widetilde{u}$ in X as $n \to \infty$.*

To prove this statement, we establish the boundedness of $\{u_n\}$ in X. Let v_0 be a given element in K, and let $n \in \mathbb{N}$. We test with $v_0 \in K \subset \widetilde{K}$ in (2) and use the strong monotonicity of the operator A to obtain that

$$m_A \|u_n - v_0\|_X^2 \le \langle Av_0, v_0 - u_n\rangle + \frac{1}{\lambda_n}[\psi(u_n, v_0) - \psi(u_n, u_n)] \qquad (21)$$
$$+\varphi(u_n, v_0) - \varphi(u_n, u_n) + j^0(u_n; v_0 - u_n) + \langle f, u_n - v_0\rangle.$$

Next, assumption (15)(a) implies that

$$\psi(u_n, v_0) - \psi(u_n, u_n) \le 0 \qquad (22)$$

and moreover, assumptions (5)(b) and (18) yield

$$\begin{aligned} &\varphi(u_n, v_0) - \varphi(u_n, u_n) \qquad (23)\\ &= [\varphi(u_n, v_0) - \varphi(u_n, u_n) + \varphi(v_0, u_n) - \varphi(v_0, v_0)] + [\varphi(v_0, v_0) - \varphi(v_0, u_n)]\\ &\le \alpha_\varphi \|u_n - v_0\|_X^2 + L_{v_0}\|u_n - v_0\|_X. \end{aligned}$$

On the other hand, by assumption (6) and Proposition 1 (iii), we have

$$\begin{aligned} &j^0(u_n; v_0 - u_n) \qquad (24)\\ &= j^0(u_n; v_0 - u_n) + j^0(v_0; u_n - v_0) - j^0(v_0; u_n - v_0)\\ &\le j^0(u_n; v_0 - u_n) + j^0(v_0; u_n - v_0) + |j^0(v_0; u_n - v_0)|\\ &\le \alpha_j \|u_n - v_0\|_X^2 + |\max\{\langle \xi, u_n - v_0\rangle : \xi \in \partial j(v_0)\}|\\ &\le \alpha_j \|u_n - v_0\|_X^2 + (c_0 + c_1\|v_0\|_X)\|u_n - v_0\|_X, \end{aligned}$$

and obviously,

$$\langle Av_0, v_0 - u_n\rangle + \langle f, u_n - v_0\rangle \le \|Av_0 - f\|_{X^*}\|u_n - v_0\|_X. \qquad (25)$$

We now combine inequalities (21)–(25) to see that

$$m_A \|u_n - v_0\|_X^2 \le \|Av_0 - f\|_{X^*}\|u_n - v_0\|_X + L_{v_0}\|u_n - v_0\|_X \qquad (26)$$
$$+\alpha_\varphi\|u_n - v_0\|_X^2 + \alpha_j\|u_n - v_0\|_X^2 + (c_0 + c_1\|v_0\|_X)\|u_n - v_0\|_X.$$

Therefore, using the smallness assumption (7), we deduce that there is a constant $C > 0$ that does not depend of n such that $\|u_n - v_0\|_X \le C$. This implies that $\{u_n\}$ is a bounded sequence in X. Thus, from the reflexivity of X, the inclusion $\{u_n\} \subset \widetilde{K}$, and assumption (14), by passing to a subsequence, if necessary, we deduce that

$$u_n \rightharpoonup \widetilde{u} \quad \text{in } X, \quad \text{as } n \to \infty, \qquad (27)$$

with some $\widetilde{u} \in \widetilde{K}$. This concludes the proof of the claim.

(ii) *Next, we show that $\widetilde{u} \in K$ and, moreover, $\widetilde{u}$ is a solution of inequality* (1).

Let v be a given element in $\widetilde{K}$. We use (2) to obtain that

$$\frac{1}{\lambda_n}[\psi(u_n, u_n) - \psi(u_n, v)] \le \langle Au_n, v - u_n\rangle \qquad (28)$$
$$+\varphi(u_n, v) - \varphi(u_n, u_n) + j^0(u_n, v - u_n) + \langle f, u_n - v\rangle.$$

Using now assumptions (4)–(6) and (18), the boundedness of the sequence $\{u_n\}$ and the arguments similar to those used in the proof of (26), we deduce that there exists a constant $D > 0$ that does not depend on n such that

$$\psi(u_n, u_n) - \psi(u_n, v) \leq \lambda_n D.$$

Hence,

$$\limsup \left[\psi(u_n, u_n) - \psi(u_n, v)\right] \leq 0. \tag{29}$$

Recall that this inequality holds for any $v \in \widetilde{K}$. We take $v = \widetilde{u}$ in this inequality to see that

$$\limsup \left[\psi(u_n, u_n) - \psi(u_n, \widetilde{u})\right] \leq 0; \tag{30}$$

then, we use (27), (30), and assumption (15) (c) to find that

$$\psi(\widetilde{u}, \widetilde{u}) - \psi(\widetilde{u}, v) \leq \liminf \left[\psi(u_n, u_n) - \psi(u_n, v)\right] \quad \forall\, v \in X. \tag{31}$$

