# DETERMINATION OF ONE UNKNOWN THERMAL COEFFICIENT OF A SEMI-INFINITE POROUS MATERIAL THROUGH A DESUBLIMATION PROBLEM WITH COUPLED HEAT AND MOISTURE FLOWS 

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#### Abstract

An analytical model of freezing (desublimation) of moisture in a porous medium with an overspecified condition at the fixed face is considered in order to determinate one unknown thermal coefficient of a semiinfinite phase-change material. It can be considered as a free boundary problem in which coupled heat and moisture flows (Luikov type equations) with eight heat parameters. We obtain the explicit expression of the temperature of the two phases, the mass-transfer


Keywords and phrases: Stefan problem, unknown thermal coefficients, phase-change process, Luikov equations, desublimation, heat-mass flow.

This paper has been partially sponsored by the Projects PIP No. 5379 from CONICET - UA, (Argentina), ANPCYT PICT No. 03-11165 from Agencia (Argentina) and by the "Fondo de Ayuda a la Investigación" through the Project "Problemas de frontera libre para la ecuación del calor $y$ sus aplicaciones" from Universidad Austral, Rosario (Argentina).

Communicated by Chin-Hsiang Cheng
Received August 29, 2007

## 1. Nomenclature

$a_{i}=\frac{k_{i}}{\rho c_{i}} \quad$ thermal diffusivity of the phase- $i$
$a_{m} \quad$ moisture diffusivity
$c_{i} \quad$ specific heat capacity of the phase- $i$
$k_{i} \quad$ thermal conductivity of the phase- $i$
$\mathcal{K}_{0}=\frac{r u_{0}}{c_{2}\left(t_{0}-t_{v}\right)} \quad$ Kossovitch number
$\mathcal{L} u=\frac{a_{m}}{a_{2}} \quad$ Luikov number
$\mathcal{P} n=\frac{\delta\left(t_{0}-t_{v}\right)}{u_{0}} \quad$ Posnov number
$q_{0} \quad$ coefficient that characterizes the heat flux at $x=0$
$r$
$s(t)$
$t$
$T_{i}$
$t_{0}$
$t_{s}$
temperature at the fixed face $x=0$
$t_{v} \quad$ phase-change temperature $\left(t_{s}<t_{v}<t_{0}\right)$
$u$
mass-transfer potential
$u_{0}$
initial mass-transfer potential

## Greek symbols

$$
\begin{aligned}
& \beta_{1}=\frac{\mathcal{P} n}{\frac{1}{\mathcal{L} u}-1}, \quad \beta_{2}=\frac{\sqrt{\pi} q_{0}}{\sqrt{c_{2} k_{2} \rho}\left(t_{0}-t_{v}\right)}, \quad \beta_{3}=\frac{\sqrt{\pi} q_{0}}{\sqrt{c_{1} k_{1} \rho}\left(t_{v}-t_{s}\right)}, \\
& \beta_{4}=\sqrt{\frac{c_{1} k_{2}}{c_{2} k_{1}}}, \quad \beta_{5}=\frac{u_{0}}{t_{0}-t_{v}}\left(\frac{1}{\mathcal{L} u}-1\right), \quad \beta_{6}=\frac{c_{2}\left(t_{0}-t_{v}\right)}{u_{0}}, \\
& \beta_{7}=\frac{1}{\lambda \mathcal{K}_{0} \sqrt{\pi}}\left[\beta_{2} e^{\left.-\left(\beta_{4} \lambda\right)^{2}-F_{1}(\lambda)\right]-1 \text { with } \lambda=\frac{1}{\beta_{4}} \operatorname{erf}^{-1}\left(\beta_{3}\right),}\right. \\
& \beta_{8}=\frac{\pi q_{0}^{2}}{c_{1} k_{1}\left(t_{v}-t_{s}\right)^{2}}, \quad \beta_{9}=\left(\frac{t_{v}-t_{s}}{t_{0}-t_{v}}\right) \frac{\sqrt{c_{1} k_{1}}}{\sqrt{c_{2} k_{2}}}, \quad \beta_{10}=\frac{1}{\beta_{3} \sqrt{\mathcal{L} u}}, \\
& \beta_{11}=\frac{q_{0}}{r u_{0}} \sqrt{\frac{c_{1}}{\rho k_{1}}}, \quad \beta_{12}=\sqrt{\frac{k_{1}}{\rho a_{m} c_{1}}} \\
& \rho \text { mass density } \\
& \delta \text { thermal gradient coefficient } \\
& \lambda \text { constant which characterizes the evaporation front }
\end{aligned}
$$

## Subscripts

$i=1 \quad$ freezing region
$i=2$ region in which there are coupled heat and moisture flows

## 2. Introduction

Heat and mass transfer with phase change problems, taking place in a porous medium, such as evaporation, condensation, freezing, melting, sublimation and desublimation, has wide applications in separation processes, food technology, heat and mixture migration in solids and grounds, etc. Due to the non-linearity of the problem, solutions usually involve mathematical difficulties. Only a few exact solutions have been found for idealized cases (see [2-5], [7] and [18] for example). A large bibliography on free and moving boundary problems for heat-diffusion
equation was given in [20]. Mathematical formulation of the heat and mass transfer in capillary porous bodies has been established by Luikov [10, 11], and it was recently considered in [6], [8], [14], [15], [23] and [25]. Two different models were presented by Mikhailov [12] for solving the problem of evaporation of liquid moisture from a porous medium. For the problem of freezing (desublimation) of humid porous half-space, Mikhailov also presented an exact solution in [13]. In [16] an exact solution was presented for temperature and moisture distributions in a humid porous half-space with a heat flux condition on the fixed face $x=0$ of the type $\frac{q_{0}}{\sqrt{t}}$. Many experimental investigations have been conducted for determining the unknown physical coefficients of the materials (for example, [9], [22] and [24]). In this paper we consider the model developed in [13-16] with an overspecified condition on the fixed face. This allows us to consider some thermal coefficients as unknown and to calculate them, under certain specified restrictions upon data, following the idea of [19] for one phase and of [17] for two phases.

