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Optimal control of differential quasivariational inequalities with applications in contact mechanics

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A B S T R A C T

We consider a differential quasivariational inequality for which we state and prove the continuous dependence of the solution with respect to the data. This convergence result allows us to prove the existence of at least one optimal pair for an associated control problem. Finally, we illustrate our abstract results in the study of a free boundary problem which describes the equilibrium of a viscoelastic body in frictionless contact with a foundation made of a rigid body covered by a rigid-elastic layer.

1. Introduction

The present paper is motivated by the study of mathematical models which describe the time-dependent unilateral contact of a deformable body with a foundation. Under appropriate mechanical assumptions on the constitutive law and the interface conditions, such kind of models leads to a weak formulation which is in the form of a system that couples an ordinary differential equation with a variational or quasivariational inequality. Despite the fact that the solvability of such systems can be obtained by using various abstract existence and uniqueness results available in the literature, at the best of our knowledge there are very few results on the optimal control of the corresponding contact models. In this current paper we try to fill this gap and, to this end, we use arguments of variational and differential variational inequalities.

The theory of variational inequalities begun with the pioneering works [39,25,7]. Later, various extensions and applications were provided and the literature in the field is extensive. Comprehensive references on this subject are [32,24,16,5,22,11,35,2,13]. A survey of several classes of time-dependent and evolutionary variational inequalities, with our without unilateral constraints, can be found in [17]. There, results on

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existence and regularity for parabolic and hyperbolic evolutionary variational inequalities can be found. The theory plays an important role in Mechanics, Physics and Engineering Sciences where a large number of free boundary problems lead to elliptic or parabolic variational inequalities problems. Some relevant examples of such problems are the free boundary problems related to fluid flows through porous media [4], phase-change processes for the one-phase Stefan problem [14] and two-phase Stefan problem [40]. Variational inequalities arise in the study of mathematical models in Contact Mechanics too, as illustrated in the books [15,34,38,36,30,37]. Their optimal control has been studied in [23,33,21,41,10], for instance.

A differential variational inequality represents a system that couples a differential equation with a variational or quasivariational inequality. This terminology was used for the first time in [3]. Existence, uniqueness and convergence results have been obtained in [19,28,29,27], for instance. A stability result for the solution set of differential variational inequalities has been obtained in [18,20]. There, perturbations of the associated set-valued mapping and perturbations of the set of constraints have been considered and the Mosco convergence of sets, introduced in [31], has been employed. The results in [20] allow, in particular, the treatment of quasistatic contact problems with short memory viscoelastic materials and Tresca's friction law. A new class of differential quasivariational inequalities in Banach spaces has been considered in [26]. There, an existence and uniqueness result has been obtained by using a general fixed point principle. Moreover, some examples and applications have been presented, including the variational analysis of a contact problem with viscoplastic materials.

The current paper represents a continuation of [26]. Its aim is three-fold. The first one is to complete the abstract existence and uniqueness result in [26] with a general convergence result for the solution. Here we assume that all the problem data are perturbed, i.e., the second member and the initial condition of the differential equation, the monotone operator, the non-differentiable function, the convex set and the second member of the variational inequality, then we study the behaviour of the solution with respect to these perturbations. The second aim is to complete our previous work [26] with an existence result for an associated optimal control problem. Finally, our third aim is to apply these new results in the study of a viscoelastic frictionless contact problem with history-dependent hardening parameter.

The rest of the paper is structured as follows. In Section 2 we introduce the differential quasivariational inequality we are interested in, denoted by $\mathcal{P}$. Then, we recall some preliminary results which are needed later in this paper. In Section 3 we present our general convergence result, Theorem 3.1, which states the continuous dependence of the solution of Problem $\mathcal{P}$ on the data. The proof of the theorem is carried out in several steps, based on arguments on convexity, pseudomonotonicity and compactness. Then, in Section 4 we introduce an optimal control problem associated to the differential quasivariational inequality $\mathcal{P}$ and prove the existence of at least one optimal solution, Theorem 4.1. Its proof is based on arguments of compactness and lower semicontinuity. Finally, in Section 5, we present an application of our abstract results in the study of a mathematical model of contact with viscoelastic materials. We describe the model, list the assumption on the data, then we state and prove its unique weak solvability. Next, we prove the continuous dependence of the weak solution with respect to the data as well as the existence of the solution for an associated optimal control problem. We also provide the mechanical interpretation of our results.

2. Preliminaries

Throughout this paper I denotes either a bounded or an unbounded time-interval, i.e., $I = [0, T]$ with $T > 0$ or $I = \mathbb{R}_+ = [0, +\infty)$. We consider two real Banach spaces X, V and a real Hilbert space Z, endowed with the inner product $(\cdot, \cdot)_Z$. The norm on these spaces will be denoted by $\|\cdot\|_X$, $\|\cdot\|_V$ and $\|\cdot\|_Z$, respectively. The strong topological dual space of V is denoted by V^* and the duality paring of V and V^* is denoted by $\langle \cdot, \cdot \rangle$. We shall use the symbols "$\rightharpoonup$" and "$\to$" for the weak and strong convergence in various normed spaces to be specified. All the limits, upper and lower limits are considered as $n \to \infty$, even if we do not mention it explicitly. Moreover, we use the notation $C(I; V)$ and $C(I; Z)$ for the space of continuous functions on I with values in V and Z, respectively. In addition, we denote by a dot above the derivative with respect to the time and we adopt the notation $C^1(I;X)$ for the space of continuously differentiable function defined on I with values in X.

Consider the following data: $F : I \times X \times V \to X$, $x_0 \in X$, $A : X \times V \to V^*$, $j : X \times V \times V \to \mathbb{R}$, $\pi : V \to Z$, $f : I \to V$ and $K \subset V$. Then, the differential quasivariational inequality problem we consider in this paper is stated as follows.

Problem $\mathcal{P}$. *Find $x \in C^1(I;X)$ and $u \in C(I;V)$ such that*

$$\dot{x}(t) = F(t, x(t), u(t)) \qquad \forall\, t \in I, \tag{2.1}$$

$$x(0) = x_0, \tag{2.2}$$

$$\begin{aligned} u(t) \in K, \quad & \langle A(x(t), u(t)), v - u(t)\rangle + j(x(t), u(t), v) - j(x(t), u(t), u(t)) \\ & \geq (f(t), \pi v - \pi u(t))_Z \quad \forall\, v \in K,\ t \in I. \end{aligned} \tag{2.3}$$

The study of Problem $\mathcal{P}$ requires some preliminaries that we present in what follows.

Definition 2.1. An operator $B : V \to V^*$ is said to be:

(i) Lipschitz continuous, if there exists $L_B > 0$ such that

$$\|Bu_1 - Bu_2\|_{V^*} \leq L_B \|u_1 - u_2\|_V \qquad \forall\, u_1, u_2 \in V;$$

(ii) strongly monotone, if there exists $m_B > 0$ such that

$$\langle Bu_1 - Bu_2, u_1 - u_2\rangle \geq m_B \|u_1 - u_2\|_V^2 \qquad \forall\, u_1, u_2 \in V.$$

Consider now the following assumptions on the data of Problem $\mathcal{P}$.

$$\left\{\begin{array}{l} F : I \times X \times V \to X \text{ is such that:} \\ \text{(a) The mapping } t \to F(t, x, u) \text{ is continuous for all } x \in X, u \in V. \\ \text{(b) For any compact set } J \subset I \text{ there exists } L_J > 0 \text{ such that} \\ \quad \|F(t, x_1, u_1) - F(t, x_2, u_2)\|_X \leq L_J\, (\|x_1 - x_2\|_X + \|u_1 - u_2\|_V) \\ \quad \text{for all } x_1, x_2 \in X,\ u_1, u_2 \in V,\ t \in J. \end{array}\right. \tag{2.4}$$

$$x_0 \in X. \tag{2.5}$$

$$K \text{ is a nonempty closed convex subset of } V. \tag{2.6}$$

$$\left\{\begin{array}{l} A : X \times V \to V^* \text{ is such that:} \\ \text{(a) There exists } L' > 0 \text{ such that} \\ \quad \|A(x_1, u) - A(x_2, u)\|_{V^*} \leq L' \|x_1 - x_2\|_X \\ \quad \text{for all } x_1, x_2 \in X,\ u \in V. \\ \text{(b) There exists } L'' > 0 \text{ such that} \\ \quad \|A(x, u_1) - A(x, u_2)\|_{V^*} \leq L'' \|u_1 - u_2\|_V \\ \quad \text{for all } x \in X,\ u_1, u_2 \in V. \\ \text{(c) There exists } m > 0 \text{ such that} \\ \quad \langle A(x, u_1) - A(x, u_2), u_1 - u_2\rangle \geq m \|u_1 - u_2\|_V^2 \\ \quad \text{for all } x \in X,\ u_1, u_2 \in V. \end{array}\right. \tag{2.7}$$

$$\left\{\begin{array}{l} j : X \times V \times V \to \mathbb{R} \text{ is such that:} \\ \text{(a) For all } x \in X \text{ and } u \in V,\ j(x,u,\cdot) \text{ is convex} \\ \qquad \text{and lower semicontinuous (l.s.c.) on } V. \\ \text{(b) There exists } \alpha > 0 \text{ and } \beta > 0 \text{ such that} \\ \qquad j(x_1,u_1,v_2) - j(x_1,u_1,v_1) + j(x_2,u_2,v_1) - j(x_2,u_2,v_2) \\ \qquad \le \alpha \|x_1 - x_2\|_X \|v_1 - v_2\|_V + \beta \|u_1 - u_2\|_V \|v_1 - v_2\|_V, \\ \qquad \text{for all } x_1, x_2 \in X,\ u_1,u_2 \in V,\ v_1,v_2 \in V. \end{array}\right. \tag{2.8}$$

$$m > \beta. \tag{2.9}$$

$$f \in C(I;Z). \tag{2.10}$$

$$\left\{\begin{array}{l} \pi : V \to Z \text{ is a linear continuous operator, i.e.,} \\ \text{there exists } c_0 > 0 \text{ such that } \|\pi v\|_Z \le c_0 \|v\|_V \quad \forall\, v \in V. \end{array}\right. \tag{2.11}$$

Note that assumption (2.11) allows us to apply the Riesz representation theorem in order to define a function $\overline{f} : I \to V^*$ such that

$$\langle \overline{f}, v\rangle = (f(t), \pi v)_Z \qquad \forall\, v \in V,\ t \in I. \tag{2.12}$$

Furthermore, assumption (2.10) implies that $\overline{f} \in C(I;V^*)$. Hence, the following results are obtained as a direct consequence of Theorem 3.1 and Lemma 3.6 in [26], respectively.

Theorem 2.1. *Assume that X is a Banach space, V is a reflexive Banach space, Z is a Hilbert space and (2.4)–(2.11) hold. Then Problem $\mathcal{P}$ has a unique solution $(x,u) \in C^1(I;X) \times C(I;V)$.*

Lemma 2.1. *Assume that X is a Banach space, V is a reflexive Banach space and (2.6)–(2.11) hold. Then, for each $\widetilde{x}(t) \in C^1(I;X)$, there exists a unique function $u \in C(I;V)$ such that*

$$\begin{aligned} u(t) \in K, \qquad & \langle A(\widetilde{x}(t), u(t)), v - u(t)\rangle + j(\widetilde{x}(t), u(t), v) - j(\widetilde{x}(t), u(t), u(t)) \\ & \ge (f(t), \pi v - \pi u(t))_Z, \qquad \forall\, v \in K,\ t \in I. \end{aligned} \tag{2.13}$$

We now complete the previous results with the following comments.