Therefore, combining (31) and (29), it follows that

$$\psi(\widetilde{u}, \widetilde{u}) - \psi(\widetilde{u}, v) \leq 0 \quad \forall\, v \in \widetilde{K}. \tag{32}$$

Assume that (15) (b_1) holds. Then, (32) and the inclusion $K \subset \widetilde{K}$ imply that $\widetilde{u} \in K$. Next, assume that (15) (b_2) holds. Then, we use (32) and equality $\widetilde{K} = X$ to see that in this case $\widetilde{u} \in K$, too. We conclude from above that, in any case, the following inclusion holds:

$$\widetilde{u} \in K. \tag{33}$$

Consider now a given element $v \in K$, and let $n \in \mathbb{N}$. We use (2) and the inclusion $K \subset \widetilde{K}$ to obtain that

$$\begin{aligned}\langle A u_n, u_n - v\rangle \leq\ & \frac{1}{\lambda_n}\left[\psi(u_n, v) - \psi(u_n, u_n)\right] \\ & + \varphi(u_n, v) - \varphi(u_n, u_n) + j^0(u_n; v - u_n) + \langle f, u_n - v\rangle.\end{aligned}$$

Therefore, using assumption (15) (a), we find that

$$\langle A u_n, u_n - v\rangle \leq \varphi(u_n, v) - \varphi(u_n, u_n) + j^0(u_n; v - u_n) + \langle f, u_n - v\rangle. \tag{34}$$

Next, using (27) and assumption (19), we have

$$\limsup \left[\varphi(u_n, v) - \varphi(u_n, u_n)\right] \leq \varphi(\widetilde{u}, v) - \varphi(\widetilde{u}, \widetilde{u}). \tag{35}$$

On the other hand, from (27) and (20), it follows that

$$\limsup j^0(u_n; v - u_n) \leq j^0(\widetilde{u}, v - \widetilde{u}). \tag{36}$$

Moreover,

$$\langle f, u_n - v\rangle \to \langle f, \widetilde{u} - v\rangle. \tag{37}$$

We now gather the inequalities (34)–(37) to see that

$$\limsup \langle A u_n, u_n - v\rangle \leq \varphi(\widetilde{u}, v) - \varphi(\widetilde{u}, \widetilde{u}) + j^0(\widetilde{u}, v - \widetilde{u}) + \langle f, \widetilde{u} - v\rangle \quad \forall\, v \in K. \tag{38}$$

We now take $v = \widetilde{u} \in K$ in (38) and use Proposition 1 (i) to deduce that

$$\limsup \langle A u_n, u_n - \widetilde{u}\rangle \leq 0. \tag{39}$$

This inequality together with (27) and the pseudomonotonicity of A imply that

$$\langle A\widetilde{u}, \widetilde{u} - v\rangle \le \liminf \langle Au_n, u_n - v\rangle \quad \forall\, v \in X. \tag{40}$$

Finally, we use (40) and (38) to see that

$$\langle A\widetilde{u}, \widetilde{u} - v\rangle \le \varphi(\widetilde{u}, v) - \varphi(\widetilde{u}, \widetilde{u}) + j^0(\widetilde{u}; v - \widetilde{u}) + \langle f, \widetilde{u} - v\rangle,$$

for all $v \in K$. Hence, recalling (33), it follows that $\widetilde{u}$ is a solution to Problem $\mathcal{P}$, as claimed.

(iii) *We now prove the weak convergence of the whole sequence* $\{u_n\}$.

Since Problem $\mathcal{P}$ has the unique solution $u \in K$, we deduce from above that $\widetilde{u} = u$. On the other part, Step (ii) shows that every weakly convergent subsequence of $\{u_n\}$ has the weak limit u. Then, since the sequence $\{u_n\} \subset X$ is bounded in X, using a standard argument, we deduce that the whole sequence $\{u_n\}$ converges weakly in X to u as $n \to \infty$.

(iv) *In the final step of the proof, we prove that* $u_n \to u$ *in* X*, as* $n \to \infty$.

We take $v = \widetilde{u} \in K$ in (40) and use (39) to obtain that

$$0 \le \liminf \langle Au_n, u_n - \widetilde{u}\rangle \le \limsup \langle Au_n, u_n - \widetilde{u}\rangle \le 0,$$

which shows that $\langle Au_n, u_n - \widetilde{u}\rangle \to 0$ as $n \to \infty$. Therefore, using the strong monotonicity of A, equality $\widetilde{u} = u$, and the convergence $u_n \rightharpoonup u$ in X, we have

$$m_A\|u_n - u\|_X^2 \le \langle Au_n - Au, u_n - u\rangle = \langle Au_n, u_n - u\rangle - \langle Au, u_n - u\rangle \to 0$$

as $n \to \infty$. Hence, it follows that $u_n \to u$ in X, which completes the proof. □

We end this section with the remark that in the statement of Theorem 2, the solvability of problems $\mathcal{P}_n$ is assumed. Nevertheless, the unique solvability of these problems can be obtained under the following assumption on the function ψ:

$$\left\{\begin{array}{l} \psi\colon X \times X \to \mathbb{R} \text{ is such that} \\ \text{(a) } \psi(\eta, \cdot)\colon X \to \mathbb{R} \text{ is convex and lower semicontinuous,} \\ \qquad \text{for all } \eta \in X. \\ \text{(b) } \psi(\eta_1, v_2) - \psi(\eta_1, v_1) + \psi(\eta_2, v_1) - \psi(\eta_2, v_2) \le 0 \\ \qquad \forall\, \eta_1, \eta_2, v_1, v_2 \in X. \end{array}\right. \tag{41}$$

More precisely, we have the following existence and uniqueness result.