Let us consider the flow of heat and moisture through a porous halfspace during freezing. The position of phase change front at time $t$ is given by $x=s(t)$. It divides the porous body into two regions. In the freezing region, $0<x<s(t)$, there is no moisture movement and the temperature distribution is described by the heat equation:

$$
\begin{equation*}
\frac{\partial T_{1}}{\partial t}(x, t)=a_{1} \frac{\partial^{2} T_{1}}{\partial x^{2}}(x, t), \quad 0<x<s(t), t>0, \quad a_{1}=\frac{k_{1}}{\rho c_{1}} \tag{1}
\end{equation*}
$$

The region $s(t)<x<+\infty$ is humid capillary porous body in which there are coupled heat and moisture flows. The process is described by the wellknown Luikov's system [11] for the case $\varepsilon=0$ ( $\varepsilon$ is the phase conversion factor of liquid into vapor) given by

$$
\begin{align*}
& \frac{\partial T_{2}}{\partial t}(x, t)=a_{2} \frac{\partial^{2} T_{2}}{\partial x^{2}}(x, t), \quad x>s(t), t>0, \quad a_{2}=\frac{k_{2}}{\rho c_{2}}  \tag{2}\\
& \frac{\partial u}{\partial t}(x, t)=a_{m} \frac{\partial^{2} u}{\partial x^{2}}(x, t), \quad x>s(t), t>0 \tag{3}
\end{align*}
$$

The initial distributions of temperature and moisture are uniform

$$
\left\{\begin{array}{l}
T_{2}(x, 0)=T_{2}(+\infty, t)=t_{0}  \tag{4}\\
u(x, 0)=u(+\infty, t)=u_{0}
\end{array}\right.
$$

It is assumed that on the surface of the half-space, the temperature is constant

$$
\begin{equation*}
T_{1}(0, t)=t_{s}, \quad \text { where } t_{s}<t_{v} \tag{5}
\end{equation*}
$$

On the freezing front, there exists the continuity of the temperatures:

$$
\begin{equation*}
T_{1}(s(t), t)=T_{2}(s(t), t)=t_{v}, \quad t>0 \tag{6}
\end{equation*}
$$

where $t_{v}<t_{0}$. Heat and moisture balance at the freezing front yields

$$
\begin{align*}
& k_{1} \frac{\partial T_{1}}{\partial x}(s(t), t)-k_{2} \frac{\partial T_{2}}{\partial x}(s(t), t)=\rho r u(s(t), t) \frac{d s}{d t}(t), \quad t>0  \tag{7}\\
& \frac{\partial u}{\partial x}(s(t), t)+\delta \frac{\partial T_{2}}{\partial x}(s(t), t)=0, \quad t>0 \tag{8}
\end{align*}
$$

We also consider an overspecified condition on the fixed face $x=0$ [1], considering that the heat flux depends on the time, like in [18]

$$
\begin{equation*}
k_{1} \frac{\partial T_{1}}{\partial x}(0, t)=\frac{q_{0}}{\sqrt{t}} \tag{9}
\end{equation*}
$$

where $q_{0}>0$ is a coefficient which characterizes the heat flux at the fixed face $x=0$. The set of equations and conditions (1)-(9) is called problem $P$.

The goal of this paper is to find formulae for the determination of an unknown thermal coefficient chosen among $\rho$ (mass density), $a_{m}$ (moisture diffusivity), $c_{1}$ (specific heat of the frozen region), $c_{2}$ (specific heat of the humid region), $k_{1}$ (thermal conductivity of the frozen region), $k_{2}$ (thermal conductivity of the humid region), $\delta$ (thermal gradient coefficient), $r$ (latent heat) together with the free boundary $s(t)$, the temperatures $T_{1}$ and $T_{2}$, and the moisture $u$.

## 3. Unknown Thermal Coefficients Through a Free Boundary Problem

Following [16], for the general case $\mathcal{L} u=\frac{a_{m}}{a_{2}}=\frac{\rho a_{m} c_{2}}{k_{2}} \neq 1$ we have

$$
\begin{align*}
& T_{1}(x, t)=t_{v}-\frac{\sqrt{\pi a_{1}} q_{0}}{k_{1}}\left[\operatorname{erf}\left(\frac{\lambda}{\sqrt{a_{12}}}\right)-\operatorname{erf}\left(\frac{x}{2 \sqrt{a_{1} t}}\right)\right], \\
& 0<x<s(t), t>0  \tag{10}\\
& T_{2}(x, t)=t_{v}+\frac{t_{0}-t_{v}}{1-\operatorname{erf}(\lambda)}\left[\operatorname{erf}\left(\frac{x}{2 \sqrt{a_{2} t}}\right)-\operatorname{erf}(\lambda)\right], \quad x>s(t), t>0 \tag{11}
\end{align*}
$$