Remark 2.1. Under the assumptions of Lemma 2.1 it is easy to see that the quasivariational inequality (2.13) is equivalent with the problem of finding a function $u : I \to V$ such that

$$u(t) \in K, \qquad G(t,u,v) \ge 0 \quad \forall\, v \in K,\ t \in I \tag{2.14}$$

where $G : I \times K \times K \to \mathbb{R}$ is the function defined by

$$G(t,u,v) = \langle A(\widetilde{x}(t), u), v - u\rangle + j(\widetilde{x}(t), u, v) - j(\widetilde{x}(t), u, u) - (f(t), \pi v - \pi u)_Z$$

for all $t \in I$, u, $v \in K$. Let $t \in I$ be fixed. Then, it is easy to see that $G(t,u,u) = 0$ and $G(t,u,\cdot) : K \to \mathbb{R}$ is a convex lower semicontinuous function, for any $u \in X$. Moreover,

$$G(t,u,v) + G(t,v,u) \le -(m-\beta)\|u - v\|_V^2 \le 0 \qquad \forall\, u,\ v \in K.$$

All these properties allow us to use Theorem 1 in [6] in order to prove the solvability of the equilibrium problem (2.14). For more details, existence results and applications of equilibrium problems, we refer to [8,9] as well as to the edited volume [12].

We end this section with the following version of the Weierstrass theorem.

Theorem 2.2. *Let W be a reflexive Banach space endowed with the norm $\|\cdot\|_W$, U a weakly closed subset of W and $J : U \to \mathbb{R}$ a weakly lower semicontinuous function. Then J is bounded from below and attains its infimum on U whenever one of the following two conditions hold:*

(i) *U is bounded;*
(ii) *J is coercive, i.e., $J(p) \to \infty$ as $\|p\|_W \to \infty$.*

We shall use Theorem 2.2 in Section 4 in order to establish the existence of at least one solution of optimal control problem. Its proof can be found in many books and surveys, including [36].

3. A convergence result

The solution (x, u) to problem $\mathcal{P}$ obtained in Theorem 2.1 depends on the data F, x_0, A, K, j and f. In this section we prove a convergence result that shows the continuous dependence of (x, u) with the above-mentioned data. This result will represent a crucial ingredient in the study of the optimal control problem that we shall study in Section 4. To describe it, for each $n \in \mathbb{N}$ we consider a function F_n, an initial data x_{0n}, a convex set K_n, an operator A_n and two functions j_n and f_n that satisfy the assumptions (2.4)–(2.10), respectively, with constants L_{Jn}, L'_n, L''_n, m_n, α_n and β_n. To avoid any confusion, when used with n, we shall refer to these assumptions as $(2.4)_n$–$(2.10)_n$. The sequences $\{L_{Jn}\}$, $\{L'_n\}$, $\{L''_n\}$, $\{m_n\}$, $\{\alpha_n\}$ are assumed to be bounded and, therefore, without the loss of generality we assume that

$$L_{Jn} \le L_J, \quad L'_n \le L', \quad L''_n < L'' \quad m_n \ge m, \quad \alpha_n \le \alpha, \quad \beta_n \le \beta \quad \forall\, n \in \mathbb{N} \tag{3.1}$$

where $L_J, L', L'', m, \alpha, \beta$ are the constants associated with the assumptions (2.4)–(2.10), respectively. Then, for each $n \in \mathbb{N}$ we consider the following problem.

Problem $\mathcal{P}_n$. *Find $x_n \in C^1(I; X)$ and $u_n \in C(I; V)$ such that*

$$\dot{x}_n(t) = F_n(t, x_n(t), u_n(t)) \qquad \forall\, t \in I, \tag{3.2}$$

$$x_n(0) = x_{0n}, \tag{3.3}$$

$$\begin{aligned} u_n(t) \in K_n, \qquad & \langle A_n(x_n(t), u_n(t)), v_n - u_n(t)\rangle \\ & + j_n(x_n(t), u_n(t), v_n) - j_n(x_n(t), u_n(t), u_n(t)) \\ & \ge (f_n(t), \pi v_n - \pi u_n(t))_Z \quad \forall\, v_n \in K_n,\ t \in I. \end{aligned} \tag{3.4}$$

Note that, if $(2.4)_n$-$(2.10)_n$ and (2.11) hold, Theorem 2.1 guarantees the existence of a unique solution of problem $\mathcal{P}_n$, denoted in what follows by (x_n, u_n). We now consider the following additional assumptions.

$$\begin{cases} \text{For all } n \in \mathbb{N} \text{ there exists } \Gamma_n \ge 0, \text{ and } \gamma_n \ge 0 \text{ such that:} \\ \text{(a) } \|F_n(t, x, u) - F(t, x, u)\|_X \le \Gamma_n\,(\|x\|_X + \|u\|_V + \gamma_n) \\ \qquad \forall\, t \in I, x \in X,\ u \in V. \\ \text{(b) } \lim\limits_{n\to\infty} \Gamma_n = 0. \\ \text{(c) The sequence } \{\gamma_n\} \subset \mathbb{R} \text{ is bounded.} \end{cases} \tag{3.5}$$

$$x_{0n} \to x_0 \quad \text{in} \quad X. \tag{3.6}$$

$$\begin{cases} \{K_n\} \text{ converges to } K \text{ in the sense of Mosco [31], i.e.:} \\ \text{(a) For each } v \in K \text{ there exists a sequence } \{v_n\} \text{ such that} \\ \qquad v_n \in K_n \ \ \forall\, n \in \mathbb{N} \ \text{ and } \ v_n \to v \ \text{ in } \ V. \\ \text{(b) For each } \{v_n\} \text{ such that} \\ \qquad v_n \in K_n \ \forall\, n \in \mathbb{N} \ \text{ and } \ v_n \rightharpoonup v \ \text{ in } \ V, \text{ we have } v \in K. \end{cases} \tag{3.7}$$

$$\begin{cases} \text{For all } n \in \mathbb{N} \text{ there exists } \Lambda_n \geq 0, \text{ and } \lambda_n \geq 0 \text{ such that:} \\ \text{(a) } \|A_n(x,u) - A(x,u)\|_{V^*} \leq \Lambda_n \left(\|x\|_X + \|u\|_V + \lambda_n\right) \quad \forall\, x \in X,\ u \in V. \\ \text{(b) } \lim\limits_{n\to\infty} \Lambda_n = 0. \\ \text{(c) The sequence } \{\lambda_n\} \subset \mathbb{R} \text{ is bounded.} \end{cases} \tag{3.8}$$

$$\begin{cases} \text{(a) For all } n \in \mathbb{N} \text{ there exists } \tau_n \geq 0 \text{ and } \delta_n \geq 0 \text{ such that:} \\ \qquad j_n(x,u,v_1) - j_n(x,u,v_2) \leq [\tau_n + \delta_n(\|x\|_X + \|u\|_V)]\, \|v_1 - v_2\|_V \\ \qquad\quad \forall\, x \in X,\ u \in V,\ v_1, v_2 \in V. \\ \text{(b) There exists } \tau_0 > 0 \text{ and } \delta_0 > 0 \text{ such that } \tau_n \leq \tau_0 \text{ and } \delta_n \leq \delta_0 < m. \\ \text{(c) For any sequences } \{u_n\} \subset V,\ \{v_n\} \subset V \text{ such that} \\ \qquad u_n \rightharpoonup u \text{ in } V,\ v_n \rightharpoonup v \text{ in } V \quad \text{ we have} \\ \qquad \limsup\limits_{n\to\infty} [j_n(x,u_n,v_n) - j_n(x,u_n,u_n)] \leq j(x,u,v) - j(x,u,u) \quad \forall\, x \in X. \end{cases} \tag{3.9}$$

$$\begin{cases} \text{(a) } f_n(t) \rightharpoonup f(t) \ \text{ in } \ Z \text{ as } n \to \infty \quad \forall\, t \in I; \\ \text{(b) For any compact set } J \subset I \text{ there exists } w_J > 0 \text{ such that} \\ \qquad \|f_n(t)\|_Z \leq w_J \quad \forall\, n \in \mathbb{N},\ t \in J. \end{cases} \tag{3.10}$$

$$\begin{cases} \text{For any sequence } \{v_n\} \subset V \text{ such that} \\ \qquad v_n \rightharpoonup v \ \text{ in } \ V \quad \text{we have} \quad \pi v_n \to \pi v \ \text{ in } \ Z. \end{cases} \tag{3.11}$$

Our main result of this section is the following.

Theorem 3.1. *Assume (2.4)–(2.11) and $(2.4)_n$–$(2.10)_n$, for each $n \in \mathbb{N}$. Moreover, assume (3.1) and (3.5)–(3.11). Then, the solution (x_n, u_n) of Problem $\mathcal{P}_n$ converges to the solution (x, u) of Problem $\mathcal{P}$ as $n \to \infty$, i.e., for each $t \in I$ we have*

$$u_n(t) \to u(t) \quad \text{in} \quad V \qquad \text{and} \qquad x_n(t) \to x(t) \quad \text{in} \quad X \qquad \text{as } n \to \infty. \tag{3.12}$$

The proof of Theorem 3.1 will be carried out in several steps. To present it, everywhere in what follows we assume that the hypotheses of Theorem 3.1 are satisfied, even if we do not mention it explicitly. Moreover, for each $n \in \mathbb{N}$ we consider the following auxiliary problem in which, recall, $x \in C^1(I;X)$ is the first component of the solution (x, u) of Problem $\mathcal{P}$.

Problem $\widetilde{\mathcal{P}}_n$. *Find $\widetilde{u}_n \in C(I;V)$ such that*

$$\begin{aligned} \widetilde{u}_n(t) \in K_n, \quad & \langle A_n(x(t), \widetilde{u}_n(t)), v_n - \widetilde{u}_n(t)\rangle + j_n(x(t), \widetilde{u}_n(t), v_n) - j_n(x(t), \widetilde{u}_n(t), \widetilde{u}_n(t)) \\ & \qquad \geq (f_n(t), \pi v_n - \pi \widetilde{u}_n(t))_Z \qquad \forall\, v_n \in K_n,\ t \in I. \end{aligned} \tag{3.13}$$

The first step of the proof is the following.

Lemma 3.1. *For each $n \in \mathbb{N}$, Problem $\widetilde{\mathcal{P}}_n$ has a unique solution $\widetilde{u}_n \in C(I;V)$. Moreover, for each compact subset $J \subset I$, there exists $\widetilde{C}_J > 0$ such that*

$$\|\widetilde{u}_n(t)\|_V \le \widetilde{C}_J, \qquad \forall\, t \in J,\ n \in \mathbb{N}. \tag{3.14}$$

Proof. The existence and uniqueness of solution to problem $\widetilde{\mathcal{P}}_n$ is derived straightforward from Lemma 2.1.