Proposition 2. *Assume* (4)–(8), (14), (16), *and* (41). *Then, Problem* $\mathcal{P}_n$ *has a unique solution* $u_n \in \widetilde{K}$ *for each* $n \in \mathbb{N}$.

Proof. Let $n \in \mathbb{N}$. We use assumptions (5), (16), and (41) to see that the function $\frac{1}{\lambda_n}\psi + \varphi : X \times X \to \mathbb{R}$ satisfies assumption (5) with constant α_φ. Therefore, using (14) and assumptions (4) and (6)–(8), it turns out that Proposition 2 is a direct consequence of Theorem 1, used with $\widetilde{K}$ and $\frac{1}{\lambda_n}\psi + \varphi$ instead of K and φ, respectively. □

The following result represents a direct consequence of Theorem 2 and Proposition 2.

Corollary 1. *Assume* (3)–(8), (13)–(20), *and* (41). *Then, for each* $n \in \mathbb{N}$*, Problem* $\mathcal{P}_n$ *has a unique solution* u_n*. Moreover,* $u_n \to u$ *in* X *as* $n \to \infty$.

4. Relevant Particular Cases

We now present some relevant particular cases of penalty problems of the form (2) for which Theorem 2 holds. Our main results in this section are Corollaries 2 and 3. Corollary 2 extends some convergence results obtained in [20] and [19], while Corollary 3 recovers a result in [19]. Some comments on this matter are presented below in this section. Consider an operator G such that the following conditions hold:

$$G : X \to X^* \text{ is a bounded monotone hemicontinuous operator.} \tag{42}$$

$$\langle Gu, v-u\rangle \le 0 \quad \forall\, u \in \widetilde{K},\ v \in K. \tag{43}$$

$$\left\{\begin{array}{l}\text{One of the two implications below holds:}\\ (\mathrm{b}_1)\quad u \in \widetilde{K},\ \langle Gu, v-u\rangle \ge 0\ \forall\, v \in K \text{ implies that } u \in K.\\ (\mathrm{b}_2)\ \ \widetilde{K} = X,\ u \in X,\ Gu = 0_{X^*} \text{ implies that } u \in K.\end{array}\right. \tag{44}$$

Moreover, we define the function $\psi : X \times X \to \mathbb{R}$ by the equality

$$\psi(u,v) = \langle Gu, v\rangle \quad \forall\, u,\, v \in X. \tag{45}$$

We start with the following preliminary result.

Lemma 1. *Let $\emptyset \neq K \subset \widetilde{K} \subset X$. Then, the following statements hold:*

(a) *Conditions (42) and (43) on the operator G imply properties (15) (a), (c) for the function ψ.*
(b) *Condition (44) (b_1) on the operator G implies property (15) (b_1) for the function ψ.*
(c) *Condition (44) (b_2) on the operator G implies property (15) (b_2) for the function ψ.*

Proof. (a) Using (45), we find that

$$\psi(u,v) - \psi(u,u) = \langle Gu, v-u\rangle \quad \forall\, u,\, v \in X. \tag{46}$$

Therefore, assumption (43) implies that

$$\psi(u,v) - \psi(u,u) \le 0 \quad \forall\, u \in \widetilde{K},\ v \in K, \tag{47}$$

which shows that condition (15) (a) is satisfied.

Assume now that $\{u_n\} \subset \widetilde{K}$, $u_n \rightharpoonup u$ in X and $\limsup \big[\psi(u_n,u_n) - \psi(u_n,u)\big] \le 0$. We use (46) to see that $\limsup \langle Gu_n, u_n - u\rangle \le 0$ and, by the pseudomonotonicity of the operator G, guaranteed by assumption (42), we deduce that

$$\liminf \langle Gu_n, u_n - v\rangle \ge \langle Gu, u-v\rangle \quad \forall\, v \in X. \tag{48}$$

We now use (48) and (46) again to find that

$$\liminf \big[\psi(u_n,u_n) - \psi(u_n,v)\big] \ge \psi(u,u) - \psi(u,v) \quad \forall\, v \in X.$$

We conclude from above that condition (15) (c) is satisfied.

(b) Assume that (44) (b_1) holds. Let $u \in \widetilde{K}$ be such that $\psi(u,v) \ge \psi(u,u)$ for all $v \in K$. Then, (46) shows that

$$\langle Gu, v-u\rangle \ge 0 \quad \forall\, v \in K,$$

and using assumption (44) (b_1), we deduce that $u \in K$. This shows that implication (15) (b_1) is satisfied.