$$
u(x, t)=u_{0}-u_{0} \beta_{1}\left\{\operatorname{erf}\left(\frac{x}{2 \sqrt{a_{2} t}}\right)-\frac{\exp \left(\left(\frac{1}{\mathcal{L} u}-1\right) \lambda^{2}\right)\left(1-\operatorname{erf}\left(\frac{x}{2 \sqrt{a_{m} t}}\right)\right)}{\sqrt{\mathcal{L} u}}\right\}
$$

$$
\begin{equation*}
x>s(t), t>0 \tag{12}
\end{equation*}
$$

$$
\begin{equation*}
s(t)=2 \lambda \sqrt{a_{2} t} \tag{13}
\end{equation*}
$$

with

$$
\begin{equation*}
\beta_{1}=\frac{\mathcal{P} n}{\frac{1}{\mathcal{L} u}-1} \tag{14}
\end{equation*}
$$

where $\lambda$ (the parameter that characterizes the free boundary) and the unknown thermal coefficient must satisfy the following system of transcendental equations:

$$
\begin{align*}
& \beta_{2} \exp \left(-\left(\beta_{4} \lambda\right)^{2}\right)-F_{1}(\lambda)=\mathcal{K}_{0} \lambda \sqrt{\pi}\left\{1-\beta_{1}\left(1-\frac{Q\left(\frac{\lambda}{\sqrt{\mathcal{L} u}}\right)}{Q(\lambda)}\right)\right\}  \tag{15}\\
& \operatorname{erf}\left(\beta_{4} \lambda\right)=\frac{1}{\beta_{3}} \tag{16}
\end{align*}
$$

with

$$
\begin{equation*}
\beta_{2}=\frac{\sqrt{\pi} q_{0}}{\sqrt{c_{2} k_{2} \rho}\left(t_{0}-t_{v}\right)}, \quad \beta_{3}=\frac{\sqrt{\pi} q_{0}}{\sqrt{c_{1} k_{1} \rho}\left(t_{v}-t_{s}\right)}, \quad \beta_{4}=\sqrt{\frac{c_{1} k_{2}}{c_{2} k_{1}}}, \tag{17}
\end{equation*}
$$

where real functions $F_{1}$ and $Q$ are defined by

$$
\begin{equation*}
F_{1}(x)=\frac{\exp \left(-x^{2}\right)}{(1-\operatorname{erf}(x))}, \quad Q(x)=\sqrt{\pi} x \exp \left(x^{2}\right)(1-\operatorname{erf}(x)) \tag{18}
\end{equation*}
$$

with properties

$$
\begin{align*}
& F_{1}(0)=1, \quad F_{1}(+\infty)=+\infty, \quad F_{1}^{\prime}(x)>0, \quad \forall x>0,  \tag{19}\\
& Q(0)=0, \quad Q(+\infty)=1, \quad Q^{\prime}(x)>0, \quad \forall x>0 . \tag{20}
\end{align*}
$$

Now, we give necessary and sufficient conditions in order to obtain solution to above system (15)-(16) and we also give formulae for the parameter $\lambda$ of the phase-change interface and the unknown thermal coefficient in the following eight cases:

Case 1. Determination of the unknown coefficient $\delta$ (c.f. Theorem 2).
Case 2. Determination of the unknown coefficient $r$ (c.f. Theorem 3).
Case 3. Determination of the unknown coefficient $a_{m}$ (c.f. Theorem 4).
Case 4. Determination of the unknown coefficient $\rho$ (c.f. Theorem 5).
Case 5. Determination of the unknown coefficient $k_{1}$ (c.f. Theorem 6).
Case 6. Determination of the unknown coefficient $k_{2}$ (c.f. Theorem 7).
Case 7. Determination of the unknown coefficient $c_{1}$ (c.f. Theorem 8).
Case 8. Determination of the unknown coefficient $c_{2}$ (c.f. Theorem 9).
First, we have the following preliminary lemma:
Lemma 1. We have

$$
E(x)=\frac{m^{2}-1}{1-\frac{Q(m x)}{Q(x)}}<0, \quad \forall x>0, \forall m>0, m \neq 1
$$

Proof. By using the properties (20) of the function $Q$, if we consider $m>1$, we have that $m^{2}-1>0$ and $\frac{Q(m x)}{Q(x)}>1$, then we obtain $E(x)<0$. On the other hand, if $0<m<1$, it follows that $m^{2}-1<0$ and $\frac{Q(m x)}{Q(x)}<1$. Then, we also obtain $E(x)<0$.