Assume now that $J \subset I$ is a given compact and let $t \in J$, $u_0 \in K$. Using (3.7) there exists a sequence $\{u_{0n}\}$ such that

$$u_{0n} \in K_n \quad \forall\, n \in \mathbb{N} \quad \text{ and } \quad u_{0n} \to u_0 \ \text{ in } \ V.$$

Let $n \in \mathbb{N}$ be fixed and take $v_n = u_{0n} \in K_n$ in (3.13) to obtain

$$\begin{aligned}\langle A_n(x(t),\widetilde{u}_n(t)),\widetilde{u}_n(t)-u_{0n}\rangle &\le j_n(x(t),\widetilde{u}_n(t),u_{0n})\\ &\quad - j_n(x(t),\widetilde{u}_n(t),\widetilde{u}_n(t)) + (f_n(t),\pi\widetilde{u}_n(t)-\pi u_{0n})_Z\end{aligned}$$

and, therefore,

$$\begin{aligned}\langle A_n(x(t),\widetilde{u}_n(t)) - A_n(x(t),u_{0n}),\widetilde{u}_n(t)-u_{0n}\rangle &\le \langle A_n(x(t),u_{0n}),u_{0n}-\widetilde{u}_n(t)\rangle\\ + j_n(x(t),\widetilde{u}_n(t),u_{0n}) - j_n(x(t),\widetilde{u}_n(t),\widetilde{u}_n(t)) &+ (f_n(t),\pi\widetilde{u}_n(t)-\pi u_{0n})_Z.\end{aligned}$$

Then, using $(2.7)_n$(c) and conditions (3.9)(a) and (2.11) we find that

$$\begin{aligned}m_n\|\widetilde{u}_n(t)-u_{0n}\|_V &\le \|A_n(x(t),u_{0n})\|_{V^*}\\ &\quad + \tau_n + \delta_n\left(\|x(t)\|_X + \|\widetilde{u}_n(t)\|_V\right) + c_0\|f_n(t)\|_Z.\end{aligned} \tag{3.15}$$

Now, since

$$\begin{aligned}\|A_n(x(t),u_{0n})\|_{V^*} &\le \|A_n(x(t),u_{0n}) - A(x(t),u_{0n})\|_{V^*}\\ &\quad + \|A(x(t),u_{0n}) - A(x(t),u_0)\|_{V^*} + \|A(x(t),u_0)\|_{V^*}\end{aligned}$$

from assumptions (3.8) and (2.7)(b) we obtain that

$$\begin{aligned}\|A_n(x(t),u_{0n})\|_{V^*} &\le \Lambda_n\left(\|x(t)\|_X + \|u_{0n}\|_V + \lambda_n\right)\\ &\quad + L''\|u_{0n}-u_0\|_V + \|A(x(t),u_0)\|_{V^*}.\end{aligned} \tag{3.16}$$

Recall now that conditions (3.1) and (3.9)(b) guarantee that $m_n \ge m$, $\tau_n \le \tau_0$ and $\delta_n \le \delta_0 < m$. As $\|\widetilde{u}_n(t)\|_V \le \|\widetilde{u}_n(t)-u_{0n}\|_V + \|u_{0n}\|_V$, combining inequalities (3.15) and (3.16) it follows that

$$\begin{aligned}\|\widetilde{u}_n(t)-u_{0n}\|_V &\le \frac{1}{m-\delta_0}\Big\{\Lambda_n\left(\|x(t)\|_X + \|u_{0n}\|_V + \lambda_n\right) + L''\|u_{0n}-u_0\|_V\\ &\quad + \|A(x(t),u_0)\|_{V^*} + \tau_0 + \delta_0\left(\|x(t)\|_X + \|u_{0n}\|_V\right) + c_0\|f_n(t)\|_Z\Big\}.\end{aligned} \tag{3.17}$$

Next, since $u_{0n} \to u$, there exists $M > 0$ which does not depend on n such that $\|u_{0n}-u_0\|_V \le M$. Consequently,

$$\|u_{0n}\|_V \le M + \|u_0\|_V. \tag{3.18}$$

On the other hand, from assumptions (3.8)(b), (c), we know that $\Lambda_n \to 0$ and $\{\lambda_n\} \subset \mathbb{R}$ in bounded. Therefore, there exists $\Lambda_0 > 0$ and $\lambda_0 > 0$ such that

$$\Lambda_n \le \Lambda_0 \qquad \text{and} \qquad \lambda_n \le \lambda_0. \tag{3.19}$$

In addition, since $x \in C^1(I;X)$, there exists $M_J > 0$ which does not depend on t such that

$$\|x(t)\|_X \le M_J. \tag{3.20}$$

Moreover, taking into account (2.7)(a) we get

$$\begin{aligned}\|A(x(t),u_0)\|_{V^*} &\le \|A(x(t),u_0) - A(x_0,u_0)\|_{V^*} + \|A(x_0,u_0)\|_{V^*}\\ &\le L'\|x(t)-x_0\|_X + \|A(x_0,u_0)\|_{V^*} \le L'(M_J + \|x_0\|_X) + \|A(x_0,u_0)\|_{V^*}\end{aligned} \tag{3.21}$$

Finally, from condition (3.10)(b), there exists a constant $w_J > 0$ which does not depend on n and t such that

$$\|f_n(t)\|_Z \le w_J. \tag{3.22}$$

Therefore, from (3.17)–(3.22) we deduce that

$$\begin{aligned}\|\widetilde{u}_n(t) - u_{0n}\|_V \le \frac{1}{m-\delta_0}\Big\{&\Lambda_0\,(M_J + M + \|u_0\|_V + \lambda_0) + L''M + L'(M_J + \|x_0\|_X)\\ &+ \|A(x_0,u_0)\|_{V^*} + \tau_0 + \delta_0\,(M_J + M + \|u_0\|_V) + c_0 w_J\Big\}.\end{aligned}$$

Defining now C_J as the right hand side of the previous inequality we get that

$$\|\widetilde{u}_n(t)\|_V \le C_J + \|u_{0n}\|_V.$$

As a result we deduce (3.14) with $\widetilde{C}_J = C_J + M + \|u_0\|_V$, which concludes the proof. □

The second step of the proof is the following.

Lemma 3.2. *For each $t \in I$ the following weak convergence holds:*

$$\widetilde{u}_n(t) \rightharpoonup u(t) \quad \text{in} \;\; V \quad \text{as } n \to \infty. \tag{3.23}$$

Proof. Let $t \in I$ and consider a compact set $J \subset I$ such that $t \in J$. Using Lemma 3.1 we obtain that there exists an element $\widetilde{u}(t) \in V$ and a subsequence of $\{\widetilde{u}_n(t)\}$, still denoted by $\{\widetilde{u}_n(t)\}$, such that $\widetilde{u}_n(t) \rightharpoonup \widetilde{u}(t)$ in V as $n \to \infty$. Recalling assumption (3.7), since $\widetilde{u}_n(t) \in K_n$ $\forall\, n \in \mathbb{N}$, we deduce that $\widetilde{u}(t) \in K$.

We now prove that $\widetilde{u}(t) = u(t)$ and, by the uniqueness of the solution to (2.3), it is enough to show that $\widetilde{u}(t)$ is a solution to inequality (2.3). To this end we consider an element $v \in K$ and use (3.7) to find that there exists a sequence $\{v_n\} \subset V$ such that

$$v_n \in K_n \quad \forall\, n \in \mathbb{N} \qquad \text{and} \qquad v_n \to v \quad \text{in } V.$$

We now use (3.13) to obtain

$$\begin{aligned}\langle A_n(x(t),\widetilde{u}_n(t)),\widetilde{u}_n(t)-v_n\rangle \le\ & j_n(x(t),\widetilde{u}_n(t),v_n)-j_n(x(t),\widetilde{u}_n(t),\widetilde{u}_n(t))\\ &+(f_n(t),\pi\widetilde{u}_n(t)-\pi v_n)_Z.\end{aligned}$$

Next, writing

$$A(x(t),\widetilde{u}_n(t))=A(x(t),\widetilde{u}_n(t))-A_n(x(t),\widetilde{u}_n(t))+A_n(x(t),\widetilde{u}_n(t))$$

we find that

$$\begin{aligned}\langle A(x(t),\widetilde{u}_n(t)),\widetilde{u}_n(t)-v_n\rangle \le\ & \langle A_n(x(t),\widetilde{u}_n(t))-A(x(t),\widetilde{u}_n(t)),v_n-\widetilde{u}_n(t)\rangle\\ &+j_n(x(t),\widetilde{u}_n(t),v_n)-j_n(x(t),\widetilde{u}_n(t),\widetilde{u}_n(t))\\ &+(f_n(t),\pi\widetilde{u}_n(t)-\pi v_n)_Z.\end{aligned}$$

Adding and subtracting v in the duality paring leads to

$$\begin{aligned}\langle A(x(t),\widetilde{u}_n(t)),\widetilde{u}_n(t)-v\rangle \le\ & \langle A(x(t),\widetilde{u}_n(t)),v_n-v\rangle\\ &+\langle A_n(x(t),\widetilde{u}_n(t))-A(x(t),\widetilde{u}_n(t)),v_n-\widetilde{u}_n(t)\rangle\\ &+j_n(x(t),\widetilde{u}_n(t),v_n)-j_n(x(t),\widetilde{u}_n(t),\widetilde{u}_n(t))\\ &+(f_n(t),\pi\widetilde{u}_n(t)-\pi v_n)_Z. \qquad (3.24)\end{aligned}$$

So,

$$\langle A(x(t),\widetilde{u}_n(t)),\widetilde{u}_n(t)-v\rangle \le \sum_{i=1}^{4} S_n^i(v_n), \qquad (3.25)$$

with

$$\begin{aligned}S_n^1(v_n)&=\langle A(x(t),\widetilde{u}_n(t)),v_n-v\rangle,\\ S_n^2(v_n)&=\langle A_n(x(t),\widetilde{u}_n(t))-A(x(t),\widetilde{u}_n(t)),v_n-\widetilde{u}_n(t)\rangle,\\ S_n^3(v_n)&=j_n(x(t),\widetilde{u}_n(t),v_n)-j_n(x(t),\widetilde{u}_n(t),\widetilde{u}_n(t)),\\ S_n^4(v_n)&=(f_n(t),\pi\widetilde{u}_n(t)-\pi v_n)_Z.\end{aligned} \qquad (3.26)$$

In order to pass to the upper limit in inequality (3.25) we now estimate each of the terms S_n^i above.