(c) Assume that (44) (b_2) holds, and let $u \in X$ be such that $\psi(u,v) \ge \psi(u,u)$ for all $v \in X$. This implies that

$$\langle Gu, v-u\rangle \ge 0 \quad \forall\, v \in X,$$

and therefore, $\langle Gu, w\rangle = 0$ for all $w \in X$. We conclude from here that $Gu = 0_{X^*}$ and, using assumption (44) (b_2) it follows that $u \in K$, too. Thus, implication (15) (b_2) holds. □

We now move to the main results of this section, Corollaries 2 and 3 below, which represent consequences of Theorems 1 and 2.

Corollary 2. *Assume* (3)–(8), (13), (14), (16)–(20), (42), (43), *and* (44) (b$_1$). *Then, for each* $n \in \mathbb{N}$*, there exists a unique element* u_n *such that*

$$u_n \in \widetilde{K}, \qquad \langle Au_n, v-u_n\rangle + \frac{1}{\lambda_n}\langle Gu_n, v-u_n\rangle + \varphi(u_n,v) - \varphi(u_n,u_n) \tag{49}$$
$$+ j^0(u_n, v-u_n) \geq \langle f, v-u_n\rangle \qquad \forall\, v \in \widetilde{K}.$$

Moreover, $u_n \to u$ *in* X *as* $n \to \infty$*, where* $u \in K$ *is a unique solution to inequality* (1).

Proof. Let $n \in \mathbb{N}$. We use assumption (42) to see that G is a pseudomonotone operator. Therefore, using (4), (16), the monotonicity of G, and standard arguments, it follows that the operator $A + \frac{1}{\lambda_n} G : X \to X^*$ is strongly monotone and pseudomonotone, with the same constant m_A. The existence of a unique solution u_n to inequality (49) follows now from Theorem 1, with the operator $A + \frac{1}{\lambda_n} G$ instead of A.

Consider now the function ψ given by (45). Then, using (46), it follows that solving inequality (49) corresponds to solving Problem $\mathcal{P}_n$. On the other hand, Lemma 1 (a), (b) guarantee that the function ψ satisfies condition (15). We are now in a position to use Theorem 2 in order to deduce the convergence $u_n \to u$ in X as $n \to \infty$. □

Corollary 2 was obtained in [20] in the particular case when the nonconvex function j vanishes. This corresponds to the case of a purely variational inequality. With respect [20], the novelty of the Corollary 2 consists in the fact that we now deal with a genuine variational–hemivariational inequality, that is, j does not vanish.

Corollary 3. *Assume* (3)–(8) *and* (16)–(20)*, and moreover, assume that* G *is a penalty operator of* K*. Then, for each* $n \in \mathbb{N}$*, there exists a unique element* u_n *such that*

$$u \in X, \qquad \langle Au_n, v-u_n\rangle + \frac{1}{\lambda_n}\langle Gu_n, v-u_n\rangle \tag{50}$$
$$+\varphi(u_n,v) - \varphi(u_n,u_n) + j^0(u_n, v-u_n) \geq \langle f, v-u_n\rangle \qquad \forall\, v \in X.$$

Moreover, $u_n \to u$ *in* X *as* $n \to \infty$*, where* $u \in K$ *is a unique solution to inequality* (1).

Proof. We take $\widetilde{K} = X$, which, obviously, satisfies conditions (13) and (14). Moreover, since G is a penalty operator on K, we deduce that G satisfies conditions (42), (43), and (44) (b$_2$). The existence of a unique solution u_n to inequality (50) follows now from the same argument as that used in the proof of Corollary 2, based on Theorem 1 with the operator $A + \frac{1}{\lambda_n} G$.

The convergence $u_n \to u$ in X as $n \to \infty$ is a direct consequence of Lemma 1 (a), (c) combined with Theorem 2. □

Corollary 3 was obtained in [21] in the particular case when $\varphi(u,v) = \varphi(v)$. Its proof in the general case, when φ depends on both u and v, was given in [19]. We also remark that the result in Corollary 3 describes a classical penalty method since the constraints in the penalty inequality (50) have been completely removed. In contrast, the result in Corollary 2 seems to be new since the constraint $u \in K$ in inequality (50) has been replaced by the constraint $u_n \in \widetilde{K}$ in the penalty inequality (49). Therefore, it represents an extension of the results previously obtained in both [19] and [20].

We now provide an example of operator G for which Corollaries 2 and 3 hold.

Example 1. *Assume that* X *is a strictly convex reflexive Banach space,* (3) *and* (13) *hold, and let* G *be the operator given by* (12)*. We show that this operator satisfies conditions* (42), (43), *and* (44) (b$_1$), (b$_2$).

Indeed, as mentioned in Section 2, G is a penalty operator on K, and therefore, it satisfies condition (42). Let $u \in \widetilde{K}$ and $v \in K$. Then, $Gv = 0_{X^*}$, and therefore, using the monotonicity of G, we have

$$\langle Gu, v-u\rangle = \langle Gu - Gu, v-u\rangle \le 0,$$

which shows that condition (43) holds, too.

