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NECESSARY AND SUFFICIENT CONDITION TO OBTAIN N PHASES IN A ONE-DIMENSIONAL MEDIUM WITH A FLUX CONDITION ON THE FIXED FACE

María Cristina SANZIEL – Domingo Alberto TARZIA

PROMAR (CONICET-UNR),
Instituto de Matemática "B. Levi",
Facultad de Ciencias Exactas, Ing. y Agr.,
Avda. Pellegrini 250, (2000) Rosario, Argentina.

ABSTRACT

We consider an n-phase Stefan Problem for a semi-infinite material $x > 0$ with constant initial condition and a heat flux of the type $q(t) = -q_0/\sqrt{t}$, imposed on the fixed face $x = 0$. We determine necessary and/or sufficient conditions on the parameter q_0, in order to obtain the existence of the solution. We also show the equivalence between this problem and that in which a temperature on the fixed face is imposed.

Key words : Multi-phase Stefan problem, Similarity variable, Neumann solution, Exact solutions, Free boundary problems.

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I. INTRODUCTION

We consider a semi-infinite material represented by $(0,+\infty)$, with a constant initial temperature u_1. We apply a heat flux of the type $q(t)=-q_0/\sqrt{t}$ with $q_0 > 0$ on the fixed face $x = 0$. If the temperature's values are between u_i and u_{i+1} $(u_i < u_{i+1})$, we will consider that the material is in the phase i. Each of these phases is separated by a free boundary.

In this work, based on [4,7], we determine which conditions must verify the parameter q_0, for the n-phases to appear. We show that the necessary and sufficient condition is:

$$q_0 > \frac{\beta_{n-1}}{\operatorname{erf}\frac{G_{n-2}(x_{n-2})}{a_{n-1}}} > \frac{\beta_{n-2}}{\operatorname{erf}\frac{G_{n-3}(x_{n-3})}{a_{n-2}}} > \dots > \frac{\beta_2}{\operatorname{erf}\frac{x_1}{a_2}} > \beta_1$$

where β_i, G_i and x_i are defined by (13), (25) and (26) respectively.

II. THE PROBLEM

We wish to find conditions on q_0 for the n-phases to appear, in other words, conditions such that there exist the free boundaries $S_{n-1}(t) < S_{n-2}(t) < \dots < S_1(t)$ and the temperature $\Theta = \Theta(x,t)$, given by:

(1) $\Theta(x,t) = \Theta_i(x,t)$ if $S_i(t) < x < S_{i-1}(t)$, $t > 0$, $i = 1, \dots, n$,

and such that the following conditions be satisfied:

(2) $\frac{\partial \Theta_i}{\partial t} = \alpha_i \frac{\partial \Theta_i}{\partial x^2}$, $x \in (S_i(t), S_{i-1}(t))$, $t > 0$, $i = 1, \dots, n$,

(3) $\Theta_1(x,0) = u_1$, $x > 0$,

(4) $\Theta_1(S_0(t), t) = u_1$, $t > 0$,

(5) $\Theta_i(S_i^+(t), t) = \Theta_{i+1}(S_i^-(t), t) = u_{i+1}$, $t > 0$, $i = 1, \dots, n-1$,

(6) $k_n \frac{\partial \Theta_n}{\partial x}(0,t) = -\frac{q_0}{\sqrt{t}}$, $t > 0$,

(7) $S_i(0) = 0$, $i = 1, \dots, n-1$,

$$(8)\quad k_i \frac{\partial\Theta_i}{\partial x}(S_i^+(t), t) - k_{i+1} \frac{\partial\Theta_{i+1}}{\partial x}(S_i^-(t), t) = \delta_i \dot{S}_i(t)\,, \qquad t > 0\,, \qquad i = 1, \ldots, n\text{-}1,$$

where k_i and $\alpha_i = \frac{k_i}{\rho c_i}$ are respectively the thermal conductivity and the diffusivity for the phase i, c_i is the especific heat of the phase i, ρ is the common mass density, $\delta_i > 0$ represents the latent heat used for passing from phase i to phase i + 1, the temperatures u_i verify $u_{i+1} > u_i$ for i = 1, . . . , n-1 and we consider $S_0(t) = +\infty$ and $S_n(t) = 0$, $\forall\ t>0$. For more physical considerations see [6,7].

Remark 1: The problem (2)–(8) can be stated in a similar way for dependent variable concentration or enthalpy.