First, using (2.7)(b) we deduce that

$$\begin{aligned}S_n^1(v_n)&\le \|A(x(t),\widetilde{u}_n(t))\|_{V^*}\|v_n-v\|_V\\ &\le \big(\|A(x(t),\widetilde{u}_n(t))-A(x(t),\widetilde{u}(t))\|_{V^*}+\|A(x(t),\widetilde{u}(t))\|_{V^*}\big)\|v_n-v\|_V\\ &\le \big(L''\|\widetilde{u}_n(t)-\widetilde{u}(t)\|_V+\|A(x(t),\widetilde{u}(t))\|_{V^*}\big)\|v_n-v\|_V.\end{aligned}$$

Therefore, since $L''\|\widetilde{u}_n(t)-\widetilde{u}(t)\|_V+\|A(x(t),\widetilde{u}(t))\|_{V^*}$ is bounded and $\|v_n-v\|_V\to 0$ it follows that

$$\limsup_{n\to\infty} S_n^1(v_n)=\limsup_{n\to\infty}\ \langle A(x(t),\widetilde{u}_n(t)),v_n-v\rangle\le 0. \qquad (3.27)$$

Next, exploiting condition (3.8)(a) we find that

$$\begin{aligned}S_n^2(v_n)&\le \|A_n(x(t),\widetilde{u}_n(t))-A(x(t),\widetilde{u}_n(t))\|_{V^*}\|v_n-\widetilde{u}_n(t)\|_V\\ &\le \Lambda_n\left(\|x(t)\|_X+\|\widetilde{u}_n(t)\|_V+\lambda_n\right)\|v_n-\widetilde{u}_n(t)\|_V.\end{aligned}$$

Taking now into account the boundedness of the sequences $\|v_n\|_V$, $\|\widetilde{u}_n(t)\|_V$ and $\{\lambda_n\}$, using assumption (3.8)(b) we obtain that

$$\limsup_{n\to\infty} S_n^2(v_n) = \limsup_{n\to\infty} \langle A_n(x(t),\widetilde{u}_n(t)) - A(x(t),\widetilde{u}_n(t)), v_n - \widetilde{u}_n(t)\rangle \le 0. \tag{3.28}$$

We proceed with the term $S_n^3(v_n)$. From hypothesis (3.9)(c), since $v_n \to v$ and $\widetilde{u}_n(t) \rightharpoonup \widetilde{u}(t)$ in V we have

$$\begin{aligned}\limsup_{n\to\infty} S_n^3(v_n) &= \limsup_{n\to\infty} \left[j_n(x(t),\widetilde{u}_n(t),v_n) - j_n(x(t),\widetilde{u}_n(t),\widetilde{u}_n(t))\right]\\ &\le j(x(t),\widetilde{u}(t),v) - j(x(t),\widetilde{u}(t),\widetilde{u}(t)).\end{aligned} \tag{3.29}$$

Finally,

$$\begin{aligned}S_n^4(v_n) &= (f_n(t),\pi\widetilde{u}_n(t) - \pi\widetilde{u}(t))_Z + (f_n(t),\pi\widetilde{u}(t) - \pi v)_Z + (f_n(t),\pi v - \pi v_n)_Z\\ &\le \|f_n(t)\|_Z\|\pi\widetilde{u}_n(t) - \pi\widetilde{u}(t)\|_Z + (f_n(t),\pi\widetilde{u}(t) - \pi v) + \|f_n(t)\|_Z\|\pi v - \pi v_n\|_Z.\end{aligned}$$

Thus, by assumptions (3.10)(a) and (3.11), the weak convergences of $\widetilde{u}_n(t)$ to $\widetilde{u}(t)$ and the strong convergence of v_n to v, both in V, we deduce that

$$\limsup_{n\to\infty} S_n^4(v_n) = \limsup_{n\to\infty} (f_n(t),\pi\widetilde{u}_n(t) - \pi v_n)_Z \le (f(t),\pi\widetilde{u}(t) - \pi v)_Z. \tag{3.30}$$

We now pass to the upper limit in inequality (3.25) and use (3.27)–(3.30) to find that

$$\begin{aligned}\limsup_{n\to\infty} \langle A(x(t),\widetilde{u}_n(t)),\widetilde{u}_n(t) - v\rangle &\le j(x(t),\widetilde{u}(t),v) - j(x(t),\widetilde{u}(t),\widetilde{u}(t))\\ &\quad + (f(t),\pi\widetilde{u}(t) - \pi v)_Z \qquad \forall\, v \in K.\end{aligned} \tag{3.31}$$

On the other hand, using the monotonicity of the operator $A(x(t),\cdot)$ we have

$$\langle A(x(t),v),\widetilde{u}_n(t) - v\rangle \le A(x(t),\widetilde{u}_n(t)),\widetilde{u}_n(t) - v\rangle \qquad \forall\, v \in V$$

and, using the convergence $\widetilde{u}_n(t) \rightharpoonup \widetilde{u}(t)$ in V, we find that

$$\langle A(x(t),v),\widetilde{u}(t) - v\rangle \le \limsup_{n\to\infty} \langle A(x(t),\widetilde{u}_n(t)),\widetilde{u}_n(t) - v\rangle \qquad \forall\, v \in V. \tag{3.32}$$

We now combine the inequalities (3.31) and (3.32) to deduce that

$$\begin{aligned}\langle A(x(t),v),\widetilde{u}(t) - v\rangle &\le j(x(t),\widetilde{u}(t),v) - j(x(t),\widetilde{u}(t),\widetilde{u}(t))\\ &\quad + (f(t),\pi\widetilde{u}(t) - \pi v)_Z \qquad \forall\, v \in K.\end{aligned} \tag{3.33}$$

Consider now an arbitrary element $w \in K$ and let $\theta \in (0,1]$. We take $v = \widetilde{u}(t) + \theta(w - \widetilde{u}(t))$ in (3.33), use the convexity of the function j with respect to the third argument and divide the resulting inequality with $\theta > 0$ to find that

$$\begin{aligned}\langle A(x(t),\widetilde{u}(t) + \theta(w - \widetilde{u}(t))),\widetilde{u}(t) - w\rangle &\le j(x(t),\widetilde{u}(t),w) - j(x(t),\widetilde{u}(t),\widetilde{u}(t))\\ &\quad + (f(t),\pi\widetilde{u}(t) - \pi w)_Z.\end{aligned}$$

We now pass to the limit as $\theta \to 0$ and use assumption (2.7)(b) to conclude that $\widetilde{u}(t) \in K$ satisfies the inequality

$$\langle A(x(t), \widetilde{u}(t)), \widetilde{u}(t) - w\rangle \le j(x(t), \widetilde{u}(t), w) - j(x(t), \widetilde{u}(t), \widetilde{u}(t)) + (f(t), \pi\widetilde{u}(t) - \pi w)_Z, \qquad \forall\, w \in K,\ t \in I. \tag{3.34}$$

On the other hand, Lemma 2.1 guarantees that (3.34) has a unique solution. Therefore, (2.3) and (3.34), yield $\widetilde{u}(t) = u(t)$. This assertion reveals that each subsequence of $\{\widetilde{u}_n(t)\}$ which converges weakly in V has the same limit $u(t)$. Therefore, by a standard argument we get that the whole sequence $\{\widetilde{u}_n(t)\}$ converges weakly to $u(t)$ in V, which concludes the proof. □

We now proceed with the following result.

Lemma 3.3. *For each $t \in I$ the following strong convergence holds:*

$$\widetilde{u}_n(t) \to u(t) \quad \text{in} \quad V \quad \text{as} \quad n \to \infty. \tag{3.35}$$

Proof. Let $t \in I$ and let $J \subset I$ be a compact set such that $t \in J$. As $u(t) \in K$, assumption (3.7) and arguments similar to those used in the proof of inequality (3.25) lead to

$$\langle A(x(t), \widetilde{u}_n(t)), \widetilde{u}_n(t) - u(t)\rangle \le \sum_{i=1}^{4} S_n^i(v_n). \tag{3.36}$$

Here, for each $n \in \mathbb{N}$ and $i \in \{1, 2, 3, 4\}$, S_n^i is given by (3.26) and $\{v_n\} \subset V$ is a sequence such that

$$v_n \in K_n \quad \forall\, n \in \mathbb{N} \quad \text{and} \quad v_n \to u(t) \ \text{ in } \ V. \tag{3.37}$$

Inequality (3.36) implies that

$$\langle A(x(t), \widetilde{u}_n(t)) - A(x(t), u(t)), \widetilde{u}_n(t) - u(t)\rangle \le \langle A(x(t), u(t)), u(t) - \widetilde{u}_n(t)\rangle + \sum_{i=1}^{4} S_n^i(v_n)$$

and, using the strong monotonicity of A, (2.7)(c), yields

$$m\|\widetilde{u}_n(t) - u(t)\|_V^2 \le \langle A(x(t), u(t)), u(t) - \widetilde{u}_n(t)\rangle + \sum_{i=1}^{4} S_n^i(v_n). \tag{3.38}$$

On the other hand, the convergence (3.23) in Lemma 3.2 implies that

$$\langle A(x(t), u(t)), u(t) - \widetilde{u}_n(t)\rangle \to 0, \quad \text{as} \ \ n \to \infty. \tag{3.39}$$

Moreover, using (3.27)–(3.30), taking into account that $\widetilde{u}_n(t) \rightharpoonup \widetilde{u}(t) = u(t)$, replacing $v = u(t)$ and considering the sequence $\{v_n\}$ such that (3.37) holds, we see that

$$\limsup_{n\to\infty} \sum_{i=1}^{4} S_n^i(v_n) \le 0. \tag{3.40}$$

Therefore, passing to the upper limit in (3.38) and using (3.39), (3.40) we deduce that

$$\limsup_{n\to\infty} \|\widetilde{u}_n(t) - u(t)\|_V^2 \le 0,$$

which implies (3.35). □

We are now in a position to provide the proof of Theorem 3.1.

Proof. Let $t \in I$ and $n \in \mathbb{N}$. Moreover, consider a compact interval $J \subset I$ such that $[0, t] \subset J$ and denote by L_J the constant which arises in condition (2.4)(b). We test with $v_n = \widetilde{u}_n(t) \in K_n$ in (3.4) to see that

$$\begin{aligned}\langle A_n(x_n(t), u_n(t)), u_n(t) - \widetilde{u}_n(t)\rangle &\le j_n(x_n(t), u_n(t), \widetilde{u}_n(t)) - j_n(x_n(t), u_n(t), u_n(t)) \\ &\quad + (f_n(t), \pi u_n(t) - \pi\widetilde{u}_n(t))_Z. \end{aligned} \tag{3.41}$$

Then, taking $v_n = u_n(t) \in K_n$ in (3.13) we find that

$$\begin{aligned}\langle A_n(x(t), \widetilde{u}_n(t)), \widetilde{u}_n(t) - u_n(t)\rangle &\le j_n(x(t), \widetilde{u}_n(t), u_n(t)) - j_n(x(t), \widetilde{u}_n(t), \widetilde{u}_n(t)) \\ &\quad + (f_n(t), \pi\widetilde{u}_n(t) - \pi u_n(t))_Z. \end{aligned} \tag{3.42}$$

We now add inequalities (3.41) and (3.42) to deduce that

$$\begin{aligned}&\langle A_n(x_n(t), u_n(t)) - A_n(x(t), \widetilde{u}_n(t)), u_n(t) - \widetilde{u}_n(t)\rangle \le j_n(x_n(t), u_n(t), \widetilde{u}_n(t)) \\ &\quad - j_n(x_n(t), u_n(t), u_n(t)) + j_n(x(t), \widetilde{u}_n(t), u_n(t)) - j_n(x(t), \widetilde{u}_n(t), \widetilde{u}_n(t)).\end{aligned}$$