Assume that $u \in \widetilde{K}$ and $\langle Gu, v-u\rangle \ge 0$ for all $v \in K$. Then, (12) implies that

$$\langle \mathcal{J}(u - P_K u), v-u\rangle \ge 0 \quad \forall\, v \in K$$

, and taking $v = P_K u$ as this inequality, we deduce that

$$\langle \mathcal{J}(u - P_K u), u - P_K u\rangle \le 0. \tag{51}$$

Recall that $\mathcal{J}$ is a single-valued operator. Therefore, (10) yields

$$\langle \mathcal{J}(u - P_K u), u - P_K u\rangle = \|u - P_K u\|_X^2. \tag{52}$$

We now combine (51) and (52) to see that $u = P_K u$. Hence, (11) implies that $u \in K$. This shows that the operator G satisfies condition (44) (b_1).

Assume now that $\widetilde{K} = X$, $u \in X$, and $Gu = 0_{X^*}$. Then, (9) implies that $u \in K$. This shows that the operator G satisfies condition (44) (b_2), too. We conclude from above that Corollaries 2 and 3 can be used with the choice (12) for the operator G.

Note that results presented in this section have been obtained for functions ψ of the form (45) under appropriate assumptions on the operator G. The existence of a unique solution to the penalty problems and their convergence to the solution of the original problem was obtained using Theorems 1 and 2, respectively. Nevertheless, we underline that, in practice, this is not the only choice we can consider. Indeed, in the next section, we present an example of function ψ that is not of the form (45). In this case, the existence of a unique solution to the penalty problems and its convergence to the solution of the original problem are obtained using Corollary 1.

5. An Example

The results in Sections 3 and 4 can be applied in the study of various nonsmooth boundary value problems with unilateral constraints that, in a variational formulation, lead to a variational–hemivariational inequality of the form (1). Examples of such problems arise in contact mechanics, and we send the reader to [19] for more details. In this section, we restrict ourselves to providing an application of Corollaries 1 and 2 in the analysis of a purely hemivariational inequality with linear operators. The functional setting can be stated as follows.

Let $\Omega \subset \mathbb{R}^2$ be a regular domain with boundary Γ. We use the short-hand notation X for the Sobolev space $H_0^1(\Omega)$ endowed with the inner product

$$(u, v)_X = (\nabla u, \nabla v)_{L^2(\Omega)^2} \qquad \forall\, u,\, v \in X \tag{53}$$

and the associated norm $\|\cdot\|_X$. Recall that the Friedrichs–Poincaré inequality guarantees that $(X, (\cdot,\cdot)_X)$ is a Hilbert space. We denote in what follows by X^* the dual of X and by $\langle\cdot,\cdot\rangle$ the duality pairing between X^* and X. In addition, we recall that the inclusion $X \subset L^2(\Omega)$ is compact and there exists a constant $c_0 > 0$ that depends only on Ω such that

$$\|u\|_{L^2(\Omega)} \le c_0 \|u\|_X \qquad \forall\, u \in X. \tag{54}$$

Let p, q, and f_0 be given functions, and let g, $\widetilde{g}$, and μ be positive constants that satisfy the following conditions:

$$\left\{\begin{array}{l} p\colon \Omega\times\mathbb{R}\to\mathbb{R}_+ \text{ is such that} \\ \quad \text{(a) there exists } L_p>0 \text{ such that} \\ \qquad |p(\boldsymbol{x},r_1)-p(\boldsymbol{x},r_2)|\le L_p|r_1-r_2| \\ \qquad\quad \text{for all } r_1,\, r_2\in\mathbb{R},\ \text{a.e. } \boldsymbol{x}\in\Omega. \\ \quad \text{(b) } (p(\boldsymbol{x},r_1)-p(\boldsymbol{x},r_2))\,(r_1-r_2)\ge 0 \\ \qquad\quad \text{for all } r_1,\, r_2\in\mathbb{R},\ \text{a.e. } \boldsymbol{x}\in\Omega. \\ \quad \text{(c) } p(\cdot,r) \text{ is measurable on } \Omega \text{ for all } r\in\mathbb{R}. \\ \quad \text{(d) } p(\boldsymbol{x},r)=0 \ \text{ if and only if } \ r\le 0,\ \text{a.e. } \boldsymbol{x}\in\Omega. \end{array}\right. \tag{55}$$

$$\left\{\begin{array}{l} q\colon \Omega\times\mathbb{R}\to\mathbb{R} \text{ is such that} \\ \quad \text{(a) } q(\cdot,r) \text{ is measurable on } \Omega \text{ for all } r\in\mathbb{R} \text{ and there} \\ \qquad \text{exists } \bar{e}\in L^2(\Omega) \text{ such that } q(\cdot,\bar{e}(\cdot))\in L^1(\Omega). \\ \quad \text{(b) } q(\boldsymbol{x},\cdot) \text{ is locally Lipschitz on } \mathbb{R} \text{ for a.e. } \boldsymbol{x}\in\Omega. \\ \quad \text{(c) } |\partial q(\boldsymbol{x},r)|\le \bar{c}_0+\bar{c}_1\,|r| \text{ for a.e. } \boldsymbol{x}\in\Omega, \\ \qquad \text{for all } r\in\mathbb{R} \text{ with } \bar{c}_0,\,\bar{c}_1\ge 0. \\ \quad \text{(d) } q^0(\boldsymbol{x},r_1;r_2-r_1)+q^0(\boldsymbol{x},r_2;r_1-r_2)\le L_q\,|r_1-r_2|^2 \\ \qquad \text{for a.e. } \boldsymbol{x}\in\Omega,\ \text{all } r_1,\, r_2\in\mathbb{R} \text{ with } L_q\ge 0. \\ \quad \text{(e) either } q(\boldsymbol{x},\cdot) \text{ or } -q(\boldsymbol{x},\cdot) \text{ is regular on } \mathbb{R} \text{ for a.e. } x\in\Omega. \end{array}\right. \tag{56}$$