III. SOLUTION OF THE PROBLEM (2)–(8)

Following the Neumann's solution idea, for the two-phase Stefan problem [1,2,5] (for the multiphase Stefan problem see [6,7]) we propose:

$$(9)\quad \begin{aligned} &\Theta_i(x,t) = A_i + B_i \operatorname{erf}\Big(\frac{x}{2a_i\sqrt{t}}\Big)\,, && i = 1, \ldots, n\,, \\ &S_i(t) = 2\omega_i \sqrt{t}\,, \qquad \omega_i > 0\,, && i = 1, \ldots, n\text{-}1\,, \end{aligned}$$

where for convenience we consider $a_i = \sqrt{\alpha_i}$ and

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x \exp(-t^2)\, dt \qquad \text{and} \qquad \operatorname{erfc}(x) = 1 - \operatorname{erf}(x)$$

are the error and the complementary error functions respectively.

From conditions (3)– (6) we obtain:

$$(10)\quad A_i = \frac{u_{i+1} \operatorname{erf}(\frac{\omega_{i-1}}{a_i}) - u_i \operatorname{erf}(\frac{\omega_i}{a_i})}{\operatorname{erf}(\frac{\omega_{i-1}}{a_i}) - \operatorname{erf}(\frac{\omega_i}{a_i})}\,, \qquad B_i = \frac{u_i - u_{i+1}}{\operatorname{erf}(\frac{\omega_{i-1}}{a_i}) - \operatorname{erf}(\frac{\omega_i}{a_i})}\,, \qquad i=1,\ldots,n\text{-}1,$$

$$(11)\quad A_n = u_n + \operatorname{erf}(\frac{\omega_{n-1}}{a_n}) \frac{q_0 a_n \sqrt{\pi}}{k_n}\,, \qquad B_n = -\frac{q_0 a_n \sqrt{\pi}}{k_n}\,,$$

here we have inserted a dummy parameter $\omega_0 = +\infty$.

Conditions (8) (i = 1, ···, n-1) are verified if and only if ω_i satisfy the following system:

(12)

a) $\delta_i \omega_i = \beta_{i+1} \dfrac{\eta(\omega_i, a_{i+1})}{\phi(\omega_i, \omega_{i+1}, a_{i+1})} - \beta_i \dfrac{\eta(\omega_i, a_i)}{\phi(\omega_{i-1}, \omega_i, a_i)}$, $\quad i = 1, \ldots, n\text{-}2,$

b) $\delta_{n-1}\omega_{n-1} = q_0\, \eta(\omega_{n-1}, a_n) - \beta_{n-1} \dfrac{\eta(\omega_{n-1}, a_{n-1})}{\phi(\omega_{n-2}, \omega_{n-1}, a_{n-1})}$,

where:

(13) $$\beta_i = -\frac{k_i}{a_i\sqrt{\pi}}(u_i - u_{i+1}) > 0\ , \quad i = 1, \ldots, n\text{-}1\ ,$$

(14) $$\eta(\omega, a) = \exp(-\ \frac{\omega^2}{a^2})\ ,$$

(15) $$\phi(\alpha, \beta, \gamma) = \operatorname{erf}\,(\frac{\alpha}{\gamma}) - \operatorname{erf}(\frac{\beta}{\gamma})\ .$$

In order to show the existence of a solution of the system (12) we will define the sequences of functions $\{h_i(x)\}_{i=1}^{n-1}$, $\{H_i(x)\}_{i=1}^{n-2}$ y $\{G_i(x)\}_{i=1}^{n-1}$.

We define:

(16) $$G_1(x) = x\ , \qquad x > 0\ ,$$

(17) $$h_1(x) = \delta_1 x + \beta_1 \frac{\eta(x, a_1)}{\operatorname{erfc}\,(\frac{x}{a_1})}\ , \qquad x > 0\ .$$

Taking into account that $F_1(x) = \dfrac{\exp(-x^2)}{\operatorname{erfc}(x)}$ $(x > 0)$ is an increasing function, it results that h_1 verifies:

$$h_1(0) = \beta_1 > 0\ , \qquad h_1(+\infty) = +\infty\ , \quad h_1'(x) > 0,\ \forall x > 0\ ,$$

then the equation (12a) for $i = 1$ is equivalent to:

(18) $$h_1(\omega_1) = \beta_2 \frac{\eta(\omega_1, a_2)}{\phi(\omega_1, \omega_2, a_2)}\ ,$$

and hence $\phi(\omega_1, \omega_2, a_2) > 0$ and $\omega_1 > \omega_2$.

Now we define:

(19) $$H_1(x) = \operatorname{erf}\frac{x}{a_2} - \beta_2 \frac{\eta(x, a_2)}{h_1(x)}\ , \quad x > 0\ .$$

From properties of the function h_1 it results that H_1 satisfies:

$$H_1(0) = -\ \frac{\beta_2}{\beta_1} < 0\ , \qquad H_1(+\infty) = 1, \qquad H_1'(x) > 0,\ \forall x > 0.$$

This implies the existence of $x_1 > 0$ such that $H_1(x_1) = 0$ and then we can define the function

(20) $$G_2(x) = a_2 \operatorname{erf}^{-1}[H_1(x)] \quad , \qquad x \in (x_1, +\infty),$$

which is an increasing function and verifies $G_2(x_1) = 0$ and $G_2(+\infty) = +\infty$.