Next, writing

$$\begin{aligned}A_n(x_n(t), u_n(t)) - A_n(x(t), \widetilde{u}_n(t)) &= A_n(x_n(t), u_n(t)) - A_n(x_n(t), \widetilde{u}_n(t)) \\ &\quad + A_n(x_n(t), \widetilde{u}_n(t)) - A_n(x(t), \widetilde{u}_n(t)),\end{aligned}$$

we get

$$\begin{aligned}&\langle A_n(x_n(t), u_n(t)) - A_n(x_n(t), \widetilde{u}_n(t)), u_n(t) - \widetilde{u}_n(t)\rangle \\ &\quad \le \langle A_n(x(t), \widetilde{u}_n(t)) - A_n(x_n(t), \widetilde{u}_n(t)), u_n(t) - \widetilde{u}_n(t)\rangle \\ &\quad + j_n(x_n(t), u_n(t), \widetilde{u}_n(t)) - j_n(x_n(t), u_n(t), u_n(t)) \\ &\quad + j_n(x(t), \widetilde{u}_n(t), u_n(t)) - j_n(x(t), \widetilde{u}_n(t), \widetilde{u}_n(t)).\end{aligned}$$

Therefore, using assumptions $(2.7)_n$(c) and $(2.8)_n$(a) we obtain that

$$\begin{aligned}m_n\|u_n(t) - \widetilde{u}_n(t)\|_V^2 &\le \|A_n(x(t), \widetilde{u}_n(t)) - A_n(x_n(t), \widetilde{u}_n(t))\|_{V^*}\|u_n(t) - \widetilde{u}_n(t)\|_V \\ &\quad + \alpha_n\|x_n(t) - x(t)\|_X\|\widetilde{u}_n(t) - u_n(t)\|_V + \beta_n\|u_n(t) - \widetilde{u}_n(t)\|_V^2. \end{aligned} \tag{3.43}$$

Next, assumptions $(2.7)_n$(a), (3.1) and inequality (3.43) imply that

$$\|u_n(t) - \widetilde{u}_n(t)\|_V \le \tfrac{L'+\alpha}{m-\beta}\|x(t) - x_n(t)\|_X. \tag{3.44}$$

Therefore, from (3.44) we deduce that

$$\begin{aligned}\|u_n(t) - u(t)\|_V &\le \|u_n(t) - \widetilde{u}_n(t)\|_V + \|\widetilde{u}_n(t) - u(t)\|_V \\ &\le \tfrac{L'+\alpha}{m-\beta}\|x(t) - x_n(t)\|_X + \|\widetilde{u}_n(t) - u(t)\|_V.\end{aligned} \tag{3.45}$$

On the other hand, since $x(t)$ and $x_n(t)$ satisfy (2.1)–(2.2) and (3.2)–(3.3), respectively, we find that

$$x(t) = x_0 + \int_0^t F(s, x(s), u(s))\, ds,$$

$$x_n(t) = x_{0n} + \int_0^t F_n(s, x_n(s), u_n(s))\, ds$$

and, therefore,

$$\begin{aligned}\|x(t) - x_n(t)\|_X &\le \|x_0 - x_{0n}\|_X \\ &+ \int_0^t \|F(s, x(s), u(s)) - F_n(s, x_n(s), u_n(s))\|_X\, ds.\end{aligned} \tag{3.46}$$

Now, using $(2.4)_n$(b) and (3.5) we obtain that

$$\begin{aligned}\|F(s, x(s), u(s)) - F_n(s, x_n(s), u_n(s))\|_X &\le \|F(s, x(s), u(s)) - F_n(s, x(s), u(s))\|_X \\ &+ \|F_n(s, x(s), u(s)) - F_n(s, x_n(s), u_n(s))\|_X \\ &\le \Gamma_n\left(\|x(s)\|_X + \|u(s)\|_V + \gamma_n\right) \\ &+ L_J\left(\|x(s) - x_n(s)\|_X + \|u(s) - u_n(s)\|_V\right).\end{aligned} \tag{3.47}$$

We combine (3.45) and (3.47) to find that

$$\begin{aligned}\|F(s, x(s), u(s)) - F_n(s, x_n(s), u_n(s))\|_X &\le \Gamma_n\left(\|x(s)\|_X + \|u(s)\|_V + \gamma_n\right) \\ &+ L_J\left(1 + \tfrac{L'+\alpha}{m-\beta}\right)\|x_n(s) - x(s)\|_X + L_J\|\widetilde{u}_n(s) - u(s)\|_V.\end{aligned} \tag{3.48}$$

Then, exploiting (3.46) and taking into account (3.48) we deduce that

$$\|x(t) - x_n(t)\|_X \le g_n(t) + c\int_0^t \|x(s) - x_n(s)\|_X\, ds, \tag{3.49}$$

with $c = L_J\left(1 + \frac{L'+\alpha}{m-\beta}\right)$ and

$$\begin{aligned}g_n(t) &= \|x_0 - x_{0n}\|_X + \int_0^t \Gamma_n\left(\|x(s)\|_X + \|u(s)\|_V + \gamma_n\right)\, ds \\ &+ \int_0^t L_J\|\widetilde{u}_n(s) - u(s)\|_V\, ds.\end{aligned} \tag{3.50}$$

We now use the Gronwall argument to see that

$$\|x(t) - x_n(t)\|_X \le g_n(t)\, e^{ct}. \tag{3.51}$$

Moreover, note that assumptions (3.6), (3.5), the bound (3.14) and the convergence (3.35) allow us to use the Lebesgue dominated convergence theorem to obtain that

$$g_n(t) \to 0 \qquad \text{as} \qquad n \to \infty. \tag{3.52}$$

We now use (3.51) and (3.52) to see that $x_n(t) \to x(t)$ in X. Then, (3.45) implies $u_n(t) \to u(t)$ in V, which concludes the proof. □

We end this section with the following remarks.

Remark 3.1. Assume that

$$\theta \in C(I; \mathbb{R}), \quad \widetilde{f}_n \in Z \quad \text{and} \quad f_n(t) = \theta(t)\widetilde{f}_n \qquad \forall\, n \in \mathbb{N},\ t \in I. \tag{3.53}$$

In addition, assume that

$$\widetilde{f} \in Z, \quad f(t) = \theta(t)\widetilde{f} \qquad \forall\, t \in I \quad \text{and} \quad \widetilde{f}_n \rightharpoonup \widetilde{f} \quad \text{in } Z. \tag{3.54}$$

Then it is easy to check that (2.10), $(2.10)_n$ and (3.10) hold and, therefore, the statement of Theorem 3.1 still remains valid if we replace these assumptions by hypotheses (3.53) and (3.54).

Remark 3.2. Note that Theorem 3.1 provides a pointwise convergence result for the solution (x_n, u_n) of Problem $\mathcal{P}_n$ to the solution (x, u) of Problem $\mathcal{P}$ as $n \to \infty$, see (3.12). Extending this result to a convergence result in the space $C^1(I; X) \times C(I; V)$ remains on open problem which deserves to be investigated in the future.

4. An optimal control problem

Throughout this section we assume that $(W, \|\cdot\|_W)$ is a reflexive Banach space and U is a nonempty subset of W. For each $q \in U$ we consider a function F_q, an initial data x_{0q}, a convex set K_q, an operator A_q and two functions j_q and f_q that satisfy the assumptions (2.4)–(2.10), respectively with constants L_{Jq}, L'_q, L''_q, m_q, α_q and β_q. To avoid any confusion, when used with q we will refer to these assumptions as $(2.4)_q$–$(2.10)_q$. We now consider the following problem.

Problem $\mathcal{P}_q$. *Find $x_q \in C^1(I; X)$ and $u_q \in C(I; V)$ such that*

$$\dot{x}_q(t) = F_q(t, x_q(t), u_q(t)) \qquad \forall\, t \in I, \tag{4.1}$$

$$x_q(0) = x_{0q}, \tag{4.2}$$

$$\begin{aligned} u_q(t) \in K_q, \qquad & \langle A_q(x_q(t), u_q(t)), v_q - u_q(t)\rangle \\ & + j_q(x_q(t), u_q(t), v_q) - j_q(x_q(t), u_q(t), u_q(t)) \\ & \ge (f_q(t), \pi v_q - \pi u_q(t))_Z \quad \forall\, v_q \in K_q,\ t \in I. \end{aligned} \tag{4.3}$$

Under assumptions (2.4)–(2.10), (2.11), Theorem 2.1 guarantees that for each $q \in U$ there exists a unique solution $(x_q, u_q) \in C^1(I; X) \times C(I; V)$ to Problem $\mathcal{P}_q$.

Consider now a cost function $\mathcal{L} : X \times V \times U \to \mathbb{R}$. Then, the optimal control problem we study in this section is the following.

Problem $\mathcal{Q}$. *Given $t \in I$, find $q^* \in U$ such that*

$$\mathcal{L}(x_{q^*}(t), u_{q^*}(t), q^*) = \min_{q \in U} \ \mathcal{L}(x_q(t), u_q(t), q). \tag{4.4}$$

In the study of this problem we consider the following assumptions.

$$U \text{ is a nonempty weakly closed subset of } W. \tag{4.5}$$

$$\begin{cases} \text{For all sequences } \{x_n\} \subset X, \ \{u_n\} \subset V, \ \{q_n\} \subset U \text{ such that} \\ x_n \to x \text{ in } X, \ u_n \to u \text{ in } V, \ q_n \rightharpoonup q \text{ in } W, \quad \text{we have} \\ \liminf\limits_{n\to\infty} \mathcal{L}(x_n, u_n, q_n) \geq \mathcal{L}(x, u, q). \end{cases} \tag{4.6}$$

$$\begin{cases} \text{There exists } z : U \to \mathbb{R} \text{ such that} \\ \text{(a) } \mathcal{L}(x, u, q) \geq z(q) \quad \forall\, x \in X, \ u \in V, \ q \in U. \\ \text{(b) } \|q_n\|_W \to \infty \text{ implies that } z(q_n) \to \infty. \end{cases} \tag{4.7}$$

$$U \text{ is a bounded subset of } W. \tag{4.8}$$

Our main result of this section is the following.

Theorem 4.1. *Assume* $(2.4)_q$–$(2.10)_q$*, for each* $q \in U$*. In addition, assume* (2.11)*,* (3.11)*,* (4.5)*,* (4.6) *and either* (4.7) *or* (4.8)*. For each sequence* $\{q_n\} \subset U$ *such that* $q_n \rightharpoonup q$ *in* W *define*

$$F = F_q, \quad x_0 = x_{0q}, \quad K = K_q, \quad A = A_q, \quad j = j_q, \quad f = f_q$$

and

$$F_n = F_{q_n}, \quad x_{0n} = x_{0q_n}, \quad K_n = K_{q_n}, \quad A_n = A_{q_n}, \quad j_n = j_{q_n}, \quad f_n = f_{q_n}$$

and assume that (3.1)*,* (3.5)–(3.10) *hold. Then, for each* $t \in I$*, the optimal control problem* $\mathcal{Q}$ *has at least one solution* q^**.*

Proof. Let $t \in I$ be fixed and consider the function $J_t : U \to \mathbb{R}$ defined by

$$J_t(q) = \mathcal{L}(x_q(t), u_q(t), q) \qquad \forall\, q \in U. \tag{4.9}$$

Then, we consider the problem of finding q^* such that

$$J_t(q^*) = \min_{q \in U} J_t(q). \tag{4.10}$$

We apply Theorem 3.1 to see that $x_{q_n}(t) \to x_q(t)$ in X and $u_{q_n}(t) \to u_q(t)$ in V. Then, taking into account the convergence $q_n \rightharpoonup q$ in U, the definition (4.9) of J_t and condition (4.6) on $\mathcal{L}$ we find that

$$\liminf_{n\to\infty} J_t(q_n) = \liminf_{n\to\infty} \mathcal{L}(x_{q_n}(t), u_{q_n}(t), q_n) \geq \mathcal{L}(x_q(t), u_q(t), q) = J_t(q). \tag{4.11}$$

This means that J_t is a weakly lower semicontinuous function.