$$c_0^2 L_q<\mu. \tag{57}$$

$$f_0\in L^2(\Omega). \tag{58}$$

$$\widetilde{g}\ge g>0. \tag{59}$$

We define the sets K and $\widetilde{K}$ by the equalities

$$K=\{\,v\in X:\ v\le g\ \text{ a.e. in } \Omega\,\}, \tag{60}$$

$$\widetilde{K}=\{\,v\in X:\ v\le \widetilde{g}\ \text{ a.e. in } \Omega\,\}, \tag{61}$$

and we consider the following problem.

Problem 3. $\mathcal{Q}$. *Find a function* $u\in K$ *such that*

$$\mu\int_\Omega \nabla u\cdot(\nabla v-\nabla u)\,dx+\int_\Omega q^0(u;v-u)\,dx\ge\int_\Omega f_0(v-u)\,dx\quad \forall\, v\in K.$$

We also consider a sequence $\{\lambda_n\}$ that satisfies condition (16) and denote by r_+ the positive part of r, and for each $n\in\mathbb{N}$, we consider the following inequality problems.

Problem 4. $\mathcal{Q}_n$. *Find a function* $u_n\in\widetilde{K}$ *such that*

$$\mu\int_\Omega \nabla u_n\cdot(\nabla v-\nabla u_n)\,dx+\frac{1}{\lambda_n}\int_\Omega (v-g)_+\,dx-\frac{1}{\lambda_n}\int_\Omega (u_n-g)_+\,dx$$
$$+\int_\Omega q^0(u_n;v-u_n)\,dx\ge\int_\Omega f_0(v-u_n)\,dx\quad \forall\, v\in\widetilde{K}.$$

Problem 5. $\overline{\mathcal{Q}}_n$. *Find a function* $\overline{u}_n \in \widetilde{K}$ *such that*

$$\mu \int_\Omega \nabla \overline{u}_n \cdot (\nabla v - \nabla \overline{u}_n)\, dx + \frac{1}{\lambda_n} \int_\Omega p(\overline{u}_n - g) v\, dx$$
$$+ \int_\Omega q^0(\overline{u}_n; v - \overline{u}_n)\, dx \geq \int_\Omega f_0(v - \overline{u}_n)\, dx \quad \forall\, v \in \widetilde{K}.$$

We now state and prove the following result.

Theorem 3. *Assume* (16), (17)*, and* (55)–(59)*. Then,*

(a) *Problem* $\mathcal{Q}$ *has a unique solution* $u \in K$.
(b) *Problem* $\mathcal{Q}_n$ *has a unique solution* $u_n \in \widetilde{K}$ *and Problem* $\overline{\mathcal{Q}}_n$ *has a unique solution* $\overline{u}_n \in \widetilde{K}$, *for each* $n \in \mathbb{N}$.
(c) *The convergences* $u_n \to u$ *and* $\overline{u}_n \to u$ *hold, in* X*, as* $n \to \infty$.

Proof. The proof is divided into six steps, as follows.

Step (i). *Equivalence results.* We define the operators $A\colon X \to X^*$, $G\colon X \to X^*$, the functions $\psi : X \times X \to \mathbb{R}$, $j : X \to \mathbb{R}$, and the element $f \in X^*$ by the equalities

$$\langle Au, v\rangle = \mu \int_\Omega \nabla u \cdot \nabla v\, dx \quad \forall\, u,\, v \in X, \tag{62}$$

$$\langle Gu, v\rangle = \int_\Omega p(u - g)\, v\, dx \quad \forall\, u,\, v \in X, \tag{63}$$

$$\psi(u, v) = \int_\Omega (v - g)_+\, dx \qquad \forall\, u,\, v \in X, \tag{64}$$

$$j(v) = \int_\Omega q(v)\, dx \qquad \forall\, v \in X, \tag{65}$$

$$\langle f, v\rangle = \int_\Omega f_0 v\, dx \qquad \forall\, v \in X. \tag{66}$$

Next, we use assumption (56) and standard arguments (see [19], for instance) to see that $j : X \to \mathbb{R}$ satisfies condition (6) and, moreover,

$$j^0(u; v) = \int_\Omega q^0(u; v)\, dx \qquad \forall\, u,\, v \in X. \tag{67}$$

We now use the definitions above and equality (67) to see that Problem $\mathcal{Q}$ is equivalent to the problem of finding a function u such that

$$u \in K, \qquad \langle Au, v - u\rangle + j^0(u; v - u) \geq \langle f, v - u\rangle \quad \forall\, v \in K. \tag{68}$$