We can write equation (18) as:

(21) $$\omega_2 = G_2(\omega_1) \ ,$$

or equivalently by

(21bis) $$\frac{\beta_2}{\phi(\omega_1,\omega_2,a_2)} = \frac{h_1(\omega_1)}{\eta(\omega_1,a_2)}.$$

Hence $G_2(x) < G_1(x)$, $\forall\ x \in (x_1, +\infty)$. Following [7] and proceeding inductively we can define now the functions:

(22) $$h_i(x) = \delta_i\ G_i(x) + \beta_i \frac{\eta(G_i(x), a_i)}{\phi(G_{i-1}(x), G_i(x), a_i)} \ , \qquad x \in (x_{i-1}, +\infty) \quad , i = 2, \ldots, n\text{-}2,$$

(23) $$h_{n-1}(x) = \delta_{n-1}\ G_{n-1}(x) + h_{n-2}(x) \frac{\eta(G_{n-1}(x), a_{n-1})}{\eta(G_{n-2}(x), a_{n-1})}, \quad x \in (x_{n-2}, +\infty)\ ,$$

(24) $$H_i(x) = \operatorname{erf} \frac{G_i(x)}{a_{i+1}} - \beta_{i+1} \frac{\eta(G_i(x), a_{i+1})}{h_i(x)}, \quad x \in (x_{i-1}, +\infty), \qquad i = 2, \ldots, n\text{-}2,$$

(25) $$G_i(x) = a_i \operatorname{erf}^{-1}[H_{i-1}(x)]\ , \quad x \in (x_{i-1}, +\infty)\ , \quad i = 3, \ldots, n\text{-}1\ ,$$

where

(26) $$x_i > x_{i-1} \ / \ H_i(x_i) = 0, \ i = 2, \ldots, n.$$

Having in mind that:

$$\frac{\beta_i}{\phi(G_{i-1}(x), G_i(x), a_i)} = \frac{h_{i-1}(x)}{\eta(G_{i-1}(x), a_i)}\ , \qquad i = 3, \ldots, n\text{-}1,$$

it results that $G_{i-1}(x) > G_i(x)$, $\forall\ x \in (x_{i-1}, +\infty)$ and then the functions defined below are increasing functions and satisfy these conditions:

$$h_i(x_{i-1}) > 0\ , \qquad h_i(+\infty) = +\infty,$$
$$H_i(x_{i-1}) < 0\ , \qquad H_i(x_i) = 0\ , \qquad H_i(+\infty) = 1\ ,$$
$$G_i(x_{i-1}) = 0\ , \qquad G_i(+\infty) = +\infty\ .$$

We can also define the function:

(27) $$Q(x) = q_0\ \eta(G_{n-1}(x), a_n), \quad x > x_{n-2},$$

which is a decreasing function and satisfies:

$$Q(x_{n-2}) = q_0 \quad , \quad Q(+\infty) = 0.$$

From (23), (25) and (27), the system ((12)a-b) can be written as:

$$(28) \quad \begin{cases} \omega_i = G_i(\omega_1), \quad i = 2, \ldots, n\text{-}1, \\ h_{n-1}(\omega_1) = Q(\omega_1), \end{cases}$$

and consecuently there will be a solution if and only if there exists an x^* such that $h_{n-1}(x^*) = Q(x^*)$. Note that because of the properties of functions h_{n-1} and Q this will happen if and only if $h_{n-1}(x_{n-2}) < Q(x_{n-2})$ or equivalently (on account of the properties of each G_i):

$$(29) \quad q_0 > \frac{\beta_{n-1}}{\mathrm{erf}(\frac{G_{n-2}(x_{n-2})}{a_{n-1}})} > \frac{\beta_{n-2}}{\mathrm{erf}(\frac{G_{n-3}(x_{n-3})}{a_{n-2}})} > \ldots > \frac{\beta_2}{\mathrm{erf}(\frac{x_1}{a_2})} > \beta_1 .$$

Remark 2 : For the particular case n = 2 we find the result obtained in [3,4] for the existence of two phases.