Theorem 2 (Determination of the unknown coefficient $\delta$ ). If

$$
\begin{equation*}
\max \left(\frac{1}{\beta_{3} \operatorname{erf}\left(\beta_{4} \lambda_{1}\right)} ; \frac{1}{\beta_{2}}\right)<1, \tag{21}
\end{equation*}
$$

with $\beta_{2}$ and $\beta_{3}$ are defined in (17), where $\lambda_{1}>0$ is the unique solution to equation

$$
\begin{equation*}
g_{1}(x)=g_{2}(x), \quad x>0 \tag{22}
\end{equation*}
$$

with

$$
\begin{equation*}
g_{1}(x)=F_{1}(x)+\mathcal{K}_{0} \sqrt{\pi} x, \quad g_{2}(x)=\beta_{2} e^{-\left(\beta_{4} x\right)^{2}} \tag{23}
\end{equation*}
$$

and $\beta_{2}$ and $\beta_{4}$ are defined in (17), then there exists a unique solution to problem $P$ which is given by (10)-(13), where $\lambda$ and $\delta$ are given by:

$$
\begin{align*}
& \lambda=\frac{1}{\beta_{4}} \operatorname{erf}^{-1}\left(\frac{1}{\beta_{3}}\right)>0,  \tag{24}\\
& \delta=\frac{\beta_{5}}{1-\frac{Q\left(\frac{\lambda}{\sqrt{\mathcal{L} u}}\right)}{Q(\lambda)}}\left\{1-\frac{1}{\lambda \mathcal{K}_{0} \sqrt{\pi}}\left(\beta_{2} e^{-\left(\beta_{4} \lambda\right)^{2}}-F_{1}(\lambda)\right)\right\}>0 \tag{25}
\end{align*}
$$

with

$$
\begin{equation*}
\beta_{5}=\frac{u_{0}}{t_{0}-t_{v}}\left(\frac{1}{\mathcal{L} u}-1\right) . \tag{26}
\end{equation*}
$$

Proof. Considering (21), it follows that

$$
\begin{equation*}
\frac{1}{\beta_{3}}<1 \tag{27}
\end{equation*}
$$

and it easily follows that there exists a unique $\lambda>0$ solution to (16) given by (24). Then, we replace $\lambda$ in (15) and after some calculations we have (25). So we have to show that $\delta>0$. First, using Lemma 1, we see that if we impose that

$$
1-\frac{1}{\lambda \mathcal{K}_{0} \sqrt{\pi}}\left(\beta_{2} e^{-\left(\beta_{4} \lambda\right)^{2}}-F_{1}(\lambda)\right)<0
$$

we have $\delta>0$. That is to say $F_{1}(\lambda)+\lambda \mathcal{K}_{0} \sqrt{\pi}<\beta_{2} e^{-\left(\beta_{4} \lambda\right)^{2}}$. According to (23) this can be written as

$$
\begin{equation*}
g_{1}(\lambda)<g_{2}(\lambda) \tag{28}
\end{equation*}
$$

The functions $g_{1}$ and $g_{2}$ have the following properties:

$$
\begin{array}{ll}
g_{1}\left(0^{+}\right)=1, & g_{1}(+\infty)=+\infty, \\
g_{1}^{\prime}(x)>0, & \forall x>0 \\
g_{2}\left(0^{+}\right)=\beta_{2}, & g_{2}(+\infty)=0,
\end{array} g_{2}^{\prime}(x)<0, \quad \forall x>0 . ~ \$
$$

We can conclude that if

$$
\begin{equation*}
\beta_{2}>1 \tag{29}
\end{equation*}
$$

then there exists a unique $\lambda_{1}>0$ such that $g_{1}\left(\lambda_{1}\right)=g_{2}\left(\lambda_{1}\right)$. Then (28) holds when

$$
\begin{equation*}
0<\lambda<\lambda_{1} \tag{30}
\end{equation*}
$$

To finish the proof, we see that the needed hypotheses (27), (29) and (30) could be written in the following way: erf is an increasing function, so (30) is equivalent to $\operatorname{erf}\left(\beta_{4} \lambda\right)<\operatorname{erf}\left(\beta_{4} \lambda_{1}\right)$. So (27) and (30) could be resumed as

$$
\begin{equation*}
\frac{1}{\beta_{3}}<\operatorname{erf}\left(\beta_{4} \lambda_{1}\right) \tag{31}
\end{equation*}
$$

Then, thanks to (29) and (31), we have

$$
1>\frac{1}{\beta_{3} \operatorname{erf}\left(\beta_{4} \lambda_{1}\right)}, \quad 1>\frac{1}{\beta_{2}},
$$

that is to say, (21) holds.

Theorem 3 (Determination of the unknown coefficient r). If

$$
\begin{equation*}
\frac{1}{\beta_{3} \operatorname{erf}\left(\beta_{4} \lambda_{2}\right)}<1, \tag{32}
\end{equation*}
$$

where $\lambda_{2}>0$ is the unique solution to equation

$$
\begin{equation*}
F_{1}(x)=g_{2}(x), \quad x>0 \tag{33}
\end{equation*}
$$

with $F_{1}$ and $g_{2}$ given by (18) and (23), respectively, and $\beta_{3}$ and $\beta_{4}$ are defined in (17), then there exists a unique solution to problem $P$ which is given by (10)-(13), where $\lambda$ is given by (24) and $r$ is given by:

$$
\begin{equation*}
r=\frac{\beta_{6}}{\lambda \sqrt{\pi}} \frac{\beta_{2} \exp \left(-\left(\beta_{4} \lambda\right)^{2}\right)-F_{1}(\lambda)}{1-\beta_{1}\left(1-\frac{Q\left(\frac{\lambda}{\sqrt{\mathcal{L} u}}\right)}{Q(\lambda)}\right)} \tag{34}
\end{equation*}
$$

with

$$
\begin{equation*}
\beta_{6}=\frac{c_{2}\left(t_{0}-t_{v}\right)}{u_{0}} . \tag{35}
\end{equation*}
$$