Assume now that condition (4.7) is satisfied. Then

$$J_t(q_n) = \mathcal{L}(x_{q_n}, u_{q_n}, q_n) \geq z(q_n)$$

and $\|q_n\|_W \to \infty$ implies $z(q_n) \to \infty$. It follows from here that $J_t(q_n) \to \infty$, i.e., J_t is coercive. Recalling that W is a reflexive Banach space and U is a weakly closed subset of W, the existence of at least one solution to problem (4.10) is a direct consequence of Theorem 2.2. This means that there exists a minimizer $q^* \in U$ for J_t which, in turn, guarantees that Problem $\mathcal{Q}$ has at least one solution. The same conclusions follow if we assume that condition (4.8) is satisfied since, in this case, the Weierstrass-type argument provided by Theorem 2.2 still holds. □

We end this section with the following remark.

Remark 4.1. Assume that

$$\theta \in C(I;\mathbb{R}), \quad \widetilde{f}_q \in Z \quad \text{and} \quad f_q(t) = \theta(t)\widetilde{f}_q \qquad \forall\, Q \in U,\ t \in I. \tag{4.12}$$

In addition, assume that

$$\widetilde{f}_{q_n} \rightharpoonup \widetilde{f}_q \quad \text{in } Z \ \text{ for any sequence } \{q_n\} \subset U \text{ such that } q_n \rightharpoonup q \text{ in } W. \tag{4.13}$$

Then it is easy to check that $(2.10)_q$ and (3.10) hold and, therefore, the statement of Theorem 4.1 still remains valid if we replace these assumptions by hypotheses (4.12), (4.13).

5. A frictionless contact problem

As mentioned in the Introduction, the results in Section 3–4 can be used in the analysis and control of mathematical models which describe the contact of a deformable body with a foundation. A large number of examples can be considered, in which the contact is frictional or frictionless and the material behaviour is described by an elastic, viscoelastic or viscoplastic constitutive law. In this section we provide such an example in which we assume that the contact is frictionless, the material is viscoelastic and the hardening of the foundation is taken into account. For more details on the modelling and analysis of contact problems we refer the reader to the books [36], [37].

Everywhere below $d \in \{2,3\}$, $\mathbb{S}^d$ denotes the space of second order symmetric tensors on $\mathbb{R}^d$ and "$\cdot$", $\|\cdot\|$ will represent the inner product and the Euclidean norm on $\mathbb{R}^d$ and $\mathbb{S}^d$, respectively. We use the notation $\mathbf{0}$ for the zero element of the spaces $\mathbb{R}^d$ and $\mathbb{S}^d$ and the indices i, j, k, l run from 1 to d. Let $\Omega \subset \mathbb{R}^d$ be a bounded domain with Lipschitz continuous boundary Γ and let $\overline{\Omega} = \Omega \cup \Gamma$. We denote by $\boldsymbol{\nu}$ the outward unit normal at Γ and $\boldsymbol{y} \in \Omega \cup \Gamma$ will represent the spatial variable which, sometimes, for simplicity, is skipped. Assume that $\Gamma = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3$ where Γ_1, Γ_2, Γ_3 are mutually disjoint measurable parts of Γ such that $meas\ (\Gamma_1) > 0$. For the displacement and the stress field we use the Hilbert spaces $(V, (\cdot,\cdot)_V)$ and $(Q, (\cdot,\cdot)_Q)$, respectively, defined by

$$V = \{\, \boldsymbol{v} = (v_i) \in H^1(\Omega)^d \ :\ \boldsymbol{v}|_{\Gamma_1} = \mathbf{0} \,\}, \quad (\boldsymbol{u}, \boldsymbol{v})_V = \int_\Omega \boldsymbol{\varepsilon}(\boldsymbol{u}) \cdot \boldsymbol{\varepsilon}(\boldsymbol{v})\, dy,$$

$$Q = \{\, \boldsymbol{\tau} = (\tau_{ij}) \in L^2(\Omega)^{d\times d} \ :\ \tau_{ij} = \tau_{ji} \,\}, \quad (\boldsymbol{\sigma}, \boldsymbol{\tau})_Q = \int_\Omega \boldsymbol{\sigma} \cdot \boldsymbol{\tau}\, dy.$$

Here and below $\boldsymbol{\varepsilon}$ represents the deformation operator, i.e., $\boldsymbol{\varepsilon}(\boldsymbol{v})$ denotes the symmetric part of the gradient of $\boldsymbol{v}$, for any $\boldsymbol{v} \in V$. The associate norms on the spaces V will be denoted by $\|\cdot\|_V$ and $\|\cdot\|_Q$, respectively.

For an element $\boldsymbol{v} \in V$, we use the notation v_ν and $\boldsymbol{v}_\tau$ for the normal and tangential traces of $\boldsymbol{v}$ on Γ, i.e., $v_\nu = \boldsymbol{v} \cdot \boldsymbol{\nu}$ and $\boldsymbol{v}_\tau = \boldsymbol{v} - v_\nu \boldsymbol{\nu}$. Moreover, for a regular stress field $\boldsymbol{\sigma} \in Q$ we use the notation $\sigma_\nu = (\boldsymbol{\sigma}\boldsymbol{\nu}) \cdot \boldsymbol{\nu}$

and $\boldsymbol{\sigma}_\tau = \boldsymbol{\sigma} - \sigma_\nu \boldsymbol{\nu}$. Finally, as usual, we denote by V^* the strong topological dual of V, by $\langle \cdot, \cdot \rangle$ the duality paring mapping and by I an interval of time of the form $I = [0, T]$ with $T > 0$ or $I = [0, +\infty)$.

Then, the classical formulation of the viscoelastic contact problem we consider in this section is the following.

Problem $\mathcal{P}^{ve}$. *Find a stress field* $\boldsymbol{\sigma} : \Omega \times I \to \mathbb{S}^d$, *a displacement field* $\boldsymbol{u} : \Omega \times I \to \mathbb{R}^d$ *and an interface function* $\eta_\nu : \Gamma_3 \times I \to \mathbb{R}$ *such that*

$$\dot{\boldsymbol{\sigma}}(t) = \mathcal{E}\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}(t)) + \beta(\boldsymbol{\sigma}(t) - \mathcal{F}(\boldsymbol{\varepsilon}(\boldsymbol{u}(t)))) \qquad \text{in } \Omega, \tag{5.1}$$

$$\mathrm{Div}\, \boldsymbol{\sigma}(t) + \boldsymbol{f}_0(t) = \boldsymbol{0} \qquad \text{in } \Omega, \tag{5.2}$$

$$\boldsymbol{u}(t) = \boldsymbol{0} \qquad \text{on } \Gamma_1, \tag{5.3}$$

$$\boldsymbol{\sigma}(t) \cdot \boldsymbol{\nu} = \boldsymbol{f}_2(t) \qquad \text{on } \Gamma_2, \tag{5.4}$$

$$\left.\begin{aligned} & u_\nu(t) \le g, \quad \sigma_\nu(t) + k u_\nu^+(t) + \eta_\nu(t) \le 0, \\ & (u_\nu(t) - g)(\sigma_\nu(t) + k u_\nu^+(t)) + \eta_\nu(t)) = 0, \\ & 0 \le \eta_\nu(t) \le h\Big(\int_0^t u_\nu^+(s)\, ds, u_\nu^+(t) \Big), \\ & \eta_\nu(t) = \begin{cases} 0 & \text{if } \ u_\nu(t) < 0, \\ h\Big(\int_0^t u_\nu^+(s)\, ds, u_\nu^+(t) \Big) & \text{if } \ u_\nu(t) > 0 \end{cases} \end{aligned} \right\} \qquad \text{on } \Gamma_3, \tag{5.5}$$

$$\boldsymbol{\sigma}_\tau(t) = \boldsymbol{0} \qquad \text{on } \Gamma_3, \tag{5.6}$$

$$\boldsymbol{\sigma}(0) = \boldsymbol{\sigma}_0, \qquad \boldsymbol{u}(0) = \boldsymbol{u}_0 \qquad \text{in } \Omega. \tag{5.7}$$

Note that Problem $\mathcal{P}^{ve}$ describes the equilibrium of a viscoelastic body which occupies the domain Ω, is held fixed on the part Γ_1 on his boundary, is acted upon by a time-dependent surface traction of density $\boldsymbol{f}_2$ on Γ_2 and is in contact with a foundation on Γ_3. Equation (5.1) represents the constitutive law which models the viscoelastic behaviour of the material. Here $\mathcal{E}$ is a fourth order elasticity tensor, β is a viscosity coefficient and $\mathcal{F}$ is a constitutive function. Equation (5.2) represents the equilibrium equation in which $\boldsymbol{f}_0$ denotes the density of body forces, (5.3) is the displacement boundary condition and (5.4) is the traction boundary condition.

Condition (5.5) is the contact condition which models the contact with a foundation made of a rigid body covered by a layer of rigid-elastic material. Here g represents the thickness of this layer, h is a given function which describes its rigidity, k is a stiffness coefficient and r^+ denotes the positive part of r, i.e., $r_+ = \max\{r, 0\}$. Details can be found in [37]. Here we restrict ourselves to recall that the quantity

$$\xi(\boldsymbol{y}, t) = \int_0^t u_\nu^+(\boldsymbol{y}, s)\, ds \tag{5.8}$$

represents the accumulated penetration in the point $\boldsymbol{y}$ of the contact surface at the time moment t. Assuming that the yield function h depends on the process variables ξ and u_ν^+ describes the hardening property of the foundation.

Condition (5.6) shows that the tangential component of the stress vanishes on the contact surface and, therefore, the contact is frictionless. Finally, (5.7) are the initial conditions, in which $\boldsymbol{u}_0$ and $\boldsymbol{\sigma}_0$ are given.