IV. RELATIONSHIP WITH THE PROBLEM WITH CONSTANT TEMPERATURE ON THE FIXED FACE

The temperature (9) on the fixed face x = 0 is given by:

$$(30) \quad u^* = \theta_n(0, t) = A_n(\omega_{n-1}) = u_n + \mathrm{erf}(\frac{\omega_{n-1}}{a_n}) \frac{q_0 a_n \sqrt{\pi}}{k_n}.$$

As $u^* > u_n$ we can consider the multi-phase Stefan problem which consists in finding the functions $S_{n-1}(t) < S_{n-2}(t) < \ldots < S_1(t)$ and the temperature $\Theta = \Theta(x,t)$, defined by:

$$\Theta(x,t) = \Theta_i(x,t) \quad \text{if} \quad S_i(t) < x < S_{i-1}(t), \quad t > 0, \quad i = 1, \ldots, n,$$

solutions of (2)-(5), (6bis), (7) and (8) where

$$(6\text{bis}) \quad \Theta_n(0,t) = u_{n+1}, \quad \text{with } u_{n+1} > u_n.$$

The solution of this problem , following the above method, is given by:

$$\Theta_i(x,t) = M_i + N_i \operatorname{erf}\Big(\frac{x}{2a_i\sqrt{t}}\Big), \qquad i = 1, \ldots, n,$$

(31)

$$S_i(t) = 2\sigma_i\sqrt{t}, \qquad \sigma_i > 0, \qquad i = 1, \ldots, n-1,$$

with

(32) $$M_i = \frac{u_{i+1}\operatorname{erf}(\frac{\sigma_{i-1}}{a_i}) - u_i \operatorname{erf}(\frac{\sigma_i}{a_i})}{\operatorname{erf}(\frac{\sigma_{i-1}}{a_i}) - \operatorname{erf}(\frac{\sigma_i}{a_i})}, \qquad N_i = \frac{u_i - u_{i+1}}{\operatorname{erf}(\frac{\sigma_{i-1}}{a_i}) - \operatorname{erf}(\frac{\sigma_i}{a_i})}, \qquad i=1,\ldots,n,$$

where we have inserted two dummy parameters $\sigma_0 = +\infty$ y $\sigma_n = 0$.

The parameters σ_i are the solutions of the system of equations

(33) $$\delta_i \sigma_i = \beta_{i+1} \frac{\eta(\sigma_i, a_{i+1})}{\phi(\sigma_i, \sigma_{i+1}, a_{i+1})} - \beta_i \frac{\eta(\sigma_i, a_i)}{\phi(\sigma_{i-1}, \sigma_i, a_i)}, \qquad i = 1, \ldots, n-1,$$

or equivalently:

$$\sigma_i = G_i(\sigma_1), \quad i = 2, \ldots, n-1,$$

(34)

$$h_{n-1}(\sigma_1) = K(\sigma_1),$$

where the functions G_i and h_i $(i = 1, \ldots, n-1)$ are given by (16), (20), (22), (23) and (25) respectively and the function K is defined by:

(35) $$K(x) = \beta_n \frac{\eta(G_{n-1}(x), a_n)}{\operatorname{erf}(\frac{G_{n-1}(x)}{a_n})} \qquad \text{with} \quad \beta_n = -\frac{k_n}{a_n\sqrt{\pi}}(u_n - u_{n+1}) > 0.$$

It satisfies $K(x_{n-2}) = +\infty$, $\quad K(+\infty) = 0$ and $K'(x) < 0,\ \forall x \in (x_{n-2}, +\infty)$.

Because of

(36) $$u_{n+1} = u_n + \operatorname{erf}(\frac{\omega_{n-1}}{a_n}) \frac{q_0 a_n\sqrt{\pi}}{k_n}$$

it results that $K(x) = Q(x)$ and from the uniqueness of the solution of (34) we deduce the equivalence of both problems, and then $M_i = A_i$, $N_i = B_i$, $\sigma_i = \omega_i$, $\forall i=1,\ldots,n$.

Furthermore, if we consider $u^* = u_{n+1}$ it results

$$q_0 = -\frac{N_n k_n}{a_n\sqrt{\pi}} = \frac{\beta_n}{\operatorname{erf}\frac{\sigma_{n-1}}{a_n}},$$

hence the inequality (29), obtained for the first problem, is transformed in

$$(37) \quad \frac{\beta_n}{\operatorname{erf}\frac{\sigma_{n-1}}{a_n}} > \frac{\beta_{n-1}}{\operatorname{erf}\frac{G_{n-2}(x_{n-2})}{a_{n-1}}} > \frac{\beta_{n-2}}{\operatorname{erf}\frac{G_{n-3}(x_{n-3})}{a_{n-2}}} > \dots > \frac{\beta_2}{\operatorname{erf}\frac{x_1}{a_2}} > \beta_1 \ .$$

for the second problem. Notice that this last inequality was obtained only with condition of temperature on the fixed face.

Remark 3 : The solution of problem (2)-(5), (6bis), (7) and (8) has been found before in [7] using a method similar to this.

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