Moreover, Problem $\mathcal{Q}_n$ is equivalent to the problem of finding a function u_n such that

$$u_n \in \widetilde{K}, \quad \langle Au_n, v - u_n\rangle + \frac{1}{\lambda_n}\psi(u_n, v) - \frac{1}{\lambda_n}\psi(u_n, u_n) \tag{69}$$
$$+ j^0(u_n; v - u_n) \geq \langle f, v - u_n\rangle \quad \forall\, v \in \widetilde{K},$$

and finally, Problem $\overline{\mathcal{Q}}_n$ is equivalent to the problem of finding a function $\overline{u}_n$ such that

$$\overline{u}_n \in \widetilde{K}, \quad \langle A\overline{u}_n, v - \overline{u}_n\rangle + \frac{1}{\lambda_n}\langle G\overline{u}_n, v - \overline{u}_n\rangle \qquad (70)$$
$$+ j^0(\overline{u}_n; v - \overline{u}_n) \geq \langle f, v - \overline{u}_n\rangle \quad \forall\, v \in \widetilde{K}.$$

Step (ii). *Unique solvability of inequality* (68). We apply Theorem 1 on the space $X = H_0^1(\Omega)$ with $\varphi \equiv 0$. To this end, we use definition (60) to see that condition (3) is satisfied. In addition, using (53), it is easy to see that the linear operator (62) satisfies condition (4) with $m_A = \mu$. Moreover, as already mentioned, the function (65) satisfies condition (6). To compute the constant α_j, we use equality (67), assumption (56) (d), and inequality (54) to see that

$$j^0(u; v-u) + j^0(v; u-v) = \int_\Omega \left[q^0(u; v-u) + q^0(v; u-v)\right] dx$$
$$\leq L_q \int_\Omega |u-v|^2\, dx \leq c_0^2 L_q \|u-v\|_V^2 \qquad \forall\, u,\, v \in V.$$

It follows from here that the corresponding constant α_j in (6) (d) is $\alpha_j = c_0^2 L_q$. Then, since $\alpha_\varphi = 0$, using equality $m_A = \mu$ and (57), we find that the smallness condition (7) holds, too. Finally, recall that the element f in (66) satisfies condition (8). The existence of a unique solution to inequality (68) is now a direct consequence of Theorem 1.

Step (iii). *Properties of the function ψ.* Let $u \in \widetilde{K}$ and $v \in K$. Then, $\psi(u,v) = 0$, and since $\psi(u,u) \geq 0$, we deduce that condition (15) (a) is satisfied. Assume now that $u \in \widetilde{K}$ is such that $\psi(u,v) \geq \psi(u,u)$ for all $v \in K$. This implies that

$$\int_\Omega (u-g)_+\, dx \leq 0;$$

hence, $u \leq g$ a.e. in Ω, which, in turn, implies that $u \in K$. It follows from here that the implication (15) (b_1) holds and, therefore, condition (15) (b) is satisfied. Finally, using the compactness of the embedding $X \subset L^2(\Omega)$, it follows that condition (15) (c) holds, too. We conclude from above that the function ψ satisfies conditions (15). Moreover, it is obvious that the function ψ satisfies condition (41).

Step (iv). *Properties of the operator G.* We claim that the operator G defined by (63) satisfies conditions (42), (43), and (44) (b_1).

To prove this claim, we consider three elements u, v and w in X. We use definition (63), assumption (55) (a), and inequality (54) to see that

$$|\langle Gu - Gv, w\rangle| = \Big|\int_\Omega \left[p(u-g) - p(v-g)\right] w\, dx\Big|$$
$$\leq \int_\Omega |p(u-g) - p(v-g)|\, |w|\, dx \leq L_p \int_\Omega |u-v|\, |w|\, dx \leq c_0^2 L_p \|u-v\|_X \|w\|_X.$$

This inequality implies that $\|Gu - Gv\|_{X^*} \leq c_0^2 L_p \|u-v\|_X$ and shows that the operator G is Lipschitz continuous with constant $L_G = c_0^2 L_p$. On the other hand, using again (63) and (55) (b), we deduce that

$$\langle Gu - Gv, u - v\rangle = \int_\Omega \left[p(u-g) - p(v-g)\right]\left[(u-g) - (v-g)\right] dx \geq 0,$$

which shows that G is a monotone operator. We conclude from above that G satisfies condition (42).

Assume that $u \in \widetilde{K}$ and $v \in K$. We use assumption (55) (d) to see that $p(v-g) = 0$ a.e. in Ω and, therefore,

$$\langle Gu, v-u \rangle = \int_\Omega p(u-g)(v-u)\,dx = \int_\Omega \big[p(u-g) - p(v-g)\big]\big[(v-g)-(u-g)\big]\,dx.$$

Now, it follows from assumption (55) (b) that $\langle Gu, v-u \rangle \leq 0$, which shows that condition (43) holds.