Proof. It easily follows in the same way as Theorem 1.
Theorem 4 (Determination of the unknown coefficient $a_{m}$ ). If

$$
\begin{equation*}
\beta_{3}>1 \tag{36}
\end{equation*}
$$

and

$$
\begin{equation*}
0<\frac{1}{\lambda \mathcal{K}_{0} \sqrt{\pi}}\left[\beta_{2} e^{-\left(\beta_{4} \lambda\right)^{2}}-F_{1}(\lambda)\right]-1<\mathcal{P}_{n} \tag{37}
\end{equation*}
$$

and $\beta_{3}$ and $\beta_{4}$ are defined in (17), then there exists a solution to problem $P$ which is given by (10)-(13), where $\lambda$ and $a_{m}$ are given by:

$$
\begin{equation*}
\lambda=\frac{1}{\beta_{4}} \operatorname{erf}^{-1}\left(\frac{1}{\beta_{3}}\right) \text { and } a_{m}=\frac{k_{2}}{\rho c_{2}} \frac{1}{\xi^{2}}, \tag{38}
\end{equation*}
$$

where $\xi$ is a solution to equation

$$
\begin{equation*}
\mathcal{P} n g_{3}(x)=\beta_{7}, \quad x>0 \tag{39}
\end{equation*}
$$

with

$$
\begin{equation*}
g_{3}(x)=\left(1-\frac{Q(\lambda x)}{Q(\lambda)}\right) \frac{1}{1-x^{2}}, \quad \beta_{7}=\frac{1}{\lambda \mathcal{K}_{0} \sqrt{\pi}}\left[\beta_{2} e^{-\left(\beta_{4} \lambda\right)^{2}}-F_{1}(\lambda)\right]-1 \tag{40}
\end{equation*}
$$

Proof. In the same way as we did in Theorem 1 , we get $\lambda$ given by (24) by using the hypothesis (36). Then, replacing $\lambda$ into (15) and denoting $x=\frac{1}{\sqrt{\mathcal{L} u}}=\sqrt{\frac{k_{2}}{\rho a_{m} c_{2}}} \in(0,1) \cup(1,+\infty)$, we have the following equation

$$
\mathcal{P} n g_{3}(x)=\frac{1}{\lambda \mathcal{K}_{0} \sqrt{\pi}}\left[\beta_{2} e^{-\left(\beta_{4} \lambda\right)^{2}}-F_{1}(\lambda)\right]-1
$$

that is (39), where $\beta_{7}$ is a constant with respect to $x$. The function $g_{3}$ is a differentiable and continuous function into $\mathbb{R}^{+}$which has the following properties:

$$
\begin{aligned}
& g_{3}\left(0^{+}\right)=1, \quad g_{3}(+\infty)=0, \quad g_{3}\left(1^{-}\right)=g_{3}\left(1^{+}\right)=\frac{\lambda Q^{\prime}(\lambda)}{2 Q(\lambda)}, \\
& g_{3}^{\prime}\left(1^{-}\right)=g_{3}^{\prime}\left(1^{+}\right)=\frac{\lambda}{4 Q(\lambda)}\left[\lambda Q^{\prime \prime}(\lambda)-Q^{\prime}(\lambda)\right] .
\end{aligned}
$$

So we can say that if $0<\beta_{7}<\mathcal{P} n$, then we have at least one solution of equation (39) and then the unknown coefficient $a_{m}$ is given by (38).

Theorem 5 (Determination of the unknown coefficient $\rho$ ). For any data, there exists at least one solution to problem $P$ which is given by (10)-(13), the coefficient $\rho$ is given by

$$
\begin{equation*}
\rho=\beta_{8} \operatorname{erf}^{2}\left(\beta_{4} \lambda\right) \tag{41}
\end{equation*}
$$

with $\beta_{4}$ defined in (17) and

$$
\beta_{8}=\frac{\pi q_{0}^{2}}{c_{1} k_{1}\left(t_{v}-t_{s}\right)^{2}}
$$

and the parameter $\lambda \in\left(0, \lambda_{5}\right)$ is a solution to equation

$$
\begin{equation*}
g_{4}(x)=g_{5}(x), \quad x>0 \tag{42}
\end{equation*}
$$

with

$$
\begin{gather*}
g_{4}(x)=\mathcal{P} n \frac{1-\frac{Q\left(\frac{\beta_{10} x}{\operatorname{erf}\left(\beta_{4} x\right)}\right)}{Q(x)}}{1-\frac{\beta_{10}^{2}}{\operatorname{erf}^{2}\left(\beta_{4} x\right)}} x  \tag{43}\\
g_{5}(x)=\frac{1}{\mathcal{K}_{0} \sqrt{\pi}}\left[\beta_{9} F_{2}\left(\beta_{4} x\right)-F_{1}(x)\right]-x \tag{44}
\end{gather*}
$$

with

$$
\begin{align*}
& F_{2}(x)=\frac{\exp \left(-x^{2}\right)}{\operatorname{erf}(x)}  \tag{45}\\
& \beta_{9}=\frac{t_{v}-t_{s}}{t_{0}-t_{v}} \frac{\sqrt{c_{1} k_{1}}}{\sqrt{c_{2} k_{2}}}, \quad \beta_{10}=\frac{1}{\beta_{3} \sqrt{\mathcal{L} u}} \tag{46}
\end{align*}
$$

and $\lambda_{5}>0$ is the unique solution to equation $g_{5}(x)=0, x>0$.
Proof. From (16) we have $\rho$ as a function of $\lambda$, given by (41). Replacing $\rho$ into (15) we have

$$
\beta_{9} F_{2}\left(\beta_{4} \lambda\right)-F_{1}(\lambda)=\lambda \mathcal{K}_{0} \sqrt{\pi}\left\{1-\beta_{1}\left(1-\frac{Q\left(\frac{\lambda}{\sqrt{\mathcal{L} u}}\right)}{Q(\lambda)}\right)\right\} .
$$