In the study of Problem $\mathcal{P}^{ve}$ we use the space of symmetric fourth order tensors $\mathbf{Q}_\infty$ defined by $\mathbf{Q}_\infty = \{\mathcal{C} = (c_{ijkl}) \mid c_{ijkl} = c_{jikl} = c_{klij} \in L^\infty(\Omega)\}$ and we consider the following assumption on the data.

$$\left\{\begin{array}{l}\text{(a) } \mathcal{E} \in \mathbf{Q}_\infty.\\[4pt] \text{(b) There exists } m_\mathcal{E} > 0 \text{ such that}\\ \qquad \mathcal{E}(\boldsymbol{y})\boldsymbol{\tau}\cdot\boldsymbol{\tau} \ge m_\mathcal{E}\|\boldsymbol{\tau}\|^2 \quad \text{for all } \boldsymbol{\tau} \in \mathbb{S}^d,\ \text{a.e. } \boldsymbol{y} \in \Omega.\end{array}\right. \tag{5.9}$$

$$\left\{\begin{array}{l}\text{(a) } \mathcal{F} : \Omega \times \mathbb{S}^d \to \mathbb{S}^d.\\[4pt] \text{(b) There exists } L_\mathcal{F} > 0 \text{ such that}\\ \qquad \|\mathcal{F}(\boldsymbol{y},\boldsymbol{\tau}_1) - \mathcal{F}(\boldsymbol{y},\boldsymbol{\tau}_2)\| \le L_\mathcal{F}\,(\|\boldsymbol{\tau}_1 - \boldsymbol{\tau}_2\|)\\ \qquad\quad \text{for all } \boldsymbol{\tau}_1, \boldsymbol{\tau}_2 \in \mathbb{S}^d,\ \text{a.e. } \boldsymbol{y} \in \Omega.\\[4pt] \text{(c) } \boldsymbol{y} \mapsto \mathcal{F}(\boldsymbol{y},\boldsymbol{\tau}) \text{ is measurable on } \Omega,\ \text{for any } \boldsymbol{\tau} \in \mathbb{S}^d.\\[4pt] \text{(d) } \boldsymbol{y} \mapsto \mathcal{F}(\boldsymbol{y},\boldsymbol{0}) \in Q.\end{array}\right. \tag{5.10}$$

$$\left\{\begin{array}{l}\text{(a) } h : \Gamma_3 \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}_+.\\[4pt] \text{(b) There exists } L_h > 0 \text{ such that}\\ \qquad |h(\boldsymbol{y}, r_1, s_1) - h(\boldsymbol{y}, r_2, s_2)| \le L_h\big(|r_1 - r_2| + |s_1 - s_2|\big)\\ \qquad\qquad \text{for all } r_1,\ r_2,\ s_1,\ s_2 \in \mathbb{R},\ \text{a.e. } \boldsymbol{y} \in \Gamma_3.\\[4pt] \text{(c) } \boldsymbol{y} \mapsto h(\boldsymbol{y}, r, s) \text{ is measurable on } \Gamma_3,\ \text{for any } r,\ s \in \mathbb{R}.\\[4pt] \text{(d) } \boldsymbol{y} \mapsto h(\boldsymbol{y}, 0, 0) \in L^2(\Gamma_3).\end{array}\right. \tag{5.11}$$

$$\boldsymbol{f}_0 \in C(I; L^2(\Omega)^d), \qquad \boldsymbol{f}_2 \in C(I; L^2(\Gamma_2)^d), \tag{5.12}$$

$$\beta \in L^\infty(\Omega). \tag{5.13}$$

$$k \in L^\infty(\Gamma_3), \quad k(\boldsymbol{y}) \ge 0 \quad \text{a.e. } \boldsymbol{y} \in \Gamma_3. \tag{5.14}$$

$$\boldsymbol{u}_0 \in V, \qquad \boldsymbol{\sigma}_0 \in Q. \tag{5.15}$$

$$\left\{\begin{array}{l}\text{There exist } G \in H^2(\Omega) \text{ and } M_0,\ M_1 \in \mathbb{R} \text{ such that}\\ \ g = \gamma_0(G) \text{ on } \Gamma_3 \text{ and } 0 < M_0 \le G(\boldsymbol{y}) \le M_1 \text{ for all } \boldsymbol{y} \in \overline{\Omega}.\end{array}\right. \tag{5.16}$$

Note that in (5.16) and below $\gamma_0 : H^1(\Omega) \to L^2(\Gamma)$ denotes the trace operator. Moreover, note that the condition (5.16) makes sense since $d \in \{2, 3\}$ and, therefore, $H^2(\Omega) \subset C(\overline{\Omega})$.

We turn in what follows to the variational analysis of Problem $\mathcal{P}^{ve}$ and, to this end, besides the function $\xi : \Gamma_3 \times I \to \mathbb{R}$ defined by (5.8), we consider the irreversible stress field $\boldsymbol{\sigma}^{ir} : \Omega \times I \to \mathbb{S}^d$ and the set of admissible displacements fields $K \subset V$ defined by

$$\boldsymbol{\sigma}^{ir} = \boldsymbol{\sigma} - \mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}), \tag{5.17}$$

$$K = \{\, \boldsymbol{v} \in V : v_\nu \le g \ \text{ a.e. on } \Gamma_3 \,\}. \tag{5.18}$$

Then, using standard arguments we deduce the following variational formulation of the problem.

Problem $\mathcal{P}^{ve}_V$. *Find an irreversible stress field* $\boldsymbol{\sigma}^{ir} : I \to Q$*, an accumulated penetration function* $\xi : I \to L^2(\Gamma_3)$ *and a displacement field* $\boldsymbol{u} : I \to V$ *such that*

$$\dot{\boldsymbol{\sigma}}^{ir}(t) = \beta(\mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}(t)) + \boldsymbol{\sigma}^{ir}(t) - \mathcal{F}\boldsymbol{\varepsilon}(\boldsymbol{u}(t))), \quad \dot{\xi}(t) = u_\nu^+(t) \quad \forall\, t \in I, \tag{5.19}$$

$$\boldsymbol{\sigma}^{ir}(0) = \boldsymbol{\sigma}_0 - \mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}_0), \quad \xi(0) = 0, \tag{5.20}$$

$$\begin{aligned}
\boldsymbol{u}(t) \in K, \quad & \int_{\Omega} (\mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}(t)) + \boldsymbol{\sigma}^{ir}(t)) \cdot (\boldsymbol{\varepsilon}(\boldsymbol{v}) - \boldsymbol{\varepsilon}(\boldsymbol{u}(t)) \, dy \\
& + \int_{\Gamma_3} k u_\nu^+(t)(v_\nu - u_\nu(t)) \, da + \int_{\Gamma_3} h(\xi(t), u_\nu^+(t))(v_\nu^+ - u_\nu^+(t)) \, da \\
& \geq \int_{\Omega} \boldsymbol{f}_0(t) \cdot \boldsymbol{v} - \boldsymbol{u}(t) \, dy + \int_{\Gamma_2} \boldsymbol{f}_2(t) \cdot (\boldsymbol{v} - \boldsymbol{u}(t) \, da \quad \forall \, \boldsymbol{v} \in K, \ \ t \in I.
\end{aligned} \tag{5.21}$$

The unique solvability of Problem $\mathcal{P}_V^{ve}$ is provided by the following existence and uniqueness result.

Theorem 5.1. *Assume (5.9)–(5.16). Then Problem* $\mathcal{P}_V^{ve}$ *has a unique solution which satisfies* $\boldsymbol{\sigma}^{ir} \in C^1(I;Q)$, $\xi \in C^1(I;L^2(\Gamma_3))$, $\boldsymbol{u} \in C(I;V)$.

Proof. We consider the product spaces $X = Q \times L^2(\Gamma_3)$ and $Z = L^2(\Omega)^d \times L^2(\Gamma_2)^d$ endowed with the canonical inner products $(\cdot,\cdot)_X$ and $(\cdot,\cdot)_Z$, respectively, as well as the operators $F : I \times X \times V \to X$, $A : X \times V \to V^*$, $\pi : V \to Z$ and the functions $j : X \times V \times V \to \mathbb{R}$, $\boldsymbol{f} : I \to V^*$ given by

$$F(t, \boldsymbol{x}, \boldsymbol{u}) = \Big(\beta\big(\mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u})\big) + \boldsymbol{\sigma} - \mathcal{F}\boldsymbol{\varepsilon}(\boldsymbol{u})\big), u_\nu^+\Big), \tag{5.22}$$

$$\langle A(\boldsymbol{x}, \boldsymbol{u}), \boldsymbol{v}\rangle = (\mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}) + \boldsymbol{\sigma}, \boldsymbol{\varepsilon}(\boldsymbol{v}))_Q + \int_{\Gamma_3} k u_\nu^+ v_\nu \, da, \tag{5.23}$$

$$\pi \boldsymbol{v} = (\boldsymbol{v}, \boldsymbol{v}|_{\Gamma_2}), \tag{5.24}$$

$$j(\boldsymbol{x}, \boldsymbol{u}, \boldsymbol{v}) = \int_{\Gamma_3} h(\xi, u_\nu^+) v_\nu^+ \, da, \tag{5.25}$$

$$(\boldsymbol{f}(t), \boldsymbol{z})_Z = \int_{\Omega} \boldsymbol{f}_0(t) \cdot \boldsymbol{z}_1 \, dx + \int_{\Gamma_2} \boldsymbol{f}_2(t) \cdot \boldsymbol{z}_2 \, da \tag{5.26}$$

for all $t \in I$, $\boldsymbol{x} = (\boldsymbol{\sigma}, \xi) \in X$, $\boldsymbol{u}, \boldsymbol{v} \in V$, $\boldsymbol{z} = (\boldsymbol{z}_1, \boldsymbol{z}_2) \in Z$. Note that in (5.24) notation $\boldsymbol{v}|_{\Gamma_2}$ represents the trace of $\boldsymbol{v}$ in Γ_2. Moreover, consider the element of X given by

$$\boldsymbol{x}_0 = (\boldsymbol{\sigma}_0 - \mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}_0), 0). \tag{5.27}$$

Then, it is easy to see that Problem $\mathcal{P}_V^{ve}$ is equivalent to the problem of finding two functions $\boldsymbol{x} = (\boldsymbol{\sigma}^{ir}, \xi) : I \to X$ and $\boldsymbol{u} : I \to V$ such that

$$\dot{\boldsymbol{x}}(t) = F(t, \boldsymbol{x}(t), \boldsymbol{u}(t)) \qquad \forall \, t \in I, \tag{5.28}$$

$$\boldsymbol{x}(0) = \boldsymbol{x}_0, \tag{5.29}$$

$$\begin{aligned}
\boldsymbol{u}(t) \in K, \qquad & \langle A(\boldsymbol{x}(t), \boldsymbol{u}(t)), \boldsymbol{v} - \boldsymbol{u}(t)\rangle + j(\boldsymbol{x}(t), \boldsymbol{u}(t), \boldsymbol{v}) - j(\boldsymbol{x}(t), \boldsymbol{u}(t), \boldsymbol{u}(t)) \\
& \geq \langle \boldsymbol{f}(t), \pi\boldsymbol{v} - \pi\boldsymbol{u}(t)\rangle_Z \qquad \forall \, \boldsymbol{v} \in K, \ t \in I.
\end{aligned} \tag{5.30}$$

Remark that, with the previous notation, all the conditions in Theorem 2.1 are satisfied for the differential variational inequality (5.28)–(5.30) For instance, it is easy to see that assumptions (5.9), (5.10), (5.13) and (5.14) imply that the operators (5.22) and (5.23) satisfy conditions (2.4) and (2.7), respectively, the later with $m = m_{\mathcal{E}}$. Moreover, the regularity (5.15) and (5.12) imply that (2.5) and (2.10) hold, too. In addition, assumption (5.16) combined with standard arguments implies that the set (5.18) satisfies condition