Assume now that $u \in \widetilde{K}$ and $\langle Gu, v-u \rangle = 0$ for all $v \in K$. Then,

$$\int_\Omega p(u-g)(v-g)\,dx = \int_\Omega p(u-g)(u-g)\,dx \quad \forall\, v \in K. \tag{71}$$

Now, recall that (55)(b) and (d) guarantee that

$$p(u-g)(v-g) \leq 0 \quad \text{a.e. in } \Omega, \quad \forall\, v \in K, \tag{72}$$

$$p(u-g)(u-g) \geq 0 \quad \text{a.e. in } \Omega. \tag{73}$$

We now use equality (71) and inequalities (72) and (73) to find that

$$\int_\Omega p(u-g)(u-g)\,dx = 0. \tag{74}$$

Next, (73) and (74) imply that $p(u-g)(u-g) = 0$ a.e. in Ω, and using condition (55) (d), again, we find that $u \leq g$ a.e. in Ω. This shows that (44) (b_1) holds and concludes the proof of this step.

Step (v). *A property of the function j.* We claim that condition (20) is satisfied. Indeed, if $u_n \rightharpoonup u \in X$ and $v \in X$, using equality (67), a standard compactness argument, and Proposition (1) (ii), we have

$$\limsup j^0(u_n; v-u_n) = \limsup \int_\Omega q^0(u_n; v-u_n)\,dx$$
$$\leq \int_\Omega \limsup q^0(u_n; v-u_n)\,dx \leq \int_\Omega q^0(u; v-u)\,dx = j^0(u; v-u),$$

which proves that condition (20) is satisfied.

Step (vi). *End of proof.* Note that the existence of a unique solution u to inequality (68) was proved in Step (ii). Moreover, we use the properties of the function ψ in Step (iii), the properties of the function j in Steps (ii) and (v), and Corollary 1 to deduce the existence of a unique solution u_n to inequality (69) for each $n \in \mathbb{N}$, as well as the convergence $u_n \to u$ in X as $n \to \infty$. In addition, we use the properties of the operator G in Step (iv), the properties of the function j in Steps (ii), and (v) and Corollary 2 to deduce the existence of a unique solution $\overline{u}_n$ to inequality (70) for each $n \in \mathbb{N}$, as well as the convergence $\overline{u}_n \to u$ in X as $n \to \infty$. Finally, we use the equivalence results in Step (i) to conclude the proof.

□

We end this section with some comments and mechanical interpretations. First, Problem $\mathcal{Q}$ models the equilibrium of an elastic membrane that occupies the domain Ω, is fixed on its boundary, and is in contact along its surface with an obstacle, the so-called foundation. The unknown u is the vertical displacement of the membrane, μ is the Lamé coefficient, and f_0 represents the density of the applied body force. The obstacle is assumed to be made of a rigid body covered by a layer of elastic material with thickness g. The behavior of this layer is described with the normal compliance function q. The inequality in Problem $\mathcal{Q}$ is obtained by taking into account the equilibrium equation, the normal compliance contact condition for the deformable layer, and the Signorini contact condition for the rigid body. It represents a two-dimensional version of a variational contact model studied in [19], for instance.

Next, Problems $\mathcal{Q}_n$ and $\overline{\mathcal{Q}}_n$ are mathematical models that describe similar physical settings. Nevertheless, here, we assume that the obstacle is made by a rigid body covered with two layers: a first layer of thickness $\widetilde{g} - g$ and a second layer of thickness g, assumed to be elastic. The model represented by Problem $\mathcal{Q}_n$ is obtained by assuming that the first layer is rigid-elastic. Here, $\frac{1}{\lambda_n}$ represent its yield limit, and q is a normal compliance function. The model represented by Problem $\overline{\mathcal{Q}}_n$ is obtained by assuming that the first layer is elastic. Here, p and q are normal compliance functions, and $\frac{1}{\lambda_n}$ represents a stiffness coefficient.

Finally, we note that the convergence results in Theorem 3 are important from a mechanical point of view since they establish the link between three models of contact constructed using three different mechanical assumptions to describe the reaction of the obstacle.

6. Conclusions

In this paper, we considered an elliptic variational–hemivariational inequality with constraints, together with a sequence of penalty problems, constructed using a different set of constraints, a function ψ, and a sequence of parameters $\{\lambda_n\}$. Our main result is Theorem 2, which states that if the penalty problems are solvable, then any sequence of solutions of these problems converges to the unique solution of the original inequality. We exploited this theorem to deduce convergence results for various penalty problems associated to the considered variational–hemivariational inequality. Then, we used these abstract results in the study of a two-dimensional elastic contact problem. The novelty of our manuscript arises in the fact that, in contrast with the previous results in the literature, the penalty method we introduce here is governed by a function, and the constraints are only partially relaxed.

Our manuscript opens the way for more research in the future. First, it would be interesting to apply the results in this paper to the analysis of three-dimensional mathematical models of contact for elastic materials with or without a looking effect. For such models, the constraints can appear either in the constitutive law or in the boundary conditions. In this way, new penalty methods can be considered, and the link between various mathematical models of elastic contact can be established. Second, it would be interesting to extend the results presented here to the study of history-dependent variational–hemivariational inequalities. In this way, various convergence results for contact problems with viscoelastic materials with long memory, as well as contact problems with slip-dependent coefficient of friction, can be obtained. Finally, a last extension can be achieved in the study of abstract differential hemivariational inequalities and/or evolutionary hemivariational inequalities with or without history-dependent operators, together with the corresponding applications in contact mechanics.

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