Taking into account $\frac{1}{\sqrt{\mathcal{L} u}}=\sqrt{\frac{k_{2}}{\rho a_{m} c_{2}}}=\frac{\beta_{10}}{\operatorname{erf}\left(\beta_{4} \lambda\right)}$, we have

$$
\begin{array}{r}
\frac{1}{\mathcal{K}_{0} \sqrt{\pi}}\left[\beta_{9} F_{2}\left(\beta_{4} \lambda\right)-F_{1}(\lambda)\right] \\
=\lambda\left\{1+\mathcal{P} n \frac{1-\frac{Q\left(\frac{\beta_{10} \lambda}{\operatorname{erf}\left(\beta_{4} \lambda\right)}\right)}{Q(\lambda)}}{1-\frac{\beta_{10}^{2}}{\operatorname{erf}^{2}\left(\beta_{4} \lambda\right)}}\right\}
\end{array}
$$

That is to say,

$$
\begin{equation*}
g_{4}(\lambda)=g_{5}(\lambda), \quad \lambda>0 \tag{47}
\end{equation*}
$$

The function $g_{5}$ is a strictly decreasing function with $g_{5}\left(0^{+}\right)=+\infty$ and $g_{5}(+\infty)=-\infty$, then $g_{5}$ has a unique positive zero $\lambda_{5}$. The function $g_{4}=g_{4}(x)$ is a continuous differentiable function which starts at zero with value 0 , and when $x$ tends to $+\infty$, it goes to $+\infty$ or to a non negative value. So we have that both functions will meet each other in at least one $\lambda \in\left(0, \lambda_{5}\right)$, and then we find a solution for equation (47) or equivalently equation (42).

Theorem 6 (Determination of the unknown coefficient $k_{1}$ ). If (29) holds and $\lambda_{6}>0$ is the solution to equation

$$
\begin{equation*}
g_{6}(x)=1, \quad x>0 \tag{48}
\end{equation*}
$$

with

$$
\begin{equation*}
g_{6}(x)=\frac{1}{\beta_{2}}\left\{F_{1}(x)+\mathcal{K}_{0} \sqrt{\pi} x\left(1-\beta_{1}\left(1-\frac{Q\left(\frac{x}{\sqrt{\mathcal{L} u}}\right)}{Q(x)}\right)\right)\right\} \tag{49}
\end{equation*}
$$

where $\beta_{1}$ is defined in (14) and $\beta_{2}$ is defined in (17), then there exists a unique solution to problem $P$ which is given by (10)-(13), the thermal coefficient $k_{1}$ is given by

$$
\begin{equation*}
k_{1}=\frac{c_{1} k_{2}}{c_{2}} \frac{\lambda^{2}}{\log \left(\frac{1}{g_{6}(\lambda)}\right)} \tag{50}
\end{equation*}
$$

where $\lambda \in\left(0, \lambda_{6}\right)$ is the unique solution to equation

$$
\begin{equation*}
g_{8}\left(g_{7}(x)\right)=\frac{\beta_{4}}{\beta_{3}} x, \quad 0<x<\lambda_{6} \tag{51}
\end{equation*}
$$

with

$$
\begin{align*}
& g_{7}(x)=\sqrt{\log \left(\frac{1}{g_{6}(x)}\right)}, \quad 0<x<\lambda_{6}  \tag{52}\\
& g_{8}(x)=x \operatorname{erf}(x) \tag{53}
\end{align*}
$$

Proof. From (15) we have

$$
\begin{equation*}
\exp \left(-\left(\beta_{4} \lambda\right)^{2}\right)=g_{6}(\lambda) \tag{54}
\end{equation*}
$$

and easily follows (50). Note that $k_{1}>0$ if and only if $0<g_{6}(\lambda)<1$. The function $g_{6}$ has the following properties:

$$
\begin{equation*}
g_{6}\left(0^{+}\right)=\frac{1}{\beta_{2}}, \quad g_{6}(+\infty)=+\infty, \quad g_{6}^{\prime}(x)>0, \quad \forall x>0 \tag{55}
\end{equation*}
$$

Therefore, if we first consider (29) and then we get $\lambda_{6}>0$ as the solution to (48) it naturally follows that $k_{1}>0$ if we take $\lambda \in\left(0, \lambda_{6}\right)$. Then, we replace $k_{1}$ in (16) and after some computations we have that $\lambda$ must verify the equation $\sqrt{\log \left(\frac{1}{g_{6}(\lambda)}\right)} \operatorname{erf}\left(\sqrt{\log \left(\frac{1}{g_{6}(\lambda)}\right)}\right)=\frac{\beta_{4}}{\beta_{3}} \lambda$, that is to say, equation (51), which has a unique solution by considering the properties of functions $g_{6}, g_{7}$ and $g_{8}$.