(2.6) and, using the assumption (5.11) and the Sobolev trace inequality it is easy to see that condition (2.8) holds with $\beta = 0$. To conclude, we deduce from Theorem 2.1 the existence of a unique solution $\boldsymbol{x} = (\boldsymbol{\sigma}^{ir}, \xi) \in C^1(I;X)$, $\boldsymbol{u} \in C(I;V)$ which satisfies (5.28)–(5.30). Then, using the equivalence between Problem $\mathcal{P}_V^{ve}$ and the differential quasivariational inequality (5.28)–(5.30), we deduce that $(\boldsymbol{\sigma}^{ir}, \xi, \boldsymbol{u})$ is the unique solution to Problem $\mathcal{P}_V^{ve}$ with regularity $\boldsymbol{\sigma}^{ir} \in C^1(I;Q)$, $\xi \in C^1(I;L^2(\Gamma_3))$, $\boldsymbol{u} \in C(I;V)$, which concludes the proof. □

We now study the continuous dependence of the solution to Problem $\mathcal{P}_V^{ve}$ with respect to the data. Various cases can be considered and various convergence results can be obtained, based on Theorem 3.1. Here, for simplicity, we restrict ourselves to provide only one example, which concerns the dependence of the solution with respect to the density of surface tractions and the thickness g. Therefore, we assume in what follows that (5.9)–(5.16) hold and, moreover, we assume that there exists two functions θ and $\widetilde{\boldsymbol{f}}_2$ such that

$$\theta \in C(I;\mathbb{R}), \quad \widetilde{\boldsymbol{f}}_2 \in L^2(\Gamma_2)^d, \tag{5.31}$$

$$\boldsymbol{f}_2(t) = \theta(t)\widetilde{\boldsymbol{f}}_2 \qquad \forall\, t \in I. \tag{5.32}$$

In addition, for each $n \in \mathbb{N}$ we consider a perturbation $\boldsymbol{f}_{2n}$ and $g_n = \gamma_0(G_n)$ of $\boldsymbol{f}_2$ and $g = \gamma_0(G)$, respectively, such that

$$\boldsymbol{f}_{2n}(t) = \theta(t)\widetilde{\boldsymbol{f}}_{2n} \qquad \forall\, t \in I \quad \text{with} \quad \widetilde{\boldsymbol{f}}_2 \in L^2(\Gamma_2)^d. \tag{5.33}$$

$$G_n \in H^2(\Omega) \ \text{ and } \ 0 < M_0 \le G_n(\boldsymbol{y}) \le M_1 \ \text{ for all } \boldsymbol{y} \in \overline{\Omega}. \tag{5.34}$$

$$\widetilde{\boldsymbol{f}}_{2n}(t) \rightharpoonup \boldsymbol{f}_2(t) \quad \text{in} \quad L^2(\Gamma_2)^d \quad \forall\, t \in I. \tag{5.35}$$

$$G_n \rightharpoonup G \quad \text{in} \quad H^2(\Omega). \tag{5.36}$$

For each $n \in \mathbb{N}$ we consider Problem $\mathcal{P}_{Vn}^{ve}$ obtained by replacing in Problem $\mathcal{P}_V^{ve}$ the data $\boldsymbol{f}_2$ and g with $\boldsymbol{f}_{2n}$ and g_n, respectively. Then, Theorem 5.1 guarantees that $\mathcal{P}_{Vn}^{ve}$ has a unique solution $(\boldsymbol{\sigma}_n^{ir}, \xi_n, \boldsymbol{u}_n)$, with regularity $\boldsymbol{\sigma}_n^{ir} \in C(I;Q)$, $\xi_n \in C(I;L^2(\Gamma_3)$, $\boldsymbol{u}_n \in C(I;V)$. Moreover we have the following convergence result.

Theorem 5.2. *Assume (5.9)–(5.16) and (5.31)–(5.36). Then, the solution $(\boldsymbol{\sigma}_n^{ir}, \xi_n, \boldsymbol{u}_n)$ of Problem $\mathcal{P}_{Vn}^{ve}$ converges to the solution $(\boldsymbol{\sigma}^{ir}, \xi, \boldsymbol{u})$ of Problem $\mathcal{P}_V^{ve}$ as $n \to \infty$, i.e., for each $t \in I$ we have*

$$\boldsymbol{\sigma}_n^{ir}(t) \to \boldsymbol{\sigma}^{ir}(t) \ \ \text{in } Q, \quad \xi_n(t) \to \xi(t) \ \ \text{in } L^2(\Gamma_3), \quad \boldsymbol{u}_n(t) \to \boldsymbol{u}(t) \ \ \text{in } V \quad \text{as } n \to \infty.$$

Proof. First, we remark that the set of constraints associated to Problem $\mathcal{P}_{Vn}^{ve}$ is given by

$$K_n = \{\boldsymbol{v} \in V : v_\nu \le g_n \ \text{ a.e. on } \Gamma_3\}. \tag{5.37}$$

Let $v \in K$. Then, assumptions (5.16) and (5.34) allow us to consider the sequence $\{v_n\} \subset V$ defined by $v_n = \frac{G_n}{G}\, v$, for each $n \in \mathbb{N}$ We now use definitions (5.18), (5.37) and equalities $g_n = \gamma_0(G_n)$, $g = \gamma_0(G)$ to see that $v_n \in K_n$ for each $n \in \mathbb{N}$. Moreover, using (5.34), (5.36) and the compactness of the inclusion $H^2(\Omega) \subset H^1(\Omega)$ (see, for instance [1]) it is easy to see that $v_n \to v$ in V. We conclude from here that condition (3.7)(a) is satisfied.

Assume now that $\{v_n\}$ is a sequence of elements of V such that $v_n \in K_n$ for all $n \in \mathbb{N}$ and $v_n \rightharpoonup v$ in V. Then,

$$v_{n\nu} \le g_n \quad \text{ a.e. on } \Gamma_3, \ \text{ for all } n \in \mathbb{N}. \tag{5.38}$$

Moreover, compactness arguments guarantee that the convergences $v_n \rightharpoonup v$ in V and $G_n \rightharpoonup G$ in $H^2(\Omega)$ imply that $v_{n\nu} \to v_\nu$ and $g_n \to g$, both in $L^2(\Gamma_3)$. Therefore, passing to some subsequences, again denoted by $\{v_n\}$ and $\{g_n\}$, we can assume that

$$v_{n\nu} \to v_\nu, \quad g_n \to g \quad \text{a.e. on } \Gamma_3. \tag{5.39}$$

It follows now from (5.38) and (5.39) that $v_\nu \le g$ a.e. on Γ_3 which shows that $v \in K$ and, hence, (3.7)(b) holds. The proof of Theorem 5.2 is now a direct consequence of Theorem 3.1 and Remark 3.1. □

We now turn to the optimal control of Problem $\mathcal{P}_V^{ve}$ and, to this end, we shall use Theorem 4.1. For simplicity, we restrict ourselves to provide the following example.

Assume that (5.9)–(5.16) hold and denote by W the product space $W = L^2(\Gamma_2)^d \times H^2(\Omega)$ endowed with the canonical Hilbertian structure. Moreover, consider the set $U \subset W$ defined by

$$\begin{aligned} U = \{\, q = (\widetilde{\boldsymbol{f}}_2, G) \in W \ : \ \|\widetilde{\boldsymbol{f}}_2\|_{L^2(\Gamma_2)^d} \le M_2 \ , \ \|G\|_{H^2(\Omega)} \le M_3, \\ M_0 \le G(\boldsymbol{y}) \le M_1 \ \text{ for all } \boldsymbol{y} \in \overline{\Omega} \,\} \end{aligned} \tag{5.40}$$

where M_0, M_1, M_2 and M_3 are given positive constants such that $M_0 \le M_1$ and $M_3 \ge M_0(mes(\Omega))^{\frac{1}{2}}$. Note that the set U is nonempty since, for instance, $(\boldsymbol{0}_{L^2(\Gamma_2)^d}, M_0) \in U$. For any $q = (\widetilde{\boldsymbol{f}}_2, G) \in U$ we consider Problem $\mathcal{P}_{Vq}^{ve}$ obtained by replacing in Problem $\mathcal{P}_V^{ve}$ the data $\boldsymbol{f}_2$ and g with $\boldsymbol{f}_{2q}$ and g_q, respectively, where

$$\boldsymbol{f}_{2q}(t) = \theta(t)\widetilde{\boldsymbol{f}}_2 \qquad \forall\, t \in I, \qquad g_q = \gamma_0(G)$$

and $\theta \in C(I;\mathbb{R})$. Then, Theorem 5.1 guarantees that $\mathcal{P}_{Vq}^{ve}$ has a unique solution $(\boldsymbol{\sigma}_q^{ir}, \xi_q, \boldsymbol{u}_q)$, we regularity $\boldsymbol{\sigma}_q^{ir} \in C^1(I;Q)$, $\xi_q \in C^1(I;L^2(\Gamma_3))$, $\boldsymbol{u}_q \in C(I;V)$. Consider now the following optimal control problem in which, for any $q \in U$, $u_{q\nu}$ represents the normal component of the function $\boldsymbol{u}_q$.

Problem $\mathcal{Q}^{ve}$. *Given $t \in I$ and $\phi \in L^2(\Gamma_3)$, find $q^* = (\boldsymbol{f}_2^*, G^*) \in U$ such that*

$$\int_{\Gamma_3} |u_{q^*\nu}(t) - \phi|^2 \, da \le \int_{\Gamma_3} |u_{q\nu}(t) - \phi|^2 \, da \qquad \forall\, q \in U. \tag{5.41}$$

We have the following existence result.

Theorem 5.3. *Under the previous assumptions, the optimal control problem $\mathcal{Q}^{ve}$ has at least one solution $q^* = (\boldsymbol{f}_2^*, G^*) \in U$.*

Proof. It is easy to see that the set (5.40) satisfies condition (4.5) and (4.8) on the space $W = L^2(\Gamma_2)^d \times H^2(\Omega)$. Moreover, the function $\mathcal{L} : X \times V \times U \to \mathbb{R}$ defined by

$$\mathcal{L}(\boldsymbol{x}, \boldsymbol{u}, q) = \int_{\Gamma_3} |u_{q\nu}(t) - \phi|^2 \, da \qquad \forall\, \boldsymbol{x} \in X,\ \boldsymbol{u} \in V, q \in U$$

satisfies condition (4.6) with $X = Q \times L^2(\Gamma_3)$. We now use Theorem 4.1 and Remark 4.1 to conclude the proof. □

We end this section with some comments and mechanical interpretation of our results. First, the variational formulation $\mathcal{P}_V^{ve}$ of Problem $\mathcal{P}^{ve}$, in terms of the irreversible stress, accumulated penetration and displacement field, is new and nonstandard. Nevertheless, we refer to solution $(\boldsymbol{\sigma}^{ir}, \xi, \boldsymbol{u})$ of $\mathcal{P}_V^{ve}$ as the weak solution of the frictionless contact problem $\mathcal{P}^{ve}$. Therefore, Theorem 5.1 provides the unique solvability of this viscoelastic contact problem. Next, Theorem 5.2 shows that the weak solution depends continuously on the density of surface tractions and the thickness of the rigid-elastic layer. Finally, the mechanical interpretation of the optimal control problem $\mathcal{Q}^{ve}$ is the following: given a contact process of the form (5.1)–(5.7), (5.31), (5.32) and a time moment $t \in I$, we are looking for a pair $q = (\widetilde{\boldsymbol{f}}_2^*, G^*) \in U$ such that the corresponding penetration of the viscoelastic body at t is as close as possible to the "desired penetration" ϕ. Theorem 5.3 guarantees the existence of at least one solution to this problem.

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