Theorem 7 (Determination of the unknown coefficient $k_{2}$ ). If (36) holds, and

$$
\begin{equation*}
\beta_{11} \frac{\exp \left(-\gamma_{0}^{2}\right)}{\gamma_{0}}-\frac{1}{\mathcal{K}_{0}}-\mathcal{P} n\left(1-Q\left(\beta_{12} \gamma_{0}\right)\right)>1 \tag{56}
\end{equation*}
$$

where

$$
\begin{equation*}
\beta_{11}=\frac{q_{0}}{r u_{0}} \sqrt{\frac{c_{1}}{\rho k_{1}}} \text { and } \beta_{12}=\sqrt{\frac{k_{1}}{\rho a_{m} c_{1}}} \tag{57}
\end{equation*}
$$

with

$$
\begin{equation*}
\gamma_{0}=\operatorname{erf}^{-1}\left(\frac{1}{\beta_{3}}\right)>0 \tag{58}
\end{equation*}
$$

then there exists a solution to problem $P$ which is given by (10)-(13), the thermal coefficient $k_{2}$ is given by

$$
\begin{equation*}
k_{2}=\frac{k_{1} c_{2}}{c_{1}} \frac{\gamma_{0}^{2}}{\lambda^{2}} \tag{59}
\end{equation*}
$$

and the parameter $\lambda \in\left(\lambda_{5},+\infty\right)$ is a solution to equation

$$
\begin{equation*}
g_{9}(x)=g_{10}(x), \quad x>0 \tag{60}
\end{equation*}
$$

with

$$
\begin{align*}
& g_{9}(x)=1-\frac{\mathcal{P} n x^{2}}{\beta_{12}^{2} \gamma_{0}^{2}-x^{2}}\left(1-\frac{Q\left(\beta_{12} \gamma_{0}\right)}{Q(x)}\right), \\
& g_{10}(x)=\beta_{11} \frac{\exp \left(-\gamma_{0}^{2}\right)}{\gamma_{0}}-\frac{1}{\mathcal{K}_{0} Q(x)}, \tag{61}
\end{align*}
$$

where $\lambda_{10}$ is the unique positive zero of $g_{10}(x)$.
Proof. Considering (36), from (16) easily follows (59). Replacing it into (15) we get

$$
\beta_{11} \frac{\exp \left(-\gamma_{0}^{2}\right)}{\gamma_{0}}-\frac{1}{\mathcal{K}_{0} Q(x)}=1-\frac{\mathcal{P} n x^{2}}{\beta_{12}^{2} \gamma_{0}^{2}-x^{2}}\left(1-\frac{Q\left(\beta_{12} \gamma_{0}\right)}{Q(x)}\right) .
$$

So we have for $\lambda$ equation (60). The functions $g_{9}$ and $g_{10}$ have the following properties:

$$
\begin{align*}
& g_{9}\left(0^{+}\right)=1, \quad g_{9}(+\infty)=1+\mathcal{P n}\left(1-Q\left(\beta_{12} \gamma_{0}\right)\right)>1  \tag{62}\\
& g_{10}\left(0^{+}\right)=-\infty, \quad g_{10}(+\infty)=\beta_{11} \frac{\exp \left(-\gamma_{0}^{2}\right)}{\gamma_{0}}-\frac{1}{\mathcal{K}_{0}}, g_{10}^{\prime}(x)>0, \quad \forall x>0 \tag{63}
\end{align*}
$$

Then, if we consider $\beta_{11} \frac{\exp \left(-\gamma_{0}^{2}\right)}{\gamma_{0}}-\frac{1}{\mathcal{K}_{0}}>1+\mathcal{P} n\left(1-Q\left(\beta_{12} \gamma_{0}\right)\right)$, we know that there exists at least one $\lambda>0$ solution of equation (60). It is easy to see that $\lambda>\lambda_{10}$, where $\lambda_{10}$ is the unique solution of

$$
\begin{equation*}
g_{10}(x)=0, \quad x>0 \tag{64}
\end{equation*}
$$

Theorem 8 (Determination of the unknown coefficient $c_{1}$ ). If

$$
\begin{equation*}
\max \left\{\frac{1}{\beta_{2}} ; \frac{1}{2 \beta_{3} \beta_{4} \lambda_{6}}\right\}<1 \tag{65}
\end{equation*}
$$

$\beta_{2}, \beta_{3}$ and $\beta_{4}$ are defined in (17) and $\lambda_{6}$ is the solution of equation (48), then there exists a unique solution to problem $P$ which is given by (10)-(13), the thermal coefficient $c_{1}$ is given by

$$
\begin{equation*}
c_{1}=\frac{k_{1} c_{2}}{k_{2}} \frac{\log \left(\frac{1}{g_{6}(\lambda)}\right)}{\lambda^{2}} \tag{66}
\end{equation*}
$$

where $\lambda \in\left(0, \lambda_{6}\right)$ is the unique solution to equation

$$
\begin{equation*}
g_{11}\left(g_{7}(x)\right)=\frac{1}{\beta_{3} \beta_{4} \sqrt{\pi} x}, \quad 0<x<\lambda_{6} \tag{67}
\end{equation*}
$$

with $g_{7}$ defined in (52), and $g_{11}$ is defined by

$$
\begin{equation*}
g_{11}(x)=\frac{\operatorname{erf}(x)}{x} \tag{68}
\end{equation*}
$$

Proof. As in Theorem 6, from (15) we have (54) and easily follows (66). Note that $c_{1}>0$ if and only if $0<g_{6}(\lambda)<1$. Therefore, if we first consider (65), condition (29) holds and then we get $\lambda_{6}>0$ as the solution to equation (48); it naturally follows that $c_{1}>0$ if we take $\lambda \in\left(0, \lambda_{6}\right)$. Replacing (66) into (16) we have

$$
\frac{\operatorname{erf}\left(\sqrt{\log \left(\frac{1}{g_{6}(\lambda)}\right)}\right)}{\sqrt{\log \left(\frac{1}{g_{6}(\lambda)}\right)}}=\frac{1}{\beta_{3} \beta_{4} \sqrt{\pi} \lambda}
$$

that is, $\lambda$ must verify equation (67). Considering (29) and $\lambda_{6}$ as before, the function $g_{7}$, defined over the domain $\left[0, \lambda_{6}\right)$, has the following properties:

$$
\begin{equation*}
g_{7}\left(0^{+}\right)=\sqrt{\log \left(\beta_{2}\right)}, \quad g_{7}\left(\lambda_{6}^{-}\right)=0, \quad g_{7}^{\prime}(x)<0, \quad \forall x>0 \tag{69}
\end{equation*}
$$

The function $g_{11}$ has the following properties:

$$
\begin{equation*}
g_{11}\left(0^{+}\right)=\frac{2}{\sqrt{\pi}}, \quad g_{11}(+\infty)=0, \quad g_{11}^{\prime}(x)<0, \quad \forall x>0 \tag{70}
\end{equation*}
$$

Furthermore, the function $g_{11} \circ g_{7}$ is an increasing function. Then we can assure that if

$$
\begin{equation*}
g_{11}\left(g_{7}\left(\lambda_{6}\right)\right)=\frac{2}{\sqrt{\pi}}>\frac{1}{\beta_{3} \beta_{4} \sqrt{\pi} \lambda_{6}} \tag{71}
\end{equation*}
$$

which is included in hypothesis (65), then there exists a unique $\lambda \in\left(0, \lambda_{6}\right)$ such that (67) holds.

Theorem 9 (Determination of the unknown coefficient $c_{2}$ ). If

$$
\begin{equation*}
\max \left\{\frac{1}{\beta_{3}} ; \frac{\gamma_{0} \exp \left(\gamma_{0}^{2}\right)}{\beta_{11}}\right\}<1, \tag{72}
\end{equation*}
$$

where $\beta_{3}$ is defined in (17), $\beta_{11}$ is defined in (57) and $\gamma_{0}$ is defined in (58), then there exists a solution to problem $P$ which is given by (10)-(13), the thermal coefficient $c_{2}$ is given by

$$
\begin{equation*}
c_{2}=\frac{c_{1} k_{2}}{k_{1}} \frac{\lambda^{2}}{\gamma_{0}^{2}} \tag{73}
\end{equation*}
$$

and the parameter $\lambda \in\left(0, \lambda_{12}\right)$ is a solution to equation

$$
\begin{equation*}
g_{9}(x)=g_{12}(x), \quad x>0 \tag{74}
\end{equation*}
$$

with

$$
\begin{equation*}
g_{12}(x)=\beta_{11} \frac{\exp \left(-\gamma_{0}^{2}\right)}{\gamma_{0}}-\frac{\beta_{4}^{2} \mathcal{K}_{0}}{\gamma_{0}^{2} \sqrt{\pi}} x F_{1}(x) \tag{75}
\end{equation*}
$$

$g_{9}$ was defined in (61) and $\lambda_{12}>0$ is the unique solution to equation

$$
\begin{equation*}
g_{12}(x)=0, \quad x>0 \tag{76}
\end{equation*}
$$

Proof. Considering (72), from (16) we have $c_{2}$ as function of $\lambda$ given by (73). Replacing it into (15), after some computations we have that $\lambda$ must verify equation (74). The function $g_{12}$ has the following properties:

$$
\begin{equation*}
g_{12}\left(0^{+}\right)=\beta_{11} \frac{\exp \left(-\gamma_{0}^{2}\right)}{\gamma_{0}}, \quad g_{12}(+\infty)=-\infty, \quad g_{12}^{\prime}(x)<0 \tag{77}
\end{equation*}
$$

Then, taking into account (62), we have that if $\beta_{11} \frac{\exp \left(-\gamma_{0}^{2}\right)}{\gamma_{0}}>1$ (which is verified thanks to (72)), there exists at least one $\lambda \in\left(0, \lambda_{12}\right)$ that verifies equation (74), where $\lambda_{7}>0$ is the unique solution to equation (76).

## 4. Conclusions

We considered an analytical model of freezing (desublimation) of moisture in a porous medium with an overspecified condition at the fixed face in order to determinate one unknown thermal coefficient of a semiinfinite phase-change material. This model has Luikov type equations with eight heat parameters, and it can be considered as a free boundary problem in which coupled heat and moisture flows. We obtained the explicit expression of the temperature of the two phases $T_{1}$ and $T_{2}$, the mass-transfer potential in the humid region $u$ and the phase-change interface $s(t)$, and we also gave formulae for the unknown thermal coefficients chosen among $\rho$ (mass density), $a_{m}$ (moisture diffusivity), $c_{1}$ (specific heat of the frozen region), $c_{2}$ (specific heat of the humid region), $k_{1}$ (thermal conductivity of the frozen region), $k_{2}$ (thermal conductivity of the humid region), $\delta$ (thermal gradient coefficient), $r$ (latent heat), together with the necessary and sufficient conditions for the existence of such a solution.

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